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Coefficient Estimates for a Family of Starlike Functions Endowed with Quasi Subordination on Conic Domain

Arzu Akgül [1,*] and Luminita-Ioana Cotîrlă [2]

1 Department of Mathematics, Faculty of Arts and Science, Umuttepe Campus, Kocaeli University, 41001 Kocaeli, Turkey
2 Department of Mathematics, Technical University of Cluj-Napoca, 400114 Cluj-Napoca, Romania; luminita.cotirla@math.utcluj.ro
* Correspondence: akgul@kocaeli.edu.tr

Abstract: In 1999, for $(0 \leq k < \infty)$, the concept of conic domain by defining k-uniform convex functions were introduced by Kanas and Wisniowska and then in 2000, they defined related k-starlike functions denoted by $k - UCV$ and $k - ST$ respectively. Motivated by their studies, in our work, we define the class of k-parabolic starlike functions, denoted $k - \mathcal{S}_{\mathcal{H}_m,q}$, by using quasi-subordination for m-fold symmetric analytic functions, making use of conic domain Ω_k. We determine the coefficient bounds and estimate Fekete–Szegö functional by the help of m-th root transform and quasi subordination for functions belonging the class $k - \mathcal{S}_{\mathcal{H}_m,q}$.

Keywords: analytic function; subordination; quasi-subordination; m-fold symmetric; Fekete–Szegö inequality

MSC: 30C45; 30C50; 33C10

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1. Introduction

Assume that $\mathcal{A}$ is the family of functions given by:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are normalized analytic functions with:

$$f(0) = f'(0) - 1 = 0$$

in the open unit disc $\Lambda = \{z : |z| < 1\}$ and let $\mathcal{H} = \{f \in \mathcal{A} : f \text{ is univalent in } \Lambda\}$. For the function $\Psi \in \mathcal{H}$ if:

$$\Psi(0) = 0 \text{ and } |\Psi(z)| < 1, (z \in \Lambda)$$

then Ψ is said to be a Schwarz function which is self-mapping of the unit disc Λ.

For the functions $f, g \in \mathcal{H}$, it is said that the function f is subordinate to g in Λ, and write:

$$f(z) \prec g(z)(z \in \Lambda),$$

if there exists a Schwarz function Ψ, such that:

$$f(z) = g(\Psi(z))(z \in \Lambda).$$

It is well known that if $f(z) \prec g(z)$, then $f(0) = g(0)$ and $f(\Lambda) \subset g(\Lambda), (z \in \Lambda)$. Moreover, if the function g is univalent in Λ, then we get:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\Lambda) \subset g(\Lambda), (z \in \Lambda).$$

In 1970, Robertson [1] developed the notions of quasi-subordination and majorization as follows:

For functions f and g, analytic in Λ, the function f is quasi-subordinate to g, writen as follows:

$$f(z) \prec_q g(z), (z \in \Lambda)$$

if there exist analytic functions Φ and Ψ,

$$|\Phi(z)| \leq 1, \Psi(0) = 0 \text{ and } |\Psi(z)| < 1,$$

such that:

$$f(z) = \Phi(z)g(\Psi(z))(z \in \Lambda).$$

If we choose the function Φ as $\Phi(z) \equiv 1$, then $f(z) = g(\Psi(z))$, so that $f(z) \prec g(z)$ in Λ. Moreover, by taking the Schwarz function Ψ as $\Psi(z) = z$, then:

$$f(z) = \Phi(z)g(z) \ (z \in \Lambda)$$

and it is said that f is majorized by g and denoted:

$$f(z) \prec\prec g(z)(z \in \Lambda).$$

Thus, it is clear that quasi-subordination is a generalization of subordination besides majorization. For a brief history and brilliant examples of univalent functions endowed with quasi-subordination, in conjunction with several other features, see [2] and references therein (see also [3–10]).

The fact that the n-th coefficient of a univalent function $f \in$ is bounded by n is well known (see [11]). The bounds for the coefficient provide knowledge the about numerous geometric features of the function. In Geometric Function Theory, the Fekete-Szegö functional for normalized univalent functions represented by the Equation (1), is well known for its rich history. The Fekete–Szegö problem is the problem of maximizing the value of the non-linear functional $\left|a_2 a_4 - \mu a_3^2\right|$ [12]. The equality is valid for the Koebe function. The sharp upper bound for Fekete-Szegö functional was found by Keogh and Merkes [13] for some subclasses of univalent function classes. Indeed, very recently, the Fekete-Szegö problem has gained importance thanks to the work of Srivastava et al. [14] (see also [15,16]). Several other authors have examined the bounds for the Fekete-Szegö functional for functions in numerous subclasses of $\mathcal{H}$. Related studies can be found in [17–22].

Let m be a positive integer and $\mathcal{D}$ be a domain. If a rotation of $\mathcal{D}$ about the origin through an angle $\frac{2\pi}{m}$ carries $\mathcal{D}$ to itself then it is said that $\mathcal{D}$ is a m-fold symmetric domain. A function $f(z)$ is said to be m-fold symmetric in Λ, if for every $z \in \Lambda$ following equality holds:

$$f\left(e^{\frac{2\pi i}{m}} z\right) = e^{\frac{2\pi i}{m}} f(z).$$

We show by $\mathcal{H}_m$ the class of m-fold symmetric univalent functions in Λ, which are normalized by the series expansion (1). Actually, the functions in the class $\mathcal{H}$ are one-fold symmetric.

In 1916, Gronwall presented that $f(z)$ has a power series expansion given by:

$$f(z) = b_1 z + b_{m+1} z^{m+1} + b_{2m+1} z^{2m+1} + \cdots = \sum_{n=0}^{\infty} b_{nm+1} z^{nm+1}. \quad (2)$$

on the condition that it is regular and m-fold symmetric in Λ.

On the contrary, if $f(z)$ is expressed by (2), then $f(z)$ is m-fold symmetric inside the circle of convergence of the series. For a univalent function $f(z)$ given by (1), the m-th root transformation is presented with:

$$F(z) = [f(z^m)]^{\frac{1}{m}} = z + \sum_{n=1}^{\infty} b_{mn+1} z^{mn+1}. \tag{3}$$

The concepts of $k-SP$ and $k-UCV$ were introduced by Kanas and Wisniowska [23,24] as follows:

$$\begin{aligned} k-SP &= \left\{ f : f \in \mathcal{H}, Re\left(\frac{zf'z)}{f(z)}\right) > k\left|\frac{zf'z)}{f(z)} - 1\right|, z \in \Lambda, 0 \le k \le \infty \right\}, \\ k-UCV &= \left\{ f : f \in \mathcal{H}, Re\left(1 + \frac{zf''z)}{f'(z)}\right) > k\left|\frac{zf''z)}{f'(z)}\right|, z \in \Lambda, 0 \le k \le \infty \right\}. \end{aligned}$$

