symmetry

Article

Several Isospectral and Non-Isospectral Integrable Hierarchies of Evolution Equations

Shiyin Zhao [1,2,*], Yufeng Zhang [2] and Jian Zhou [1,2]

1 College of Mathematics, Suqian University, Suqian 223800, China; zyfxz@cumt.edu.cn (Y.Z.); 23009@sqc.edu.cn or zym0809@126.com (J.Z.)
2 School of Mathematics, China University of Mining and Technology, Xuzhou 221116, China
* Correspondence: 23004@sqc.edu.cn

Abstract: By introducing a 3×3 matrix Lie algebra and employing the generalized Tu scheme, a AKNS isospectral–nonisospectral integrable hierarchy is generated by using a third-order matrix Lie algebra. Through a matrix transformation, we turn the 3×3 matrix Lie algebra into a 2×2 matrix case for which we conveniently enlarge it into two various expanding Lie algebras in order to obtain two different expanding integrable models of the isospectral–nonisospectral AKNS hierarchy by employing the integrable coupling theory. Specially, we propose a method for generating nonlinear integrable couplings for the first time, and produce a generalized KdV-Schrödinger integrable system and a nonlocal nonlinear Schrödinger equation, which indicates that we unite the KdV equation and the nonlinear Schrödinger equation as an integrable model by our method. This method presented in the paper could apply to investigate other integrable systems.

Keywords: 3×3 AKNS spectral problem; nonlinear integrable coupling; nonisospectral integrable hierarchy

check for updates

Citation: Zhao, S.; Zhang, Y.; Zhou, J. Several Isospectral and Non-Isospectral Integrable Hierarchies of Evolution Equations. *Symmetry* **2022**, *14*, 402. https://doi.org/10.3390/sym14020402

Academic Editor: Alexander Zaslavski

Received: 21 January 2022
Accepted: 11 February 2022
Published: 17 February 2022

1. Introduction

It has been an important issue to generate new integrable hierarchies and further investigate their related properties, such as symmetries, Bäcklund transformations, algebraic–geometric solutions, covering, etc. [1–7]. Blaszak and Ma [8] started from AKNS 3×3 matrix Lax pairs to discuss the Liouville integrable noncanonical systems with variable coefficient symplectic form by using binary symmetry constraints of the AKNS hierarchy, and they further provided a class of integrable factorization for every AKNS system in the hierarchy. Ma, Fuchssteiner and Oevel [9] adopted 3×3 matrix spectral problems

$$\Phi_x = U\Phi, U = \begin{pmatrix} -2\lambda & \sqrt{2}q & 0 \\ \sqrt{2}r & 0 & \sqrt{2}q \\ 0 & \sqrt{2}r & 2\lambda \end{pmatrix}, \tag{1}$$

$$\Phi_t = V\Phi, V = \sum_{i=0}^{\infty} \begin{pmatrix} 2a_i & \sqrt{2}b_i & 0 \\ \sqrt{2}c_i & 0 & \sqrt{2}b_i \\ 0 & \sqrt{2}c_i & -2a_i \end{pmatrix} \lambda^{-i}, \tag{2}$$

and employed zero curvature equations for deriving the standard AKNS hierarchy. Furthermore, they exploited the binary nonlinearization theory to extend a case of 3×3 matrix spectral problems for AKNS hierarchy. Fuchssteiner [10] proposed the notation on integrable couplings of some known integrable systems while investigating properties of Virasora algebras. Later, Ma and Fuchssteiner [11] employed the perturbation technique for generating the integrable couplings of the KdV equation. However, this method is too tedious and only obtains the integrable couplings of single integrable equations. In 2002, Zhang et al. [12–14] adopted finite dimensional Lie algebras to introduce spectral problems, then employed the Tu Scheme [15] to have generated integrable couplings of some known integrable hierarchy. In order to produce Hamiltonian structures of integrable couplings, Guo and Zhang [16] introduced a s-dimensional vector space V and defined a Lie bracket to make V a Lie algebra $\bar{V}$. On $\bar{V}$, a linear functional was again introduced, by employing the variational method, a formula called the quadratic-form identity was obtained. Later, Ma and Chen [17,18] further improved the formula to get a variational identity, which is a powerful method to generate Hamiltonian structures of some integrable couplings of the known integrable hierarchies. The quadratic-form identity is generalized form of the trace identity proposed by Tu Guizhang [15]. It is remarkable that not all integrable couplings of the known integrable hierarchies possess Hamiltonian structures deduced by the quadratic-form identity or the variational identity. For example, the Lie algebra

$$
\begin{aligned}
&A_{2,1} = \mathbf{span}\{e_1, e_2, e_3, e_4, e_5\},\\
&e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\\
&e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},
\end{aligned}
$$

has a resulting loop algebra:

$$\widetilde{A_{2,1}} = \mathbf{span}\{e_i(n), i = 1, \ldots, 5\}, e_i(n) = e_i \lambda^n.$$

By applying the loop algebra $\widetilde{A_{2,1}}$, some integrable couplings of the AKNS hierarchy, the TD hierarchy, etc., could be obtained by the use of the Tu Scheme. However, the Hamiltonian structures of such the integrable couplings cannot be generated by the quadratic-form identity or the variational identity. Therefore, it is necessary to choose appropriate Lie algebras for deducing their Hamiltonian structures. In addition, the Lie algebras for generating integrable couplings are usually enlarged by the basis of the Lie algebra A_1. In the paper, we want to start from the 3×3 AKNS spectral problems which are represented by 3×3 matrix Lie algebras to derive integrable couplings of the known integrable hierarchies through turning the 3×3 matrix Lie algebras to the 2×2 Lie algebra by choosing proper matrix commutative transformations. Based on those, we choose two appropriate enlarged Lie algebras for which the integrable couplings of the AKNS hierarchy from the 3×3 spectral problems are generated, respectively. Besides, their Hamiltonian structures are also obtained by employing the quadratic-form identity and the trace identity. Specifically, we obtain new nonlinear isospectral and nonisospectral integrable couplings and their initial symmetries of the nonlinear Schröger equation, the KdV equations. In particular, we obtain a nonisospectral nonlinear Schröger equation.

2. Isospectral and Nonisospectral 3 × 3 ANKS Hierarchies

Based on the 3×3 spectral problem (1) and (2), we take the spatial spectral equation

$$\Psi_x = U\Psi, U = \begin{pmatrix} -\lambda & q & 0 \\ r & 0 & q \\ 0 & r & \lambda \end{pmatrix}, \tag{3}$$

Set

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and define a commutative operation as follows:

$$[A, B] = AB - BA,$$

then we have

$$[e_1, e_2] = e_2, [e_1, e_3] = -e_3, [e_2, e_3] = e_1. \tag{4}$$

Making a matrix commutative transformation

$$e_1 \sim \frac{1}{2}h, e_2 \sim e, e_3 \sim \frac{1}{2}f, \tag{5}$$

where

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{6}$$

is a basis of the Lie algebra A_1.

By using (6), an enlarge Lie algebra is given by

$$H =: \mathbf{span}\{h_1, h_2, h_3, h_4, h_5, h_6\},$$

where

$$h_1 = \frac{1}{2}\begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}, h_2 = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}, h_3 = \frac{1}{2}\begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix},$$
$$h_4 = \frac{1}{2}\begin{pmatrix} 0 & h \\ 0 & h \end{pmatrix}, h_5 = \begin{pmatrix} 0 & e \\ 0 & e \end{pmatrix}, h_6 = \frac{1}{2}\begin{pmatrix} 0 & f \\ 0 & f \end{pmatrix}.$$

It is easy to see that

$$\begin{aligned}
&[h_1, h_2] = h_2, [h_1, h_3] = -h_3, [h_1, h_4] = 0, [h_1, h_5] = h_5, \\
&[h_1, h_6] = -h_6, [h_2, h_3] = h_1, [h_2, h_4] = -h_5, [h_2, h_5] = 0, \\
&[h_2, h_6] = h_4, [h_3, h_4] = h_6, [h_3, h_5] = -h_4, [h_3, h6] = 0, \\
&[h_4, h_5] = h_5, [h_4, h_6] = -h_6, [h_5, h_6] = h_6.
\end{aligned}$$

Denoting

$$H_1 = \mathbf{span}\{h_1, h_2, h_3\}, H_2 = \mathbf{span}\{h_4, h_5, h_6\},$$

then

$$H = H_1 \oplus H_2, [H_1, H_1] \subset H_1, [H_1, H_2] \subset H_2, [H_2, H_2] \subset H_2, \tag{7}$$

hence, H is a semi-simple Lie algebra. Specially, we find the Lie algebra $A_1 = \mathbf{span}\{e_1, e_2, e_3\}$ has the same commutative relations with the Lie algebra H_1. Therefore, $\mathbf{span}\{e_1, e_2, e_3\}$ is isomorphic to H_1. Thus, if employing the loop algebras $\widetilde{A_1}$ and $\widetilde{H_1}$ with the same degrations and the Tu Scheme, we could generate the common integrable hierarchies. In what follows, we first apply the loop algebra $\widetilde{A_1}$ and the Tu Scheme to deduce the 3×3 isospectral and nonisospectral hierarchy.

