symmetry

Article

Estimates of Eigenvalues of a Semiperiodic Dirichlet Problem for a Class of Degenerate Elliptic Equations

Mussakan Muratbekov *,† and Sabit Igissinov *,†

Department of Mathematics, M.Kh. Dulaty Taraz Regional University, Taraz 080000, Kazakhstan
* Correspondence: musahan_m@mail.ru (M.M.); igisinovsabit@mail.ru (S.I.); Tel.: +7-705-7892552 (M.M.)
† These authors contributed equally to this work.

Abstract: In this paper, we consider a class of degenerate elliptic equations with arbitrary power degeneration. The issues about the existence, uniqueness, and smoothness of solutions of the semiperiodic Dirichlet problem for a class of degenerate elliptic equations with arbitrary power degeneration are studied. The two-sided estimates for singular numbers (s-numbers) are obtained. Note that estimates of singular numbers (s-numbers) show the rate of approximation of the found solutions by finite-dimensional subspaces. Here, we also obtain estimates for the eigenvalues. We note that, in this paper, apparently, two-sided estimates of singular numbers (s-numbers) for degenerate elliptic operators are obtained for the first time. At the end of the paper, a symmetric operator is considered, i.e., a self-adjoint case.

Keywords: degenarate elliptic operator; boundary value problem; singular numbers; power degeneration; solution; uniqueness

check for updates

Citation: Muratbekov, M.; Igissinov, S. Estimates of Eigenvalues of a Semiperiodic Dirichlet Problem for a Class of Degenerate Elliptic Equations. *Symmetry* **2022**, *14*, 692. https://doi.org/10.3390/sym14040692

Academic Editor: Eduard Marusic-Paloka

Received: 22 February 2022
Accepted: 17 March 2022
Published: 28 March 2022

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

1. Introduction

Let $\Omega = \{(x, y) : -\pi < x < \pi, 0 < y < 1\}$. Consider the following problem

$$Lu = -k(y)u_{xx} - u_{yy} + a(y)u_x + c(y)u = f(x, y) \in L_2(\Omega), \tag{1}$$

$$u(-\pi, y) = u(\pi, y), u_x(-\pi, y) = u_x(\pi, y), \tag{2}$$

$$u(x, 0) = u(x, 1) = 0, \tag{3}$$

where $a(y)$ and $c(y)$ are piecewise continuous functions in $[0, 1]$, $k(y) > 0$ as $y \in (0, 1]$ and $k(0) = 0$. Let $C^{\infty}_{0,\pi}(\bar{\Omega})$ be a class of infinitely differentiable finite functions in $\bar{\Omega}$ and satisfying the conditions (2) and (3).

Closure of the operator L by the norm of $L_2(\Omega)$ we also denote by L.

In the study of the smoothness and approximation properties of solutions of boundary value problems for some nonlinear equations we encounter questions of the spectral properties of linear degenerate elliptic equations. In contrast to elliptic operators, spectral questions for degenerate elliptic operators are poorly understood. Known results on this topic or those close to it in content are contained in the papers of M. Smirnov [1], M. Keldysh [2], T. Kalmenov, M. Otelbaev [3], A.A. Nakhushev [4], G. Huang [5], A. Sbai, Y. El hadfi [6], and others.

As is known, when studying the spectral properties of boundary value problems for degenerate elliptic equations, a completely different situation arises compared to studying the spectral properties of boundary value problems for elliptic equations. In this case, the main difficulties are that the equation changes type and the solutions do not retain their smoothness at degeneracy points. Therefore, in this case, various difficulties arise related to the behavior of functions from the domain of the differential operator, and these difficulties, in turn, affect the spectral characteristics of boundary value problems for degenerate elliptic equations.

It can be seen from the review of the literature that, in the general case, traditional questions such as asymptotic behavior and estimates of eigenvalues are not sufficiently studied in the general case. This paper is devoted to estimates of singular values (s-numbers) and eigenvalues of the semiperiodic Dirichlet problem for a class of degenerate elliptic equations with arbitrary power degeneration.

