*symmetry*

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# Simplification of Galactic Dynamic Equations

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**Abstract:** Galactic dynamics is the foundation for simulating galactic structure and for solving other problems. However, the traditional dynamic equations include some unreasonable assumptions and are therefore scientifically invalid. In this paper, by introducing the following three working assumptions, we established the galactic dynamics of high precision and convenient formalism. 1. In the research of large-scale structure, the retarded potential of the gravitational field should be taken into account, and the weak field and low velocity approximation of Einstein's field equation should be adopted. 2. The stars in a fully developed galaxy should be zero-pressure and inviscid fluid, and the equation of motion is different from that of ordinary continuum mechanics. Stars move along geodesics. 3. The structure of the galaxy is only related to the total mass density distribution. The equation of state of dark halo is different from that of ordinary luminous interstellar matter, so their trajectories are also very different. In a galaxy, the dark halo and the ordinary matter are automatically separated. The total mass density distribution can be presupposed according to the observation data, and then it can be determined by comparing the solution of the equations with the observed data. These assumptions and treatments are supported by theory and observation. The variables of the equations of simplified galactic dynamics are separated from each other, and the equations are well-posed and can be solved according to a definite procedure. The solution explains the Tully–Fisher relation. Therefore, this simplified dynamic equation system provides a more reasonable and practical framework for the further study of galactic structure, and can solve many practical problems. In addition, it is closely related to the study of dark matter halo in galaxy.

**Keywords:** galactic dynamics; spiral structure; density wave; gravitational retarded potential; Tully–Fisher relation; dark halo; dark matter distribution



**Citation:** Gu, Y.-Q. Simplification of Galactic Dynamic Equations. *Symmetry* **2022**, *14*, 407. https://doi.org/10.3390/sym14020407

Academic Editors: Vasilis K. Oikonomou, Abraham A. Ungar and Stefano Profumo

Received: 10 January 2022
Accepted: 16 February 2022
Published: 18 February 2022



## 1. Introduction

Most fully developed galaxies have vivid spiral structures that have a profound effect on the generation and evolution of stars. In the past 70 years, a great deal of research has been done to reveal the essence of spiral structures. Although great progress has been made in understanding the mechanism of spiral arm formation, there are still many mysteries to be revealed. We do not even know whether the spiral is a long-standing phenomenon, or whether it is a transient and multiple regeneration phenomenon in the evolution of galaxies [1–3]. The prevailing view now is that the galactic spiral structure should be a quasi-stationary density wave [4]. According to the concept of density wave proposed by B. Lindblad, C. C. Lin and F. H. Shu analyzed and solved the quasi-stable solution of planetary fluid dynamics in gravitational field. However, there are other ideas that the spiral structure is a short-lived but recurring spiral pattern formed by the vibration amplification in the rotating galaxy disk [5], or caused by external forces [6]. In the literature [7,8], by comparing the distant galaxies observed by the Hubble Space Telescope with those observed on the ground, the results suggest that the spiral structure in the range of $0 < z < 1$ in the universe may be a long-standing quasi-stationary pattern.

In density wave theory, Lin and Shu assume that the spiral structure is quasi-stationary in the form of density waves on the disk of a galaxy. The key points of this assumption are as follows: 1. The spiral galaxy has an invisible background gravitational field, which has a regular spiral structure, and the visible optical spiral arm is only the external manifestation of this spiral force field. 2. The galactic spiral pattern is made up of flowing materials, which is not the fixed same materials on the spiral arm, but changes over time. 3. The entire spiral pattern is quasi-steady and rotates at constant angular velocity $\Omega$ like a rigid body. The above assumption is called the quasi-stable spiral structure assumption.

A spiral galaxy consists of at least three parts: disk, bulge and halo. Because of the large dispersion velocity of the two parts of the bulge and halo, it is not easy to produce density disturbance, so it is usually treated as rigid body, and its function is to form a part of the fundamental gravitational field. Only the galactic disk is part of the density wave. In terms of composition, most of the matter in the galactic disk is stars, but there is also a small amount of gas. Because of the great distance between stars, the average 'collision interval' is several orders of magnitude larger than the age of the galaxy, so the collision can be omitted.

If the spiral arm of a spiral galaxy is made of fixed matter, and the matter near the center of the galaxy rotates faster than the matter at the edge of the galaxy, then the spiral arm becomes tighter and tighter. After only a few cycles, the arms become entangled and indistinguishable. Density wave theory has succeeded in avoiding this contradiction, with the arm matter not made up of fixed planets but of regions with higher star densities, similar to traffic jams on highways. For dynamic reasons, luminous matter such as a star moves slowly within the spiral arm, lingering for more time and gathering together, but the stars outside the spiral arm move fast and have lower densities.

Despite the great success of density wave theory, there are still some defects and problems. 1. nonlinear effects; 2. analysis of the open arm and barred spiral system; 3. detailed study of the near-central region; 4. dynamical classification of galaxies and so on. When using density wave theory to make galaxy arm pattern, for some cases, the rotary arm pattern is in good agreement with the actual one. However, some other galaxies are not the case, and the barred spiral structure is not clearly explained. The better coincident part is mainly on the outer side of the spiral galaxy, but the results are often far from the actual observations when approaching the center.

The orbit of a star in a non-central potential field is usually a non-closed and complicated spatial curve, which is closely related to the initial velocity and its direction [1]. Therefore, it is more reasonable to take the star system in a galaxy as fluid rather than mass points. Although the dynamics of both models are essentially the same Newtonian mechanics, the initial and boundary conditions are set differently. In the case of fluid, the streamlines are consistent fields, which are a natural result of stars generating from nebula.

W. Dehnen and J. Binney fitted the mass distribution within the galaxy with the observed data (such as the rotation curve and the Oort constant [9]), but found that the mass distribution of the fitting was uncertain. In [10,11], the authors numerically simulated the two-dimensional stellar fluid dynamics embedded in a planar galactic disk in dark matter halo and solved the Boltzmann equation to the second-order moment [1]. The references [12] propose a kinetic approach to explain the formation of spirals and rings in galaxy clusters based on spiral arms, rings, and pseudo-rings driven near the unstable equilibrium point of a given rotating barred potential, and so on, associated with the invariant manifold composed of periodic orbitals around the equilibrium point. By adjusting the kinetic parameters of the main galaxy, the spiral and ringed structures are obtained. The dark matter is beyond the standard model, how to disclose its nature is one of the most challenging problem in astrophysics.

In this paper, we propose a more reasonable and practical galactic dynamics to study the structure and properties of galaxies, which is closely linked to the study of dark matter halo. This framework has the following advantages: 1. The galactic dynamic equations are highly accurate and convenient to solve for the solution, but the previously used equations is not suitable due to unreasonable assumptions. 2. We derived the structure equations for a stable non-warped galaxy. 3. The galactic dynamics is the foundation to analyze and explain other properties of galaxies, on which other problems can be easily solved. We derived and confirmed the experimental law—the Tully–Fisher relation according to the model. 4. By calculating the high precision total density distribution $\rho$, we can study the dynamic properties of dark matter halo such as the state functions. In simplified dynamical equations, the physical variables are separated from each other. The equations can be solved by fixed procedure, and give a clearer explanation for the spiral and barred structure of galaxies. Thus, this simplified system of equations provides a more reasonable and practical framework for further study of the galactic structure and properties of the dark matter halo.

The dark matter halo provides a dynamical background for the generation and development of stars and planets in a galaxy, and the relationship is similar to that of the amniotic fluid and the fetus. Therefore, the mechanical properties of dark matter are very different from those of ordinary luminous matter, and their moving trajectories are also very different. Technical schemes for detecting dark matter particles by means of detecting ordinary particles are not feasible, because dark matter particles do not interact with ordinary particles at all and any other interaction will result in instable structure of a galaxy. Dark matter detection can only be performed by dynamic methods.