Set

$$\widetilde{A_1} = \mathbf{span}\{e_1(n), e_2(n), e_3(n)\}, e_i(n) = e_i\lambda^n, i = 1, 2, 3, n \in \mathbb{Z}.$$

Taking

$$\begin{aligned}
V &= \sum_{i \geq 0}(a_i e_1(-i) + b_i e_2(-i) + c_i e_3(-i)) + \sum_{j \geq 0}(\bar{a}_j e_1(-j) + \bar{b}_j e_2 e_2(-j) + \bar{c}_j e_3(-j)) \\
&=: V_1 + V_2,
\end{aligned}$$

then the compatibility condition of the spectral problems

$$\varphi_x = U(u, \lambda)\varphi, \varphi_t = V(u, \lambda)\varphi, \lambda_t \neq 0 \tag{8}$$

reads that

$$\frac{\partial U}{\partial u}u_t + \frac{\partial U}{\partial \lambda}\lambda_t - V_x + [U, V] = 0.$$

According to the scheme called the generalized Tu Scheme presented in [19–21], we first solve the equation for V

$$\frac{\partial U}{\partial \lambda}\lambda_t - V_x + [U,V] = 0, \lambda_t = \sum_{j\geq 0} k_j(t)\lambda^{-j}, \tag{9}$$

which admits that

$$\begin{aligned}
&a_i = \partial^{-1}(qc_i - rb_i) - \alpha_i(t),\\
&\bar{a}_j = \partial^{-1}(q\bar{c}_j - r\bar{b}_j) - k_j(t)x,\\
&b_{i+1} = (-\partial + q\partial^{-1}r)b_i - q\partial^{-1}qc_i + \alpha_i(t)q,\\
&\bar{b}_{j+1} = (-\partial + q\partial^{-1}r)\bar{b}_j - q\partial^{-1}q\bar{c}_j + k_j(t)xq,\\
&c_{i+1} = (\partial - r\partial^{-1}q)c_i + r\partial^{-1}rb_i + \alpha_i(t)r,\\
&\bar{c}_{j+1} = (\partial - r\partial^{-1}q)\bar{c}_j + r\partial^{-1}r\bar{b}_j + k_j(t)xr.
\end{aligned}$$

Noting

$$\begin{aligned}
V_+^{(n,m)} &= \sum_{i=0}^{n}(a_ie_1(n-i) + b_ie_2(n-i) + c_ie_3(n-i))\\
&\quad + \sum_{j=0}^{m}(\bar{a}_je_1(m-j) + \bar{b}_je_2(m-j) + \bar{c}_je_3(m-j))\\
&= \lambda^n V_1 + \lambda^m V_2 - V_-^{(n,m)},\\
\lambda_{t,+}^{(m)} &= \lambda^m\lambda_t - \lambda_{t,-}^{(m)},
\end{aligned}$$

then (9) can be decomposed into

$$-V_{+,x}^{(n,m)} + [U, V_+^{(n,m)}] - \frac{\partial U}{\partial \lambda}\lambda_{t,+}^{(m)} = V_{-,x}^{(n,m)} - [U, V_-^{(n,m)}] - \frac{\partial U}{\partial \lambda}\lambda_{t,-}^{(m)}. \tag{10}$$

The gradations of the left-hand side in (10) are more than 0, while the right-hand side less than 1. Thus, the gradations of (10) read $0, 1$, which lead us to the following

$$-V_{+,x}^{(n,m)} + [U, V_+^{(n,m)}] + \frac{\partial U}{\partial \lambda}\lambda_{t,+}^{(m)} = (b_{n+1} + \bar{b}_{m+1})e_2(0) - (c_{n+1} + \bar{c}_{m+1})e_3(0).$$

The zero curvature equation

$$\frac{\partial U}{\partial \lambda}\lambda_{t,+}^{(m)} + \frac{\partial U}{\partial u}u_t - V_{+,x}^{(n,m)} + [U, V_+^{(n,m)}] = 0$$

gives rise to the following isospectra and nonisospectral AKNS hierarchy

$$\begin{aligned}
u_t =: \begin{pmatrix} q \\ r \end{pmatrix}_t &= \begin{pmatrix} -b_{n+1} - \bar{b}_{m+1} \\ c_{n+1} + \bar{c}_{m+1} \end{pmatrix}\\
&= \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 2c_{n+1} + 2\bar{c}_{m+1} \\ 2b_{n+1} + 2\bar{b}_{m+1} \end{pmatrix} =: J_1\begin{pmatrix} 2c_{n+1} + 2\bar{c}_{m+1} \\ 2b_{n+1} + 2\bar{b}_{m+1} \end{pmatrix},
\end{aligned} \tag{11}$$

where $J_1 = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$ is a Hamiltonian operator. Equation (11) can be written again

$$\begin{aligned}
u_t =& \frac{1}{2}\begin{pmatrix} -q\partial^{-1}q & \partial - q\partial^{-1}q \\ \partial - r\partial^{-1}q & r\partial^{-1}r \end{pmatrix}\begin{pmatrix} 2c_n + 2\bar{c}_m \\ 2b_n + 2\bar{b}_m \end{pmatrix}\\
&+ \alpha_n(t)\begin{pmatrix} -q \\ r \end{pmatrix} + k_m(t)\begin{pmatrix} -xq \\ xr \end{pmatrix}
\end{aligned}. \tag{12}$$

It is easy to compute that

$$\begin{aligned}\begin{pmatrix} c_{n+1} \\ b_{n+1} \end{pmatrix} &= \begin{pmatrix} \partial - r\partial^{-1}r & r\partial^{-1}r \\ -q\partial^{-1}q & -\partial + q\partial^{-1}r \end{pmatrix}\begin{pmatrix} c_n \\ b_n \end{pmatrix} + \alpha_n\begin{pmatrix} r \\ q \end{pmatrix} \\ &=: L\begin{pmatrix} c_n \\ b_n \end{pmatrix} + \alpha_n\begin{pmatrix} r \\ q \end{pmatrix} = \dots \\ &= L^n\begin{pmatrix} c_1 \\ b_1 \end{pmatrix} + [\alpha_1 L^{n-1} + \alpha_2 L^{n-2} + \dots + \alpha_{n-1}L + \alpha_n]\begin{pmatrix} r \\ q \end{pmatrix}.\end{aligned}$$

Hence, (11) can be written as

$$\begin{aligned}u_t = \begin{pmatrix} q \\ t \end{pmatrix}_t &= \Phi^n J\begin{pmatrix} 2\alpha_0 r \\ 2\alpha_0 q \end{pmatrix} + (2\alpha_1\Phi^{n-1} + 2\alpha_2\phi^{n-2} + \dots + 2\alpha_n)J\begin{pmatrix} r \\ q \end{pmatrix} \\ &\quad + \Phi^m J\begin{pmatrix} 2k_0 xr \\ 2k_0 xq \end{pmatrix} + (2k_1\Phi^{m-1} + 2k_2\Phi^{m-2} + \dots + 2k_m)J\begin{pmatrix} xr \\ xq \end{pmatrix} \\ &= \sum_{i=0}^{n} 2\alpha_{n-i}\Phi^i J\begin{pmatrix} r \\ q \end{pmatrix} + \sum_{j=0}^{m} 2k_{m-j}\Phi^i J\begin{pmatrix} xr \\ xq \end{pmatrix}.\end{aligned}$$

Due to

$$J\begin{pmatrix} r \\ q \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -q \\ r \end{pmatrix},$$

we have the 3×3 AKNS isospectral–nonisospectral hierarchy:

$$u_t = \sum_{i=0}^{n} \alpha_{n-i}\left(\frac{1}{4}\right)^i \bar{\Phi}^i\begin{pmatrix} -q \\ r \end{pmatrix} + \sum_{j=0}^{m} k_{m-j}\left(\frac{1}{4}\right)^j \bar{\Phi}^j\begin{pmatrix} -xq \\ xr \end{pmatrix}, \tag{13}$$

where

$$\bar{\Phi} = \begin{pmatrix} -\partial + q\partial^{-1}r & q\partial^{-1}q \\ -r\partial^{-1}r & \partial - r\partial^{-1}q \end{pmatrix}.$$

Let $j = 0, i = n, \alpha_0(t) = 4^n$, (13) reduces to the isospectral hierarchy:

$$u_t = \bar{\Phi}^n\begin{pmatrix} -q \\ r \end{pmatrix} =: K_n. \tag{14}$$

It is easy to verify that

$$T_0^n = ntK_{n-1} + x\begin{pmatrix} -q \\ r \end{pmatrix}$$

are nonlocal symmetries of (14). Since $\bar{\Phi}$ is a strong symmetric operator, $\tau_n^m = \bar{\Phi}^m\tau_0^n = ntK_{m+n-1} + \bar{\Phi}^n x\begin{pmatrix} -q \\ r \end{pmatrix}$ are still symmetries of (14).

Remark 1. *Through the discussion as above, we declare that the integrable hierarchies derived from* 3×3 *matrix Lie algebras can be worked out by employing* 2×2 *matrix Lie algebras via such the transformation* (5)*. Actually, some other* 3×3 *spectral problems can also transform to* 2×2 *cases by the transformation* (5)*. For example, the following* 3×3 *spectral problem* [22]*:*

$$\Psi_x = \begin{pmatrix} 2\lambda - 2s & \sqrt{2}q & 0 \\ -\sqrt{2}\lambda r & 0 & \sqrt{2}q \\ 0 & -\sqrt{2}\lambda r & -2\lambda - 2s \end{pmatrix}\Psi.$$

The advantage for turning 3×3 *spectral problems to* 2×2 *cases by using* (5) *lies in further investigating integrable couplings of the associating integrable hierarchies derived from* 3×3

spectral problems. In what follows, we still take the 3×3 *AKNS integrable hierarchy for example to illustrate the question.*

3. Two Kinds of Integrable Models

The so-called integrable models in the paper imply integrable couplings of some known integrable hierarchies.

Case 3.1: The first kind of integrable model

Now, we shall employ the enlarged Lie algebra H (7) obtained by using the transformation (5) for deducing a kind of integrable coupling of (13) and discuss its Hamiltonian structure.

Set

$$\varphi_x = U\varphi, U = -h_1(1) + qh_2(0) + rh_3(0) + u_1h_5(0) + u_2h_6(0), \tag{15}$$

$$\varphi_t = V\varphi, V = V_1 + V_2, \tag{16}$$

$$V_1 = \sum_{i\geq 0}(a_ih_1(-i) + b_ih_2(-i) + c_ih_3(-i) + d_ih_4(-i) + e_ih_5(-i) + g_ih_6(-i),$$

$$V_2 = \sum_{j\geq 0}(\bar{a}_ih_1(-j) + \bar{b}_jh_2(-j) + \bar{c}_jh_3(-j) + d_jh_4(-j) + e_jh_5(-j) + g_jh_6(-j),$$

$$\lambda_t = \sum_{j\geq 0} k_j(t)\lambda^{-j}.$$

By using (9) along with (15) and (16), we have

$$\begin{cases} a_i = \partial^{-1}(qc_i - rb_i) - \alpha_i(t), \\ \bar{a}_j = \partial^{-1}(q\bar{c}_j - r\bar{b}_j) - k_j(t)x, \\ b_{i+1} = -b_ix - qa_i, \\ \bar{b}_{j+1} = -\bar{b}_jx - q\bar{a}_j, \\ c_{i+1} = c_ix - ra_i, \\ \bar{c}_{j+1} = \bar{c}_jx - r\bar{a}_j, \end{cases} \tag{17}$$

$$\begin{cases} d_i = \partial^{-1}[(q+u_1)g_i - (r+u_2)e_i + u_1c_i - u_2b_i] - \beta_i(t), \\ \bar{d}_j = \partial^{-1}[(q+u_1)\bar{g}_j - (r+u_2)\bar{e}_j + u_1\bar{c}_j - u_2\bar{b}_j] - \gamma_j(t)x, \\ e_{i+1} = -e_ix - (q+u_1)d_i - u_1a_i, \\ \bar{e}_{j+1} = -\bar{e}_jx - (q+u_1)\bar{d}_j - u_1\bar{a}_j, \\ g_{i+1} = g_ix - (r+u_2)d_i - u_2a_i, \\ \bar{g}_{j+1} = \bar{g}_jx - (r+u_2)\bar{d}_j - u_2\bar{a}_j. \end{cases} \tag{18}$$

