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Existence of Attractive Solutions for Hilfer Fractional Evolution Equations with Almost Sectorial Operators

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Abstract: The purpose of this paper is to investigate the existence of attractive solutions for a Cauchy problem of fractional evolution equations with Hilfer fractional derivative, which is a generalization of both the Riemann–Liuoville and Caputo fractional derivatives. Our methods are based on the generalized Ascoli–Arzela theorem, Schauder's fixed point theorem, the Wright function and Kuratowski's measure of noncompactness. The symmetric structure of the spaces and the operators defined by us plays a crucial role in showing the existence of fixed points. We obtain the global existence and attractivity results of mild solutions when the semigroup associated with an almost sectorial operator is compact as well as noncompact.

Keywords: fractional evolution equations; existence; attractivity; hilfer derivative; almost sectorial operator

MSC: 26A33; 34A08; 34K37

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Citation: Zhou, M.; Ahmad, B.; Zhou, Y. Existence of Attractive Solutions for Hilfer Fractional Evolution Equations with Almost Sectorial Operators. *Symmetry* **2022**, *14*, 392. https://doi.org/10.3390/sym14020392

Academic Editors: Garik Petrosyan and Palle E. T. Jorgensen

Received: 29 December 2021
Accepted: 3 February 2022
Published: 16 February 2022

1. Introduction

Fractional calculus is considered as a generalization of classical calculus. The order of the fractional derivative can be an arbitrary (noninteger) positive real number or even a complex number. In the past two decades, fractional calculus has been a research focus and attracted the attention of many researchers all over the world. It has been mainly due to the extensive development in the theory of fractional calculus. Moreover, fractional calculus is widely used in various disciplines, especially in fluid mechanics, physics, signal processing, materials science, electrochemistry, biology and so on.

In recent years, fractional differential equations are found to be of great interest in the mathematical modeling of real-world phenomena. These applications have motivated many researchers in the field of differential equations to investigate fractional differential equations of different order, for instance, see the monographs [1–5].

The main motivation of studying fractional evolution equations comes from two aspects: (i) One is that many mathematical models in physics and fluid mechanics are characterized by fractional partial differential equations; (ii) many types of fractional partial differential equations, such as fractional diffusion equations, wave equations, Navier-Stokes equations, Rayleigh-Stokes equations, Fokker–Planck equations, fractional Schrödinger equations, and so on, can be abstracted as fractional evolution equations [6–8]. Therefore, the study of fractional evolution equations is of great significance both in terms of theory and practical application.

The well-posed nature of fractional evolution equations is an important research topic of evolution equations, see [9–11]. However, it seems that there are few works dealing with the existence of fractional evolution equations on infinite intervals. Almost all of these results involve the existence of solutions for fractional evolution equations on a finite interval $[0, T]$, where $T \in (0, \infty)$ (see [9–11]). Recently, several research papers have been published on the attractivity for fractional ordinary differential equations [12], fractional functional differential equations [13], Volterra fractional integral equations [14] and fractional evolution equations [15]. On the other hand, the Hilfer fractional derivative is a natural generalization of fractional derivatives which include the Caputo derivative and Riemann–Liouville derivative [2]. The evolution equations with Hilfer fractional derivative received great attention from several researchers (see [16,17]). However, it seems that there are few works concerned with the attractivity of Hilfer fractional evolution equations. Almost all these results involve the existence of Hilfer fractional evolution equations on a finite interval $[0, T]$.

Consider the initial value problem in a Banach space X

$$\begin{cases} {}^{H}D^{\mu,\beta}_{0+}y(t) = Ay(t) + G(t, y(t)), \quad t \in (0, \infty), \\ I^{(1-\mu)(1-\beta)}_{0+}y(0) = y_0, \end{cases} \tag{1}$$

where ${}^{H}D^{\mu,\beta}_{0+}$ is the Hilfer fractional derivative of order $0 < \beta < 1$ and type $0 \leq \mu \leq 1$, $I^{(1-\mu)(1-\beta)}_{0+}$ is a Riemann–Liouville fractional integral of order $(1-\mu)(1-\beta)$, A is an almost sectorial operator in Banach space X, $G : [0, \infty) \times X \to X$ is a function to be defined later.

In this paper, we obtain the global existence and attractivity results for mild solutions of the initial value problem (1) when the semigroup associated with the almost sectorial operator is compact as well as noncompact. The considerations of this paper are based on the generalized Ascoli–Arzela theorem, Schauder's fixed point theorem and Kuratowski's measure of noncompactness. The symmetric structure of the spaces and the operators defined by us plays a crucial role in proving the existence of fixed points.

2. Preliminaries

We first introduce some notations and definitions about almost sectorial operators, fractional calculus and the measure of noncompactness. For more details, we refer to [2,3,18,19].

Denote by $D(A)$ the domain of A, by $\sigma(A)$ its spectrum, while $\rho(A) := \mathbb{C} - \sigma(A)$ is the resolvent set of A. We denote by $\mathcal{L}(X)$ the space of all bounded linear operators from X to X with the usual operator norm $\| \cdot \|_{\mathcal{L}(X)}$. Let $S^0_\lambda = \{z \in \mathbb{C}\backslash\{0\} : |\arg z| < \lambda\}$ be the open sector for $0 < \lambda < \pi$, and S_λ be its closure, i.e., $S_\lambda = \{z \in \mathbb{C}\backslash\{0\} : |\arg z| \leq \lambda\} \cup \{0\}$.

Definition 1. *Let $0 < k < 1$ and $0 < \omega < \frac{\pi}{2}$. We denote $\Theta^{-k}_\omega(X)$ as a family of all closed linear operators $A : D(A) \subset X \to X$ such that*

(i) $\sigma(A) \subset S_\omega = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| \leq \omega\} \cup \{0\}$ *and*

(ii) $\forall \lambda \in (\omega, \pi)$, $\exists C_\lambda$ *such that*

$$\|R(z; A)\|_{\mathcal{L}(X)} \leq C_\lambda |z|^{-k}, \text{ for all } z \in \mathbb{C} \setminus S_\lambda,$$

where $R(z; A) = (zI - A)^{-1}, z \in \rho(A)$ is the resolvent operator of A. The linear operator A will be called an almost sectorial operator on X if $A \in \Theta^{-k}_\omega(X)$.