This is a fascinating association of the notion of univalent convex functions [25] and uniformily convex functions [26]. Kanas and Wiśniowska [27] considered the geometric definition of $k-UCV$ and its relations with the conic domains. The class $k-SP$ was studied in [23]. The class $k-SP$, composing of k-parabolic starlike functions, is defined from $k-UCV$ by means of the well-known Alexandar's transforms [24]. That is,

$$f \in k-UCV \Leftrightarrow zf'(z) \in k-SP, z \in \Lambda.$$

According to the one variable characterization theorem [23] of the class $k-UCV$, $f \in k-UCV$ (in turn $f \in k-SP$) if the values of $p(z) = 1 + \frac{zf''(z)}{f'(z)}$(in turn $p(z) = \frac{zf\prime(z)}{f(z)}$) lie in the conic region Ω_k in the $w-$plane, where:

$$\Omega_k = \left\{ w = u + iv \in \mathbb{C} : u^2 > k^2(u-1)^2 + k^2v^2, u > 0, 0 \le k < \infty \right\}.$$

This property allows us to obviously determine the domain Ω_k, as a convex domain contained in the right half-plane. Moreover, if we specify the parameter k, then Ω_k denotes certain interesting domain regions. We know that for Ω_k, if $0 < k < 1$ then it is an hyperbolic, if $k > 1$ then it is elliptic, if $k = 1$ then it is parabolic region, and after all Ω_0 is the whole right half-plane.

Assume that $\mathcal{B} = \{\Psi \in \mathcal{H} : \Psi(0) = 0$ and $|\Psi(1)| < 1\}$. In sequel, we will use the next lemmas to obtain our results.

Lemma 1 ([13])**.** *Let $\Psi \in \mathcal{B}$ is given by:*

$$\Psi(z) = \Psi_1 z + \Psi_2 z^2 + \cdots, z \in \Lambda.$$

Then for every $t \in \mathbb{C}$,

$$\left|\Psi_2 - s\Psi_1^2\right| \le \max\{1, |s|\}, \forall s \in \mathbb{C}.$$

Lemma 2 ([28])**.** *If $\Psi \in \mathcal{B}$ and:*

$$\Psi(z) = \Psi_1 z + \Psi_2 z^2 + \cdots, z \in \Lambda,$$

then:

$$\left|\Psi_2 - s\Psi_1^2\right| \le \begin{cases} -s & , \quad s \le -1 \\ 1 & , \quad -1 \le s \le 1 \\ s & , \quad s \ge 1 \end{cases}.$$

Lemma 3 ([29])**.** *If $\Psi \in \mathcal{B}$ and:*

$$\Psi(z) = \Psi_1 z + \Psi_2 z^2 + \cdots, z \in \Lambda,$$

then:

$$\left|\Psi_2 - s\Psi_1^2\right| \leq 1 + (|s| - 1)\left|\Psi_1^2\right|, \forall s \in \mathbb{C}.$$

Lemma 4 ([30]). *Let $0 \leq k \leq \infty$ be fixed and $\mathfrak{R}_k(z)$ be the Riemann map of Λ on to Ω_k fulfilling $\mathfrak{R}_k(0) = 1, \mathfrak{R}_k'(0) > 0$. If:*

$$\mathfrak{R}_k(z) = 1 + \mathfrak{R}_1(k)z + \mathfrak{R}_2(k)z^2 + \mathfrak{R}_3(k)z^3 + \cdots, z \in \Lambda \tag{4}$$

then:

$$\mathfrak{R}_1 = \mathfrak{R}_1(k) = \begin{cases} \frac{2\mathcal{T}^2}{1-k^2} & , \quad 0 \leq k < 1 \\ \frac{8}{\pi^2} & , \quad k = 1 \\ \frac{\pi^2}{4\mathrm{K}^2(r)(k^2-1)(1-r)\sqrt{r}} & , \quad k > 1 \end{cases}$$

$$\mathfrak{R}_2 = \mathfrak{R}_2(k) = D(k)\mathfrak{R}_1(k)$$

where:

$$\mathfrak{R}_1 = \mathfrak{R}_1(k) = \begin{cases} \frac{2+\mathcal{T}^2}{3} & , \quad 0 \leq k < 1 \\ \frac{2}{3} & , \quad k = 1 \\ \frac{4\mathrm{K}^2(r)(r^2+6r+1)-\pi^2}{24\mathrm{K}^2(r)(1+r)\sqrt{r}} & , \quad k > 1 \end{cases}$$

$$\mathcal{T} = \frac{2}{\pi}\arccos k,$$

$\mathrm{K}(r)$ is complex elliptc integral of first kind (see [31–33]).

Very recently, Çağlar et al. [34] presented the coefficient estimate by $m-$throot transform for a family defined by Hohlov operator using quasi-subordination for conic domains. In this study, motivated by works of Kanas and Wisniowska [26,27] and Çağlar et al. [34] as well as earlier studies mentioned above, we define the class of k-parabolic starlike functions via $m-$fold symmetric functions, denoted $k - \mathcal{S}_{\mathcal{H}_m,q}$. We use the concepts of quasi-subordination and majorization to define our new classes. Moreover, coefficient bounds and Fekete–Szegö inequality are examined.

Definition 1. *For $f \in \mathcal{A}$ defined by (1), f is said to be in the class $k - \mathcal{S}_{\mathcal{H}_m,q}$ if it provides the quasi-subordination:*

$$\frac{zf'z)}{f(z)} - 1 \prec_q \mathfrak{R}_k(z) - 1, z \in \Lambda. \tag{5}$$

We consider throughout this study that $\Phi(z) = c_0 + c_1z + c_2z^2 + c_3z^3 + \ldots$ and $|c_n| \leq 1$. Additionally, we will assume that F is the m-th root transform of f presented by (2) in next theorems and corrollaries.

2. Main Results

Theorem 1. *If $f \in k - \mathcal{S}_{\mathcal{H}_m,q}$, then:*

$$\begin{aligned}|b_{m+1}| &\leq \frac{\mathfrak{R}_1}{m},\\ |b_{2m+1}| &\leq \frac{1}{2m}\left[\mathfrak{R}_1+\max\{\mathfrak{R}_1,\left(\frac{m-1}{m}+1\right)\mathfrak{R}_1^2+|\mathfrak{R}_2|\}\right],\\ \left|b_{2m+1}-\mu b_{m+1}^2\right| &\leq \frac{1}{2m}\left[\mathfrak{R}_1+\max\left\{\mathfrak{R}_1,\left|\frac{2\mu+m-1}{m}+1\right|\mathfrak{R}_1^2+|\mathfrak{R}_2|\right\}\right]\end{aligned}$$