The results of this work are close to those of M.B. Muratbekov [7–10], where differential operators of mixed and hyperbolic types were investigated. In contrast to the above papers, here, we investigate previously unconsidered degenerate elliptic equations with an arbitrary power-law degeneracy on the degeneracy line.

2. Results

Definition 1. *The function $u \in L_2(\Omega)$ is called a solution of (1)–(3) if there exists a sequence $\{u_k(x,y)\}_{k=1}^{\infty} \subset C_{0,\pi}^{\infty}(\overline{\Omega})$ such that*

$$\|u_k - u\|_2 \to 0, \quad \|Lu_k - f\|_2 \to 0 \text{ as } k \to \infty.$$

$W_2^1(\Omega)$ is the Sobolev space with norm

$$\|u\|_{2,1,\Omega} = \left[\|u_x\|_2^2 + \|u_y\|_2^2 + \|u\|_2^2\right]^{\frac{1}{2}},$$

where $\|\cdot\|$ is a norm of $L_2(\Omega)$.

Theorem 1. *Let $a(y)$ and $c(y)$ be piecewise continuous functions in $[0,1]$ and satisfying the conditions*

$$i) |a(y)| \geq \delta_0 > 0, c(y) \geq \delta > 0.$$

Then there exists a unique solution $u(x,y)$ of (1)–(3) such that

$$\|u\|_{2,1,\Omega} \leq c\|f\|_2$$

for all $f \in L_2(\Omega)$, where c is a constant.

In what follows, the operator corresponding to problems (1)–(3) is denoted by L.

Definition 2. *[11] Let A be a completely continuous linear operator and $|A| = \sqrt{A^*A}$. Any eigenvalues of $|A|$ are called s-numbers of A.*

Any nonzero s-numbers of L^{-1} will be numbered in descending order, taking into account their multiplicity, such that

$$s_k(L^{-1}) = \lambda_k(|L^{-1}|), k = 1, 2, 3, \ldots.$$

Theorem 2. *Let the condition (i) be fulfilled. Then for the singular numbers (s-numbers) of L^{-1}, the following estimate*

$$c^{-1}\frac{1}{k} \leq s_k \leq c\frac{1}{k^{\frac{1}{2}}}, k = 1, 2, 3, \ldots \tag{4}$$

holds, where c is any constant and s_k is a singular number (s-numbers) of L^{-1}.

Theorem 3. *Let the condition (i) be fulfilled. Then for the eigenvalues of L^{-1}, the following estimate*

$$|\lambda_k| \leq \frac{c \cdot e^{\frac{1}{2}}}{k^{\frac{1}{2}}}, k = 1, 2, 3, \ldots,$$

holds, where λ_k are the eigenvalues of the operator L^{-1}.

Example 1. *Let the equation be given:*

$$Lu = -y^3 u_{xx} - u_{yy} + (y^2+1)u_x + (y^5+1)u = f(x,y) \in L_2(\Omega),$$

Let's consider problems (2) and (3) for this equation. It is easy to check that all the conditions of Theorems 1–3 are satisfied. Therefore, for this problem, there is a unique solution and for s_k and λ_k where the following estimates

$$\frac{1}{2} \cdot \frac{1}{k} \le s_k \le 2 \cdot \frac{1}{k^{\frac{1}{2}}}, k = 1,2,3,...;$$

$$|\lambda_k| \le 2 \cdot \frac{e^{\frac{1}{2}}}{k^{\frac{1}{2}}}, k = 1,2,3,...$$

hold.

2.1. Auxiliary Lemmas

Lemma 1. *The estimate*

$$\|Lu\|_2 \ge c_0 \|u\|_2 \tag{5}$$

holds for all $u \in D(L)$, where $c_0 > 0$ is a constant.

Proof. Let $C^\infty_{0,\pi}(\Omega)$. Integrating by parts and taking into account the boundary conditions, we have

$$< Lu, u > \ge \int\limits_\Omega (u_y^2 + c(y)u^2)dxdy + \int\limits_\Omega k(y)u_x^2 dxdy$$

and

$$< Lu, u_x > = \int\limits_\Omega a(y)u_x^2 dxdy.$$

From these relations, we obtain (5) for any c_0 using the Cauchy inequality with "ϵ" and taking into account the condition (i). Lemma 1 is proved. □

We denote the closure of the operator by l_n such that

$$l_n u(y) = -u'' + (n^2 k(y) + ina(y) + c(y))u, n = 0, \pm 1, \pm 2, ...$$

defined on $C_0^\infty[0,1]$, where $C_0^\infty[0,1]$ is the set infinitely differentiable functions satisfying the conditions (3).