Denoting

$$\begin{aligned} V_+^{(n,m)} &= \sum_{i=0}^{n}(a_ih_1(n-i) + b_ih_2(n-i) + c_ih_3(n-i) + d_ih_4(n-i) + e_ih_5(n-i) \\ &\quad + g_ih_6(n-i)) + \sum_{j=0}^{m}(\bar{a}_jh_1(m-j) + \bar{b}_jh_2(m-j) + \bar{c}_jh_3(m-j) \\ &\quad + \bar{d}_jh_4(m-j) + \bar{e}_jh_5(m-j) + \bar{g}_jh_6(m-j)) =: V_{1,+}^{(n)} + V_{2,+}^{(m)}, \\ \lambda_{t,+}^{(n,m)} &= \sum_{j=0}^{m} k_j(t)\lambda^{m-j} = \lambda^m\lambda_t - \lambda_{t,-}^{(m)}. \end{aligned}$$

A direct calculation gives rise to

$$-V_{+,x}^{(n,m)}+[U,V_{+}^{(n,m)}]+\frac{\partial U}{\partial\lambda}\lambda_{t,+}^{(m)}=(b_{n+1}+\bar{b}_{m+1})h_2(0)-(c_{n+1}\bar{c}_{m+1})h_3(0)$$
$$+(e_{n+1}+\bar{e}_{m+1})h_5(0)-(g_{n+1}+\bar{g}_{m+1})h_6(0).$$

Noting $V^{(n,m)}=V_{+}^{(n,m)}$, then the nonisospectral zero curvature equation

$$\frac{\partial U}{\partial u}u_t+\frac{\partial U}{\partial\lambda}\lambda_{t,+}^{(n,m)}-V_x^{n,m}+[U,V^{(n,m)}]=0$$

admits an integrable model

$$u_t=:\begin{pmatrix} q\\ r\\ u_1\\ u_2\end{pmatrix}_t=\begin{pmatrix}-b_{n+1}-\bar{b}_{m+1}\\ c_{n+1}+\bar{c}_{m+1}\\ -e_{n+1}-\bar{e}_{m+1}\\ g_{n+1}+\bar{g}_{m+1}\end{pmatrix}. \tag{19}$$

Obviously, when $u_1=u_2=0$, (19) reduces to (13). Therefore, (19) is an integrable coupling of the isospectral–nonisospectral hierarchy (13). In the following, we consider some reductions of (19).

Set

$$b_0=c_0=g_0=e_0=0, a_0=\alpha_0(t), d_0=-\beta_0(t),$$

then from (17) and (18) we get that

$$\begin{aligned}
&b_1=\alpha_0 q, c_1=\alpha_0 r, a_1=-\alpha_1(t),\\
&b_2=-\alpha_0 q_x+\alpha_1 q, c_2=\alpha_0 r_x+\alpha_1 r, a_2=\alpha_0 qr-\alpha_2(t),\\
&b_3=\alpha_0 q_{xx}-\alpha_1 q_x-\alpha_0 q^2 r+\alpha_2 q,\\
&c_3=\alpha_0 r_{xx}+\alpha_1 r_x-\alpha_0 qr^2+\alpha_2 r,\\
&a_3=\alpha_0(qr_x-q_x r)+\alpha_1 qr-\alpha_3,\\
&b_4=-\alpha_0 q_{xxx}+3\alpha_0 qq_x r+\alpha_1 q_{xx}-\alpha_1 q^2 r-\alpha_2 q_x+\alpha_3 q,\\
&c_4=\alpha_0 r_{xxx}-3\alpha_0 qrr_x+\alpha_1 r_{xx}-\alpha_1 qr^2+\alpha_2 r_x+\alpha_3 r,\\
&\ldots
\end{aligned}$$

$$\begin{aligned}
&g_1=\beta_0(r+u_2)+\alpha_0 u_2,\\
&e_1=\beta_0(q+u_1)+\alpha_0 u_1,\\
&d_1=-\beta_1(t),\\
&e_2=-\beta_0(q+u_1)_x-\alpha_0 u_{1,x}+\beta_1(q+u_1)+\alpha_1 u_1,\\
&g_2=\beta_0(r+u_2)_x+\alpha_0 u_{2,x}+\beta_1(r+u_2)+\alpha_1 u_2,\\
&d_2=\beta_0(q+u_1)(r+u_2)+\alpha_0 u_1 r+\alpha_0 u_1 u_2+\alpha_0 u_2 q-\beta_2,\\
&\ldots
\end{aligned}.$$

When $n=1, m=0$, (19) reduces to

$$\begin{cases}
q_t=\alpha_0 q_x-\alpha_1 q,\\
r_t=\alpha_0 r_x+\alpha_1 r,\\
u_{1t}=\beta_0(q+u_1)_x+\alpha_0 u_{1,x}-\beta(q+u_1)-\alpha_1 u_1,\\
u_{2t}=\beta_0(r+u_2)_x+\alpha_0 u_{2,x}+\beta_1(r+u_2)+\alpha_1 u_2.
\end{cases}$$

When $n = 2, m = 0$, (19) becomes

$$\begin{cases} q_t = -\alpha_0 q_{xx} + \alpha_1 q_x + \alpha_0 q^2 r - \alpha_2 q, \\ r_t = \alpha_0 r_{xx} + \alpha_1 r_x - \alpha_0 q r^2 + \alpha_2 r, \\ u_{1t} = -\beta_0 (q+u_1)_{xx} - \alpha_0 u_{1,xx} + \beta_1 (q+u_1)_x + \alpha_1 u_{1,x}, \\ \qquad + \beta_0 (q+u_1)^2 (r+u_2) + \alpha_0 (q+u_1)(u_1 r + u_2 q + u_1 u_2) \\ \qquad - \beta_2 (q+u_1) + \alpha_0 q r u_1 - \alpha_2 u_1, \\ u_{2t} = \beta_0 (r+u_2)_{xx} + \alpha_0 u_{2,xx} + \beta_1 (r+u_2)_x + \alpha_1 u_{2,x} \\ \qquad - \beta_0 (q+u_1)(r+u_2)^2 - \alpha_0 (r+u_2)(u_1 r + u_1 u_2 + u_2 q) \\ \qquad + \beta_0 (r+u_2) - \alpha_0 q r u_2 + \alpha_2 u_2. \end{cases} \tag{20}$$

Specially, set $\alpha_1 = \alpha_2 = 0, \alpha_0 = 1, \beta_0 = \beta_1 = \beta_2 = 0, u_1 = u_2 = 0$, (20) reduces to the well-known nonlinear Schrödinger system

$$\begin{cases} q_t = -q_{xx} + q^2 r, \\ r_t = r_{xx} - q r^2. \end{cases} \tag{21}$$

Remark 2. *We first time obtained such the nonlinear integrable coupling of the nonlinear Schrödinger equation. The so-called nonlinear integrable coupling means that if $u_t = K(u)$ is a known integrable hierarchy, $v_t = S(u, v)$ is also integrable and is nonlinear with respect to the new potential variable v, then the integrable system*

$$\begin{cases} u_t = K(u), \\ v_t = S(u, v) \end{cases}$$

is called a nonlinear integrable coupling.

When $n = 3, m = 0$, (19) again reduces to

$$\begin{cases} q_t = -b_4 = \alpha_0 q_{xxx} - 3\alpha_0 q q_x r - \alpha_1 q_{xx} - \alpha_1 q^2 r + \alpha_2 q_x - \alpha_3 q, \\ r_t = c_4 = \alpha_0 r_{xxx} + \alpha_1 r_{xx} - 3\alpha_0 q r r_x - \alpha_1 q r^2 + \alpha_2 r_x + \alpha_3 r, \end{cases} \tag{22}$$

$$\begin{aligned} u_{1t} = -e_4 = &\beta_0 (q+u_1)_{xxx} + \alpha_0 u_{1,xxx} - \beta_1 (q+u_1)_{xx} - \alpha_1 u_{1,xx} \\ &- \alpha_0 [(q+u_1)(u_1 r + u_2 q + u_1 u_2)]_x + \beta_2 (q+u_1)_x - \alpha_0 (q r u_1)_x \\ &+ \alpha_2 u_{1,x} - 3\beta_0 (q+u_1)(q+u_1)_x (r+u_2) + \alpha_0 (q+u_1)^2 u_{2,x} \\ &- \alpha_0 (q+u_1)(q+u_1)_x u_2 + \alpha_0 q u_{2,x} (q+u_1) - \alpha_0 q_x u_2 (q+u_1) \\ &+ \alpha_0 (q+u_1)(u_1 u_{2,x} - u_{1,x} u_2 + u_1 r_x - u_{1,x} r) + \beta_1 (q+u_1)^2 (q+u_2) \\ &+ \alpha_1 (q+u_1)(q + u_2 + u_1 u_2 + u_1 r) - \beta_3 (q+u_1) + \alpha_0 u_1 (q r_x - q_x r) \\ &+ \alpha_1 q r u_1 - \alpha_3 u_1, \end{aligned} \tag{23}$$

$$\begin{aligned} u_{2t} = g_4 = &\beta_0 (r+u_2)_{xxx} + \alpha_0 u_{2,xxx} + \beta_1 (r+u_2)_{xx} + \alpha_1 u_{2,xx} \\ &- 3\beta_0 (q+u_1)(r+u_2)(r+u_2)_x - \alpha_0 [(r+u_2)(u_1 r + u_1 u_2 + u_2 q)]_x \\ &+ \beta_0 (r+u_2)_x - \alpha_0 (q r u_2)_x + \alpha_2 u_{2,x} - \alpha_0 (r+u_2)(q+u_1) u_{2,x} \\ &+ \alpha_0 (r+u_2)(q+u_1)_x u_2 - \alpha_0 (r+u_2) q u_{2,x} + \alpha_0 q_x u_2 (r+u_2) \\ &- \alpha_0 (r+u_2)(u_1 u_{2,x} - u_{1,x} u_2 + u_1 r_x - u_{1,x} r) - \beta_1 (r+u_2)(q+u_1) \\ &\times (q+u_2) - \alpha_1 (r+u_2)(q u_2 + u_1 u_2 + u_1 r) + \beta_3 (r+u_2) \\ &- \alpha_0 u_2 (q r_x - q_x r) - \alpha_1 q r u_2 + \alpha_3 u_2. \end{aligned} \tag{24}$$

When $\alpha_1 = \alpha_2 = \alpha_3 = 0, \alpha_0 = 1$, (22) reduces to

$$\begin{cases} q_t = q_{xxx} + 3 q q_x r, \\ r_t = r_{xxx} - 3 q r r_x. \end{cases} \tag{25}$$

When $r = 1$, (25) turns to the KdV equation

$$q_t = q_{xxx} + 3qq_x.$$

When $r = -iq$, (25) becomes

$$q_t = q_{xxx} - 3iq^2 q_x, \tag{26}$$

which is a complex modified KdV equation.