Proof. Let $f\in k-k-\mathcal{S}_{\mathcal{H}_{m},q}$. Then, there are two analytic functions Φ and Ψ with $|\Phi(z)|\leq 1$, $\Psi(0)=0$ and $|\Psi(z)|<1$ such that:

$$\frac{zf'z)}{f(z)}-1=\Phi(z)[\mathfrak{R}_k(\Psi(z))-1], \tag{6}$$

$$\Phi(z)[\mathfrak{R}_k(\Psi(z))-1]=\mathfrak{R}_1c_0\Psi_1z+\left[\mathfrak{R}_1c_1\Psi_1+c_0\left(\mathfrak{R}_1\Psi_2+\mathfrak{R}_2\Psi_1^2\right)\right]z^2+\cdots \tag{7}$$

By using (7) in (6), we have:

$$a_2=\mathfrak{R}_1c_0\Psi_1 \tag{8}$$

$$\begin{aligned}2a_3-a_2^2 &= \mathfrak{R}_1c_1\Psi_1+c_0\left(\mathfrak{R}_1\Psi_2+\mathfrak{R}_2\Psi_1^2\right)\\ &\Rightarrow\ a_3=2\left(\mathfrak{R}_1c_1\Psi_1+\mathfrak{R}_1c_0\Psi_2+\left(\mathfrak{R}_1^2c_0^2+\mathfrak{R}_2c_0\right)\Psi_1^2\right).\end{aligned} \tag{9}$$

For f given by (1), we can easily compute that:

$$F(z)=[f(z^m)]^{\frac{1}{m}}=z+\frac{1}{m}a_2z^{m+1}+\left[\frac{1}{m}a_3-\frac{1}{2}\left(\frac{m-1}{m^2}\right)a_2^2\right]z^{2m+1}+\cdots. \tag{10}$$

Upon equating the coefficients of z^{m+1} and z^{2m+1} in view of (2) and (10), we have:

$$b_{m+1}=\frac{1}{m}a_2 \tag{11}$$

and:

$$b_{2m+1}=\frac{1}{m}a_3-\frac{1}{2}\left(\frac{m-1}{m^2}\right)a_2^2. \tag{12}$$

From (8), (9), (11) and (12), we obtain:

$$b_{m+1}=\frac{\mathfrak{R}_1c_0\Psi_1}{m} \tag{13}$$

and by using $|c_n|\leq 1, |\Psi_n(z)|<1$ in (13),

$$|b_{m+1}|\leq\frac{\mathfrak{R}_1}{m}, \tag{14}$$

$$\begin{aligned}b_{2m+1} &= \frac{1}{m}a_3-\frac{m-1}{2m^2}a_2^2\\ &= \frac{1}{2m}\left[\mathfrak{R}_1c_1\Psi_1+c_0\mathfrak{R}_1\Psi_2+\left(\mathfrak{R}_1^2c_0^2+c_0\mathfrak{R}_2\right)\Psi_1^2-\frac{m-1}{2m^2}\mathfrak{R}_1^2c_0^2\Psi_1^2\right]\\ &= \frac{\mathfrak{R}_1}{2m}\left[c_1\Psi_1+c_0\left\{\Psi_2-\left(\frac{m-1}{m}\mathfrak{R}_1c_0-\mathfrak{R}_1c_0-\frac{\mathfrak{R}_2}{\mathfrak{R}_1}\right)\Psi_1^2\right\}\right]\\ &= \frac{\mathfrak{R}_1}{2m}\left[c_1\Psi_1+c_0\left\{\Psi_2-s\Psi_1^2\right\}\right].\end{aligned} \tag{15}$$

Due to the fact that $D=\frac{\mathfrak{R}_2}{\mathfrak{R}_1}$, we can write:

$$s=\frac{m-1}{m}\mathfrak{R}_1c_0-\mathfrak{R}_1c_0-D. \tag{16}$$

Additionally, by using $|c_n| \leq 1, |\Psi_n(z)| < 1$ we obtain:

$$\begin{aligned} |s| &= \left| \frac{m-1}{m} \mathfrak{R}_1 c_0 - \mathfrak{R}_1 c_0 - D \right| \\ &\leq \left(\frac{m-1}{m} + 1 \right) \mathfrak{R}_1 + |D|. \end{aligned}$$

By taking taking the modulus of both sides of the Equation (15) and applying Lema 1 to the $\left|\Psi_2 - s\Psi_1^2\right|$, we have:

$$\begin{aligned} |b_{2m+1}| &\leq \frac{\mathfrak{R}_1}{2m}(1 + \max\{1, |s|\}) \\ &\leq \frac{1}{2m}\left(\mathfrak{R}_1 + \max\{\mathfrak{R}_1, \left(\frac{m-1}{m} + 1\right)\mathfrak{R}_1^2 + |\mathfrak{R}_2|\}\right). \end{aligned} \tag{17}$$

Therefore, for any $\mu \in \mathbb{C}$,

$$\begin{aligned} b_{2m+1} - \mu b_{m+1}^2 &= \frac{\mathfrak{R}_1}{2m}\left[c_1\Psi_1 + c_0\left\{\Psi_2 - \left(\frac{m-1}{m}\mathfrak{R}_1 c_0 - \mathfrak{R}_1 c_0 - D\right)\Psi_1^2\right\}\right] \\ &\quad -\mu\frac{\mathfrak{R}_1^2 c_0^2 \Psi_1^2}{m^2} \\ &= \frac{\mathfrak{R}_1}{2m}\left[c_1\Psi_1 + c_0\left\{\Psi_2 - \left(s + 2\mu\frac{\mathfrak{R}_1 c_0}{m}\right)\Psi_1^2\right\}\right] \\ &= \frac{\mathfrak{R}_1}{2m}\left[c_1\Psi_1 + c_0\left(\Psi_2 - l\Psi_1^2\right)\right] \end{aligned} \tag{18}$$

where s is given by (16) and:

$$l = \left(s + 2\mu\frac{\mathfrak{R}_1 c_0}{m}\right). \tag{19}$$

Using inequalities $|c_n| \leq 1$, $|\Psi_n(z)| < 1$ and applying Lemma 1 to $\left|\Psi_2 - l\Psi_1^2\right|$, we obtain:

$$\begin{aligned} \left|b_{2m+1} - \mu b_{m+1}^2\right| &\leq \frac{\mathfrak{R}_1}{2m}\left[1 + \left|\Psi_2 - l\Psi_1^2\right|\right] \\ &\leq \frac{\mathfrak{R}_1}{2m}[1 + \max\{1, |l|\}]. \end{aligned}$$

Further, because of:

$$\begin{aligned} |l| &= \left|s + 2\mu\frac{\mathfrak{R}_1 c_0}{m}\right| \\ &= \left|\frac{(m-1)}{m}\mathfrak{R}_1 c_0 - \mathfrak{R}_1 c_0 - D + 2\mu\frac{\mathfrak{R}_1 c_0}{m}\right| \\ &\leq \left|\frac{2\mu + m - 1}{m}\right|\mathfrak{R}_1 + \mathfrak{R}_1 + |D|, \end{aligned}$$

we conclude that:

$$\left|b_{2m+1} - \mu b_{m+1}^2\right| \leq \frac{1}{2m}\left[\mathfrak{R}_1 + \max\left\{\mathfrak{R}_1, \left|\frac{2\mu + m - 1}{m} + 1\right|\mathfrak{R}_1^2 + |\mathfrak{R}_2|\right\}\right]$$