Lemma 2. *The estimates*

$$\|l_n u\|_{L_2(0,1)} \ge c_1(\|u'\|_{L_2(0,1)} + \|u\|_{L_2(0,1)}); \tag{6}$$

$$\|l_n u\|_{L_2(0,1)} \ge c_2 \|u\|_{C[0,1]}, \tag{7}$$

hold for all $u(y) \in D(l_n)$, where $c_1 > 0$ and $c_2 > 0$ are constants.

Proof. Let's compose the quadratic form $(l_n u, u)$, $u \in C_0^\infty[0,1]$. Integrating by parts, we obtain

$$|(l_n u, u)| = \left| \int_0^1 (l_n u)\bar{u} dy \right| = \left| \int_0^1 (u'^2 + (n^2 k(y) + ina(y) + c(y))|u|^2) dy \right|.$$

Hence, using the inequality $|\alpha + i\beta| \ge max(|\alpha|, |\beta|)$ $(\alpha, \beta \in R)$, the inequality Schwartz and the Cauchy inequality with "$\epsilon > 0$", we obtain

$$c\|l_n u\|^2_{L_2(0,1)} \ge c_3 \int_0^1 (|u'|^2 + c(y)|u|^2)dy + \int_0^1 n^2 k(y)|u|^2)dy. \tag{8}$$

From (8), taking into account $k(y) \geq 0$ and the condition (i), we obtain

$$\|l_n u\|^2_{L_2(0,1)} \geq c_1(\|u\|^2_{L_2(0,1)} + \|u'\|^2_{L_2(0,1)}) \geq c_1\|u\|^2_{W_2^1(0,1)}.$$

Since the embedding operator of continuous functions on $[0,1]$ of the Sobolev space $W_2^1(0,1)$ to $C[0,1]$ is bounded, it follows that

$$\|l_n u\|_{L_2(0,1)} \geq c_1\|u\|_{C[0,1]},$$

which is true for all $u \in D(L)$. □

Lemma 3. *The operator l_n is continuously invertible.*

Proof. Taking into account (6), it is enough if we show the density of $D(l_n)$ in $L_2(\Omega)$. Assume the contrary. Consider that the set $D(l_n)$ is not density in $L_2(0,1)$. Then there exists a nonzero element $w \in L_2(0,1)$, such that $(l_n u, w) = 0$ for $u \in D(l_n)$. Hence, since the set $D(l_n)$ is not density in $L_2(0,1)$, we obtain that w is a solution of $l_n^* w = -w'' + (n^2 k(y) + ina(y) + c(y))w = 0$. From this equality, it follows that $w'' \in L_2(0,1)$ by virtue of the continuous coefficients on $[0,1]$. Now we show that $w(y)$ satisfies the condition $w(0) = w(1) = 0$. Integrating by parts, we obtain

$$0 = (u, l_n^* w) = (l_n u, w) - u'(1)\bar{w}(1) + u'(0)\bar{w}(0)$$

for all $u \in D(l_n)$. The last equality holds if $w(0) = w(1) = 0$. Therefore, $w \in D(l_n)$. Then we obtain

$$\|l_n^* w\|_{L_2(0,1)} \geq c_1\|w\|_{L_2(0,1)}$$

which is the same as (6). It is show that $w = 0$. The resulting contradiction proves Lemma 3. □

Lemma 4. *The following estimate holds for l_n^{-1}*

$$\|l_n^{-1}\|_{L_2(0,1) \to L_2(0,1)} \leq \frac{1}{|n|\delta_0}, n = \pm 1, \pm 2, \ldots.$$