## 2. Simplification of Galactic Dynamic Equation

In the current theoretical analysis, the commonly used galactic dynamics is similar to the following one [13]

$$\nabla^2\Phi = \kappa\rho, \tag{1}$$
$$(\partial_t + \vec{V}\cdot\nabla)\vec{V} + 2\Omega\times\vec{V} = -\nabla\Phi - \tfrac{1}{\rho}\nabla P, \tag{2}$$
$$\partial_t P + \nabla\cdot(P\vec{V}) = -P(\gamma-1)\nabla\cdot\vec{V} - \Lambda, \tag{3}$$
$$\partial_t\rho + \nabla\cdot(\rho\vec{V}) = 0, \tag{4}$$

in which $\kappa = 4\pi G$, $\Omega$ is the angular speed of the galactic disk, $\gamma$ adiabatic index, $\Lambda$ the cooling term reflecting the gravitational disturbance. Some more complicated models also introduce viscous term.

Next we take the dynamic Equations (1)–(4) as example to analyze several problems in current galactic dynamics. 1. The Einstein's field equation of gravity is essentially a wave equation. In [14], the measurement discloses that gravitational waves travel at $c$ to a precision of $10^{-15}$. The diameters of galaxies are over 100,000 light-years, so the effect of retarded potential of gravity should be taken into account in solving the dynamic equations in such cases. The stars near the center of a galaxy turn around the center several times before the change of the corresponding gravitational field reaches the edge of the galaxy. Whereas the Newton's gravity is an action at a distance and ignores this time delay. So there is a hidden danger to use (1) to calculate the gravitational field in such an occasion, and the relativistic effect must be considered in the large-scale structure.

The weak field and low velocity approximation of the Einstein's field equation is as follows (see Appendix A)

$$\partial_\alpha\partial^\alpha\Phi = -\kappa\rho, \tag{5}$$

in which $\rho$ is the total equivalent mass density. The solution of (5) is retarded potential

$$\Phi(t,\vec{x}) = -\frac{\kappa}{4\pi c^2}\int_{r\le ct}\frac{\rho(ct-r,\vec{y})}{r}\mathrm{d}^3y, \qquad r = |\vec{x}-\vec{y}|.$$

In contrast with the Newtonian gravitational potential of (1), the retarded potential has finite propagating speed $c$ for perturbation. Not only in the cases of high-speed we need relativity, but also in the cases of large distance or large structure we need relativity. The galactic dynamics must be discussed by general relativity, because the characteristic time of gravity propagating from the center to the edge is much larger than the moving period of the stars near the center, and this time difference has strong effect on the structure of the galaxy.

2. In the early stages of galaxy formation, luminous matter consists of hot plasma. The evolution process is complex at this time, and the dynamics should indeed take into account the influence of pressure, temperature, viscosity, and even magneto fluid effect. However as galaxies mature, luminous matter forms stars, and the motion of the stellar system becomes more and more orderly, which is similar to what happened in the solar system. The stars and ordinary luminous matter in the galaxy are mainly moving along geodesics, and these materials form zero-pressure and non-viscous fluid, so the energy-momentum tensor of the fluid should be as follows

$$T_s^{\mu\nu} = \rho_s U^\mu U^\nu, \tag{6}$$

where $\rho_s$ is the comoving mass density of stars and ordinary luminous particles, $U^\mu$ is the 4-vectors speed distribution of ordinary matter. The ordinary matter satisfies the energy conservation law $T^{\mu\nu}_{s;\nu} = 0$ independent of dark halo. In the form of continuity equations and equations of motion, we obtain the dynamic equations of the stellar system as

$$U^\mu \partial_\mu \rho_s + \rho_s U^\mu_{;\mu} = 0, \tag{7}$$

$$U^\nu U^\mu_{;\nu} = 0. \tag{8}$$

For convenience, we take $c = 1$ as unit of speed. The Equations (2) and (3) derived from continuum theory and thermodynamics are not consistent with the actual conditions as mentioned above. The variables in such equations are strongly coupled, which brings great inconvenience and complexity to the analysis and solution of the equations.

3. The equation of state of the dark halo is different from that of ordinary luminous interstellar matter. For dark halo with self-action nonlinear potential, its energy-momentum tensor has the following form [15,16]

$$T^{\mu\nu} = (\rho_{\text{tot}} + P)\mathcal{U}^\mu \mathcal{U}^\nu + (W - P)g^{\mu\nu}, \tag{9}$$

where $W = W(\rho_{\text{tot}})$ corresponds to the nonlinear potential of the dark particles, which has the effect of negative pressure and causes the moving trajectories of the dark halo to deviate from geodesics. For dark spinors, we have $W \sim \rho \gg P$, so it also acts like dark energy. According to the energy-momentum conservation law $T^{\mu\nu}_{;\nu} = 0$, we obtained the continuity equation $\mathcal{U}_\mu T^{\mu\nu}_{;\nu} = 0$ and the equation of motion of the dark halo

$$(\rho_{\text{tot}} + P)\mathcal{U}^\nu \mathcal{U}^\mu_{;\nu} = (g^{\mu\nu} - \mathcal{U}^\mu \mathcal{U}^\nu)\partial_\nu (P - W), \tag{10}$$

which is quite different from the geodesic equation (8). Thus the dark halo is automatically separated from ordinary matter during galaxy formation. So near the solar system, there are few traces of the dark matter that dominates galaxies. Dark halo consists of diffuse gases, and the equation of motion (10) is a complete 1 + 3 dimensional nonlinear field equation. To analyze or solve such equation system is extremely difficult. However, the equation system (6) can be regarded as a planar dynamics in the domain that deviates slightly from the center of the galaxy, and that greatly simplifies the analysis and solving process for the equations. The galactic dynamics currently used confuses the two different kinds of concepts in (6) and (9). For example, in Equations (1)–(4), the mass density $\rho$ in each equation should be different concept. Therefore, the generality and effectiveness of the traditional galactic dynamic equations are quite weak, and it is difficult to obtain the right results in accordance with the actual situation.

Taking into account all above factors, we obtain the galactic dynamic equations in Minkowski space-time under weak field and low speed approximations of relative error $\sim 10^{-6}$(see Appendix A)

$$\partial_\alpha \partial^\alpha \Phi = -\kappa\rho, \tag{11}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{V} \equiv (\partial_t + \vec{V}\cdot\nabla)\vec{V} = -\nabla\Phi, \tag{12}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_s \equiv (\partial_t + \vec{V}\cdot\nabla)\rho_s = -\rho_s \nabla\cdot\vec{V}, \tag{13}$$

where $\vec{V}(t,\vec{x})$ is the velocity field of ordinary matter, and the total equivalent mass density is given by

$$\rho \doteq \rho_{\mathrm{tot}} - 2W + 3P. \tag{14}$$

In the above galactic dynamic equations, the variables $(\rho, \vec{V}, \rho_s)$ are relatively separated from each other, and the equations for $\Phi$ and $\rho_s$ are linear and easy to be solved. These features are significant advantages over traditional dynamical Equations (1)–(4). If the total mass density $\rho$ is known, the gravitational potential of $\Phi(t,\vec{x})$ can be integrated, and the velocity field of $\vec{V}$ or stellar orbits can be obtained from (12)

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\vec{X}(t) = -\nabla\Phi(t,\vec{X}). \tag{15}$$

If the gravitational potential $\Phi$ is independent of time, the mechanical energy of each star is conserved. Integrating the above equation, we obtained

$$\frac{1}{2}\vec{V}^2 + \Phi(\vec{X}) = E(\vec{X}_0). \tag{16}$$

Thus, the inappropriate approximation in traditional galactic dynamics brings unnecessary complexity of the equations.

In theory, the complete dynamic equation should include the continuity equation and the dynamic equation of the dark halo. However these equations rely on the equations of state for dark matter and dark energy, which is now an unsolved puzzle. The following analysis shows, the lack of this knowledge can be compensated by observational data. For example, by the rotational velocity curves of the stars, we can then conversely derive some important information about the total density distribution and spiral structure of the dark halo. In some sense, we study the inverse problem of the literature [9,12].