Letting $m = 1$ in Theorem 1, we obtain next: □

Corollary 1. *If* $f \in k - \mathcal{S}_{\mathcal{H}_{m},q}$, *then:*

$$\left|b_3-\mu b_2^2\right| \leq \frac{1}{2}\left[\mathfrak{R}_1+\max\left\{\mathfrak{R}_1, |2\mu+1|\mathfrak{R}_1^2+|\mathfrak{R}_2|\right\}\right].$$

Taking values of $\mathfrak{R}_1=\mathfrak{R}_1(k)$ and $D=D(k)$ also k in Theorem 1, we obtain next:

Corollary 2. *If* $f\in k-\mathcal{S}_{\mathcal{H}_m,q}, (0\leq k<\infty)$ *and* $0\leq k<1$, *then:*

$$|b_{m+1}| \leq \frac{1}{m}\left(\frac{2\mathcal{T}^2}{1-k^2}\right),$$

$$\begin{aligned}|b_{2m+1}| \quad \leq \quad & \frac{1}{2m}\left[\left(\frac{2\mathcal{T}^2}{1-k^2}\right)+\max\left\{\left(\frac{2\mathcal{T}^2}{1-k^2}\right),\left(\frac{m-1}{m}+1\right)\left(\frac{2\mathcal{T}^2}{1-k^2}\right)^2\right.\right.\\ & \left.\left.+\left(\frac{2\mathcal{T}^2}{1-k^2}\right)\left(\frac{\mathcal{T}^2+2}{3}\right)\right\}\right],\end{aligned}$$

and for any $\mu\in\mathbb{C}$*:*

$$\begin{aligned}\left|b_{2m+1}-\mu b_{m+1}^2\right| \quad \leq \quad & \frac{1}{2m}\left[\left(\frac{2\mathcal{T}^2}{1-k^2}\right)+\max\left\{\left(\frac{2\mathcal{T}^2}{1-k^2}\right),\left|\frac{2\mu+m-1}{m}+1\right|\left(\frac{2\mathcal{T}^2}{1-k^2}\right)^2\right.\right.\\ & \left.\left.+\left|\left(\frac{2\mathcal{T}^2}{1-k^2}\right)\left(\frac{\mathcal{T}^2+2}{3}\right)\right|\right\}\right].\end{aligned}$$

Corollary 3. *If* $f\in k-\mathcal{S}_{\mathcal{H}_m,q}, (0\leq k<\infty)$ *and* $k=1$, *then:*

$$|b_{m+1}| \leq \frac{8}{m\pi^2},$$

$$|b_{2m+1}| \leq \frac{1}{2m}\left[\frac{8}{\pi^2}+\max\left\{\frac{8}{\pi^2},\left(\frac{m-1}{m}+1\right)\left(\frac{8}{\pi^2}\right)^2+\frac{16}{3\pi^2}\right\}\right],$$

and for any $\mu\in\mathbb{C}$:

$$\left|b_{2m+1}-\mu b_{m+1}^2\right| \leq \frac{1}{2m}\left[\frac{8}{\pi^2}+\max\left\{\frac{8}{\pi^2},\left|\frac{2\mu+m-1}{m}+1\right|\frac{64}{\pi^4}+\frac{16}{3\pi^2}\right\}\right]$$

Corollary 4. *If* $f\in k-\mathcal{S}_{\mathcal{H}_m,q}, (0\leq k<\infty)$ *and* $k>1$, *then:*

$$|b_{m+1}| \leq \frac{1}{m}B_1,$$

$$|b_{2m+1}| \leq \frac{1}{2m}(B_1+\max\{B_1, B_2+B_3\})$$

and for any $\mu\in\mathbb{C}$:

$$\left|b_{2m+1}-\mu b_{m+1}^2\right| \leq \frac{1}{2m}(B_1+\max\{B_1, B_4+B_3\}),$$

where:

$$\begin{aligned} B_1 &= \mathfrak{R}_1 = \frac{\pi^2}{4\mathrm{K}^2(r)(k^2-1)(1-r)\sqrt{r}}, \\ B_2 &= \left(\frac{m-1}{m}+1\right)\mathfrak{R}_1^2 = \left(\frac{m-1}{m}+1\right)\left(\frac{\pi^2}{4\mathrm{K}^2(r)(k^2-1)(1-r)\sqrt{r}}\right)^2, \\ B_3 &= |\mathfrak{R}_2| = \left|\frac{4\pi^2k^2(r)(r^2+6r+1)-\pi^2}{96\mathrm{K}^4(r)(k^2-1)(1-r^2)r}\right|, \\ B_4 &= \left|\frac{2\mu+m-1}{m}+1\right|\mathfrak{R}_1^2 \end{aligned}$$

Theorem 2. *If $f \in \mathcal{A}$ fulfilles:*

$$\frac{zf'z)}{f(z)} - 1 \prec\prec \mathfrak{R}_k(z) - 1, \tag{20}$$

then the following inequailies hold:

$$\begin{aligned} |b_{m+1}| &\leq \frac{\mathfrak{R}_1}{m}, \\ |b_{2m+1}| &\leq \frac{1}{2m}\left[\mathfrak{R}_1 + |\mathfrak{R}_2| + \frac{1}{m}\mathfrak{R}_1^2\right]. \end{aligned}$$

and for any $\mu \in \mathbb{C}$:

$$\left|b_{2m+1} - \mu b_{m+1}^2\right| \leq \frac{1}{2m}\left[\mathfrak{R}_1 + |\mathfrak{R}_2| + \left(1 + \frac{2\mu+m-1}{m}\right)\mathfrak{R}_1^2\right]$$

Proof. By choosing $\Psi(z) = z$ in Theorem 1, we obtain the desired result. □

Letting $\Phi(z) = 1$ and $m = 1$ in Theorem 2, we then obtain :

Corollary 5. *If $f \in \mathcal{A}$ fulfilles (20) and $\Phi(z) = 1$, we obtain:*

$$\begin{aligned} |b_{m+1}| &\leq \frac{\mathfrak{R}_1}{m} \\ |b_{2m+1}| &\leq \frac{1}{2m}\left[|\mathfrak{R}_2| + \frac{1}{m}\mathfrak{R}_1^2\right] \end{aligned}$$

and for any $\mu \in \mathbb{C}$:

$$\left|b_{2m+1} - \mu b_{m+1}^2\right| \leq \frac{1}{2m}\left[\mathfrak{R}_1 + |\mathfrak{R}_2| + \frac{2\mu+1}{m}\mathfrak{R}_1^2\right].$$