Proof. Taking into account the condition (i), we obtain

$$|(l_n u, u)| \geq \left| \int_0^1 ina(y)|u|^2 dy \right| \geq |n|\delta_0\|u\|^2_{L_2(0,1)}$$

for any function $u \in C_0^\infty[0,1]$. Hence, using the Cauchy inequality, we obtain

$$\|l_n u\|_{L_2(0,1)} \geq |n|\delta_0\|u\|_{L_2(0,1)}.$$

From the last estimate, it follows Lemma 4. □

2.2. Proofs of Main Theorems

Proof of Theorem 1. The existence and continuity of l_n^{-1} follows from Lemma 3. Let $u_n(y) = (l_n^{-1} f_n)(y)$. By direct verification, we make sure that the function

$$u_k(x,y) = \sum_{n=-k}^{k} u_n(y)e^{inx} = \sum_{n=-k}^{k} (l_n^{-1} f_n)(y)e^{inx} \tag{9}$$

is a solution of (1) with the right side

$$f_k(x,y) = \sum_{n=-k}^{k} f_n(y)e^{inx}$$

which satisfies the conditions (2) and (3). Moreover, the following equality

$$\|u_k(x,y)\|^2_{L_2(\Omega)} = 2\pi \sum_{n=-k}^{k} \|u_n(y)\|_2^2,$$

holds, where $\|\cdot\|_2$ is a norm in $L_2(0,1)$. Then from the estimate (6), it follows that

$$\|u_k(x,y)\|^2_{L_2(\Omega)} = 2\pi \sum_{n=-k}^{k} \|u_n(y)\|_2^2 \leq$$

$$\leq \frac{1}{c_1} 2\pi \sum_{n=-k}^{k} \|l_n u\|^2_{L_2(0,1)} \leq \frac{1}{c_1} 2\pi \sum_{n=-k}^{k} \|f_n(y)\|^2_{L_2(0,1)} = c\|f_k(x,y)\|^2_{L_2(\Omega)}, \tag{10}$$

where $c = \frac{1}{c_1}, c_1 > 0$.

From Lemma 4, we have

$$\|\frac{\partial u_k(x,y)}{\partial x}\|^2_{L_2(\Omega)} = \|\frac{\partial}{\partial x} \sum_{n=-k}^{k} (l_n^{-1} f_n)(y) e^{inx})\|^2_{L_2(\Omega)} = \|in \sum_{n=-k}^{k} (l_n^{-1} f_n)(y) e^{inx}\|^2_{L_2(0,1)} \leq$$

$$\leq \sum_{n=-k}^{k} |n|^2 \|l_n^{-1}\|^2_{L_2(0,1)\to L_2(0,1)} \|f_n\|^2_{L_2(0,1)} \leq \frac{1}{\delta_0^2} \sum_{n=-k}^{k} \|f_n\|^2_{L_2(0,1)} = \frac{1}{\delta_0^2} \|f_k(x,y)\|^2_{L_2(\Omega)}. \tag{11}$$

Similarly, using estimates (6) and (7), we obtain

$$\|\frac{\partial u_k(x,y)}{\partial y}\|^2_{L_2(\Omega)} + \|u_k\|^2_{L_2(\Omega)} = \|\frac{\partial}{\partial y} \sum_{n=-k}^{k} (l_n^{-1} f_n)(y) e^{inx})\|^2_{L_2(\Omega)} +$$

$$+ \| \sum_{n=-k}^{k} (l_n^{-1} f_n)(y) e^{inx})\|_2^2 \leq \frac{1}{c_1} \sum_{n=-k}^{k} \|f_n\|^2_{L_2(0,1)} +$$

$$+ \sum_{n=-k}^{k} \frac{1}{c_1} \|f_n\|^2_{L_2(0,1)} \leq c_0 \|f_k(x,y)\|^2_{L_2(\Omega)}, \tag{12}$$

where $c_0 = \frac{2}{c_1}, c_1 > 0$.