## 3. Structural Equations of Non-Warped Stationary Galaxies

In the context of general relativity, the structure of galaxies depends on the distribution and properties of dark halo, which is unknown in the present situation [15–17]. Because the gravity in the galactic disk is very weak, except for the region near the center, the single gravitational potential $\Phi$ can describe the spiral structure precisely enough. We express equations in spherical coordinate system $(t, r, \theta, \phi)$. In this case, we have $g_{kl} = \mathrm{diag}(1, r^2, r^2\sin^2\theta)$ and the non-zero Christoffel symbols

$$\begin{cases} \Gamma^r_{\theta\theta} = -r, \ \Gamma^r_{\phi\phi} = -r\sin^2\theta, \ \Gamma^\theta_{r\theta} = \Gamma^\phi_{r\phi} = \frac{1}{r}, \\ \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \ \Gamma^\phi_{\theta\phi} = \cot\theta. \end{cases} \tag{17}$$

By 3-*d* tensor calculus, (11) and (12) becomes

$$c^{-2}\partial_t^2\Phi - g^{kl}(\partial_{kl} - \Gamma^m_{kl}\partial_m)\Phi = -\kappa\rho, \tag{18}$$

$$\partial_t V^m + V^k(\partial_k V^m + \Gamma^m_{kl}V^l) = -g^{ml}\partial_l\Phi. \tag{19}$$

Substituting (17) into the above equations, we obtained the dynamic system in spherical coordinate system as

$$c^{-2}\partial_t^2\Phi - (\partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\hat{L}^2)\Phi + \kappa\rho = 0, \tag{20}$$

$$\mathrm{D}_t V_r - rV_\theta^2 - r\sin^2\theta V_\phi^2 + \partial_r\Phi = 0, \tag{21}$$

$$\mathrm{D}_t V_\theta + \frac{2}{r}V_r V_\theta - \sin\theta\cos\theta V_\phi^2 + \frac{\partial_\theta\Phi}{r^2} = 0, \tag{22}$$

$$\mathrm{D}_t V_\phi + \frac{2}{r}V_r V_\phi + 2\cot\theta V_\theta V_\phi + \frac{\partial_\phi\Phi}{(r\sin\theta)^2} = 0, \tag{23}$$

where the derivative operators are defined as

$$\begin{aligned}\hat{L}^2 &= \partial_\theta^2 + \cot\theta\partial_\theta + \frac{1}{\sin^2\theta}\partial_\varphi^2, \\ \mathrm{D}_t &= \partial_t + V_r\partial_r + V_\theta\partial_\theta + V_\phi\partial_\phi,\end{aligned}$$

and the velocity of the stars is defined in the form of a three-dimensional contra-variant vector

$$V_r \equiv \frac{\mathrm{d}}{\mathrm{d}t}r(t), \qquad V_\theta \equiv \frac{\mathrm{d}}{\mathrm{d}t}\theta(t), \qquad V_\phi \equiv \frac{\mathrm{d}}{\mathrm{d}t}\phi(t). \tag{24}$$

Equations (21)–(23) is the Newton's second law for stellar motion.

For the stars moving in the galactic disk, we have

$$\theta = \frac{\pi}{2}, \qquad V_\theta \equiv 0. \tag{25}$$

In this case (22) holds automatically, so we obtained the two-dimensional equation of motion and continuity equation for stellar fluid in non-warped disk as [3,10,15]

$$(\partial_t + V_r\partial_r + V_\phi\partial_\phi)V_r - rV_\phi^2 + \partial_r\Phi = 0, \tag{26}$$

$$(\partial_t + V_r\partial_r + V_\phi\partial_\phi)V_\phi + \frac{2}{r}V_r V_\phi + \frac{1}{r^2}\partial_\phi\Phi = 0, \tag{27}$$

$$(\partial_t + V_r\partial_r + V_\phi\partial_\phi)\Sigma + (\partial_r V_r + \partial_\phi V_\phi + \frac{1}{r}V_r)\Sigma = 0, \tag{28}$$

where $\Sigma \geq 0$ is the mass surface density of stars in the disk.

For quasi-stationary spiral structures, the galactic structure moves as a whole at constant angular velocity $\Omega$ around the $z$-axis. Therefore, under the coordinate transformation $\varphi = \phi - \Omega t$, the solution of the structural equation in the new coordinate system $(t, r, \theta, \varphi)$ is static, that is, the solution is independent of time $t$. So we obtained the following structural equations for non-warped galaxies

$$(\partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\hat{L}^2 - \frac{\Omega^2}{c^2}\partial_\varphi^2)\Phi = \kappa\rho, \tag{29}$$

$$(V_r\partial_r + V_\varphi\partial_\varphi)V_r - r(V_\varphi + \Omega)^2 + \partial_r\Phi = 0, \tag{30}$$

$$(V_r\partial_r + V_\varphi\partial_\varphi)V_\varphi + \frac{2}{r}V_r(V_\varphi + \Omega) + \frac{1}{r^2}\partial_\varphi\Phi = 0, \tag{31}$$

$$(V_r\partial_r + V_\varphi\partial_\varphi)\Sigma + (\partial_r V_r + \partial_\varphi V_\varphi + \frac{1}{r}V_r)\Sigma = 0. \tag{32}$$

The terms related with $\Omega$ in the above formulas have obvious physical significance, $r(V_\varphi + \Omega)^2$ is centrifugal force, and $\frac{2}{r}V_r\Omega$ Coriolis force. Because of the background gravitational force, Equations (29)–(31) is represented in spherical coordinate system, but (32) is described in polar coordinate system.

**4. Second-Order Approximation of Structural Equation**

The Equations (29)–(32) can be solved according to the following procedure: At first, a reasonable total mass density $\rho$ is assumed, which can be expressed as series expansion of spherical harmonic function $Y_{lm}(\theta, \varphi)$. The distribution of the dark halo can be determined inversely by comparing the calculated solutions with the observed data. Second, for given $\rho$, (29) is a linear equation with respect to $\Phi$, which can be easily solved with natural boundary conditions. Third, by (30) and (31), the velocity function can be represented by the trigonometric series of $\varphi$. Due to symmetry, the trigonometric series has only the terms with an integer multiple of a fixed base frequency, usually only even terms. The fourth, equation (32) for the surface mass density is a first-order linear equation, which is also easy to solve. In this way, the stellar distribution and movement are perfectly determined for non-warped stationary galaxies.

In what follows, we demonstrate the solving process of using the triangular series expansion according to this procedure. Consider the following second-order approximation of mass and gravitational potential distributions

$$\rho = \rho_0 + (\rho_1 + \rho_2 \cos 2\varphi + \rho_3 \sin 2\varphi) \sin^2 \theta, \tag{33}$$
$$\Phi = \Phi_0 + (\Phi_1 + \Phi_2 \cos 2\varphi + \Phi_3 \sin 2\varphi) \sin^2 \theta, \tag{34}$$

in which all $(\rho_n, \Phi_n)$ are functions of $r$ and satisfy $\rho \geq 0$. The relative error of (33) and (34) is about $3^{-2s} < 12\%$, where $s \geq 1$ reflects the degree of mass distribution similar to (33). The corresponding second order velocity of the stars in the disk is given by

$$V_r = v_1 \cos 2\varphi + v_2 \sin 2\varphi, \qquad V_\theta = 0, \tag{35}$$
$$V_\varphi = \omega_0 + \omega_1 \cos 2\varphi + \omega_2 \sin 2\varphi, \tag{36}$$

in which all $(v_n, \omega_n)$ are functions of $r$.