Further, by letting $m = 1$, in last inequalities, we conclude that:

$$\begin{aligned} |b_2| &\leq \mathfrak{R}_1 \\ |b_3| &\leq |\mathfrak{R}_2| + \mathfrak{R}_1^2 \end{aligned}$$

and for any $\mu \in \mathbb{C}$:

$$\left|b_3 - \mu b_2^2\right| \leq \frac{1}{2}\left[\mathfrak{R}_1 + |\mathfrak{R}_2| + |2\mu+1|\mathfrak{R}_1^2\right].$$

Theorem 3. *If $f \in k - \mathcal{S}_{\mathcal{H}_m,q}$, then:*

$$\left|b_{2m+1}-\mu b_{m+1}^2\right| \leq \begin{cases} \frac{\mathfrak{R}_1}{2m}\left[1+\mathfrak{R}_1 c_0 + D - \frac{2\mu+m-1}{m}\mathfrak{R}_1 c_0\right] & , \quad \mu \leq \zeta_1 \\ \frac{\mathfrak{R}_1}{m} & , \quad \zeta_1 \leq \mu \leq \zeta_2 \\ \frac{\mathfrak{R}_1}{2m}\left[1-\mathfrak{R}_1 c_0 - D + \frac{2\mu+m-1}{m}\mathfrak{R}_1 c_0\right] & , \quad \mu \geq \zeta_2 \end{cases},$$

where:

$$\begin{aligned} \zeta_1 &= \frac{m}{2\mathfrak{R}_1 c_0}(D+\mathfrak{R}_1 c_0 - \frac{m-1}{m}\mathfrak{R}_1 c_0 - 1) \\ \zeta_2 &= \frac{m}{2\mathfrak{R}_1 c_0}(D+\mathfrak{R}_1 c_0 - \frac{m-1}{m}\mathfrak{R}_1 c_0 + 1). \end{aligned}$$

Proof. From the equality (18), we have:

$$b_{2m+1}-\mu b_{m+1}^2 = \frac{\mathfrak{R}_1}{2m} c_1 \Psi_1 + \frac{\mathfrak{R}_1}{2m}\left[c_0\left\{\Psi_2 - \left(s+\frac{2\mu}{m}\mathfrak{R}_1 c_0\right)\Psi_1^2\right\}\right]$$

Using the inequalities $|c_n| \leq 1, |\Psi_n(z)| < 1$, we have:

$$\begin{aligned} \left|b_{2m+1}-\mu b_{m+1}^2\right| &\leq \frac{\mathfrak{R}_1}{2m}\left[c_1 + c_0\left\{\left|\Psi_2 - \left(s+\frac{2\mu}{m}\mathfrak{R}_1 c_0\right)\Psi_1^2\right|\right\}\right] \\ &\leq \frac{\mathfrak{R}_1}{2m}\left[1+\left|\Psi_2 - l\Psi_1^2\right|\right] \end{aligned}$$

where l is presened by (19). For $\mu \in \mathbb{C}$, according to Lemma 3, there are three situations:

Case 1: If:

$$\mu \leq \frac{1}{2\mathfrak{R}_1 c_0}(D+\mathfrak{R}_1 c_0 - \frac{m-1}{m}\mathfrak{R}_1 c_0 - 1) = \zeta_1.$$

which implies $l \leq -1$, we have:

$$\begin{aligned} \left|b_{2m+1}-\mu b_{m+1}^2\right| &\leq \frac{\mathfrak{R}_1}{2m}\left[1+\left|\Psi_2 - l\Psi_1^2\right|\right] \\ &\leq \frac{\mathfrak{R}_1}{2m}[1-l] = \frac{\mathfrak{R}_1}{2m}\left[1+\mathfrak{R}_1 c_0 + D - \frac{2\mu+m-1}{m}\mathfrak{R}_1 c_0\right]. \end{aligned}$$

Case 2: If:

$$\mu \geq \frac{1}{2\mathfrak{R}_1 c_0}(D+\mathfrak{R}_1 c_0 - \frac{m-1}{m}\mathfrak{R}_1 c_0 + 1) = \zeta_2,$$

which implies that $l \geq 1$, thus:

$$\begin{aligned} \left|b_{2m+1}-\mu b_{m+1}^2\right| &\leq \frac{\mathfrak{R}_1}{2m}\left[1+\left|\Psi_2 - l\Psi_1^2\right|\right] \\ &\leq \frac{\mathfrak{R}_1}{2m}[1+l] = \frac{\mathfrak{R}_1}{2m}\left[1-\mathfrak{R}_1 c_0 - D + \frac{2\mu+m-1}{m}\mathfrak{R}_1 c_0\right]. \end{aligned}$$

Case 3: If:

$$\zeta_1 \leq \mu \leq \zeta_2,$$

which implies $-1 \leq l \leq 1$,thus:

$$\begin{aligned}\left|b_{2m+1}-\mu b_{m+1}^{2}\right| &\leq \frac{\mathfrak{R}_1}{2m}\left[1+\left|\Psi_2-l\Psi_1^2\right|\right]\\ &\leq \frac{\mathfrak{R}_1}{m}.\end{aligned}$$

Letting $\Phi(z)=1$ and $m=1$ in Theorem 3, we then obtain: □

Corollary 6. *if $f\in k-\mathcal{S}_{\mathcal{H}_m,q}$, and $\Phi(z)=1$, then:*

$$\left|b_{2m+1}-\mu b_{m+1}^{2}\right| \leq \begin{cases} \frac{\mathfrak{R}_1}{2m}\left[1+D+\mathfrak{R}_1-\frac{2\mu-m+1}{m}\mathfrak{R}_1\right] & , \quad \mu\leq\zeta_1' \\ \frac{\mathfrak{R}_1}{m} & , \quad \zeta_1'\leq\mu\leq\zeta_2' \\ \frac{\mathfrak{R}_1}{2m}\left[1-D-\mathfrak{R}_1+\frac{2\mu-m+1}{m}\mathfrak{R}_1\right] & , \quad \mu\geq\zeta_2' \end{cases},$$

where:

$$\begin{aligned}\zeta_1' &= \frac{m}{2\mathfrak{R}_1}(D+\frac{1}{m}\mathfrak{R}_1-1)\\ \zeta_2' &= \frac{m}{2\mathfrak{R}_1}(D+\frac{1}{m}\mathfrak{R}_1+1).\end{aligned}$$

Moreover, letting $m=1$ in the last three inequalities, we obtain:

$$\left|b_3-\mu b_2^2\right|\leq\begin{cases}\frac{\mathfrak{R}_1}{2}[1+D+(1-2\mu)\mathfrak{R}_1] & , \quad \mu\leq\zeta_1''\\ \mathfrak{R}_1 & , \quad \zeta_1''\leq\mu\leq\zeta_2''\\ \frac{\mathfrak{R}_1}{2m}[1-D-(1-2\mu)\mathfrak{R}_1] & , \quad \mu\geq\zeta_2''\end{cases},$$

where:

$$\begin{aligned}\zeta_1'' &= \frac{1}{\mathfrak{R}_1}(D+\mathfrak{R}_1-1)\\ \zeta_2'' &= \frac{1}{2\mathfrak{R}_1}(D+\mathfrak{R}_1+1).\end{aligned}$$