When $\alpha_0 = \alpha_2 = \alpha_3 = 0, \alpha_1 = 1$, (22) reduces to

$$\begin{cases} q_t = -q_{xx} + q^2 r, \\ r_t = r_{xx} - qr^2, \end{cases} \tag{27}$$

which represents the Schrödinger equation.

Therefore, we call (22) a generalized KdV-Schrödinger integrable system. Obviously, when $\beta_i = 0 (i = 1, 2, 3), u_1 = u_2 = 0$, Equations (23) and (24) identically hold. Hence, (22)–(24) consist of an nonlinear integrable coupling of the generalized KdV-Schrödinger integrable system (22).

Remark 3. *From the above discussion, we conclude that a simple and efficient approach for generating nonlinear integrable couplings just right take multiple parameter functions.*

In what follows, we look for the Hamiltonian structure and the symmetries of the isospectral integrable hierarchy:

$$u_t = \begin{pmatrix} q \\ r \\ u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} -b_{n+1} \\ c_{n+1} \\ -e_{n+1} \\ g_{n+1} \end{pmatrix}, \tag{28}$$

which is the integrable coupling of the hierarchy (14). For arbitrary elements a, b in the Lie algebra H, we represent them as

$$a = \sum_{i=1}^{6} a_i h_i, b = \sum_{i=1}^{6} b_i h_i$$

which can be used to define a commutative operation in the vector space R^6 as follows:

$$[a, b] = \begin{pmatrix} [a, b]_1 \\ [a, b]_2 \end{pmatrix}, \tag{29}$$

where

$$\begin{aligned} [a, b]_1^T =& (a_2 b_3 - a_3 b_2, a_1 b_2 - a_2 b_1, a_3 b_1 - a_1 b_3), \\ [a, b]_2^T =& (a_2 b_6 - a_6 b_2 + a_5 b_3 - a_3 b_5 + a_5 b_6 - a_6 b_5, \\ & a_1 b_5 - a_5 b_1 + a_4 b_2 - a_2 b_4 + a_4 b_5 - a_5 b_4, \\ & a_3 b_4 - a_4 b_3 + a_6 b_1 - a_1 b_6 + a_6 b_4 - a_4 b_6). \end{aligned}$$

It can be verified that R^6 becomes a vector Lie algebra if equipped with (29). Besides, (29) can be written as

$$[a, b] = a^T R(b) = a^T \begin{pmatrix} R_1 & R_2 \\ 0 & R_3 \end{pmatrix},$$

where

$$R_1 = \begin{pmatrix} 0 & b_2 & -b_3 \\ b_3 & -b_1 & 0 \\ -b_2 & 0 & b_1 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & b_5 & -b_6 \\ b_6 & -b_4 & 0 \\ b_5 & 0 & b_4 \end{pmatrix},$$
$$R_3 = \begin{pmatrix} 0 & b_2 + b_5 & -b_3 - b_6 \\ b_3 + b_6 & -b_1 - b_4 & 0 \\ -b_2 - b_5 & 0 & b_1 + b_4 \end{pmatrix},$$

$R(b)$ requires satisfy

$$R(b)M = -(R(b)M)^T, M^T = M, M_{xt} = 0. \tag{30}$$

Solving (30) obtains

$$M = \begin{pmatrix} \eta_1 & 0 & 0 & \eta_2 & 0 & 0 \\ 0 & 0 & \eta_1 & 0 & 0 & \eta_2 \\ 0 & \eta_1 & 0 & 0 & \eta_2 & 0 \\ \eta_2 & 0 & 0 & \eta_2 & 0 & 0 \\ 0 & 0 & \eta_2 & 0 & 0 & \eta_2 \\ 0 & \eta_2 & 0 & 0 & \eta_2 & 0 \end{pmatrix},$$

here η_2 and η_2 are constants.

According to Refs. [16,17], a linear functional is defined by

$$\{a, b\} = a^T M b. \tag{31}$$

Taking

$$U = (-\lambda, q, r, 0, u_1, u_2) \in R^6,$$
$$V = (A, B, C, D, E, G) \in R^6,$$
$$A = \sum_{i \geq 0} a_i \lambda^{-i}, B = \sum_{i \geq 0} b_i \lambda^{-i}, \ldots$$

In terms of (31), it is easy to calculate that

$$\{V, \frac{\partial U}{\partial q}\} = C\eta_1 + G\eta_2, \{V, \frac{\partial U}{\partial r}\} = B\eta_1 + E\eta_2,$$
$$\{V, \frac{\partial U}{\partial u_1}\} = C\eta_2 + G\eta_2, \{V, \frac{\partial U}{\partial u_2}\} = (B + E)\eta_2,$$
$$\{V, \frac{\partial U}{\lambda}\} = A\eta_1 + D\eta_2.$$

Substituting the above results into the quadratic-form identity reads that

$$\frac{\delta}{\delta u}(A\eta_1 + D\eta_2) = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \begin{pmatrix} C\eta_1 + G\eta_2 \\ B\eta_1 + E\eta_2 \\ C\eta_2 + G\eta_2 \\ B\eta_2 + E\eta_2 \end{pmatrix}. \tag{32}$$

Comparing the coefficients of λ^{-n-1} of both sides in (32) gives

$$\frac{\delta}{\delta u}(a_{n+1}\eta_1 + d_{n+1}\eta_2) = (-n) \begin{pmatrix} c_n\eta_1 + g_n\eta_2 \\ b_n\eta_1 + e_n\eta_2 \\ c_n\eta_2 + g_n\eta_2 \\ b_n\eta_2 + e_n\eta_2 \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} c_n\eta_1 + g_n\eta_2 \\ b_n\eta_1 + e_n\eta_2 \\ c_n\eta_2 + g_n\eta_2 \\ b_n\eta_2 + e_n\eta_2 \end{pmatrix}. =: \frac{\delta H_n}{\delta u},$$

where $H_n = -\frac{a_{n+1}\eta_1 + d_{n+1}\eta_2}{n}$ *are the Hamiltonian function. We can take* $\eta_1 = 1$, *then the integrable coupling* (28) *can be written as*

$$
\begin{aligned}
u_t = \begin{pmatrix} q \\ r \\ u_1 \\ u_2 \end{pmatrix}_t &= \begin{pmatrix} 0 & \frac{1}{\eta_2 - 1} & 0 & \frac{1}{1-\eta_2} \\ \frac{1}{1-\eta_2} & 0 & \frac{1}{\eta_2-1} & 0 \\ 0 & \frac{1}{1-\eta_2} & 0 & -\frac{1}{\eta_2-\eta_2^2} \\ \frac{1}{\eta_2-1} & 0 & \frac{1}{\eta_2-\eta_2^2} & 0 \end{pmatrix} \begin{pmatrix} c_{n+1}\eta_1 + g_{n+1}\eta_2 \\ b_{n+1}\eta_1 + e_{n+1}\eta_2 \\ c_{n+1}\eta_2 + g_{n+1}\eta_2 \\ b_{n+1}\eta_2 + e_{n+1}\eta_2 \end{pmatrix} \\
&=: \bar{J} \begin{pmatrix} c_{n+1} + g_{n+1}\eta_2 \\ b_{n+1} + e_{n+1}\eta_2 \\ c_{n+1}\eta_2 + g_{n+1}\eta_2 \\ b_{n+1}\eta_2 + e_{n+1}\eta_2 \end{pmatrix} = \bar{J}\frac{\delta H_{n+1}}{\delta u} + \begin{pmatrix} (r+u_2)\beta_n\eta_2 + u_2\alpha_n\eta_2 + \alpha_n r \\ -(q+u_1)\beta_n\eta_2 + u_1\alpha_n\eta_2 + \alpha_n q \\ (r+u_2)\beta_n\eta_2 + u_2\alpha_n\eta_2 + \alpha_n r\eta_2 \\ -(q+u_1)\beta_n\eta_2 + u_1\alpha_n\eta_2 + q\alpha_n\eta_2 \end{pmatrix},
\end{aligned}
\tag{33}
$$

which is the Hamiltonian form of (28). *It can be found that*

$$
\begin{pmatrix} c_{n+1} + g_{n+1}\eta_2 \\ b_{n+1} + e_{n+1}\eta_2 \\ (c_{n+1} + g_{n+1})\eta_2 \\ (b_{n+1} + e_{n+1})\eta_2 \end{pmatrix} = L \begin{pmatrix} c_n + g_n\eta_2 \\ b_n + e_n\eta_2 \\ (c_n + g_n)\eta_2 \\ (b_n + e_n)\eta_2 \end{pmatrix},
$$

where

$$
L = \begin{pmatrix} \partial - r\partial^{-1}q & r\partial^{-1}r & -r(r+u_2)\partial^{-1}u_1 - u_2\partial^{-1}q & (r+u_2)\partial^{-1}u_2 + u_2\partial^{-1}r \\ -q\partial^{-1}q & -\partial + q\partial^{-1}r & -(q+u_1)\partial^{-1}u_1 - u_1\partial^{-1}q & (q+u_1)\partial^{-1}u_2 + u_1\partial^{-1}r \\ (r+u_2)\partial^{-1}(u_1-u_2) & 0 & \partial - (r+u_2)\partial^{-1}(q+u_1) & (r+u_2)\partial^{-1}(r+u_2) \\ 0 & 0 & -(q+u_1)\partial^{-1}(q+u_1) & -\partial + (q+u_1)\partial^{-1}(r+u_2) \end{pmatrix}
$$