Substituting (33) and (34) into (29), we obtained the equation of $\Phi_n$ as

$$\Phi_0'' + \frac{2}{r}\Phi_0' + \frac{4}{r^2}\Phi_1 = \kappa\rho_0, \tag{37}$$
$$\Phi_1'' + \frac{2}{r}\Phi_1' - \frac{6}{r^2}\Phi_1 = \kappa\rho_1, \tag{38}$$
$$\frac{4\Omega^2}{c^2}\Phi_k + \Phi_k'' + \frac{2}{r}\Phi_k' - \frac{6}{r^2}\Phi_k = \kappa\rho_k, \tag{39}$$

where $k = 2, 3$. The solution is given by

$$\Phi_1 = -\frac{\kappa}{5}\Big[\frac{1}{r^3}\int_0^r \rho_1(\chi)\chi^4 \mathrm{d}\chi + r^2 \int_r^\infty \frac{\rho_1(\chi)}{\chi}\mathrm{d}\chi\Big], \tag{40}$$
$$\Phi_0 = -\frac{1}{r}\int_0^r (\chi^2\kappa\rho_0 - 4\Phi_1)\mathrm{d}\chi - \int_r^\infty \frac{1}{\chi}(\chi^2\kappa\rho_0 - 4\Phi_1)\mathrm{d}\chi \tag{41}$$

and

$$\Phi_k = \frac{\kappa c^5}{32\Omega^5 r^2}\Big[A(r)\int_0^r B(\chi)\rho_k(\chi)\mathrm{d}\chi + B(r)\int_r^\infty A(\chi)\rho_k(\chi)\mathrm{d}\chi\Big], \tag{42}$$

in which

$$A(r) = \frac{6\Omega}{c}\cos\Big(\frac{2\Omega r}{c}\Big) + \frac{4\Omega^2 r^2 - 3c^2}{rc^2}\sin\Big(\frac{2\Omega r}{c}\Big),$$
$$B(r) = \frac{4\Omega^2 r^2 - 3c^2}{rc^2}\cos\Big(\frac{2\Omega r}{c}\Big) - \frac{6\Omega}{c}\sin\Big(\frac{2\Omega r}{c}\Big).$$

Substituting (34)–(36) into (30) and (31), and restricting $\theta = \frac{1}{2}\pi$, we obtained

$$\begin{aligned} 0 \quad = \quad & \frac{\mathrm{d}}{\mathrm{d}r}[\Phi_0 + \Phi_1 + \frac{1}{4}(v_1^2 + v_2^2)] - \\ & \frac{1}{2}R[2(\omega_0 + \Omega)^2 + \omega_1^2 + \omega_2^2] + \\ & \omega_1 v_2 - \omega_2 v_1 + [2\omega_0 v_2 - 2r(\omega_0 + \Omega)\omega_1 + \Phi_2'] \cos 2\varphi - \\ & [2\omega_0 v_1 + 2r(\omega_0 + \Omega)\omega_2 - \Phi_3'] \sin 2\varphi + \cdots \end{aligned} \tag{43}$$

as well as

$$\begin{aligned} 0 \quad = \quad & \frac{1}{2}[(r\omega_1' + 2\omega_1)v_1 + (r\omega_2' + 2\omega_2)v_2] + \\ & [(r\omega_0' + 2\omega_0 + 2\Omega)v_1 + 2r\omega_0\omega_2 + \frac{2}{r}\Phi_3] \cos 2\varphi + \\ & [(r\omega_0' + 2\omega_0 + 2\Omega)v_2 - 2r\omega_0\omega_1 - \frac{2}{r}\Phi_2] \sin 2\varphi. \end{aligned} \tag{44}$$

Let the coefficients of the terms $(\sin 2\varphi, \cos 2\varphi)$ in (43) and (44) be 0, we obtained

$$v_1 \quad = \quad -K[\omega_0 r^2 \Phi_3' + 2r(\omega_0 + \Omega)\Phi_3], \tag{45}$$

$$v_2 \quad = \quad K[\omega_0 r^2 \Phi_2' + 2r(\omega_0 + \Omega)\Phi_2], \tag{46}$$

$$\omega_1 \quad = \quad K[(\frac{1}{2}r^2\omega_0' + r(\omega_0 + \Omega))\Phi_2' + 2\omega_0\Phi_2], \tag{47}$$

$$\omega_2 \quad = \quad K[(\frac{1}{2}r^2\omega_0' + r(\omega_0 + \Omega))\Phi_3' + 2\omega_0\Phi_3], \tag{48}$$

$$K \quad \equiv \quad \{r^2[r(\omega_0 + \Omega)\omega_0' + 2\Omega(2\omega_0 + \Omega)]\}^{-1}. \tag{49}$$

Substituting the above results into (44), that is, into the equation

$$(r\partial_r \omega_1 + 2\omega_1)v_1 + (r\partial_r \omega_2 + 2\omega_2)v_2 = 0, \tag{50}$$

we obtained a first-order linear ordinary differential equation for $\omega_0$

$$\begin{aligned} \omega_0' \quad = \quad & \frac{1}{r}(3\omega_0 + \Omega) + \frac{\omega_0 r}{2} \cdot \frac{\Phi_3'\Phi_2'' - \Phi_2'\Phi_3''}{\Phi_3\Phi_2' - \Phi_2\Phi_3'} + \\ & (\omega_0 + \Omega)\frac{\Phi_3\Phi_2'' - \Phi_2\Phi_3''}{\Phi_3\Phi_2' - \Phi_2\Phi_3'}, \end{aligned} \tag{51}$$

and another solution with no physical meaning $\omega_0' = -\frac{2}{r}(\omega_0 + \Omega)$. It is important to note that, while $\Phi_2 \equiv 0$ or $\Phi_3 \equiv 0$, $\omega_0$ cannot be determined by above Equation (51). In this case (50) holds automatically. Substituting (45)–(49) into (43), i.e., substituting them into

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}[\Phi_0 + \Phi_1 + \frac{1}{4}(v_1^2 + v_2^2)] = & -\omega_1 v_2 + \omega_2 v_1 + \\ & \frac{1}{2}r[2(\omega_0 + \Omega)^2 + \omega_1^2 + \omega_2^2], \end{aligned} \tag{52}$$

we obtained a constraint between the mass density distribution and $\Omega$, so $\Omega$ is determined by the total mass density. From (45)–(51) we learn that, the stellar speed $\vec{V}$ is completely determined by $(\rho_2, \rho_3, \Omega)$. After the velocity distribution is determined, the surface mass density $\Sigma(r, \varphi)$ of the stars can be solved from (32). At this point, the second-order approximation of galactic dynamics is completely solved. The above process only involves solving ordinary differential equations.

## 5. Simple Solutions to Stationary Galaxy Structure

The calculations in the preceding section show that the structure of a stationary galaxy is determined by $(\rho_2, \rho_3, \Omega)$. If the total mass density distribution is known, it is easy to determine the structure of the galaxy according to the above procedure. However, it is difficult to directly determine the density distribution. The data observed are mainly the moving speeds of the stars. High-precision observation show that the stellar speeds in galaxies in a larger range is approximately equal. To illustrate the above solving procedure in detail and obtained some enlightening conclusions, we concretely solve two of the simplest examples. As a working hypothesis, we assume [18,19],

$$\omega_0 = \frac{v}{r} - \Omega, \qquad (r \in [R_0, R_1]), \tag{53}$$

where $v$ is a constant velocity with a typical value of $|v| = 200{\sim}400$ km/s, $[R_0, R_1]$ is the effective domain of assumption (53). Usually we have $R_0 = 100{\sim}500$ pc and $R_1 = 10{\sim}60$ kpc, which are about the radius of the bulge and the radius of the visible range, respectively. In the bulge region $r < R_0$, the mass distribution can be approximated by a rigid sphere. No loss of generality, we can set $(\omega_0 > 0, v > 0)$, otherwise we can make a transformation $\phi \to -\phi$ to obtained this condition. So we always have $v > r\Omega$ in the effective domain. As shown below, using the star's velocity curve $v = v(r)$ to calculate the total mass density is a good method to study the properties of dark halo [15,20–23].

### 5.1. Barred Spiral Galaxy

We consider the simple case $\rho_2 \neq 0, \rho_3 = 0$. In this case we have $P_3 = 0$, and $P_2(r)$ can be calculated by (42). By (45)–(49), we obtained

$$v_1 = 0, \qquad v_2 = -\frac{r(v - r\Omega)P_2' + 2vP_2}{2(v - r\Omega)^2 - v^2}, \tag{54}$$

$$\omega_2 = 0, \qquad \omega_1 = -\frac{rvP_2' + 4(v - r\Omega)P_2}{2r[2(v - r\Omega)^2 - v^2]}. \tag{55}$$

The effective region of the above formulas is $r < \frac{(2-\sqrt{2})v}{2\Omega}$. Substituting (53)–(55) into (32), we obtained the surface density of stars as

$$\Sigma = [\Sigma_0 + C_2 r v_2 \cos(2\varphi)]^{-1}, \tag{56}$$

in which

$$\Sigma_0 = r v_2 e^{-\int \frac{2\omega_1}{v_2} dr}[C_1 + 2C_2 \int \frac{v - r\Omega}{r v_2} e^{\int^r \frac{2\omega_1(\chi)}{v_2(\chi)} d\chi} dr],$$

$(C_1, C_2)$ are integration constants.