Denote the semigroup $\{Q(t)\}_{t \geq 0}$ by

$$Q(t) = e^{-tz}(A) = \frac{1}{2\pi i} \int_{\Gamma_\rho} e^{-tz} R(z; A) dz, \ t \in S^0_{\frac{\pi}{2}-\omega},$$

where $\Gamma_\rho = \{\mathbb{R}^+ e^{i\rho}\} \bigcup \{\mathbb{R}^+ e^{-i\rho}\}$ with $\omega < \rho < \lambda < \frac{\pi}{2} - |\arg t|$ is oriented counter-clockwise.

Lemma 1 (see [5]). *Assume that $0 < k < 1$ and $0 < \omega < \frac{\pi}{2}$. Set $A \in \Theta^{-k}_\omega(X)$. Then*

(i) $Q(s + t) = Q(s)Q(t)$, *for $\forall s, t \in S^0_{\frac{\pi}{2}-\omega}$;*

(ii) $\exists$ *a constant* $C_0 > 0$ *such that* $\|Q(t)\|_{\mathcal{L}(X)} \leq C_0 t^{k-1}$, *for* $\forall\, t > 0$.

Definition 2 (see [3]). *The fractional integral of order* β *for a function* $u : [0, \infty) \to \mathbb{R}$ *is defined as*

$$I^{\beta}_{0+}u(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} u(s) ds, \quad \beta > 0,\ t > 0,$$

provided the right side is point-wise defined on $[0, \infty)$, *where* $\Gamma(\cdot)$ *is the gamma function.*

Definition 3 (Hilfer fractional derivative, see [2]). *Let* $0 < \beta < 1$ *and* $0 \leq \mu \leq 1$. *The Hilfer fractional derivative of order* β *and type* μ *for a function* $u : [0, \infty) \to \mathbb{R}$ *is defined as*

$$^{H}D^{\mu,\beta}_{0+}u(t) = I^{\mu(1-\beta)}_{0+} \frac{d}{dt} I^{(1-\mu)(1-\beta)}_{0+} u(t).$$

Remark 1.

(i) *If* $\mu = 0$, $0 < \beta < 1$, *then*

$$^{H}D^{0,\beta}_{0+}u(t) = \frac{d}{dt} I^{1-\beta}_{0+} u(t) =: {}^{L}D^{\beta}_{0+}u(t),$$

where $^{L}D^{\beta}_{0+}$ *is the Riemann–Liouville derivative.*

(ii) *If* $\mu = 1$, $0 < \beta < 1$, *then*

$$^{H}D^{1,\beta}_{0+}u(t) = I^{1-\beta}_{0+} \frac{d}{dt} u(t) =: {}^{C}D^{\beta}_{0+}u(t),$$

where $^{C}D^{\beta}_{0+}$ *is the Caputo derivative.*

Assume that X is a Banach space with the norm $|\cdot|$. Let D be a nonempty subset of X. The Kuratowski's measure of noncompactness χ is defined by

$$\chi(D) = \inf\left\{ d > 0 :\ D \subset \bigcup_{j=1}^{n} M_j \text{ and } \operatorname{diam}(M_j) \leq d \right\},$$

where the diameter of M_j is given by $\operatorname{diam}(M_j) = \sup\{|x-y| :\ x, y \in M_j\}$, $j = 1, \ldots, n$.

Lemma 2 ([20]). *Let* X *be a Banach space, and let* $\{u_n(t)\}_{n=1}^{\infty} : [0, \infty) \to X$ *be a continuous function family. If there exists* $\xi \in L^1[0, \infty)$ *such that*

$$|u_n(t)| \leq \xi(t), \quad t \in [0, \infty),\ n = 1, 2, \ldots.$$

Then $\beta(\{u_n(t)\}_{n=1}^{\infty})$ *is integrable on* $[0, \infty)$, *and*

$$\chi\left(\left\{ \int_0^t u_n(t) dt :\ n = 1, 2, \ldots \right\}\right) \leq 2 \int_0^t \chi(\{u_n(t) :\ n = 1, 2, \ldots\}) dt.$$

Definition 4 ([21]). *Define the Wright function* $W_\beta(\theta)$ *by*

$$W_\beta(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-\beta n)},\ 0 < \beta < 1,\ \theta \in \mathbb{C},$$

with the following property

$$\int_0^{\infty} \theta^{\delta} W_\beta(\theta) d\theta = \frac{\Gamma(1+\delta)}{\Gamma(1+\beta\delta)},\ for\ \delta \geq 0.$$

Lemma 3. *The problem* (1) *is equivalent to the integral equation*

$$
\begin{aligned}
y(t) =& \frac{y_0}{\Gamma(\mu(1-\beta)+\beta)} t^{(\mu-1)(1-\beta)} \\
&+ \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}[Ay(s) + G(s, y(s))]ds, \; t \in (0, \infty).
\end{aligned} \tag{2}
$$

Proof. This proof is similar to [22], so we omit it. □

Lemma 4. *Assume that $y(t)$ satisfies integral Equation (2). Then*

$$
y(t) = \mathcal{K}_{\mu,\beta}(t)y_0 + \int_0^t \mathcal{Q}_\beta(t-s)G(s, y(s))ds, \; t \in (0, \infty), \tag{3}
$$

where

$$
\mathcal{K}_{\mu,\beta}(t) = I_{0+}^{\mu(1-\beta)} \mathcal{Q}_\beta(t), \;\; \mathcal{Q}_\beta(t) = t^{\beta-1}\mathcal{T}_\beta(t), \;\; and \;\; \mathcal{T}_\beta(t) = \int_0^\infty \beta\theta W_\beta(\theta) Q(t^\beta\theta)d\theta.
$$

Proof. This proof is similar to [16], so we omit it. □

In view of Lemma 3, we have the following definition.