Theorem 4. *If $f\in k-\mathcal{S}_{\mathcal{H}_m,q}$, then:*

$$\left|b_{2m+1}-\mu b_{m+1}^{2}\right| \leq \begin{cases} \frac{\mathfrak{R}_1}{2m}\left[1-\frac{2\mu+m-1}{m}\mathfrak{R}_1c_0+D+\mathfrak{R}_1c_0\right] & , \quad \mu\leq\alpha_2 \\ \frac{\mathfrak{R}_1}{m} & , \quad \alpha_2\leq\mu\leq\alpha_1 \\ \frac{\mathfrak{R}_1}{2m}\left[1+\frac{2\mu+m-1}{m}\mathfrak{R}_1c_0-D-\mathfrak{R}_1c_0\right] & , \quad \mu\geq\alpha_1 \end{cases},$$

where:

$$\begin{aligned}\alpha_1 &= \frac{m}{2\mathfrak{R}_1c_0}\left(D+\mathfrak{R}_1c_0-\frac{m-1}{m}\mathfrak{R}_1c_0+1\right),\\ \alpha_2 &= \frac{m}{2\mathfrak{R}_1c_0}\left(D+\mathfrak{R}_1c_0-\frac{m-1}{m}\mathfrak{R}_1c_0-1\right).\end{aligned}$$

Proof. From the equality (18), we have:

$$\begin{aligned}
b_{2m+1}-\mu b_{m+1}^2 &= \frac{\mathfrak{R}_1}{2m}c_1\Psi_1+\frac{\mathfrak{R}_1}{2m}\left[c_0\left\{\Psi_2-\left(s+\frac{\mu}{m^2}\mathfrak{R}_1^2c_0\right)\Psi_1^2\right\}\right] \\
&= \frac{\mathfrak{R}_1}{2m}c_1\Psi_1 \\
&\quad +\frac{\mathfrak{R}_1}{2m}\left[c_0\left\{\Psi_2_\Psi_1^2-\left(s+\frac{\mu}{m^2}\mathfrak{R}_1^2c_0-1\right)\Psi_1^2\right\}\right] \\
&= \frac{\mathfrak{R}_1}{2m}c_1\Psi_1 \\
&\quad +\frac{\mathfrak{R}_1}{2m}\left[c_0\left\{\Psi_2_\Psi_1^2+\left(1-s-\frac{\mu}{m^2}\mathfrak{R}_1^2c_0\right)\Psi_1^2\right\}\right] \qquad (21)
\end{aligned}$$

Where s is given by (16) . Using the inequalities $|c_n|\le 1, |\Psi_n(z)|<1$, we have:

$$\left|b_{2m+1}-\mu b_{m+1}^2\right|\le\frac{\mathfrak{R}_1}{2m}+\frac{\mathfrak{R}_1}{2m}\left[\left|\Psi_2\_\Psi_1^2\right|+\left|1-s-\frac{2\mu}{m}\mathfrak{R}_1c_0\right||\Psi_1|^2\right] \qquad (22)$$

Case 1: Choosing the representation of second pharantezis on the right hand side of (21):

$$-s-\frac{\mu}{m}\mathfrak{R}_1c_0=\frac{2\mu+m(m-1)}{m}\mathfrak{R}_1c_0-D-\mathfrak{R}_1c_0\le -1$$

then we obtain:

$$\mu\ge\frac{m}{2\mathfrak{R}_1c_0}(D+\mathfrak{R}_1c_0-\frac{m-1}{2m}\mathfrak{R}_1c_0+1)=\alpha_1.$$

Let $\mu\ge\alpha_1$. According the Lemma 2 we can write that:

$$\left|\Psi_2\_\Psi_1^2\right|\le 1 \qquad (23)$$

Utilizing (23), Lemmas 2 and 3 and putting expression of $s=\frac{(m-1)}{2m}\mathfrak{R}_1c_0-\mathfrak{R}_1c_0-\frac{\mathfrak{R}_2}{\mathfrak{R}_1}$, given in (16), in (22) we obtain:

$$\begin{aligned}
\left|b_{2m+1}-\mu b_{m+1}^2\right| &\le \frac{\mathfrak{R}_1}{2m}+\frac{\mathfrak{R}_1}{2m}\left[1+\left(\frac{2\mu+m-1}{m}\mathfrak{R}_1c_0-D-\mathfrak{R}_1c_0-1\right)|\Psi_1|^2\right] \\
&\le \frac{\mathfrak{R}_1}{2m}\left[1+\frac{2\mu+m-1}{m}\mathfrak{R}_1c_0-D-\mathfrak{R}_1c_0\right].
\end{aligned}$$

Case 2: Choosing the expression of second pharantezis on the right hand side of (21):

$$-s-\frac{\mu}{m}\mathfrak{R}_1c_0=\frac{2\mu+m-1}{m}\mathfrak{R}_1c_0-D-\mathfrak{R}_1c_0\ge 1$$

then we obtain:

$$\mu\le\frac{m}{2\mathfrak{R}_1c_0}(D+\mathfrak{R}_1c_0-\frac{m-1}{m}\mathfrak{R}_1c_0-1)=\alpha_2.$$

Let $\mu\le\alpha_2$. Then, from (22) we can write that:

$$\left|b_{2m+1}-\mu b_{m+1}^2\right|\le\frac{\mathfrak{R}_1}{2m}+\frac{\mathfrak{R}_1}{2m}\left[1+\left(1-s-\frac{2\mu}{m}\mathfrak{R}_1c_0\right)|\Psi_1|^2\right]. \qquad (24)$$

According to Lemma 3 we obtain:

$$\Psi_2\le 1-|\Psi_1|^2 \text{ and } |\Psi_1|\le 1,$$

If we apply to (24) inequalities above, also putting:

$$-s-\frac{\mu}{m^2}\mathfrak{R}_1^2c_0=\frac{2\mu+m-1}{m}\mathfrak{R}_1c_0-D-\mathfrak{R}_1c_0$$

in (24), we have:

$$\begin{aligned}\left|b_{2m+1}-\mu b_{m+1}^2\right| &\leq \frac{\mathfrak{R}_1}{2m} \\ &+\frac{\mathfrak{R}_1}{2m}\left[1-|\Psi_1|^2+\left(D+\mathfrak{R}_1c_0-\frac{2\mu+m-1}{m}\mathfrak{R}_1c_0\right)|\Psi_1|^2\right].\end{aligned}$$

This implies that:

$$\left|b_{2m+1}-\mu b_{m+1}^2\right| \leq \frac{\mathfrak{R}_1}{2m}\left[1-\frac{2\mu+m-1}{m}\mathfrak{R}_1c_0+D+\mathfrak{R}_1c_0\right].$$