is a recurrence operator. Hence, (33) *can be written again as*

$$
\begin{aligned}
u_t = \begin{pmatrix} q \\ r \\ u_1 \\ u_2 \end{pmatrix}_t &= \bar{J}L \begin{pmatrix} c_n + g_n\eta_2 \\ b_n + e_n\eta_2 \\ (c_n + g_n)\eta_2 \\ (b_n + e_n)\eta_2 \end{pmatrix} + \begin{pmatrix} (r+u_2)\beta_n\eta_2 + u_2\alpha_n\eta_2 + \alpha_n r \\ -(q+u_1)\beta_n\eta_2 + u_1\alpha_n\eta_2 + \alpha_n q \\ (r+u_2)\beta_n\eta_2 + u_2\alpha_n\eta_2 + \alpha_n r\eta_2 \\ -(q+u_1)\beta_n\eta_2 + u_1\alpha_n\eta_2 + q\alpha_n\eta_2 \end{pmatrix} \\
&=: P_1 + P_2 = \Phi^n \bar{J} \begin{pmatrix} c_1 + g_1\eta_2 \\ b_1 + e_1\eta_2 \\ (c_1 + g_1)\eta_2 \\ (b_1 + e_1)\eta_2 \end{pmatrix} + P_2,
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi =& \bar{J}L\bar{J}^{-1} \\
=& \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & \frac{1}{\eta_2} \\ 1 & 0 & -\frac{1}{\eta_2} & 0 \end{pmatrix} \\
& \begin{pmatrix} \partial - r\partial^{-1}q & r\partial^{-1}r & -(r+u_2)\partial^{-1}u_1 - u_2\partial^{-1}q & (r+u_2)\partial^{-1}u_2 + u_2\partial^{-1}r \\ -q\partial^{-1}q & -\partial + q\partial^{-1}r & -(q+u_1)\partial^{-1}u_1 - u_1\partial^{-1}q & (q+u_1)\partial^{-1}u_2 + u_1\partial^{-1}r \\ (r+u_2)\partial^{-1}(u_1-u_2) & 0 & \partial - (r+u_2)\partial^{-1}(q+u_1) & (r+u_2)\partial^{-1}(r+u_2) \\ 0 & 0 & -(q+u_1)\partial^{-1}(q+u_1) & \partial + (q+u_1)\partial^{-1}(r+u_2) \end{pmatrix} \\
& \begin{pmatrix} 0 & -\frac{1}{\eta_2}\left(\frac{1}{\eta_2} - 1\right) & 0 & 1 - \frac{1}{\eta_2} \\ \frac{1}{\eta_2}\left(\frac{1}{\eta_2} - 1\right) & 0 & 1 - \frac{1}{\eta_2} & 0 \\ 0 & 1 - \frac{1}{\eta_2} & 0 & \frac{1}{\eta_2} - 1 \\ \frac{1}{\eta_2} - 1 & 0 & \frac{1}{\eta_2} - 1 & 0 \end{pmatrix}.
\end{aligned}
$$

Obviously,

$$
\begin{aligned}
K_0 =& \bar{J}\begin{pmatrix} c_1+g_1\eta_2 \\ b_1+e_1\eta_2 \\ (c_1+g_1)\eta_2 \\ (b_1+e_1)\eta_2 \end{pmatrix} \\
=& \frac{1}{\eta_2-1}\begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & \frac{1}{\eta_2} \\ 1 & 0 & -\frac{1}{\eta_2} & 0 \end{pmatrix}\begin{pmatrix} \alpha_0 r+(\beta_0 r+\beta_0 u_2+\alpha_0 u_1)\eta_2 \\ \alpha_0 q+(\beta_0 q+\beta_0 u_1+\alpha_0 u_1)\eta_2 \\ (\alpha_0 r+\beta_0 r+\beta_0 u_2+\alpha_0 u_1)\eta_2 \\ (\alpha_0 q+\beta_0 q+\beta_0 u_1+\alpha_0 u_1)\eta_2 \end{pmatrix} \\
=& \begin{pmatrix} -\alpha_0 q \\ \alpha_0 r \\ -(\beta_0 q+\beta_0 u_1+\alpha_0 u_1) \\ (\beta_0 r+\beta_0 u_2+\alpha_0 u_1) \end{pmatrix}
\end{aligned}
$$

is an initial symmetry of the integrable coupling (33)*. The isospectral–nonisospectral integrable hierarchy* (19) *also contains the nonisospectral integrable hierarchy*

$$
u_t = \begin{pmatrix} q \\ r \\ u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} -\bar{b}_{m+1} \\ \bar{c}_{m+1} \\ -\bar{e}_{m+1} \\ \bar{g}_{m+1} \end{pmatrix}. \tag{34}
$$

In what follows, we consider some reductions of (34)*. Set* $\bar{b}_0 = \bar{c}_0 = \bar{g}_0 = \bar{e}_0 = 0, \bar{a}_0 = -k_0(t)x, \bar{d}_0 = -\gamma_0(t)x$*, then we have from* (17) *and* (18) *that*

$$
\begin{aligned}
&\bar{b}_1 = k_0 xq, \bar{c}_1 = k_0 xr, \bar{g}_1 = \gamma_0 x(r+u_2) + k_0 x u_2, \\
&\bar{e}_1 = \gamma_0 x(q+u_1) + k_0 x u_1, \bar{a}_1 - k_1(t)x, \bar{d}_1 = -\gamma_1(t)x, \\
&\bar{b}_2 = -k_0(xq)_x + k_1 xq, \bar{c}_2 = k_0(xr)_x + k_1 xr, \\
&\bar{g}_2 = \gamma_0(x(r+u_2))_x + k_0(xu_2)_x + \gamma_1 x(r+u_2) + k_1 x u_2, \\
&\bar{e}_2 = -\gamma_0(x(q+u_1))_x - k_0(xu_1)_x + \gamma_1 x(q+u_1) + k_1 x u_1, \\
&\bar{a}_2 = k_0 xqr + k_0\partial^{-1}(qr) - k_2 x, \\
&\bar{b}_3 = k_0(xq)_{xx} + k_1(xq)_x - k_0 xq^2 r - k_0 q\partial^{-1}(qr) + k_2 xq, \\
&\bar{c}_3 = k_0(xr)_{xx} + k_1(xr)_x - k_0 xqr^2 - k_0 r\partial^{-1}(qr) + k_2 xr, \\
&\bar{d}_2 = \gamma_0 x(q+u_1)(r+u_2) + \gamma_0\partial^{-1}(q+u_1)(r+u_2) \\
&\qquad + k_0\partial^{-1}[(q+u_1)(xu_2)_x + (r+u_2)(xu_1)_x] \\
&\qquad + k_0\partial^{-1}(u_1 r + u_1 x r_x + u_2 q + u_2 x q_x) - \gamma_2 x,
\end{aligned}
$$

$$
\begin{aligned}
\bar{g}_3 =& \gamma_0(x(r+u_2))_{xx} + k_0(xu_2)_{xx} + \gamma_1(x(r+u_2))_x \\
&+ k_1(xu_2)_x - u_2[k_0 xqr + k_0\partial^{-1}(qr) - k_2 x] - (r+u_2)\bar{d}_2,
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
\bar{e}_3 =& \gamma_0(x(q+u_1))_{xx} + k_0(xu_1)_{xx} - \gamma_1(x(q+u_1))_x \\
&- k_1(xu_1)_x - u_1[k_0 xqr + k_0\partial^{-1}(qr) - k_2 x] - (q+u_1)\bar{d}_2,
\end{aligned} \tag{36}
$$

$$\cdots\cdots$$

When $m = 1$, (34) *reduces to*

$$
\begin{cases}
q_t = k_0(xq)_x - k_1 xq, \\
r_t = k_0(xr)_x + k_1 xr, \\
u_{1t} = \gamma_0(x(q+u_1))_x + k_0(xu_1)_x - \gamma_1 x(q+u_1) - k_1 x u_1, \\
u_2 t = \gamma_0(x(r+u_2))_x + k_0(xu_2)_x + \gamma_1 x(r+u_2) + k_1 x u_2.
\end{cases} \tag{37}
$$

Obviously, When $\gamma_0 = \gamma_1 = u_1 = u_2 = 0$, (37) *just reduces to*

$$\begin{cases} q_t = k_0(xq)_x - k_1 xq, \\ r_t = k_0(xr)_x + k_1 xr. \end{cases} \tag{38}$$

Hence, (37) *is a nonisospectral integrable coupling of* (38).

When $m = 2$, (34) *becomes*

$$\begin{cases} q_t = -k_0(xq)_{xx} - k_1(xq)_x + k_0 xq^2 r + k_0 q\partial^{-1}(qr) - k_2 xq, \\ r_t = k_0(xr)_{xx} + k_1(xr)_x - k_0 xqr^2 - k_0 r\partial^{-1}(qr) + k_2 xr, \\ u_{1t} = -\bar{e}_3, \\ u_2 t = \bar{g}_3, \end{cases} \tag{39}$$

where $\bar{e}_3, \bar{g}_3$ *are presented by* (36) *and* (35), *respectively.*

When $\gamma_0 = \gamma_1 = \gamma_2 = 0, u_1 = u_2 = 0$, (39) *reduces to*

$$\begin{cases} q_t = -k_0(xq)_{xx} - k_1(xq)_x + k_0 xq^2 r + k_0 q\partial^{-1}(qr) - k_2 xq, \\ r_t = k_0(xr)_{xx} + k_1(xr)_x - k_0 xqr^2 - k_0 r\partial^{-1}(qr) + k_2 xr. \end{cases} \tag{40}$$

Therefore, (39) *is a nonlinear nonisospectral integrable coupling of* (40).