Definition 5. *By the mild solution of the initial value problem* (1)*, we mean that the function $y \in C((0, \infty), X)$ satisfies*

$$
y(t) = \mathcal{K}_{\mu,\beta}(t)y_0 + \int_0^t \mathcal{Q}_\beta(t-s)G(s, y(s))ds, \; t \in (0, \infty).
$$

Definition 6. *The mild solution $y(t)$ of the initial value problem* (1) *is called attractive if $y(t) \to 0$ as $t \to \infty$.*

Lemma 5 ([17])**.** *For any fixed $t > 0$, $\{\mathcal{T}_\beta(t)\}_{t>0}$, $\{\mathcal{Q}_\beta(t)\}_{t>0}$ and $\{\mathcal{K}_{\mu,\beta}(t)\}_{t>0}$ are linear operators, and for any $y \in X$,*

$$
|\mathcal{T}_\beta(t)y| \le L_1 t^{\beta(k-1)}|y|, \;\; |\mathcal{Q}_\beta(t)y| \le L_1 t^{-1+\beta k}|y|, \;\; and \;\; |\mathcal{K}_{\mu,\beta}(t)y| \le L_2 t^{-1+\mu-\beta\mu+\beta k}|y|,
$$

where

$$
L_1 = \beta C_0 \frac{\Gamma(1+k)}{\Gamma(1+\beta k)} \;\; and \;\; L_2 = \frac{L_1\Gamma(\beta k)}{\Gamma(\mu(1-\beta)+\beta k)}.
$$

Lemma 6 ([17])**.** *$\{\mathcal{T}_\beta(t)\}_{t>0}$, $\{\mathcal{Q}_\beta(t)\}_{t>0}$ and $\{\mathcal{K}_{\mu,\beta}(t)\}_{t>0}$ are strongly continuous, that is, for $\forall\, y \in X$ and $t'' > t' > 0$, we have*

$$
|\mathcal{T}_\beta(t')y - \mathcal{T}_\beta(t'')y| \to 0, \; |\mathcal{Q}_\beta(t')y - \mathcal{Q}_\beta(t'')y| \to 0, \;\; as \; t'' \to t'
$$

and

$$
|\mathcal{K}_{\mu,\beta}(t')y - \mathcal{K}_{\mu,\beta}(t'')y| \to 0, \; as \; t'' \to t'.
$$

Let

$$
E = \{u \in C([0, \infty), X) : \lim_{t\to\infty} \frac{|u(t)|}{1+t} = 0\}.
$$

Clearly, $(E, \|\cdot\|)$ is a Banach space with the norm $\|u\| = \sup_{t\in[0,\infty)} |u(t)|/(1+t) < \infty$.

In the following, we state the generalized Ascoli–Arzela theorem [23].

Lemma 7. *The set $\Lambda \subset E$ is relatively compact if and only if the following conditions hold:*

(a) *for any $h > 0$, the set $V = \{v : \; v(t) = x(t)/(1+t), x \in \Lambda\}$ is equicontinuous on $[0, h]$;*

(b) $\lim_{t\to\infty} |x(t)|/(1+t) = 0$ *uniformly for* $x \in \Lambda$;

(c) *for any* $t \in [0,\infty)$, $V(t) = \{v(t) : v(t) = x(t)/(1+t), x \in \Lambda\}$ *is relatively compact in* E.

3. Some Lemmas

Define

$$C_\beta([0,\infty),X) = \Big\{ y \in C((0,\infty),X) : \lim_{t\to 0+} t^{1-\mu+\beta\mu-\beta k} y(t) = 0, \text{ and } \lim_{t\to\infty} \frac{t^{1-\mu+\beta\mu-\beta k}|y(t)|}{1+t} = 0 \Big\},$$

with the norm

$$\|y\|_\beta = \sup_{t\in[0,\infty)} \frac{t^{1-\mu+\beta\mu-\beta k}|y(t)|}{1+t}.$$

Then $(C_\beta([0,\infty),X), \|\cdot\|_\beta)$ is a Banach space (see Lemma 3.2 of [24]).

For any $y \in C_\beta([0,\infty),X)$, define the mapping Ψ by

$$(\Psi y)(t) = (\Psi_1 y)(t) + (\Psi_2 y)(t),$$

where

$$(\Psi_1 y)(t) = \mathcal{K}_{\mu,\beta}(t) y_0, \quad (\Psi_2 y)(t) = \int_0^t \mathcal{Q}_\beta(t-s) G(s,y(s)) ds, \quad for\ t \in (0,\infty).$$

Clearly, the problem (1) has a mild solution $y^* \in C_\beta([0,\infty),X)$ if and only if Ψ has a fixed point $y^* \in C_\beta([0,\infty),X)$.

Let

$$y(t) = t^{-1-\mu+\beta\mu-\beta k} u(t), \quad \text{for any } u \in E, t \in (0,\infty).$$

Clearly, $y \in C_\beta([0,\infty),X)$. Define an operator Φ by

$$(\Phi u)(t) = (\Phi_1 u)(t) + (\Phi_2 u)(t),$$

where

$$(\Phi_1 u)(t) = \begin{cases} t^{1-\mu+\beta\mu-\beta k}(\Psi_1 y)(t), & \text{for } t \in (0,\infty), \\ 0, & \text{for } t = 0. \end{cases}$$

$$(\Phi_2 u)(t) = \begin{cases} t^{1-\mu+\beta\mu-\beta k}(\Psi_2 y)(t), & \text{for } t \in (0,\infty), \\ 0, & \text{for } t = 0. \end{cases}$$

First, we introduce the following hypotheses:

H1. *for each* $t \in [0,\infty)$, *the function* $G(t,\cdot) : X \to X$ *is continuous and for each* $y \in X$, *the function* $G(\cdot,y) : [0,\infty) \to X$ *is strongly measurable.*

H2. *there exist* $L \geq 0$ *and* $\alpha \in (\beta k, 1-\mu(1-\beta))$ *such that*

$$|G(t,y)| \leq L t^{-\alpha}, \quad \text{for a.e. } t \in (0,\infty) \text{ and all } y \in X.$$