The above results show the structural characteristics of a barred spiral galaxy. For given total mass density $\rho$ and boundary values, all the above formulas can be calculated concretely, but the analytical expression is a little complicated. If a restriction $v_2 = 0$ is added, that is $V_r \equiv 0$, then the stars move around circles, the results are relatively simple. In this case, by (54), we have

$$\Phi_2 = -\frac{q}{r^2}(v - r\Omega)^2, \qquad \Phi_3 = 0, \tag{57}$$

in which $q \geq 0$ is an integral constant. Substituting (57) into (45)–(49) we obtained

$$V_r = 0, \qquad V_\varphi = \frac{1}{r^3}(v - r\Omega)(r^2 + q\cos 2\varphi). \tag{58}$$

Again by (32), we obtained the surface mass density of stars

$$\Sigma = \frac{\varrho(r)}{r^2 + q\cos 2\varphi}, \tag{59}$$

in which $\varrho(r)$ is determined by boundary condition. Substituting the above results into (52), we obtained a restriction between $\Omega$ and potential

$$\Phi_0 + \Phi_1 = v^2 \ln\left(\frac{r}{r_0}\right) - \frac{q^2}{24r^4}(3v^2 - 8vr\Omega + 6r^2\Omega^2), \tag{60}$$

in which $r_0 > 0$. In the case of $q = 0$, the solution (57)–(60) is exact, which correspond to elliptical galaxy. However, as long as the conditions deviate a little, the elliptical galaxy will evolve into a spiral one. The spiral structure should be an inevitable result for the evolution of ordinary galaxies.

As an application of (60), we derive the Tully–Fisher relation $L \propto v^4$ [24]. In astrophysics, we have the following empirical laws for a galaxies

$$L \propto M \propto R^2, \tag{61}$$

in which $L$ is the luminosity, $M$ the total mass and $R$ the visible radius. For an elliptical galaxy, in (60) we have

$$q = \Omega = 0, \qquad |\Phi_1| \ll |\Phi_0|. \tag{62}$$

Then, we obtained

$$4\pi G\rho = \Phi'' + \frac{2}{r}\Phi' \doteq \frac{v^2}{r^2}. \tag{63}$$

So we have relation

$$M = 4\pi \int_0^R \rho r^2 \mathrm{d}r \doteq \frac{v^2}{G} R. \tag{64}$$

Taking $v$ as known parameter and solving (61) and (64), we obtain relations

$$R \propto v^2, \qquad L \propto M \propto v^4. \tag{65}$$

In general case, if the perturbative terms $\Omega$, $V_r$ and $\hat{L}^2$ in (29) and (30) can be omitted and $v$ is almost a constant, then we have equations

$$\partial_r \Phi \doteq \frac{v^2}{r}, \qquad 4\pi G\rho = \partial_r^2 \Phi + \frac{2}{r}\partial_r \Phi \doteq \frac{v^2}{r^2}. \tag{66}$$

Equation (66) implies that the Tully–Fisher relation also approximately holds for all stable galaxies.

### *5.2. Spiral Galaxy*

Now, we calculate the following simple case of $\Omega = 0$,

$$\omega_0 = \frac{v}{r}, \qquad \Phi_2 = P^{-1}\cos(\xi r), \qquad \Phi_3 = P^{-1}\sin(\xi r), \tag{67}$$

where $\xi > 0$ is a constant, $P(r)$ is a function of $r$. Substituting (67) into (51), we obtained a linear differential eguation for $P(r)$

$$P'' - \frac{4}{r}P' + (\xi^2 + \frac{8}{r^2})P = 0. \tag{68}$$

The solution is given by

$$P = \sqrt{r^5}[C_1 J_\alpha(\xi r) + C_2 J_{-\alpha}(\xi r)], \qquad \alpha = \frac{1}{2}\sqrt{7}, \tag{69}$$

in which $(C_1, C_2)$ are constants, $(J_\alpha, J_{-\alpha})$ is Bessel function of imaginary parameter

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k}, \ (\nu = \pm\alpha).$$

Substituting the above relations into (45)–(49), we have

$$\begin{aligned} \Phi &= \Phi_0 + [\Phi_1 + P^{-1}\cos\gamma]\sin^2\theta, \\ V_r &= \tfrac{1}{\upsilon P^2}[\xi r P\cos\gamma - (rP' - 2P)\sin\gamma], \\ rV_\varphi &= \tfrac{1}{2\upsilon P^2}[(rP' - 4P)\cos\gamma + \xi r P\sin\gamma] + \upsilon, \\ \Sigma &= \tfrac{C_3 P^2}{\xi r^5}\left[(rP' - 2P)\cos\gamma + \xi r P\sin\gamma + 2r^2\upsilon^2 \int \tfrac{P^2}{r^3}\mathrm{d}r\right], \end{aligned} \tag{70}$$

in which $\gamma = \xi r - 2\varphi$. All of these variables are characterized by the structure of a spiral galaxy, which are closer to the Archimedes spiral rather than the logarithmic one. Since the pitch angle of galaxy is approximately $10^\circ\sim 40^\circ$ [25], we have estimation $\xi = (2\sim 11)R_1^{-1}$.

## 6. Discussion and Conclusions

To explain the galaxy structure and flat velocity curves, [26] declared that it is unnecessary to introduce 'hidden mass hypothesis', and the following modified Newtonian dynamics was proposed,

$$M_g \mu(a/a_0)\mathbf{a} = \mathbf{F}. \tag{71}$$

In order to obtain the Tully–Fisher relation, in [27] the centripetal acceleration $a = V^2/r$ and the following gravity were used,

$$g_N = MGr^{-2}. \tag{72}$$

Since the visible mass distribution in galaxies is disc-like, if the dark matter is absent, then the main part of the Newtonian gravitational potential of a galaxy should be similar to that of a disc mass distribution [28]

$$\Phi_d = -\frac{2GM}{R_1}\sum_{n=0}^{\infty} C_{1/2}^n \left(\frac{r}{R_1}\right)^{2n} P_{2n}(\cos\theta) + \frac{2GMr}{R_1^2}|\cos\theta|, \quad r \le R_1, \tag{73}$$

$$\Phi_d = -\frac{2GM}{R_1}\sum_{n=0}^{\infty} C_{1/2}^{n+1} \left(\frac{R_1}{r}\right)^{2n+1} P_{2n}(\cos\theta), \quad r > R_1, \tag{74}$$

where $C_{1/2}^n$ is defined by $\sqrt{1+x} = \sum C_{1/2}^n x^n$. In the disc $(r \le R_1, \theta = \frac{1}{2}\pi)$, we have $\cos\theta = 0$ and

$$\Phi_d = -\frac{2GM}{R_1}[1 - \frac{1}{4}\left(\frac{r}{R_1}\right)^2 + \cdots], \qquad g_N = \frac{GMr}{R_1^3} + \cdots. \tag{75}$$

How can we obtain (72)? Equation (72) is actually a proof of the existence of dark matter, because it is just the Newtonian gravity of the spherically symmetric distribution of mass. The reasonable explanation for the Tully–Fisher relation $L \propto \upsilon^4$ should be (63)–(66).

In addition, Equation (71) is also not a good model in physics, because it obviously violates some fundamental principles of physics such as covariance and the Poisson equation or wave equation. For problems as complex as galactic dynamics, a systematic research based on the first principles is needed. Ref. [26] only used 5 simple algebraic formulas without any logical relation between them: partial assumptions, partial fitting to the observed data and most of the content is stating facts, opinions and conjectures. For complex problems, this style of research is difficult to effectively solve the problem, and is easily disoriented. One without sharp logic can hardly check the validity of the results or find out where a bug exists. From the review paper [29], we can find a number of studies misled by the model, much of time and resource was wasted by the professional barriers. The simple treatment is feasible for organizing experimental data and laws, but the galactic dynamics and structure involve large scale space-time geometry and partial differential equations of gravity and matter distribution, so it is impossible to obtain profound results by simple formula and conjecture. In this case, open mind and multidisciplinary collaboration is important, because some problems that are difficult for the experts in the subject but simple for others.