Taking $r = 0$, (40) *again reduces to*

$$q_t = -k_0(xq)_{xx} - k_1(xq)_x - k_2 xq.$$

A simple symmetry of the nonisospectral integrable coupling (34) *is given by*

$$\begin{aligned} \tau_0 &= \bar{J}\begin{pmatrix} \bar{c}_1 + \bar{g}_1\eta_2 \\ \bar{b}_1 + \bar{e}_1\eta_2 \\ (\bar{c}_1 + \bar{g}_1)\eta_2 \\ (\bar{b}_1 + \bar{g}_1)\eta_2 \end{pmatrix} \\ &= \frac{1}{\eta_2 - 1}\begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & \frac{1}{\eta_2} \\ 1 & 0 & -\frac{1}{\eta_2} & 0 \end{pmatrix}\begin{pmatrix} k_0 xr + [\gamma_0 x(r + u_2) + k_0 x u_2]\eta_2 \\ k_0 xq + [\gamma_0 x(q + u_1) + k_0 x u_1]\eta_2 \\ [k_0 xr + \gamma_0 x(r + u_2) + k_0 x u_2]\eta_2 \\ [k_0 xq + \gamma_0 x(q + u_1) + k_0 x u_1]\eta_2 \end{pmatrix} \\ &= \begin{pmatrix} -k_0 xq \\ k_0 xr \\ -\gamma_0 x(q + u_1) - k_0 x u_1 \\ \gamma_0 x(r + u_2) + k_0 x u_2 \end{pmatrix}. \end{aligned}$$

Case 2: The second kind of integrable model

In the section, we take the basis of the Lie algebra A_1:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

which is enlarged to the following

$$G =: \mathbf{span}\{f_1, f_2, f_3, f_4, f_5, f_6\},$$

where

$$f_1 = \frac{1}{2}\begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}, f_2 = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}, f_3 = \frac{1}{2}\begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix},$$

$$f_4 = \frac{1}{2}\begin{pmatrix} 0 & h \\ h & 0 \end{pmatrix}, f_5 = \begin{pmatrix} 0 & e \\ e & 0 \end{pmatrix}, f_6 = \frac{1}{2}\begin{pmatrix} 0 & f \\ f & 0 \end{pmatrix},$$

along with the following commutative relations

$$\begin{aligned}
&[f_1, f_2] = f_2, [f_1, f_3] = -f_3, [f_1, f_4] = 0, [f_1, f_5] = f_5, \\
&[f_1, f_6] = -f_6, [f_2, f_3] = f_1, [f_2, f_4] = -f_5, [f_2, f_5] = 0, \\
&[f_2, f_6] = f_4, [f_3, f_4] = f_6, [f_3, f_5] = -f_4, [f_3, f_6] = 0, \\
&[f_4, f_5] = f_2, [f_4, f_6] = -f_3, [f_5, f_6] = f_1.
\end{aligned}$$

Denoting

$$G = G_1 \oplus G_2, G_1 = \mathbf{span}\{f_1, f_2, f_3\}, G_2 = \mathbf{span} = \{f_4, f_5, f_6\},$$

we find that

$$[G_1, G_1] \subset G_1, [G_1, G_2] \subset G_2, [G_2, G_2] \subset G_1. \tag{41}$$

Remark 4. *Obviously, G_1 and H_1 has the same commutators. Hence, if defining the same loop algebras $\widetilde{G_1}$ and $\widetilde{H_1}$ for G_1 and H_1, and introducing the same spectral equations, then it follows that the same integrable hierarchy can be generated by the $\widetilde{G_1}$ and $\widetilde{H_1}$. However,* (41) *is different from* (7)*, we conclude that we could obtain various integrable coupling by using the Lie algebra G. In what follows, we shall employ the Lie algebra G for generating another kind of expanding integrable model of* (14)*.*

Set

$$\varphi_x = U\varphi, U = -f_1(1) + qf_2(0) + rf_3(0) + s_1h_5(0) + s_2h_6(0), \tag{42}$$

$$\varphi_t = V\varphi, V = V_1 + V_2, \tag{43}$$

$$V_1 = \sum_{i\geq 0}(a_{1i}f_1(-i) + a_{2i}f_2(-i) + a_{3i}f_3(-i) + a_{4i}f_4(-i) + a_{5i}f_5(-i) + a_{6i}f_6(-i)),$$

$$V_2 = \sum_{j\geq 0}\sum_{l=1}^{6} b_{l_j}f_l(-j), \lambda_t = \sum_{j\geq 0} k_j(t)\lambda^{-j}.$$

Denoting

$$V_+^{(n,m)} = V_{1,+}^{(n)} + V_{2,+}^{(m)} = \sum_{l=1}^{6}\left(\sum_{i=0}^{n} a_{li}f_l(-i) + \sum_{j=0}^{m} b_{lj}f_l(-j)\right),$$

$$\lambda_{t,+}^{(m)} = \sum_{j=0}^{m} k_j(t)\lambda^{m-j} = \lambda^m\lambda_t - \lambda_{t,-}^{(m)},$$

then equation

$$V_x = \frac{\partial U}{\partial \lambda}\lambda_t + [U, V]$$

admits that

$$\begin{aligned}
&a_{1i} = \partial^{-1}(qa_{3i} - ra_{2i} + s_1a_{6i} - s_2a_{5i}) - \alpha_i(t), \\
&a_{2,i+1} = -(a_{2i})_x - qa_{1i} - s_1a_{4i}, \\
&a_{3,i+1} = (a_{3i})_x - ra_{1i} - s_2a_{4i}, \\
&a_{4i} = \partial^{-1}(qa_{6i} - ra_{5i} + s_1a_{3i} - s_2a_{2i}) - \beta_i(t), \\
&a_{5,i+1} = -(a_{5i})_x - qa_{4i} - s_1a_{1i}, \\
&a_{6,i+1} = (a_{6i})_x - ra_{4i} - s_2a_{1i},
\end{aligned}$$

$$\begin{aligned}
\bar{a}_{1j} &= \partial^{-1}(q\bar{a}_{3j} - r\bar{a}_{2j} + s_1\bar{a}_{6j} - s_2\bar{a}_{5j}) - k_j(t)x, \\
\bar{a}_{2,j+1} &= -(\bar{a}_{2j})_x - q\bar{a}_{1j} - s_1\bar{a}_{4j}, \\
a_{3,j+1} &= (\bar{a}_{3j})_x - r\bar{a}_{1j} - s_2\bar{a}_{4j}, \\
\bar{a}_{4j} &= \partial^{-1}(q\bar{a}_{6j} - r\bar{a}_{5j} + s_1\bar{a}_{3j} - s_2\bar{a}_{1j}) - \gamma_j(t), \\
\bar{a}_{5,j+1} &= -(\bar{a}_{5j})_x - q\bar{a}_x - q\bar{a}_{4j} - s_1\bar{a}_{1j}, \\
\bar{a}_{6,j+1} &= (\bar{a}_{6j})_x - r\bar{a}_{4j} - s_2\bar{a}_{1j}, i, j \geq 0.
\end{aligned}$$

Taking

$$\begin{aligned}
&a_{20} = a_{30} = a_{50} = a_{60} = \bar{a}_{20} = \bar{a}_{30} = \bar{a}_{50} = \bar{a}_{60} = 0, \\
&a_{10} = -\alpha_0(t), a_{40} = -\beta_0(t), \bar{a}_{10} = -k_0(t)x, \bar{a}_{40} = -\gamma_0(t),
\end{aligned}$$

then we get from the above equations that

$$\begin{aligned}
&a_{21} = \alpha_0 q + \beta_0 s_1, a_{31} = \alpha_0 \gamma + \beta_0 s_2, a_{51} = \alpha_0 s_1 + \beta_0 q, \\
&a_{61} = \alpha_0 s_2 + \beta_0 \gamma, a_{11} = -\alpha_1(t), a_{41} = -\beta_1(t),
\end{aligned}$$

$$\begin{aligned}
a_{22} &= -\alpha_0 q_x - \beta_0 s_{1x} + \alpha_1 q + \beta_1 s_1, \\
a_{32} &= -\alpha_0 \gamma_x + \beta_0 s_{2x} + \alpha_1 r + \beta_1 s_2, \\
a_{52} &= -\alpha_0 s_{1x} - \beta_0 q_x + \beta_1 q + \alpha_1 s_1, \\
a_{62} &= \alpha_0 s_{2x} + \beta_0 \gamma_x + \beta_1 r + \alpha_1 s_1, \\
a_{12} &= \alpha_0(qr + s_1 s_2) + \beta_0(qs_2 + rs_1) - \alpha_2(t), \\
a_{42} &= \alpha_0(qs_2 + rs_1) + \beta_0(qr + s_1 s_2) - \beta_2(t),
\end{aligned}$$

$$\begin{aligned}
a_{23} &= \alpha_0(q_{xx} - q^2 r - 2qs_1 s_2 - rs_1 s_2) + \beta_0(s_{1xx} - q^2 s_2 - 2qrs_1 - s_1^2 s_2) \\
&\quad - \alpha_1 q_x - \beta_1 s_{1x} + \alpha_2 q + \beta_2 \gamma, \\
a_{33} &= \alpha_0(r_{xx} - qr^2 - 2rs_1 s_2 - qs_2^2) + \beta_0(s_{2xx} - 2qrs_2 - r^2 s_1 + s_1 s_2^2) \\
&\quad + \alpha_1 r_x + \beta_1 s_{2x} + \alpha_2 r + \beta_2 s_2, \\
a_{53} &= \alpha_0(s_{1xx} - 2qrs_1 - s_1^2 s_2 - q^2 s_2) + \beta_0(q_{xx} - q^2 r - 2qs_1 s_2 - rs_1^2) \\
&\quad - \alpha_1 s_{1x} - \beta_1 q_x + \alpha_2 s_1 + \beta_2 q, \\
a_{63} &= \alpha_0(s_{2xx} - 2qrs_2 - r^2 s_1 - s_1 s_2^2) + \beta_0(r_{xx} - qr^2 - 2rs_1 s_2 - qs_2^2) \\
&\quad + \alpha_1 s_{2x} + \beta_1 r_x + \beta_2 r + \alpha_2 s_2,
\end{aligned}$$

$$\begin{aligned}
&\bar{a}_{21} = k_0 xq + \gamma_0 s_1, \bar{a}_{31} = k_0 xr + \gamma_0 s_2, \\
&\bar{a}_{51} = \gamma_0 q + k_0 x s_1, \bar{a}_{61} = \gamma_0 r + k_0 x s_2, \bar{a}_{11} = -k_1(t)x, \\
&\bar{a}_{41} = -\gamma_1(t), \bar{a}_{52} = -\gamma_0 q_x - k_0(xs_1)_x + q\gamma_1 + k_1 x s_1, \\
&\bar{a}_{62} = \gamma_0 \gamma_x + k_0(xs_2)_x + \gamma_1 r + k_1 x s_2, \\
&\bar{a}_{32} = k_0(xr)_x + \gamma_0 s_{2x} + k_1 xr + \gamma_1 s_2, \\
&\bar{a}_{22} = -k_0(xq)_x - \gamma_0 s_{1x} + k_1 xq + \gamma_1 s_1, \\
&\bar{a}_{12} = k_0 \partial^{-1}(qr + s_1 s_2) + k_0 x(qr + s_1 s_2) + \gamma_0(qs_2 + rs_1) - k_2(t)x, \\
&\bar{a}_{42} = k_0 \partial^{-1}(qs_2 + rs_1) + \gamma_0(qr + s_1 s_2) + k_0 x(qs_2 + rs_1) - \gamma_2(t),
\end{aligned}$$