Since

$$\lim_{t\to 0} \left\{ \frac{L_1 L \Gamma(\beta k)\Gamma(1-\alpha)}{\Gamma(\beta k + 1 - \alpha)} \frac{t^{1-\alpha-\mu(1-\beta)}}{1+t} \right\} = 0,$$

and

$$\lim_{t\to\infty} \left\{ \frac{L_1 L \Gamma(\beta k)\Gamma(1-\alpha)}{\Gamma(\beta k + 1 - \alpha)} \frac{t^{1-\alpha-\mu(1-\beta)}}{1+t} \right\} = 0,$$

there exists a constant $r > 0$ such that

$$\max_{t\in[0,\infty)} \left\{ L_2|y_0| + \frac{L_1 L\Gamma(\beta k)\Gamma(1-\alpha)}{\Gamma(\beta k+1-\alpha)} \frac{t^{1-\alpha-\mu(1-\beta)}}{1+t} \right\} \le r. \tag{4}$$

Let

$$\Omega_1 = \{u : u \in E, \|u\| \le r\}. \tag{5}$$

Then, Ω_1 is a nonempty, convex and closed subset of E.
Let

$$V := \{v : v(t) = (\Phi u)(t)/(1+t), u \in \Omega_1\}.$$

To prove the results in this paper we need the following lemmas.

Lemma 8. *Suppose that H1 and H2 hold. Then, the set V is equicontinuous.*

Proof. Step I. We first prove that $\{v : v(t) = (\Phi_1 u)(t)/(1+t), u \in \Omega_1\}$ is equicontinuous. Since

$$\begin{aligned} t^{1-\mu+\beta\mu-\beta k}\mathcal{K}_{\mu,\beta}(t)y_0 &= \frac{t^{1-\mu+\beta\mu-\beta k}}{\Gamma(\mu(1-\beta))}\int_0^t (t-s)^{\mu(1-\beta)-1}s^{\beta-1}\mathcal{T}_\beta(s)y_0 ds \\ &= \int_0^1 (1-z)^{\mu(1-\beta)-1}z^{\beta-1}t^{\beta(1-k)}\mathcal{T}_\beta(tz)y_0 dz. \end{aligned}$$

Noting that $\lim_{t\to0+} t^{\beta(1-k)}\mathcal{T}_\beta(tz)y_0 = 0$ and $\int_0^1 (1-z)^{\mu(1-\beta)-1}z^{\beta-1}dz$ exists, we have

$$\lim_{t\to0+} t^{1-\mu+\beta\mu-\beta k}\mathcal{K}_{\mu,\beta}(t)y_0 = 0.$$

Hence, for $t_1 = 0$, $t_2 \in (0,\infty)$, we obtain

$$\Big|\frac{(\Phi_1 u)(t_2)}{1+t_2} - (\Phi_1 u)(0)\Big| \le \Big|\frac{1}{1+t_2}{t_2}^{1-\mu+\beta\mu-\beta k}\mathcal{K}_{\mu,\beta}(t_2)y_0 - 0\Big| \to 0, \quad \text{as } t_2 \to 0. \tag{6}$$

For any $t_1, t_2 \in (0,\infty)$ with $t_1 < t_2$, we have

$$\begin{aligned} &\Big|\frac{(\Phi_1 u)(t_2)}{1+t_2} - \frac{(\Phi_1 u)(t_1)}{1+t_1}\Big| \\ \le&\Big|\frac{{t_2}^{1-\mu+\beta\mu-\beta k}\mathcal{K}_{\mu,\beta}(t_2)y_0}{1+t_2} - \frac{{t_1}^{1-\mu+\beta\mu-\beta k}\mathcal{K}_{\mu,\beta}(t_1)y_0}{1+t_1}\Big| \\ \le&\Big|\frac{{t_2}^{1-\mu+\beta\mu-\beta k}\mathcal{K}_{\mu,\beta}(t_2)y_0}{1+t_2} - \frac{{t_2}^{1-\mu+\beta\mu-\beta k}\mathcal{K}_{\mu,\beta}(t_2)y_0}{1+t_1}\Big| \\ &+ \Big|\frac{{t_2}^{1-\mu+\beta\mu-\beta k}\mathcal{K}_{\mu,\beta}(t_2)y_0}{1+t_1} - \frac{{t_1}^{1-\mu+\beta\mu-\beta k}\mathcal{K}_{\mu,\beta}(t_1)y_0}{1+t_1}\Big| \\ \le&|{t_2}^{1-\mu+\beta\mu-\beta k}\mathcal{K}_{\mu,\beta}(t_2)y_0|\frac{|t_2-t_1|}{(1+t_2)(1+t_1)} \\ &+ |{t_2}^{1-\mu+\beta\mu-\beta k}\mathcal{K}_{\mu,\beta}(t_2)y_0 - {t_1}^{1-\mu+\beta\mu-\beta k}\mathcal{K}_{\mu,\beta}(t_1)y_0|\frac{1}{1+t_1} \\ \le&|{t_2}^{1-\mu+\beta\mu-\beta k}\mathcal{K}_{\mu,\beta}(t_2)y_0|\frac{|t_2-t_1|}{(1+t_2)(1+t_1)} \\ &+ |{t_2}^{1-\mu+\beta\mu-\beta k}||\mathcal{K}_{\mu,\beta}(t_2)y_0 - \mathcal{K}_{\mu,\beta}(t_1)y_0|\frac{1}{1+t_1} \\ &+ |{t_2}^{1-\mu+\beta\mu-\beta k} - {t_1}^{1-\mu+\beta\mu-\beta k}||\mathcal{K}_{\mu,\beta}(t_1)y_0|\frac{1}{1+t_1} \\ \to& 0, \quad \text{as } t_2 \to t_1. \end{aligned} \tag{7}$$

Hence, $\{v : v(t) = (\Phi_1 u)(t)/(1+t),\ u \in \Omega_1\}$ is equicontinuous.