For the mature galaxies, in the paper three working hypotheses are introduced to simplify the dynamic equations: 1. To research the Large-scale structure of galaxy, we should consider the retarded potential of gravity, which takes longer time to propagate to border of galaxy than the revolution period of the stars near the center. In this case, to use Newtonian gravity directly is unreasonable and it should be replaced by wave equation, i.e., the weak field and low velocity approximation of Einstein's field Equation (5).

2. The stellar system of mature galaxies should be zero-pressure and viscosity–Free fluid having different equations of motion from ordinary continuum mechanics. The pressure of such system has no physical meaning and cannot be defined, and the introduction of pressure will only bring errors and troubles. The star moves along geodesic and the dynamic equation of stellar system should be (7) and (8).

3. The equation of state of dark halo is different from that of ordinary luminous interstellar matter, so their trajectories are very different, and the dark halo is automatically separated from ordinary matter in galaxies. The structure of galaxies is only determined by the total mass density, which can be assumed beforehand based on observations. Then, the total mass density can be determined by comparing the solution of structural equations with the observed data.

By these processes, we find that the variables in the dynamical equations of galaxies are separated from each other. The equations are well-posed and can be solved according to a fixed procedure. Traditional equations include unreasonable assumptions and are too complicated to be studied in depth. In fact, the basic objects of Nature are designed elaborately, and the corresponding equations are simple and symmetrical. If our research is trapped in complex computation and incomprehensible dilemma, this situation reflects that the model used has gone wrong with big probability.

The dynamic equation system derived above provides a more reasonable and practical framework for simulating the structure of galaxies and is closely linked to the study of dark matter halo. The physical significance of the above framework is as follows: 1. A system of highly accurate and convenient galactic dynamic Equations (11)–(13) is established, but the previously used equations such as (1)–(4) is not suitable for galactic structure. 2. For a mature galaxy, its structure is stable, and (29)–(32) gives the structure equations for a stable non-warped galaxy. 3. Equations (11)–(13) is the foundation to analyze and explain other properties of galaxies, on which other problems can be easily solved, such as the Tully–Fisher relation derived by (66). It is clear that the discreteness of the Tully–Fisher relation between different types of galaxies can also be derived by calculating the errors of (66) for various galaxies. 4. By calculating the fine structure of the total density distribution $\rho$ in (11), we can study the dynamic properties of dark matter halo, such as the state function $W(\rho)$ and so on. In conclusion, galactic dynamics is the theoretical foundation for the study of galactic structure and properties, and a prerequisite for solving other problems. The previous dynamic equations included unreasonable assumptions, while the framework in the paper provides a reasonable dynamic model of higher accuracy and more convenience.

**Funding:** This research received no external funding.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** I am grateful to an anonymous reviewer; this paper has been modified according to their suggestions.

**Conflicts of Interest:** The authors declare no conflict of interest.

## Appendix A. Linearization of Einstein Field Equation

In this section, by weak-field and low-speed approximation of Einstein field equation, we will establish the dynamic equations describing the galactic system. Some fundamental contents can be found in [30], but here we make a more systematic and detailed derivation for galactic dynamics.

In the context of general relativity, the whole system of the dynamic equations for galactic evolution should be Einstein field equation

$$G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = -\bar{\kappa}T^{\mu\nu}, \qquad (\bar{\kappa} \equiv \frac{8\pi G}{c^4}), \tag{A1}$$

combined with the law of energy-momentum conservation and the equation of state of the gravitating source. In most cases, the classical approximation for the total energy-momentum tensor takes the following form [16]

$$T^{\mu\nu} = (\rho + P)\mathcal{U}^\mu\mathcal{U}^\nu + (W - P)g^{\mu\nu}, \tag{A2}$$

where $W$ corresponds to the nonlinear potentials, which acts as negative pressure and leads to the deviation from geodesic. According to the law of energy-momentum conservation or the Bianchi identity $T^{\mu\nu}_{;\nu} = 0$, we can derive the continuity equation $\mathcal{U}_\mu T^{\mu\nu}_{;\nu} = 0$ and the equation of motion for the source as follows

$$\mathcal{U}^\mu\partial_\mu(\rho + W) = -(\rho + P)\mathcal{U}^\mu_{;\mu}, \tag{A3}$$

$$(\rho + P)\mathcal{U}^\nu\mathcal{U}^\mu_{;\nu} = (g^{\mu\nu} - \mathcal{U}^\mu\mathcal{U}^\nu)\partial_\nu(P - W). \tag{A4}$$

In the case of $W \sim \rho \gg P$, such as for the nonlinear spinor, we find that the stream lines are quite different from the geodesics $\mathcal{U}^\nu\mathcal{U}^\mu_{;\nu} = 0$. Therefore, a fully relativistic simulation of the galactic evolution should include such terms. However, in the following non-relativistic approximation, the effects of $(P, W)$ are merged into an effective mass-energy density $\rho$, which becomes much simpler.

For convenience, we take $c = 1$ as the unit of velocity. Noting the facts that collisions between stars rarely occur, the trajectories of the ordinary matter, such as atoms, are almost geodesics, so for stars, the following zero-pressure and inviscid energy-momentum tensor holds

$$T_s^{\mu\nu} = \rho_s U^\mu U^\nu,$$

in which $\rho_s$ is the comoving mass density of stars, and $U^\mu$ is the 4-vector speed of the stellar flow. For energy-momentum tensor of a compound system, we have the following useful theorem [31], which means the energy-momentum of any independent subsystem is conserved, respectively.

**Theorem A1.** *Assume matter consists of two subsystems I and II, namely $\mathcal{L}_m = \mathcal{L}_I(\phi) + \mathcal{L}_{II}(\psi)$, then we have*

$$T^{\mu\nu} = T_I^{\mu\nu} + T_{II}^{\mu\nu}. \tag{A5}$$

*If the subsystems I and II have not interaction with each other, namely,*

$$\frac{\delta}{\delta\psi}\mathcal{L}_I(\phi) = \frac{\delta}{\delta\phi}\mathcal{L}_{II}(\psi) = 0, \tag{A6}$$

*then the two subsystems have independent energy-momentum conservation laws, respectively,*

$$T^{\mu\nu}_{I;\nu} = 0, \qquad T^{\mu\nu}_{II;\nu} = 0. \tag{A7}$$

According to Theorem A1, ordinary matter satisfies the law of energy-momentum conservation independent of that of the dark halo, so we have $T^{\mu\nu}_{s;\nu} = 0$. Expressing it in the form of equations of continuity and motion, we obtain the dynamic equations for the stars

$$U^\mu\partial_\mu\rho_s + \rho_s U^\mu_{;\mu} = 0, \qquad U^\nu U^\mu_{;\nu} = 0. \tag{A8}$$

The total energy-momentum tensor of the galaxy is still given by (A2), and satisfies the dynamic Equations (A3) and (A4). Using (A1) and (A2), we obtain

$$R = \bar{\kappa}(\rho + 4W - 3P), \tag{A9}$$

where $R = g_{\mu\nu}R^{\mu\nu}$ is the scalar curvature. Substituting (A9) into (A1), we obtain

$$R^{\mu\nu} = -\bar{\kappa}(\rho + P)\mathcal{U}^\mu\mathcal{U}^\nu + \frac{1}{2}\bar{\kappa}(\rho + 2W - P)g^{\mu\nu}, \tag{A10}$$

where $\mathcal{U}^\mu$ is the average 4-vector speed of all gravitating source.