Case 3: Choosing the representation in second pharanthezis on the right hand side of (21):

$$-1\leq -s-\frac{\mu}{m^2}\mathfrak{R}_1c_0=\frac{2\mu+m-1}{m}\mathfrak{R}_1c_0-D-\mathfrak{R}_1c_0\leq 1,$$

we get $\alpha_2\leq\mu\leq\alpha_1$. Under this condition,

$$\left|b_{2m+1}-\mu b_{m+1}^2\right| \leq \frac{\mathfrak{R}_1}{2m}+\frac{\mathfrak{R}_1}{2m}=\frac{\mathfrak{R}_1}{m}.$$

By choosing $m=1$ in Theorem 4, we get next: □

$$\left|b_3-\mu b_2^2\right|\leq\begin{cases}\frac{\mathfrak{R}_1}{2}[D+(1-2\mu)\mathfrak{R}_1c_0] & , \quad \mu\leq\alpha_2' \\ \mathfrak{R}_1 & , \quad \alpha_2'\leq\mu\leq\alpha_1' \\ \frac{\mathfrak{R}_1}{2}[D-(1-2\mu)\mathfrak{R}_1c_0] & , \quad \mu\geq\alpha_1'\end{cases},$$

where:

$$\begin{aligned}\alpha_1' &= \frac{1}{2\mathfrak{R}_1c_0}(D+\mathfrak{R}_1c_0+1) \\ \alpha_2' &= \frac{1}{2\mathfrak{R}_1c_0}(D+\mathfrak{R}_1c_0-1).\end{aligned}$$

Letting $\Phi(z)=1$ and $m=1$ in Theorem 4, we get next:

Corollary 7. *If $f\in k-\mathcal{S}_{\mathcal{H}_{m,q}}$ and $\Phi(z)=1$, then we have:*

$$\left|b_{2m+1}-\mu b_{m+1}^2\right|\leq\begin{cases}\frac{\mathfrak{R}_1}{2m}\left[D+\left(1-\frac{2\mu+m-1}{m}\right)\mathfrak{R}_1+1\right] & , \quad \mu\leq\alpha_2'' \\ \frac{\mathfrak{R}_1}{m} & , \quad \alpha_2''\leq\mu\leq\alpha_1'' \\ \frac{\mathfrak{R}_1}{2m}\left[D-\left(1-\frac{2\mu+m-1}{m}\right)\mathfrak{R}_1+1\right] & , \quad \mu\geq\alpha_1''\end{cases},$$

where:

$$\begin{aligned}\alpha_1'' &= \frac{m}{2\mathfrak{R}_1}\left(D+\left(1-\frac{m-1}{m}\right)\mathfrak{R}_1c_0+1\right), \\ \alpha_2'' &= \frac{m}{2\mathfrak{R}_1c_0}\left(D+\left(1-\frac{m-1}{m}\right)\mathfrak{R}_1c_0-1\right).\end{aligned}$$

Furthermore, choosing $m=1$, we get:

$$\left|b_3-\mu b_2^2\right| \leq \begin{cases} \frac{\mathfrak{R}_1}{2}[D+(1-2\mu)\mathfrak{R}_1+1] & , \quad \mu \leq \alpha_2''' \\ \frac{\mathfrak{R}_1}{m} & , \quad \alpha_2''' \leq \mu \leq \alpha_1''' \\ \frac{\mathfrak{R}_1}{2m}[D-(1-2\mu)\mathfrak{R}_1+1] & , \quad \mu \geq \alpha_1''' \end{cases},$$

where:

$$\begin{aligned} \alpha_1''' &= \frac{1}{2\mathfrak{R}_1}(D+\mathfrak{R}_1 c_0+1), \\ \alpha_2''' &= \frac{1}{2\mathfrak{R}_1 c_0}(D+\mathfrak{R}_1 c_0-1). \end{aligned}$$

Putting the values of $\mathfrak{R}_1 = \mathfrak{R}_1(k), \mathfrak{R}_2 = \mathfrak{R}_2(k) = D(k)\mathfrak{R}_1(k)$ and k from Lemma 4, we obtain some new results of Theorem 4:

Corollary 8. *Let $f \in k-\mathcal{S}_{\mathcal{H}_m,q}$ and $0 \leq k < 1$. Then:*

$$\left|b_{2m+1}-\mu b_{m+1}^2\right| \leq$$

$$\begin{cases} \frac{\mathcal{T}^2}{m(1-k^2)}\left[\frac{2+\mathcal{T}^2}{3}-\left(\frac{2\mu+m-1}{m}-1\right)\left(\frac{2\mathcal{T}^2}{1-k^2}\right)c_0+1\right] & , \quad \mu \leq \beta_2 \\ \frac{2\mathcal{T}^2}{m(1-k^2)} & , \quad \beta_2 \leq \mu \leq \beta_1 \\ \frac{\mathcal{T}^2}{m(1-k^2)}\left[\left(\frac{2\mu+m-1}{m}-1\right)\left(\frac{2\mathcal{T}^2}{1-k^2}\right)c_0-\frac{2+\mathcal{T}^2}{3}+1\right] & , \quad \mu \geq \beta_1 \end{cases},$$

where:

$$\begin{aligned} \beta_1 &= \frac{m(1-k^2)}{4\mathcal{T}^2 c_0}\left[\frac{2+\mathcal{T}^2}{3}+\frac{2\mathcal{T}^2}{m(1-k^2)}c_0+1\right], \\ \beta_2 &= \frac{m(1-k^2)}{4\mathcal{T}^2 c_0}\left[\frac{2+\mathcal{T}^2}{3}+\frac{2\mathcal{T}^2}{m(1-k^2)}c_0-1\right]. \end{aligned}$$

Moreover, taking $m = 1$ in Corollary 8, we obtain:

$$\left|b_3-\mu b_2^2\right| \leq \begin{cases} \frac{2\mathcal{T}^2}{2(1-k^2)}\left[1+\frac{2+\mathcal{T}^2}{3}+(1+2\mu)\left(\frac{2\mathcal{T}^2}{1-k^2}\right)c_0\right] & , \quad \mu \leq \beta_2' \\ \frac{2\mathcal{T}^2}{m(1-k^2)} & , \quad \beta_2' \leq \mu \leq \beta_1' \\ \frac{2\mathcal{T}^2}{2(1-k^2)}\left[1-\frac{2+\mathcal{T}^2}{3}-(1-2\mu)\left(\frac{2\mathcal{T}^2}{1-k^2}\right)c_0\right] & , \quad \mu \geq \beta_1' \end{cases},$$

where:

$$\begin{aligned} \beta_1' &= \frac{(1-k^2)}{2\mathcal{T}^2 c_0}\left[\frac{2+\mathcal{T}^2}{3}-\frac{2\mathcal{T}^2}{1-k^2}c_0+1\right], \\ \beta_2' &= \frac{(1-k^2)}{2\mathcal{T}^2 c_0}\left[\frac{2+\mathcal{T}^2}{3}+\frac{2\mathcal{T}^2}{1-k^2}c_0-1\right]. \end{aligned}$$