Step II. We prove that $\{v : v(t) = (\Phi_2 u)(t)/(1+t),\ u \in \Omega_1\}$ is equicontinuous.

Let $y(t) = t^{-1-\mu+\beta\mu-\beta k}u(t)$, for any $u \in \Omega_1$, $t \in (0,\infty)$. Then $y \in \widetilde{\Omega}_1$, where $\widetilde{\Omega}_1$ is nonempty, convex and closed set defined by

$$\widetilde{\Omega}_1 = \{y \in C_\beta([0,\infty), X) :\ \|y(t)\|_\beta \le r\}.$$

For $\varepsilon > 0$, in view of $\beta k < \alpha < 1 - \mu(1-\beta)$, there exists $T_2 > 0$ such that

$$\frac{L_1 L\Gamma(\beta k)\Gamma(1-\alpha)}{\Gamma(\beta k + 1 - \alpha)} \frac{t^{1-\alpha-\mu(1-\beta)}}{1+t} < \frac{\varepsilon}{2}, \text{ for } t \ge T_2. \tag{8}$$

For $t_1, t_2 > T_2$, by virtue of H2 and (8), we obtain

$$\begin{aligned}
&\Big|\frac{(\Phi_2 u)(t_2)}{1+t_2} - \frac{(\Phi_2 u)(t_1)}{1+t_1}\Big| \\
\le&\Big|\frac{{t_2}^{1-\mu+\beta\mu-\beta k}}{1+t_2}\int_0^{t_2} \mathcal{Q}_\beta(t_2-s)G(s,y(s))ds\Big| \\
&+\Big|\frac{{t_1}^{1-\mu+\beta\mu-\beta k}}{1+t_1}\int_0^{t_1} \mathcal{Q}_\beta(t_1-s)G(s,y(s))ds\Big| \\
\le&\frac{L_1 L{t_2}^{1-\mu+\beta\mu-\beta k}}{1+t_2}\int_0^{t_2}(t_2-s)^{\beta k-1}s^{-\alpha}ds \\
&+\frac{L_1 L{t_1}^{1-\mu+\beta\mu-\beta k}}{1+t_1}\int_0^{t_1}(t_1-s)^{\beta k-1}s^{-\alpha}ds \\
=&\frac{L_1 L\Gamma(\beta k)\Gamma(1-\alpha)}{\Gamma(\beta k+1-\alpha)}\frac{{t_2}^{1-\alpha-\mu(1-\beta)}}{1+t_2} \\
&+\frac{L_1 L\Gamma(\beta k)\Gamma(1-\alpha)}{\Gamma(\beta k+1-\alpha)}\frac{{t_1}^{1-\alpha-\mu(1-\beta)}}{1+t_1} \\
<&\varepsilon.
\end{aligned} \tag{9}$$

When $t_1 = 0$, $0 < t_2 \le T_2$, we have

$$\begin{aligned}
\Big|\frac{(\Phi_2 u)(t_2)}{1+t_2} - (\Phi_2 u)(0)\Big| =&\Big|\frac{{t_2}^{1-\mu+\beta\mu-\beta k}}{1+t_2}\int_0^{t_2} \mathcal{Q}_\beta(t_2-s)G(s,y(s))ds\Big| \\
\le&\frac{L_1 L{t_2}^{1-\mu+\beta\mu-\beta k}}{1+t_2}\int_0^{t_2}(t_2-s)^{\beta k-1}s^{-\alpha}ds \\
=&\frac{L_1 L\Gamma(\beta k)\Gamma(1-\alpha)}{\Gamma(\beta k+1-\alpha)}\frac{{t_2}^{1-\alpha-\mu(1-\beta)}}{1+t_2} \\
&\to 0 \quad \text{as } t_2 \to 0.
\end{aligned} \tag{10}$$

When $0 < t_1 < t_2 \le T_2$, we obtain

$$\begin{aligned}
&\Big|\frac{(\Phi_2 u)(t_2)}{1+t_2} - \frac{(\Phi_2 u)(t_1)}{1+t_1}\Big| \\
\le&\Big|\frac{{t_1}^{1-\mu+\beta\mu-\beta k}}{1+t_1}\int_{t_1}^{t_2}(t_2-s)^{\beta-1}\mathcal{T}_\beta(t_2-s)G(s,y(s))ds\Big| \\
&+\Big|\frac{{t_1}^{1-\mu+\beta\mu-\beta k}}{1+t_1}\int_0^{t_1}\big((t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}\big)\mathcal{T}_\beta(t_2-s)G(s,y(s))ds\Big| \\
&+\Big|\frac{{t_1}^{1-\mu+\beta\mu-\beta k}}{1+t_1}\int_0^{t_1}(t_1-s)^{\beta-1}\big(\mathcal{T}_\beta(t_2-s)-\mathcal{T}_\beta(t_1-s)\big)G(s,y(s))ds\Big|
\end{aligned} \tag{11}$$

$$
\begin{aligned}
&+\left|\frac{t_2{}^{1-\mu+\beta\mu-\beta k}}{1+t_2}-\frac{t_1{}^{1-\mu+\beta\mu-\beta k}}{1+t_1}\right|\left|\int_0^{t_2}(t_2-s)^{\beta-1}\mathcal{T}_\beta(t_2-s)G(s,y(s))ds\right| \\
\leq& I_1+I_2+I_3+I_4,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1=&\frac{L_1Lt_1{}^{1-\mu+\beta\mu-\beta k}}{1+t_1}\left|\int_0^{t_2}(t_2-s)^{\beta k-1}s^{-\alpha}ds-\int_0^{t_1}(t_1-s)^{\beta k-1}s^{-\alpha}ds\right|, \\
I_2=&\frac{L_1Lt_1{}^{1-\mu+\beta\mu-\beta k}}{1+t_1}\int_0^{t_1}\left((t_1-s)^{\beta-1}-(t_2-s)^{\beta-1}\right)(t_2-s)^{\beta(k-1)}s^{-\alpha}ds, \\
I_3=&\frac{t_1{}^{1-\mu+\beta\mu-\beta k}}{1+t_1}\left|\int_0^{t_1}(t_1-s)^{\beta-1}\left(\mathcal{T}_\beta(t_2-s)-\mathcal{T}_\beta(t_1-s)\right)G(s,y(s))ds\right|, \\
I_4=&\left|\frac{t_2{}^{1-\mu+\beta\mu-\beta k}}{1+t_2}-\frac{t_1{}^{1-\mu+\beta\mu-\beta k}}{1+t_1}\right|\frac{L_1L\Gamma(\beta k)\Gamma(1-\alpha)}{\Gamma(\beta k+1-\alpha)}t_2{}^{\beta k-\alpha}.
\end{aligned}
$$