It is known that the set of functions

$$f_k(x,y) = \sum_{n=-k}^{k} f_n(y) e^{inx} \quad (k = 1, 2, ...)$$

is dense in $L_2(\Omega)$. Therefore, we can assume that $\|f_k(x,y) - f(x,y)\|_{L_2(\Omega)} \to 0$ as $k \to \infty$. Then the sequence $\{f_k\}_{k=1}^{\infty}$ is fundamental, and by virtue of estimates (10)–(12)

$$\|u_k(x,y) - u_m(x,y)\|_{2,1,\Omega} \leq c_0 \|f_k(x,y) - f_m(x,y)\|_{L_2(\Omega)} \to 0$$

as $k, m \to \infty$. Hence, since the space $W_2^1(\Omega)$ is complete, it follows that the sequence $\{u_k(x,y)\}_{k=-\infty}^{\infty}$ has the limit $u(x,y)$, for which, by virtue of (10)–(12), the estimate

$$\|u\|_{2,1,\Omega} \leq c_0 \|f\|_{2,\Omega}$$

holds, where $c_0 > 0$ is a constant.

Hence, and from (9), it follows that if $f(x,y) = \sum_{k=-\infty}^{\infty} f_k(y) \cdot e^{inx}$ then $u(x,y) = L^{-1} f = \sum_{k=-\infty}^{\infty} l_n^{-1} f_k(y) \cdot e^{inx}$ is a solution of (1)–(3). □

Let us introduce the sets

$$M = \{u \in L_2(\Omega) : \|Lu\|_{2,\Omega} + \|u\|_{2,\Omega} \leq 1\},$$

$$\widetilde{M}_c = \{u \in L_2(\Omega) : (\|u_x\|^2_{2,\Omega} + \|u_y\|^2_{2,\Omega} + \|u\|^2_{2,\Omega})^{\frac{1}{2}} \leq c\},$$

$$\dot{M}_{c^{-1}} = \{u \in L_2(\Omega); (\|u_{xx}\|^2_{2,\Omega} + \|u_{yy}\|^2_{2,\Omega} + \|u_x\|^2_{2,\Omega} + \|u_y\|^2_{2,\Omega} + \|u\|^2_{2,\Omega})^{\frac{1}{2}} \leq c^{-1}\},$$

where $c = \max\limits_{y\in[0,1]} \{k(y), |a(y)|, c(y), c_0\}$, and $\|\cdot\|_{2,\Omega}$ is a norm in $L_2(\Omega)$.

The following lemma holds

Lemma 5. *Let condition (i) be satisfied. Then for some constant $c_1 > 1$, the inclusions*

$$\dot{M}_{c^{-1}} \subseteq M \subseteq \widetilde{M}_c$$

hold.

Proof. Let $u(x,y) \in \dot{M}_{c^{-1}}$. Then, taking into account condition (i), we obtain

$$\|Lu\|^2_{2,\Omega} + \|u\|^2_{2,\Omega} \leq c_2(\|u_{xx}\|^2_{2,\Omega} + \|u_{yy}\|^2_{2,\Omega} + \|u_x\|^2_{2,\Omega} + \|u_y\|^2_{2,\Omega} + \|u\|^2_{2,\Omega})^{\frac{1}{2}} \leq c_2^{-1}c_2 \leq 1,$$

where $c_2 = \max\limits_{y\in[0,1]} \{|k(y)|, |a(y)|, |c(y)|\}$.

Hence, we have $\dot{M}_{c^{-1}} \subseteq M$.

Let $u \in M$. Then it follows from Theorem 1 that

$$(\|u_x\|^2_{2,\Omega} + \|u_y\|^2_{2,\Omega} + \|u\|^2_{2,\Omega})^{\frac{1}{2}} \leq c(\|Lu\|^2_{2,\Omega} + \|u\|^2_{2,\Omega}) \leq c,$$

i.e., $M \subseteq \widetilde{M}_c$. □

Definition 3. *[11] The Kolmogorov k-width of a set M in $L_2(\Omega)$ is called the quantity*

$$d_k = \inf_{\{y_k\}} \sup_{u\in M} \inf_{v\in y_k} \|u - v\|_{L_2(\Omega)},$$

where $\{y_k\}$ are the sets of all subspaces in $L_2(\Omega)$ whose dimensions do not exceed k.

Lemma 6. *Let condition (i) be satisfied. Then the estimates*

$$c^{-1}\dot{d}_k \leq d_k \leq c\widetilde{d}_k, \quad k = 1, 2, \ldots \tag{13}$$

hold, where $c > 0$ is any constant, and $\widetilde{d}_k, d_k$, and $\dot{d}_k$ are the k-widths of the $\widetilde{M}_c, M, \dot{M}_{c^{-1}}$ sets, respectively.