In order to make a weak-field approximation, we choose the harmonic coordinate system, which leads to the usual Cartesian coordinate system when linearizing of metric. Then we have the de Donder coordinate condition

$$\Gamma^\mu \equiv g^{\alpha\beta}\Gamma^\mu_{\alpha\beta} = -\frac{1}{\sqrt{g}}\partial_\nu(\sqrt{g}g^{\mu\nu}) = 0,$$

where $g = |\det(g)|$. Denote the Minkowski metric by $\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. For the weak-field approximation, we have the linearization for the metric

$$\begin{aligned}
g_{\mu\nu} &\equiv \eta_{\mu\nu} + h_{\mu\nu}, \quad g^{\mu\nu} \doteq \eta^{\mu\nu} - h^{\mu\nu}, \\
h^{\mu\nu} &= \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}, \quad h = h^\mu{}_\mu = \eta^{\mu\nu}h_{\mu\nu}, \\
g &\doteq 1 + h, \qquad \sqrt{g} \doteq 1 + \frac{1}{2}h.
\end{aligned}$$

In the above approximation we have a precision $O(|h_{\mu\nu}|^2) \sim \Phi^2 + V^2/c^2 \sim 10^{-6}$, which is sufficient for calculations of galactic dynamics. Next, we directly use $=$ to replace $\doteq$ for convenience. By straightforward calculation, we obtain the linearization for other parameters

$$\begin{aligned}
\Gamma^\mu_{\alpha\beta} &= \frac{1}{2}\eta^{\mu\nu}(\partial_\alpha h_{\nu\beta} + \partial_\beta h_{\alpha\nu} - \partial_\nu h_{\alpha\beta}), \\
\Gamma^\mu &= \partial_\nu(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h), \\
R_{\mu\nu} &= \frac{1}{2}\partial_\alpha\partial^\alpha h_{\mu\nu} - \frac{1}{2}(\eta_{\mu\alpha}\partial_\nu\Gamma^\alpha + \eta_{\nu\alpha}\partial_\mu\Gamma^\alpha), \\
R^{\mu\nu} &= \frac{1}{2}\partial_\alpha\partial^\alpha h^{\mu\nu} - \frac{1}{2}(\eta^{\mu\alpha}\partial_\alpha\Gamma^\nu + \eta^{\nu\alpha}\partial_\alpha\Gamma^\mu), \\
R &= \frac{1}{2}\partial_\alpha\partial^\alpha h - \partial_\alpha\Gamma^\alpha.
\end{aligned}$$

In which $\partial_\alpha\partial^\alpha = \partial_t^2 - \nabla^2$ is the d'Alembert operator.

In the harmonic coordinate system, we have

$$\Gamma^\mu = \partial_\nu(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h) = 0, \tag{A11}$$

$$R_{\mu\nu} = \frac{1}{2}\partial_\alpha\partial^\alpha h_{\mu\nu}, \quad R^{\mu\nu} = \frac{1}{2}\partial_\alpha\partial^\alpha h^{\mu\nu}, \tag{A12}$$

$$R = \frac{1}{2}\partial_\alpha\partial^\alpha h, \quad G^{\mu\nu} = \frac{1}{2}\partial_\alpha\partial^\alpha(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h). \tag{A13}$$

From (A11) and (A13), we find that if $\Gamma^\mu = 0$, $\partial_t\Gamma^\mu = 0$ at any given time $t = t_0$, it will always hold due to the Bianchi identity $G^{\mu\nu}_{;\nu} = 0$.

To compare with electromagnetism and to understand the physical meaning of the parameters, denote

$$\Phi = \frac{1}{2}h_{tt} = \frac{1}{2}h^{tt}, \quad \vec{A} = (h^{tx}, h^{ty}, h^{tz}) = -(h_{tx}, h_{ty}, h_{tz}), \tag{A14}$$

$$H = (h_{ab}) = (h^{ab}), \ (\{a, b\} \in \{1, 2, 3\}), \quad \vec{B} = \nabla \times \vec{A}. \tag{A15}$$

In the International System of Units, we have the order of magnitude of the metric components

$$c^2|h_{ab}| \sim c|A_k| \sim |\Phi| \ll 1, \ \ (a \neq b), \tag{A16}$$

which means $|h_{ab}| \ll |A_k| \ll |\Phi| \ll 1$ if we take $c = 1$ as the unit.

For the present purpose, we define the stellar speed $\vec{V}$ by

$$\vec{V} \equiv \frac{1}{U^0}(U^1, U^2, U^3), \tag{A17}$$

which is approximately equivalent to the usual definition. For galaxies, we have the following order of magnitude

$$|\vec{V}| \sim 300\ \mathrm{km/s} = 10^{-3}c, \quad \vec{A} \sim \bar{\kappa}\vec{V}, \quad h_{ab} \sim \bar{\kappa}|\vec{V}|^2, \ (a \neq b),$$

in which the coefficient $\bar{\kappa}$ is also a number of small value. Then, according to

$$1 = \sqrt{g_{\mu\nu}U^\mu U^\nu} = (1 + 2\Phi - 2\vec{A}\cdot\vec{V} + g_{ab}V^aV^b)^{\frac{1}{2}}U^0,$$

by omitting the $O(V^2)$ terms, the low-speed assumption gives

$$U^0 = 1 - \Phi + \vec{A}\cdot\vec{V}. \tag{A18}$$

Substituting (A17) and (A18) into (A8) and omitting the higher-order terms, we obtain the continuity equation and motion equation for the stars

$$(\partial_t + \vec{V}\cdot\nabla)\rho_s = -\rho_s[\nabla\cdot\vec{V} + (\partial_t\Phi + \nabla\cdot\vec{A})], \tag{A19}$$

$$(\partial_t + \vec{V}\cdot\nabla)\vec{V} = -\nabla\Phi + (-\partial_t\vec{A} + \vec{V}\partial_t\Phi) + \vec{V}\times\vec{B} + \vec{V}\cdot\partial_t H. \tag{A20}$$

In (A19), we used the de Donder condition $\Gamma^0 = 0$ in the form

$$\frac{1}{2}\partial_t(h_{xx} + h_{yy} + h_{zz}) = -(\partial_t\Phi + \nabla\cdot\vec{A}). \tag{A21}$$

The equation of motion (A20) is similar to electrodynamics. From it we learn that, $\Phi$ gives the Newtonian gravitational potential, and $\vec{A}$ leads to the gravimagnetic field $\vec{B}$.

By (A9) and (A13), we have

$$\partial_\alpha\partial^\alpha h = 2\bar{\kappa}(\rho + 4W - 3P). \tag{A22}$$

By (A10), (A12) and (A22), we obtain the dynamic equations for $h^{\mu\nu}$

$$\partial_\alpha\partial^\alpha h^{\mu\nu} = -2\bar{\kappa}(\rho+P)\mathcal{U}^\mu\mathcal{U}^\nu + \bar{\kappa}(\rho+2W-P)\eta^{\mu\nu}, \tag{A23}$$
$$\partial_\alpha\partial^\alpha \chi^{\mu\nu} = -2\bar{\kappa}[(\rho+P)\mathcal{U}^\mu\mathcal{U}^\nu + (W-P)\eta^{\mu\nu}], \tag{A24}$$

where $\chi^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h$. If the average speed of the dark halo is also small, omitting $O(\vec{\mathcal{U}}^2)$ from (A23), we obtain $h^{xx} = h^{yy} = h^{zz} \equiv 2\Psi$, $h^{ab} = 0, (a \neq b)$ and

$$\partial_\alpha\partial^\alpha \Phi = \frac{1}{2}\partial_\alpha\partial^\alpha h^{00} = -4\pi G\rho, \tag{A25}$$
$$\partial_\alpha\partial^\alpha \Psi = \frac{1}{2}\partial_\alpha\partial^\alpha h^{kk} = -4\pi G\widetilde{\rho}, \tag{A26}$$

where $\rho$ and $\widetilde{\rho}$ are the effective mass densities with little difference. Their zeroth-order approximation gives

$$\rho = \rho[2(\mathcal{U}^0)^2-1] - 2W + P[2(\mathcal{U}^0)^2+1] \doteq \rho - 2W + 3P. \tag{A27}$$
$$\widetilde{\rho} \doteq \rho + 2W - P. \tag{A28}$$

In the following discussion, only the zeroth-order approximation of (A19), (A20) and (A25) are involved. The precision of these equations is sufficient for calculations of galactic dynamics in the effective domain $r \in [R_0, R_1]$.