By direct calculation, we obtain

$$
I_1=\frac{L_1L\Gamma(\beta k)\Gamma(1-\alpha)}{\Gamma(\beta k+1-\alpha)}\frac{t_1{}^{1-\mu+\beta\mu-\beta k}\left|t_2{}^{\beta k-\alpha}-t_1{}^{\beta k-\alpha}\right|}{1+t_1}\to 0, \quad \text{as } t_2\to t_1.
$$

Note that

$$
((t_1-s)^{\beta-1}-(t_2-s)^{\beta-1})(t_2-s)^{\beta(k-1)}s^{-\alpha}\leq(t_1-s)^{\beta k-1}s^{-\alpha}, \quad \text{for } s\in[0,t_1],
$$

then Lebesgue dominated convergence theorem implies that

$$
\int_0^{t_1}\left((t_1-s)^{\beta-1}-(t_2-s)^{\beta-1}\right)(t_2-s)^{\beta(k-1)}s^{-\alpha}ds\to 0, \quad \text{as } t_2\to t_1,
$$

So, $I_2\to 0$ as $t_2\to t_1$.
By H2, for $\varepsilon>0$, we have

$$
\begin{aligned}
I_3\leq&\frac{t_1{}^{1-\mu+\beta\mu-\beta k}}{1+t_1}\int_0^{t_1-\varepsilon}(t_1-s)^{\beta-1}\left|\left(\mathcal{T}_\beta(t_2-s)-\mathcal{T}_\beta(t_1-s)\right)G(s,y(s))\right|ds \\
&+\frac{t_1{}^{1-\mu+\beta\mu-\beta k}}{1+t_1}\int_{t_1-\varepsilon}^{t_1}(t_1-s)^{\beta-1}\left|\left(\mathcal{T}_\beta(t_2-s)-\mathcal{T}_\beta(t_1-s)\right)G(s,y(s))\right|ds \\
\leq&\frac{t_1{}^{1-\mu+\beta\mu-\beta k}}{1+t_1}\int_0^{t_1}(t_1-s)^{\beta-1}|G(s,y(s))|ds\sup_{s\in[0,t_1-\varepsilon]}|\mathcal{T}_\beta(t_2-s)-\mathcal{T}_\beta(t_1-s)| \\
&+\frac{2L_1t_1{}^{1-\mu+\beta\mu-\beta k}}{1+t_1}\int_{t_1-\varepsilon}^{t_1}(t_1-s)^{\beta k-1}|G(s,y(s))|ds \\
\leq& I_{31}+I_{32}+I_{33},
\end{aligned}
$$

where

$$
\begin{aligned}
I_{31}=&\frac{t_1{}^{1-\mu+\beta\mu-\beta k}}{1+t_1}\int_0^{t_1}(t_1-s)^{\beta-1}|G(s,y(s))|ds\sup_{s\in[0,t_1-\varepsilon]}\|\mathcal{T}_\beta(t_2-s)-\mathcal{T}_\beta(t_1-s)\|_{\mathcal{L}(X)}, \\
I_{32}=&\frac{2L_1Lt_1{}^{1-\mu+\beta\mu-\beta k}}{1+t_1}\left|\int_0^{t_1}(t_1-s)^{\beta k-1}s^{-\alpha}ds-\int_0^{t_1-\varepsilon}(t_1-\varepsilon-s)^{\beta k-1}s^{-\alpha}ds\right|, \\
I_{33}=&\frac{2L_1Lt_1{}^{1-\mu+\beta\mu-\beta k}}{1+t_1}\int_0^{t_1-\varepsilon}((t_1-\varepsilon-s)^{\beta k-1}-(t_1-s)^{\beta k-1})s^{-\alpha}ds.
\end{aligned}
$$

Lemma 6 implies that $I_{31}\to 0$ as $t_2\to t_1$. Using the method employed to prove I_1, I_2 tend to zero, we obtain $I_{32}\to 0$ and $I_{33}\to 0$ as $\varepsilon\to 0$. Hence, I_3 tends to zero as $t_2\to t_1$. We can also prove that $I_4\to 0$ as $t_2\to t_1$ which is similar to (7).

For $0 \leq t_1 < T_2 < t_2$, if $t_2 \to t_1$, then $t_2 \to T_2$ and $t_1 \to T_2$. Thus, for $u \in \Omega_1$,

$$\begin{aligned} &\left|\frac{(\Phi_2 u)(t_2)}{1+t_2} - \frac{(\Phi_2 u)(t_1)}{1+t_1}\right| \\ \leq &\left|\frac{(\Phi_2 u)(t_2)}{1+t_2} - \frac{(\Phi_2 u)(T_2)}{1+T_2}\right| + \left|\frac{(\Phi_2 u)(T_2)}{1+T_2} - \frac{(\Phi_2 u)(t_1)}{1+t_1}\right| \to 0, \quad \text{as } t_2 \to t_1. \end{aligned} \tag{12}$$

Consequently,

$$\left|\frac{(\Phi_2 u)(t_2)}{1+t_2} - \frac{(\Phi_2 u)(t_1)}{1+t_1}\right| \to 0, \quad \text{as } t_2 \to t_1.$$

Therefore, $\{v : v(t) = (\Phi_2 u)(t)/(1+t),\ u \in \Omega_1\}$ is equicontinuous. Furthermore, V is equicontinuous. □

Lemma 9. *Assume that H1 and H2 hold. Then,* $\lim_{t\to\infty} |(\Phi u)(t)|/(1+t) = 0$ *uniformly for* $u \in \Omega_1$.