Proof. The proof of this lemma follows from Lemma 5 and the properties of the Kolmogorov k-widths. □

Let us introduce the functions

$$N(\lambda) = \sum_{d_k>\lambda} 1, \widetilde{N}(\lambda) = \sum_{\widetilde{d}_k>\lambda} 1, \dot{N}(\lambda) = \sum_{\dot{d}_k>\lambda} 1,$$

equal, respectively, to the number of widths $d_k(M)$, where $\widetilde{d}_k$ and $\dot{d}_k$ are greater than $\lambda > 0$. From (10), it follows the following inequalities

$$\dot{N}(c\lambda) \leq N(\lambda) \leq \widetilde{N}(c^{-1}\lambda).$$

Proof of Theorem 2. It is known that the estimates [12,13]

$$c^{-1}\lambda^{-2} \leq \widetilde{N}(\lambda) \leq c\lambda^{-2}, \tag{14}$$

$$c^{-1}\lambda^{-1} \leq N(\lambda) \leq c\lambda^{-1} \tag{15}$$

hold for the functions $\widetilde{N}(\lambda)$ and $N(\lambda)$. Let $\lambda = \widetilde{d}_k$. Then $\widetilde{N}(\widetilde{d}_k) = k$. Therefore, from (14) and (15), it follows that

$$c^{-1}\frac{1}{\sqrt{k}} \leq \widetilde{d}_k \leq c\frac{1}{\sqrt{k}}, \quad c^{-1}\frac{1}{k} \leq \dot{d}_k \leq c\frac{1}{k} \tag{16}$$

respectively. Hence, taking into account estimates (16) and the equality $s_k(L^{-1}) = d_k$, we obtain

$$c^{-1}\frac{1}{k} \leq s_k \leq c\frac{1}{k^{\frac{1}{2}}}, \ \ k = 1, 2, 3, \ldots$$

□

Proof of Theorem 3. It follows from Theorem 1 and equality (13) that if λ is an eigenvalue of L^{-1}, then λ is an eigenvalue of one of the operators $l_n^{-1}(n = 0, \pm 1, \pm 2, \ldots)$ and vice versa. Consequently, it follows from equality (13) that the operator L^{-1} has an infinite number of eigenvalues, where the last statement follows from the fact that the operator l_n^{-1} as $n = 0$, i.e., operator $l_0^{-1}(l_0 = \frac{d^2}{dy^2} + c(y))$ is a self-adjoint and compact operator. Now, using the estimate (4) and the Weyl inequality [11], as well as the inequality $e^k \cdot k! \geq k^k, k = 1, 2, 3, \ldots$, we obtain

$$|\lambda_k|^k \leq \prod_{j=1}^{k} |\lambda_j| \leq \prod_{j=1}^{k} s_j \leq \frac{c^k \cdot e^{\frac{1}{2}k}}{k^{\frac{1}{2}k}}.$$

Hence,

$$|\lambda_k| \leq \frac{c \cdot e^{\frac{1}{2}}}{k^{\frac{1}{2}}}.$$

□

3. Conclusions

In conclusion, we consider the self-adjoint case. Let $a(y) \equiv 0$. Consider the operator

$$Lu = -k(y)u_{xx} - u_{yy} + c(y)u$$

initially defined on $C^{\infty}_{0,\pi}(\bar{\Omega})$, where $C^{\infty}_{0,\pi}(\bar{\Omega})$ is the set consisting of infinitely differentiable functions and satisfying conditions (2) and (3).

It is easy to check that a closure of the operator L in $L_2(\Omega)$ is self-adjoint and this operator satisfies the following estimate

$$c^{-1}\sum_{n=-\infty}^{\infty} \lambda^{-\frac{1}{2}} mes(y \in [0,1] : (n^2 + c(y)) \leq c^{-1}\lambda^{-1}) \leq N(\lambda) \leq$$

$$\leq c\sum_{n=-\infty}^{\infty} \lambda^{-1} mes(y \in [0,1] : (k(y)n^2 + c(y)) \leq c^{-1}\lambda^{-2}),$$

where $c > 0$ is a constant number, and $N(\lambda) = \sum_{\lambda_k > \lambda} 1$, where λ_k are the eigenvalues of L^{-1}.