In order to obtain the the Hamiltonian formalism of Dirac equation in curved soace-time, we need the natural coordinate system(NCS) [32]

$$ds^2 = g_{tt}dt^2 - \bar{g}_{kl}dx^k dx^l, \qquad d\tau = \sqrt{g_{tt}}dt, \qquad dV = \sqrt{\bar{g}}d^3x. \tag{A29}$$

In which $ds$ is the 4-$d$ length of line element, $d\tau$ is the Newton's absolute cosmic time element and $dV$ is the absolute volume element of space at time $t$. The NCS generally exists and the global simultaneity is unique. Only in NCS can we clearly establish the Hamiltonian formalism and calculate the Noether charges.

The NCS is a wonderful coordinate system different from the Gaussian coordinate system that is valid only in the neighborhood of the initial Cauchy hypersurface. The isodistant translating hypersurface will deform soon, so that the metric $ds^2 = dt^2 + g_{kl}dx^k dx^l$ becomes invalid. The NCS is also different from the Einstein's lift moving along a geodesic, namely the co-moving coordinate system, as this requires the lift to be an infinitesimal volume in curved space-time. While NCS holds unconditionally and globally, and its time is objective cosmic time. $g_{tt}$ represents gravity and cannot be merged into the time coordinate $t$.

However, the NCS is not suitable for the linearization of the Einstein's field equations, because for a rotating galaxy the coordinate transformation from harmonic coordinate system to NCS is similar to

$$r' = r, \qquad \theta' = \theta, \qquad \varphi' = \omega(r,\theta)t + \varphi, \tag{A30}$$

and such transformation with rotation cannot be linearized.

## References

1. Binney, B.; Tremaine, S. *Galactic Dynamics*, 2nd ed.; Princeton University Press: Princeton, NJ, USA, 2008.
2. Orlova, N.; Korchagin, V.; Kikuchi, N.; Miyama, S.M.; Moiseev, A. Application of the global modal approach to the spiral galaxies. *Astrophys. Space Sci.* **2003**, *284*, 739. [CrossRef]
3. Meidt, S.E.; Rand, R.J.; Merrifield, M.R.; Debattista, V.P.; Shen, J. Tests of the Radial Tremaine-Weinberg Method. *Astrophys. J.* **2008**, *676*, 899. [CrossRef]
4. Lin, C.C.; Shu, F.H. On the Spiral Structure of Disk Galaxies. *Astrophys. J.* **1964**, *140*, 646. [CrossRef]

5. Goldreich, P.; Lynden-Bell, D., II. Spiral Arms as Sheared Gravitational Instabilities. *Mon. Not. R. Astron. Soc.* **1965**, *130*, 125. [CrossRef]
6. Julian, W.H.; Toomre, A. Non-Axisymmetric Responses of Differentially Rotating Disks of Stars. *Astrophys. J.* **1966**, *146*, 810. [CrossRef]
7. Somerville, R.S.; Barden, M.; Rix, H.W.; Bell, E.F.; Beckwith, S.V.; Borch, A.; Wolf, C. An Explanation for the Observed Weak Size Evolution of Disk Galaxies. *Astrophys. J.* **2008**, *672*, 776. [CrossRef]
8. Kendall, S.; Kennicutt, R.C.; Clarke, C.; Thornley, M.D. Tracing Spiral Density Waves in M81. *Mon. Not. R. Astron. Soc.* **2008**, *387*, 1007. [CrossRef]
9. Dehnen, W.; Binney, J. Mass Models of the Milky Way. *Mon. Not. R. Astron. Soc.* **1998**, *294*, 429. [CrossRef]
10. Vorobyov, E.I.; Theis, C. Boltzmann Moment Equation Approach for the Numerical Study of Anisotropic Stellar Discs. *Mon. Not. R. Astron. Soc.* **2006**, *373*, 197. [CrossRef]
11. Vorobyov, E.I.; Theis, C. Shape and Orientation of Stellar velocity Ellipsoids in Spiral Galaxies. *Mon. Not. R. Astron. Soc.* **2008**, *383*, 817. [CrossRef]
12. Romero-Gómez, M.; Athanassoula, E.M.; Masdemont, J.J.; García-Gómez, C. The Formation of Spiral Arms and Rings in Barred Galaxies. *Astron. Astrophys.* **2007**, *472*, 63. [CrossRef]
13. Riols, A.; Latter, H.N. Spiral Density Waves and Vertical Circulation in Protoplanetary Discs. *Mon. Not. R. Astron. Soc.* **2018**, *476*, 5115. [CrossRef]
14. Virgo & LIGO Collaborations. GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral. *Phys. Rev. Lett.* **2017**, *119*, 161101. [CrossRef] [PubMed]
15. Gu, Y.Q. Stationary Spiral Structure and Collective Motion of the Stars in a Spiral Galaxy. In *Old Problems and New Horizons in World Physics*; Christianto, V., Smarandache, F., Eds.; Nova Science Publishers: New York, NY, USA, 2019; p. 323.
16. Gu, Y.Q. *Clifford Algebra and Unified Field Theory*; LAP LAMBERT Academic Publishing: Chisinau, Moldova, 2020.
17. Mori, M.; Umemura, M. The Evolution of Galaxies from Primeval Irregulars to Present-Day Ellipticals. *Nature* **2006**, *440*, 644. [CrossRef] [PubMed]
18. Sofue, Y.; Tutui, Y.; Honma, M.; Tomita, A.; Takeda, Y. Central Rotation Curves of Spiral Galaxies. *Astrophys. J.* **2009**, *523*, 136. [CrossRef]
19. Bownstein, J.R.; Moffat, J.W. Galaxy Rotation Curves without Nonbaryonic Dark Matter. *Astrophys. J.* **2006**, *636*, 721. [CrossRef]
20. Salucci, P.; Martins, C.F. The Mass Distribution in Spiral Galaxies. *EAS Publ.* **2009**, *36*, 133. [CrossRef]
21. Persic, M.; Salucci, P.; Stel, F. Rotation Curves of 967 Spiral Galaxies: Implications for Dark Matter. *AIP Conf. Proc.* **1995**, *336*, 137.
22. Persic, M.; Salucci, P.; Stel, F. The Universal Rotation Curve of Spiral Galaxies—I. The Dark Matter Connection. *Mon. Not. R. Astron. Soc.* **1996**, *281*, 27. [CrossRef]
23. Widrow, L.M.; Pym, B.; Dubinski, J. Dynamical Blueprints for Galaxies. *Astrophys. J.* **2008**, *679*, 1239. [CrossRef]
24. Tully, R.B.; Fisher, J.R. A new method of determining distance to galaxies. *Astron. Astrophys.* 1977, *54*, 661–673.
25. Yu, S.Y.; Ho, L.C. On the Connection between Spiral Arm Pitch Angle and Galaxy Properties. *Astrophys. J.* **2019**, *871*, 194. [CrossRef]
26. Milgrom, M. A Modification of the Newtonian Dynamics as a Possible Alternative to the Hidden mass Hypothesis. *Astrophys. J.* **1983**, *270*, 365. [CrossRef]
27. Milgrom, M. A Modification of the Newtonian Dynamics—Implications for Galaxies. *Astrophys. J.* **1983**, *270*, 371. [CrossRef]
28. Croxton, C.A. *Introductory Eigenphysics*; John Wiley & Sons Ltd.: New York, NY, USA, 1974.
29. Banik, I.; Zhao, H.S. From galactic bars to the Hubble tension—Weighing up the astrophysical evidence for Milgromian gravity. *arXiv* **2021**, arXiv:2110.06936.
30. Weinberg, S. *Gravitation and Cosmology*; (Ch5.4, Ch7, Ch.9.8, Ch.10.1); Wiley: New York, NY, USA, 1972.
31. Gu, Y.Q. Dynamical Reason for a Cyclic Universe. *Symmetry* **2021**,*13*, 2272. [CrossRef]
32. Gu, Y.Q. Natural Coordinate System in Curved Space-time. *J. Geom. Symmetry Phys.* **2018**, *47*, 51–62. [CrossRef]