Proof. In fact, for any $u \in \Omega_1$, by H2 and Lemma 5, we obtain

$$\begin{aligned} |(\Phi u)(t)| \leq &\left|t^{1-\mu+\beta\mu-\beta k}\mathcal{K}_{\mu,\beta}(t)y_0\right| + \left|t^{1-\mu+\beta\mu-\beta k}\int_0^t \mathcal{Q}_\beta(t-s)G(s,y(s))ds\right| \\ \leq &L_2|y_0| + L_1 L t^{1-\mu+\beta\mu-\beta k}\int_0^t (t-s)^{\beta k-1}s^{-\alpha}ds \\ \leq &L_2|y_0| + \frac{L_1 L\Gamma(\beta k)\Gamma(1-\alpha)}{\Gamma(\beta k+1-\alpha)}t^{1-\alpha-\mu(1-\beta)}. \end{aligned} \tag{13}$$

Dividing (13) by $(1+t)$, we get

$$\frac{|(\Phi u)(t)|}{1+t} \leq \frac{L_2|y_0|}{1+t} + \frac{L_1 L\Gamma(\beta k)\Gamma(1-\alpha)}{\Gamma(\beta k+1-\alpha)}\frac{t^{1-\alpha-\mu(1-\beta)}}{1+t}. \tag{14}$$

Consequently,

$$\frac{|(\Phi u)(t)|}{1+t} \to 0, \quad \text{as } t \to \infty,$$

which implies that $\lim_{t\to\infty} |(\Phi u)(t)|/(1+t) = 0$ uniformly for $u \in \Omega_1$. This completes the proof. □

Lemma 10. *Assume that H1 and H2 hold. Then* $\Phi\Omega_1 \subset \Omega_1$.

Proof. Let $y(t) = t^{-1-\mu+\beta\mu-\beta k}u(t)$, for $u \in \Omega_1$, $t \in (0,\infty)$. Then $y \in \widetilde{\Omega}_1$.

For $t > 0$, from (4) and (14), we have

$$\frac{|(\Phi u)(t)|}{1+t} \leq L_2|y_0| + \frac{L_1 L\Gamma(\beta k)\Gamma(1-\alpha)}{\Gamma(\beta k+1-\alpha)}\frac{t^{1-\alpha-\mu(1-\beta)}}{1+t} \leq r, \tag{15}$$

For $t = 0$, $|(\Phi u)(0)| = 0 < r$. Therefore, $\Phi\Omega_1 \subset \Omega_1$. □

Lemma 11. *Suppose that H1 and H2 hold. Then* Φ *is continuous.*

Proof. Let $\{u_m\}_{m=1}^{\infty}$ be a sequence in Ω_1 converging to $u \in \Omega_1$. Consequently,

$$\lim_{m\to\infty} u_m(t) = u(t), \text{ and } \lim_{m\to\infty} t^{-1-\mu+\beta\mu-\beta k}u_m(t) = t^{-1-\mu+\beta\mu-\beta k}u(t), \text{ for } t \in (0,\infty).$$

Let $y(t) = t^{-1-\mu+\beta\mu-\beta k}u(t)$, $y_m(t) = t^{-1-\mu+\beta\mu-\beta k}u_m(t)$ $t \in (0,\infty)$. Then $y, y_m \in \widetilde{\Omega}_1$. In view of H1, we have

$$\lim_{m\to\infty} G(t, y_m(t)) = \lim_{m\to\infty} G(t, t^{-1-\mu+\beta\mu-\beta k} u_m(t)) = G(t, t^{-1-\mu+\beta\mu-\beta k} u(t)) = G(t, y(t)).$$

For any $\varepsilon > 0$, there exists $T_2 > 0$ such that (8) holds. Thus, for $t > T_2$,

$$\Big|\frac{(\Phi u_m)(t)}{1+t} - \frac{(\Phi u)(t)}{1+t}\Big| \leq \frac{2L_1 L\Gamma(\beta k)\Gamma(1-\alpha)}{\Gamma(\beta k+1-\alpha)} \frac{t^{1-\alpha-\mu(1-\beta)}}{1+t} < \varepsilon. \tag{16}$$

which implies that $\|\Phi u_m - \Phi u\| \to 0$ as $m \to \infty$.

For each $t \in [0, T_2]$, $(t-s)^{\beta k-1}|G(s, y_m(s)) - G(s, y(s))| \leq 2L_1(t-s)^{\beta k-1}s^{-\alpha}$. By the Lebesgue-dominated convergence theorem, we obtain

$$\int_0^t (t-s)^{\beta k-1}|G(s, y_m(s)) - G(s, y(s))|ds \to 0, \quad \text{as } m \to \infty.$$

Thus, for $t \in [0, T_2]$,

$$\begin{aligned}
&\Big|\frac{(\Phi u_m)(t)}{1+t} - \frac{(\Phi u)(t)}{1+t}\Big| \\
\leq &\frac{t^{1-\mu+\beta\mu-\beta k}}{1+t} \int_0^t |\mathcal{Q}_\beta(t-s)(G(s, y_m(s)) - G(s, y(s)))|ds \\
\leq &L_1 \frac{t^{1-\mu+\beta\mu-\beta k}}{1+t} \int_0^t (t-s)^{\beta k-1}|G(s, y_m(s)) - G(s, y(s))|ds \to 0, \quad \text{as } m \to \infty.
\end{aligned}$$

So, $\|\Phi u_m - \Phi u\| \to 0$ as $m \to \infty$. Hence, Φ is continuous. The proof is completed. □

4. Main Results

Theorem 1. *Suppose that $Q(t)$ is compact for $t > 0$. Further assume that H1 and H2 hold. Then (i) there is at least one mild solution of (1); (ii) all mild solutions of (1) are attractive.*