This statement is proved by repeating the calculation and reasoning used in the proof of Theorems 1–3 of this paper and Theorem 1.4 of [14].

Regarding the results of this paper, the following results are obtained for a class of degenerate elliptic operators:

- bounded invertibility is proved;
- the two-sided estimates of singular numbers (s-numbers) are obtained;
- the estimate of the eigenvalues is obtained.

The results obtained in this paper make it possible to study the non-linear degenerate operator of elliptic type

$$Lu = -k(y)u_{xx} - u_{yy} + a(x,y,u)u_x + c(x,y,u)u,$$

where $u \in D(L)$, and $D(L)$ is the domain of L.

Author Contributions: Conceptualization, M.M. and S.I.; methodology, M.M.; validation, M.M. and S.I.; formal analysis, S.I.; investigation, M.M.; resources, M.M.; writing—original draft preparation, M.M.; writing—review and editing, S.I.; visualization, S.I.; supervision, M.M.; project administration, M.M.; funding acquisition, M.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the grant AP08855802 of the Ministry of Education and Science of the Republic of Kazakhstan.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Smirnov, M.M. *Degenerate Elliptic and Hyperbolic Equations*; Nauka: Moscow, Russia, 1966. (In Russian)
2. Keldysh, M.V. On some cases of degeneration of equations of elliptic type on the boundary of the domain. *DAN USSR* **1951**, *77*, 181–183. (In Russian)
3. Kalmenov, T.S.; Otelbaev, M. On the smoothness of solutions of a class of degenerate elliptic equations. *Differ. Equ.* **1977**, *13*, 1244–1255. (In Russian)
4. Nakhushev, A.M. *Problems with Mixing for Partial Differential Equations*; Nauka: Moscow, Russia, 2006. (In Russian)
5. Sbai, A.; El hadfi, Y. Degenerate elliptic problem with a singular nonlinearity. *Complex Var. Elliptic Equ.* **2021**, *2021*, 1–18. [CrossRef]
6. Huang, G. A Liouville theorem of degenerate elliptic equation and its application. *Discret. Contin. Dyn. Syst.* **2013**, *33*, 4549–4566. [CrossRef]
7. Muratbekov, M.B.; Muratbekov, M.M. Estimates of the spectrum of semiperiodical Dirichlet problem for a class of degenerate elliptic equations. *Complex Var. Elliptic Equ.* **2012**, *57*, 1135–1146. [CrossRef]
8. Muratbekov, M.B.; Muratbekov, M.M.; Abylayeva, A.M. On existence of the resolvent and discreteness of the spectrum of a class of differential operators of hyperbolic type. *Electron. J. Theory Differ. Equ.* **2013**, *63*, 1–10. [CrossRef]
9. Muratbekov, M.B.; Muratbekov, M.M. Estimates of spectrum for a class of mixed type operators. *Differ. Equ.* **2007**, *43*, 143–146. [CrossRef]
10. Muratbekov, M.B.; Muratbekov, M.M. On the compactness of the resolvent of a Schrodinger type singular operator with a negative parameter. *Chaos Solitons Fractals* **2021**, *151*, 111248. [CrossRef]
11. Gohberg, I.C.; Krein, M.G. *Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space*; American Mathematical Society: Providence, RI, USA, 1969.
12. Ismagilov, R.S. On n-dimensional widths of compact sets in a Hilbert space. *Funct. Anal.* **1968**, *2*, 32–39. (In Russian)
13. Ismagilov, R.S. Widths of sets in linear normed spaces and approximation of functions by trigonometric polynomials. *Adv. Math. Sci.* **1974**, *29*, 161–178. (In Russian)
14. Muratbekov, M.B.; Muratbekov, M.M. Sturm-Liouville operator with a parameter and its usage to spectrum research of some differential operators. *Complex Var. Elliptic Equ.* **2019**, *64*, 1457–1476. [CrossRef]