Corollary 9. *Let $f \in k-\mathcal{S}_{\mathcal{H}_m,q}$ and $k = 1$. Then:*

$$\left|b_{2m+1} - \mu b_{m+1}^2\right| \leq \begin{cases} \frac{4}{m\pi^2}\left[\frac{1}{3} + \left(\frac{2\mu+m-1}{m} - 1\right)\frac{8}{\pi^2}c_0\right] & , \quad \mu \leq \eta_2 \\ \frac{8}{m\pi^2} & , \quad \eta_2 \leq \mu \leq \eta_1 \\ \frac{4}{m\pi^2}\left[\frac{5}{3} - \left(\frac{2\mu+m-1}{m} - 1\right)\frac{8}{\pi^2}c_0\right] & , \quad \mu \geq \eta_1 \end{cases},$$

where:

$$\begin{aligned} \eta_1 &= \frac{m\pi^2}{16c_0}\left[\frac{5}{3} + \frac{8}{m\pi^2}c_0\right], \\ \eta_2 &= \frac{m\pi^2}{16c_0}\left[\frac{8}{m\pi^2}c_0 - \frac{1}{3}\right]. \end{aligned}$$

Further, taking $m = 1$ in Corollary 9, we obtain:

$$\left|b_3 - \mu b_2^2\right| \leq \begin{cases} \frac{4}{\pi^2}\left[\frac{1}{3} + (2\mu - 1)\frac{8}{\pi^2}c_0\right] & , \quad \mu \leq \eta_2' \\ \frac{8}{\pi^2} & , \quad \eta_2' \leq \mu \leq \eta_1' \\ \frac{4}{\pi^2}\left[\frac{5}{3} - (2\mu - 1)\frac{8}{\pi^2}c_0\right] & , \quad \mu \geq \eta_1' \end{cases},$$

where:

$$\begin{aligned} \eta_1' &= \frac{\pi^2}{16c_0}\left[\frac{5}{3} + \frac{8}{\pi^2}c_0\right], \\ \eta_2' &= \frac{\pi^2}{16c_0}\left[\frac{8}{\pi^2}c_0 - \frac{1}{3}\right]. \end{aligned}$$

Corollary 10. *Let $f \in k - \mathcal{S}_{\mathcal{H}_m,q}$ and $k > 1$. Then:*

$$\left|b_{2m+1} - \mu b_{m+1}^2\right| \leq$$

$$\begin{cases} \frac{\pi^2}{8m\mathrm{K}^2(r)(k^2-1)(1-r)\sqrt{r}}\left[1 + \left(\frac{2\mu+m-1}{m} - 1\right)\frac{\pi^2}{4\mathrm{K}^2(r)(k^2-1)(1-r)\sqrt{r}}c_0 - \frac{4\mathrm{K}^2(r)(r^2+6r+1)-\pi^2}{24\mathrm{K}^2(r)(1+r)\sqrt{r}}\right] & , \quad \mu \leq \vartheta_2 \\ \frac{\pi^2}{4m\mathrm{K}^2(r)(k^2-1)(1-r)\sqrt{r}} & , \quad \vartheta_2 \leq \mu \leq \vartheta_1 \\ \frac{\pi^2}{8m\mathrm{K}^2(r)(k^2-1)(1-r)\sqrt{r}}\left[1 - \left(\frac{2\mu+m-1}{m} - 1\right)\frac{\pi^2}{4\mathrm{K}^2(r)(k^2-1)(1-r)\sqrt{r}}c_0 + \frac{4\mathrm{K}^2(r)(r^2+6r+1)-\pi^2}{24\mathrm{K}^2(r)(1+r)\sqrt{r}}\right] & , \quad \mu \geq \vartheta_1 \end{cases}$$

where:

$$\begin{aligned} \vartheta_1 &= \frac{2m\mathrm{K}^2(r)(k^2-1)(1-r)\sqrt{r}}{\pi^2 c_0}\left[\frac{4\mathrm{K}^2(r)(r^2+6r+1) - \pi^2}{24\mathrm{K}^2(r)(1+r)\sqrt{r}} + \frac{\pi^2}{4m\mathrm{K}^2(r)(k^2-1)(1-r)\sqrt{r}}c_0 + 1\right], \\ \vartheta_2 &= \frac{2m\mathrm{K}^2(r)(k^2-1)(1-r)\sqrt{r}}{\pi^2 c_0}\left[\frac{4\mathrm{K}^2(r)(r^2+6r+1) - \pi^2}{24\mathrm{K}^2(r)(1+r)\sqrt{r}} + \frac{\pi^2}{4m\mathrm{K}^2(r)(k^2-1)(1-r)\sqrt{r}} - 1\right]. \end{aligned}$$

Moreover, taking $m = 1$ in Corollary 10, we obtain:

$$\left|b_3 - \mu b_2^2\right| \leq$$

$$\begin{cases} \frac{\pi^2}{8\mathrm{K}^2(r)(k^2-1)(1-r)\sqrt{r}}\left[1+\frac{(2\mu-1)\pi^2}{4\mathrm{K}^2(r)(k^2-1)(1-r)\sqrt{r}}c_0-\frac{4\mathrm{K}^2(r)(r^2+6r+1)-\pi^2}{24\mathrm{K}^2(r)(1+r)\sqrt{r}}\right] , & \mu \le \vartheta_2' \\ \frac{\pi^2}{4m\mathrm{K}^2(r)(k^2-1)(1-r)\sqrt{r}} , & \vartheta_2' \le \mu \le \vartheta_1' , \\ \frac{\pi^2}{8\mathrm{K}^2(r)(k^2-1)(1-r)\sqrt{r}}\left[1-\frac{(2\mu-1)\pi^2}{4\mathrm{K}^2(r)(k^2-1)(1-r)\sqrt{r}}c_0+\frac{4\mathrm{K}^2(r)(r^2+6r+1)-\pi^2}{24\mathrm{K}^2(r)(1+r)\sqrt{r}}\right] , & \mu \ge \vartheta_1' \end{cases}$$

where:

$$\begin{aligned} \vartheta_1' &= \frac{2\mathrm{K}^2(r)(k^2-1)(1-r)\sqrt{r}}{\pi^2 c_0} \\ &\quad \left[\frac{4\mathrm{K}^2(r)(r^2+6r+1)-\pi^2}{24\mathrm{K}^2(r)(1+r)\sqrt{r}}+\frac{\pi^2}{4\mathrm{K}^2(r)(k^2-1)(1-r)\sqrt{r}}c_0+1\right], \\ \vartheta_2' &= \frac{2\mathrm{K}^2(r)(k^2-1)(1-r)\sqrt{r}}{\pi^2 c_0} \\ &\quad \left[\frac{4\mathrm{K}^2(r)(r^2+6r+1)-\pi^2}{24\mathrm{K}^2(r)(1+r)\sqrt{r}}+\frac{\pi^2}{4\mathrm{K}^2(r)(k^2-1)(1-r)\sqrt{r}}c_0-1\right]. \end{aligned}$$

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