$$
\begin{aligned}
\bar{a}_{23} &= k_0(xq)_{xx} + \gamma_0 s_{1xx} - k_1(xq)_x - \gamma_1 s_{1x} - k_0 q\partial^{-1}(qr + s_1s_2) \\
&\quad - k_0xq(qr + s_1s_2) - \gamma_0(qs_2 + rs_1) + k_2xq - k_0s_1\partial^{-1}(qs_2 + rs_1) \\
&\quad - \gamma_0 s_1(qr + s_1s_2) - k_0xs_1(qs_2 + rs_1) + \gamma_2 s_1, \\
\bar{a}_{33} &= k_0(xr)_{xx} + \gamma_0 s_{2xx} + k_1(xr)_x + \gamma_1 s_{2x} - k_0 r\partial^{-1}(qr + s_1s_2) \\
&\quad - k_0xr(qr + s_1s_2) - \gamma_0 r(qs_2 + rs_1) + k_2xr - k_0s_2\partial^{-1}(qs_2 + rs_1) \\
&\quad - \gamma_0 s_2(qr + s_1s_2) - k_0xs_2(qs_2 + rs_1) + \gamma_2 s_2, \\
\bar{a}_{53} &= k_0(xs_1)_{xx} + \gamma_0 q_{xx} - \gamma_1 q_x - k_1(xs_1)_x - k_0 q\partial^{-1}(qs_2 + rs_1) \\
&\quad - \gamma_0 q(qr + s_1s_2) - k_0xq(qs_2 + rs_1) + \gamma_2 q - k_0s_1\partial^{-1}(qr + s_1s_2) \\
&\quad - k_0xs_1(qr + s_1s_2) - \gamma_0 s_1(qs_2 + rs_1) + k_2xs_1, \\
\bar{a}_{63} &= k_0(xs_2)_{xx} + \gamma_0 r_{xx} + \gamma_1 r_x + k_1(xs_2)_x - k_0 r\partial^{-1}(qs_2 + rs_1) \\
&\quad - \gamma_0 r(qr + s_1s_2) - k_0xr(qs_2 + rs_2) + \gamma_2 r - k_0s_2\partial^{-1}(qr + s_1s_2) \\
&\quad - k_0xs_2(qr + s_1s_2) - \gamma_0 s_2(qs_2 + rs_1) + k_2xs_2, \\
&\dots
\end{aligned}
$$

A direct calculation reads

$$
\begin{aligned}
&-(V_+^{(n,m)})_x + [U, V_+^{(n,m)}] + \frac{\partial U}{\partial \lambda}\lambda_{t,+}^{(m)} \\
&= (V_-^{(n,m)})_x - [U, V_-^{(n,m)}] - \frac{\partial U}{\partial \lambda}\lambda_{t,-}^{(m)} \\
&= (a_{2,n+1} + \bar{a}_{2,m+1})f_2(0) - (a_{3,n+1} + \bar{a}_{3,m+1})f_3(0) \\
&\quad + (a_{5,n+1} + \bar{a}_{5,m+1})f_5(0) - (a_{6,n+1} + \bar{a}_{6,m+1})f_6(0).
\end{aligned}
$$

Therefore, the nonisospectral zero curvature equation

$$
\frac{\partial U}{\partial u}u_t + \frac{\partial U}{\partial \lambda}\lambda_{t,+}^{(m)} - V_{+,x}^{(n,m)} + [U, V_+^{(n,m)}] = 0
$$

admits an isospectral-nonisospectral Lax integrable hierarchy

$$
u_t = \begin{pmatrix} q \\ r \\ s_1 \\ s_2 \end{pmatrix}_t = \begin{pmatrix} -a_{2,n+1} - \bar{a}_{2,m+1} \\ a_{3,n+1} + \bar{a}_{3,m+1} \\ -a_{5,n+1} - \bar{a}_{5,m+1} \\ a_{6,n+1} + \bar{a}_{6,m+1} \end{pmatrix}. \tag{44}
$$

When $\bar{a}_{2,p}, \bar{a}_{3,p}, \bar{a}_{5,p}, \bar{a}_{6,p}(p = m+1) = 0$, (44) reduces to the resulting isospectral integrable hierarchy

$$
u_t = \begin{pmatrix} q \\ r \\ s_1 \\ s_2 \end{pmatrix}_t = \begin{pmatrix} -a_{2,n+1} \\ a_{3,n+1} \\ -a_{5,n+1} \\ a_{6,n+1} \end{pmatrix}. \tag{45}
$$

Specially, when $s_1 = s_2$, (45) reduces to the AKNS hierarchy.

In what follows, we deduce the Hamiltonian structure of (45). The U and V_1 in (42) and (43) can be written as

$$
\begin{aligned}
U &= \begin{pmatrix} -h(1) + qe(0) + rf(0) & s_1e(0) + s_2f(0) \\ s_1e(0) + s_2f(0) & -h(1) + qe(0) + rf(0) \end{pmatrix}, \\
V_1 &= \begin{pmatrix} a_1h(0) + a_2e(0) + a_3f(0) & a_4h(0) + a_5e(0) + a_6f(0) \\ a_4h(0) + a_5e(0) + a_6f(0) & a_1h(0) + a_2e(0) + a_3f(0) \end{pmatrix},
\end{aligned}
$$

where

$$a_i = \sum_{j\geq 0} a_{ij} f_i(j), i = 1, 2, \ldots, 6.$$

It is easy to calculate that

$$\langle V_1, \frac{\partial U}{\partial \lambda}\rangle = -4a_1, \langle V_1, \frac{\partial U}{\partial q}\rangle = 4a_2, \langle V_1, \frac{\partial U}{\partial r}\rangle = -4a_3,$$
$$\langle V_1, \frac{\partial U}{\partial s_1}\rangle = -4a_5, \langle V_1, \frac{\partial U}{\partial s_2}\rangle = -4a_6.$$

Substituting the above consequences to the trace identity gives

$$\frac{\delta}{\delta u}(-4a_1) = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \begin{pmatrix} 4a_2 \\ -4a_3 \\ 4a_5 \\ -4a_6 \end{pmatrix}. \tag{46}$$

Comparing the coefficients of λ^{-n-1} both sides of (46) leads to

$$\frac{\delta}{\delta u}(-a_{1,n+1}) = (-n+\gamma) \begin{pmatrix} a_{2,n} \\ -a_{3,n} \\ a_{5,n} \\ -a_{6,n} \end{pmatrix}.$$

It is easy to check that $\gamma = 0$. Thus, we have

$$\begin{pmatrix} a_{2,n} \\ -a_{3,n} \\ a_{5,n} \\ -a_{6,n} \end{pmatrix} = \frac{\delta H_n}{\delta u}, H_n = \frac{a_{1,n+1}}{n}.$$

The integrable hierarchy (46) can be written as Hamiltonian form

$$u_t = \begin{pmatrix} q \\ r \\ s_1 \\ s_2 \end{pmatrix}_t \begin{pmatrix} -a_{2,n+1} \\ a_{3,n+1} \\ -a_{5,n+1} \\ a_{6,n+1} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a_{2,n+1} \\ -a_{3,n+1} \\ a_{5,n+1} \\ -a_{6,n+1} \end{pmatrix}$$
$$= \widetilde{J} \begin{pmatrix} -a_{2,n+1} \\ a_{3,n+1} \\ -a_{5,n+1} \\ a_{6,n+1} \end{pmatrix} = \widetilde{J} \; \frac{\delta H_{n+1}}{\delta u}.$$

Next, we consider reductions of (44).
When $n = 2, m = 2$, we have that

$$\begin{aligned} q_t =& \alpha_0(-q_{xx} + q^2 r + 2qs_1s_2 + rs_1s_2) + \beta_0(-s_{1xx} + q^2 s_2 + 2qrs_1 + s_1^2 s_2) \\ &+ \alpha_1 q_x + \beta_1 s_{1x} - \alpha_2 q - \beta_2 r - k_0 xq(qr)_{xx} - \gamma_0 s_{1xx} + k_1(xq)_x + \gamma_1 s_{1x} \\ &+ k_0 q\partial^{-1}(qr + s_1s_2) + k_0 xq(qr + s_1s_2) + \gamma_0(qs_2 + rs_1) - k_2 xq \\ &+ k_0 s_1 \partial^{-1}(qs_2 + rs_1) + \gamma_0 s_1(qr + s_1s_2) + k_0 xs_1(qs_2 + rs_1) - \gamma_2 s_1, \end{aligned} \tag{47}$$

$$\begin{aligned} r_t =& \alpha_0(r_{xx} - qr^2 - 2rs_1s_2 - qs_2^2) + \beta_0(s_{2xx} - 2qrs_2 - r^2 s_1 + s_1 s_2^2) \\ &+ \alpha_1 r_x + \beta_1 s_{2x} + \alpha_2 r + \beta_2 s_2 + k_0(xr)_{xx} + \gamma_0 s_{2xx} + k_1(xr)_x + \gamma_1 s_{2x} \\ &- k_0 r\partial^{-1}(qr + s_1s_2) - k_0 xr(qr + s_1s_2) - \gamma_0(qs_2 + rs_1) + k_2 xr \\ &- k_0 s_2 \partial^{-1}(qs_2 + rs_1) - \gamma_0 s_2(qr + s_1s_2) - k_0 xs_2(qs_2 + rs_1) + \gamma_2 s_2, \end{aligned} \tag{48}$$

$$
\begin{aligned}
s_{1t} =&\alpha_0(-s_{1xx}+2qrs_1+s_1^2s_2+q^2s_2)+\beta_0(-q_{xx}+q^2r+2qs_1s_2+rs_1^2)\\
&+\alpha_1 s_{1x}+\beta_1 q_x-\alpha_2 s_1-\beta_2 q-k_0(xs_2)_{xx}-\gamma_0 r_{xx}-\gamma_1 r_x-k_1(xs_2)_x\\
&+k_0 r\partial^{-1}(qs_2+rs_1)+\gamma_0 r(qr+s_1s_2)+k_0 xr\partial^{-1}(qs_2+rs_1)-\gamma_2 r\\
&+k_0 s_2\partial^{-1}(qr+s_1s_2)+k_0 xs_2(qr+s_1s_2)+\gamma_0 s_2(qs_2+rs_1)-k_2 xs_2,
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
s_{2t} =&\alpha_0(s_{2xx}-2qrs_2-r^2s_1-s_1s_2^2)+\beta_0(r_{xx}-qr^2-2rs_1s_2-qs_2^2)\\
&+\alpha_1 s_{2x}+\beta_1 r_x+\beta_2 r+\alpha_2 s_2+k_0(xs_2)_{xx}+\gamma_0 r_{xx}+\gamma_1 r_x+k_1(xs_2)_x\\
&-k_0 r\partial^{-1}(qs_2+rs_1)-\gamma_0 r(qr+s_1s_2)-k_0 xr(qs_2+rs_1)\\
&+\gamma_2 r-k_0 xs_2\partial^{-1}(qr+s_1s_2)-k_0 xs_2(qr+s_1s_2)+k_2 xs_2.
\end{aligned}
\tag{50}
$$

Taking $\beta_0=s_1=s_2=0$, (47) *and* (48) *reduce to*

$$
\begin{cases}
q_t =\alpha_0(-q_{xx}+q^2r)+\alpha_1 q_x-\alpha_2 q-\beta_2 r-k_0(xq)_{xx}\\
\qquad +k_1(xq)_x+k_0 q\partial^{-1}(qr)+k_0 xq^2r-k_2 xq,\\
r_t =\alpha_0(r_{xx}-qr^2)+\alpha_1 r_x+\alpha_2 r+k_0(sr)_{xx}+k_1(xr)_x\\
\qquad -k_0 r\partial^{-1}(qr)-k_0 xqr^2+k_2 xr.
\end{cases}
\tag{51}
$$

If taking $\alpha_1=\alpha_2=\beta_2=k_i(i=0,1,2)=0, \alpha_0=1$, (51) *becomes*

$$
\begin{cases}
q_t=-q_{xx}+q^2r,\\
r_t=r_{xx}-qr^2,
\end{cases}
$$

which can be transformed to

$$
iq_t=q_x x-q|q|^2 \tag{52}
$$

by using the transformation $r\to q*, t\to -it$. *Equation* (52) *is obviously nonlinear Schrödinger equation.*

When $\alpha_0=\alpha_1=\alpha_2=\beta_2=0$, (51) *turns to*

$$
\begin{cases}
q_t=-k_0(xq)_{xx}+k_1(xq)_x+k_0 q\partial^{-1}(qr)+k_0 xq^2r-k_2 xq,\\
r_t=k_0(xr)_{xx}+k_1(xr)_x-k_0 r\partial^{-1}(qr)-k_0 xqr^2+k_2 xr,
\end{cases}
\tag{53}
$$

which is a non-local integrable system with variable coefficients.

In particular, when $k_1=k_2=0, k_0=1$, (53) *presents*

$$
\begin{cases}
q_t=-(xq)_{xx}+q\partial^{-1}(qr)+xq^2r,\\
r_t=(xr)_{xx}-r\partial^{-1}(qr)-xqr^2.
\end{cases}
\tag{54}
$$

If taking $r=q*, t\to -it$, (54) *reads that*

$$
iq_t=(xq)_{xx}+q\partial^{-1}|q|^2+xq|q|^2,
$$

which is a non-local nonlinear Schrödinger equation with variable coefficients.

Remark 5. *The first two equations in* (39) *are the same with* (53) . *When* $m=2$, *the nonisospectral integrable hierarchy of* (44) *presents*

$$
\begin{cases}
q_t=-\bar{a}_{2,3},\\
r_t=\bar{a}_{3,3},\\
s_{1t}=-\bar{a}_{5,3},\\
s_{2t}=\bar{a}_{6,3}.
\end{cases}
\tag{55}
$$

Specially, when $\beta_0 = \beta_1 = \beta_2 = \gamma_0 = \gamma_1 = \gamma_2 = 0$, (55) *reduces to*

$$\begin{aligned}
q_t =& -k_0(xq)_{xx} + k_1(xq)_x + k_0 q\partial^{-1}(qr + s_1 s_2) + k_0 xq(qr + s_1 s_2) \\
& - k_2 xq + k_0 s_1 \partial^{-1}(qs_2 + rs_1) + k_0 xs_1(qs_2 + rs_1), \\
r_t =& k_0(xr)_{xx} + k_1(xr)_x - k_0 r\partial^{-1}(qr + s_1 s_2) - k_0 xr(qr + s_1 s_2) \\
& + k_2 xr - k_0 s_2 \partial^{-1}(qs_2 + rs_1) - k_0 xs_2(qs_2 + rs_1), \\
s_{1t} =& -k_0(xs_2)_{xx} - k_1(xs_2)_x + k_0 r\partial^{-1}(qs_2 + rs_1) + k_0 xr(qs_2 + rs_1) \\
& + k_0 s_2 \partial^{-1}(qr + s_1 s_2) + k_0 xs_2(qr + s_1 s_2) - k_2 xs_2, \\
s_{2t} =& k_0(xs_2)_{xx} + k_1(xs_2)_x - k_0 r\partial^{-1}(qs_2 + rs_1) - k_0 xr(qs_2 + rs_1) \\
& - k_0 s_2 \partial^{-1}(qr + s_1 s_2) - k_0 xs_2(qr + s_1 s_2) + k_2 xs_2,
\end{aligned}$$

which are different from (39)*. Hence, the isospectral–nonisospectral integrable hierarchy* (44) *is different from the hierarchy* (19)*, they are all integrable couplings of the AKNS hierarchy* (14)*.*

4. Conclusions

In the paper, we obtained an isospectral–nonisospectral AKNS hierarchy by using a third-order matrix Lie algebras. In order to discover its expanding integrable models, i.e., integrable couplings, we turned the matrix Lie algebra into a 2×2 matrix Lie algebra so that it was enlarged into two higher order semi-simple Lie algebras for which two kinds of nonisospectral expanding integrable models were generated. Specially, we obtained nonlinear integrable couplings which has been required to overcome. An interesting result reads that we united the well-known KdV equation and the NLS as an integrable model. The aim for publishing the paper lies in a few aspects. One purpose reads how to transfer higher-order matrix Lie algebras into lower-order matrix Lie algebras. The second one presents how to generate nonlinear integrable couplings. The third one manifests how to generate multiply nonisospectral integrable couplings. The approach presented in the paper can be used to discuss other integrable systems.

Author Contributions: Writing—original draft preparation, S.Z. and J.Z.; writing—review and editing, Y.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This work is supported by the National Natural Science Foundation of China (No. 11971475).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors wish to thank the anonymous referees for their valuable suggestions.

Conflicts of Interest: The authors declare that they have no competing interests.

References

1. Newell, A.C. *Solitons in Mathematics and Physics*; Society for Industrial and Applied Mathematics: Philadelphia, PA, USA, 1985.
2. Ma, W.X. A new hierarchy of Liouville integrable generalized Hamiltonian equations and its reduction. *Chin. J. Contemp.* **1992**, *13*, 79–89.
3. Hu, X.B. A powerful approach to generate new integrable systems. *J. Phys. A Gen. Phys.* **1994**, *27*, 2497–2514. [CrossRef]
4. Geng, X.G.; Ma, W.X. A Multipotential Generalization of the Nonlinear Diffusion Equation. *J. Phys. Soc. Jpn.* **2000**, *69*, 985–986. [CrossRef]
5. Fan, E.G. Quasi-periodic waves and an asymptotic property for the asymmetrical Nizhnik-Novikov-Veselov equation. *J. Phys. A Math. Theor.* **2009**, *42*, 095206. [CrossRef]
6. Qiao, Z.J.; Fan, E.G. Negative-order Korteweg–De Vries equations. *Phys. Rev. E* **2012**, *86*, 016601. [CrossRef] [PubMed]
7. Krasilshchik, I.S.; Verbovetsky, A.M. Geometry of jet spaces and integrable systems. *J. Geom. Phys.* **2011**, *61*, 1633–1674. [CrossRef]
8. Blaszak, M.; Ma, W.X. New Liouville integrable noncanonical Hamiltonian systems from the AKNS spectral problem. *J. Math. Phys.* **2002**, *43*, 3107–3123. [CrossRef]

9. Ma, W.X.; Fuchssteiner, B.; Oevel, W. A 3×3 matrix spectral problem for AKNS hierarchy and its binary nonlinearization. *Phys. Statal Mech. Its Appl.* **1996**, *233*, 331–354. [CrossRef]
10. Fuchssteiner, B. Coupling of completely integrable systems: The perturbation bundl. In *Applications of Analytic and Geometric Methods to Nonlinear Differential Equations*; Clarkson, P.A, Ed.; Springer: Dordrecht, The Netherlands, 1993; pp. 125–138.
11. Ma, W.X.; Fuchssteiner, B. Integrable theory of the perturbation equations. *Chaos Solitons Fractals* **1996**, *7*, 1227–1250. [CrossRef]
12. Zhang, Y.F.; Zhang, H.Q. A direct method for integrable couplings of TD hierarchy. *J. Math. Phys.* **2002**, *43*, 466–472. [CrossRef]
13. Zhang, Y.F.; Zhang, H.Q.; Yan, Q.Z. Integrable couplings of Botie–Pempinelli–Tu (BPT) hierarchy. *Phys. Lett. A* **2002**, *299*, 543–548. [CrossRef]
14. Zhang, Y.F.; Guo, F.K. Matrix Lie Algebras and Integrable Couplings. *Commun. Theor. Phys.* **2006**, 46, 812–818.
15. Tu, G.Z. The trace identity, a powerful tool for constructing the Hamiltonian structure of integrable systems. *J. Math. Phys.* **1989**, *30*, 330–338. [CrossRef]
16. Guo, F.K.; Zhang, Y.F. The quadratic-form identity for constructing the Hamiltonian structure of integrable systems. *J. Phys. A* **2005**, *38*, 8537–8548. [CrossRef]
17. Ma, W.X.; Chen, M. Hamiltonian and quasi-Hamiltonian structures associated with semi-direct sums of Lie algebras. *J. Phys. A* **2006**, *39*, 10787–10801. [CrossRef]
18. Ma, W. Variational identities and applications to Hamiltonian structures of soliton equations. *Nonlinear-Anal.-Theory Methods Appl.* **2009**, *71*, 1716–1726. [CrossRef]
19. Zhang, Y.F.; Mei, J.Q.; Guan, H.Y. A method for generating isospectral and nonisospectral hierarchies of equations as well as symmetries. *J. Geom. Phys.* **2020**, *147*, 103538. [CrossRef]
20. Lu, H.H.; Zhang, Y.F.; Mei, J.Q. A generalized isospectral-nonisospectral of heat equation hierarchy and its expanding integrable model. *Adv. Differ. Equ.* **2020**, *2020*, 471. [CrossRef]
21. Wang, H.F.; Zhang, Y.F. Generating of Nonisospectral Integrable Hierarchies via the Lie-Algebraic Recursion Scheme. *Mathematics* **2020**, *8*, 621. [CrossRef]
22. Zhu, Z.; Huang, H.C.; Xue, W.M.; Wu, X.N. The bi-Hamiltonian structures of some new Lax integrable hierarchies associated with 3×3 matrix spectral problems. *Phys. Lett. A* **1997**, *235*, 227–232. [CrossRef]