Proof. (i) Clearly, the problem (1) has a mild solution $y \in \widetilde{\Omega}_1$ if and only if the operator Φ has a fixed point $u \in \Omega_1$, where $u(t) = t^{1-\mu+\beta\mu-\beta k}y(t)$. Hence, we only need to prove that the operator Φ has a fixed point in Ω_1. From Lemmas 10 and 11, we know that $\Phi\Omega_1 \subset \Omega_1$ and Φ is continuous. In order to prove that Φ is a completely continuous operator, we need to prove that $\Phi\Omega_1$ is a relatively compact set. In view of Lemmas 8 and 9, the set $V = \{v : v(t) = (\Phi u)(t)/(1+t),\ u \in \Omega_1\}$ is equicontinuous and $\lim_{t\to\infty} |(\Phi u)(t)|/(1+t) = 0$ uniformly for $u \in \Omega_1$. According to Lemma 7, we only need to prove $V(t)$ is relatively compact in X for $t \in [0, \infty)$. Clearly, $V(0)$ is relatively compact in X. We only consider the case $t > 0$. For $\forall\ \varepsilon \in (0, t)$ and $\delta > 0$, define $\Phi_{\varepsilon,\delta}$ on Ω_1 as follows

$$\begin{aligned}
&(\Phi_{\varepsilon,\delta}u)(t) := t^{1-\mu+\beta\mu-\beta k}(\Psi_{\varepsilon,\delta}y)(t) \\
:=&t^{1-\mu+\beta\mu-\beta k}\left(\frac{\mathcal{K}_{\mu,\beta}(t)y_0}{1+t} + \int_0^{t-\varepsilon}\int_\delta^\infty \beta\theta(t-s)^{\beta-1}W_\beta(\theta)Q((t-s)^\beta\theta)G(s, y(s))d\theta ds\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\Phi_{\varepsilon,\delta}u)(t)}{1+t} = &\frac{t^{1-\mu+\beta\mu-\beta k}}{1+t}\Bigg(\frac{\mathcal{K}_{\mu,\beta}(t)y_0}{1+t} \\
&+ Q(\varepsilon^\beta\delta)\int_0^{t-\varepsilon}\int_\delta^\infty \beta\theta(t-s)^{\beta-1}W_\beta(\theta)Q((t-s)^\beta\theta - \varepsilon^\beta\delta)G(s, y(s))d\theta ds\Bigg).
\end{aligned}$$

By Theorem 3 of [17], we know that $\mathcal{K}_{\mu,\beta}(t)$ is compact because $Q(t)$ is compact for $t > 0$. Furthermore, $Q(\varepsilon^\beta\delta)$ is compact. Then the set $\{V_{\varepsilon,\delta}u,\ u \in \Omega_1\}$ is relatively compact in X for any $\varepsilon \in (0, t)$ and for any $\delta > 0$. Moreover, for every $u \in \Omega_1$, we find that

$$
\begin{aligned}
&\left|\frac{(\Phi u)(t)}{1+t} - \frac{(\Phi_{\varepsilon,\delta} u)(t)}{1+t}\right| \\
\leq &\frac{t^{1-\mu+\beta\mu-\beta k}}{1+t}\left|\int_0^t \int_0^\delta \beta\theta(t-s)^{\beta-1} W_\beta(\theta) Q((t-s)^\beta\theta) G(s,y(s)) d\theta ds\right| \\
&+ \frac{t^{1-\mu+\beta\mu-\beta k}}{1+t}\left|\int_{t-\varepsilon}^t \int_\delta^\infty \beta\theta(t-s)^{\beta-1} W_\beta(\theta) Q((t-s)^\beta\theta) G(s,y(s)) d\theta ds\right| \\
\leq &\frac{\beta C_0 t^{1-\mu+\beta\mu-\beta k}}{1+t}\int_0^t (t-s)^{\beta k-1}|G(s,y(s))|ds \int_0^\delta \theta^k W_\beta(\theta) d\theta \\
&+ \frac{\beta C_0 t^{1-\mu+\beta\mu-\beta k}}{1+t}\int_{t-\varepsilon}^t (t-s)^{\beta k-1}|G(s,y(s))|ds \int_0^\infty \theta^k W_\beta(\theta) d\theta \\
\leq &\frac{\beta C_0 L\Gamma(\beta k)\Gamma(1-\alpha)}{\Gamma(\beta k+1-\alpha)}\frac{t^{1-\alpha-\mu(1-\beta)}}{1+t}\int_0^\delta \theta^k W_\beta(\theta) d\theta \\
&+ \frac{\beta C_0 L t^{1-\alpha-\mu(1-\beta)}}{1+t}\int_{1-\varepsilon/t}^1 (1-s)^{\beta k-1} s^{-\alpha} ds \int_0^\infty \theta^k W_\beta(\theta) d\theta \\
\to &\, 0, \quad \text{as } \varepsilon \to 0,\ \delta \to 0.
\end{aligned}
$$

Thus, $V(t)$ is also a relatively compact set in X for $t \in [0,\infty)$. Therefore, the Schauder fixed point theorem implies that Φ has at least a fixed point $u^* \in \Omega_1$. Let $y^*(t) = t^{-1-\mu+\beta\mu-\beta k}u^*(t)$. Then y^* is a mild solution of (1).

(ii) If $y(t)$ is a mild solution of (1), then

$$
y(t) = \mathcal{K}_{\mu,\beta}(t)y_0 + \int_0^t \mathcal{Q}_\beta(t-s)G(s,y(s))ds,\ t \in (0,\infty).
$$

By H2, noting that $-1+\mu-\beta\mu+\beta k < 0$ and $\beta k < \alpha$, we obtain

$$
|y(t)| \leq L_2|y_0|t^{-1+\mu-\beta\mu+\beta k} + \frac{L_1 L\Gamma(\beta k)\Gamma(1-\alpha)}{\Gamma(\beta k+1-\alpha)}t^{\beta k-\alpha} \to 0, \ \text{as } t \to \infty, \tag{17}
$$

which implies that $y(t)$ is an attractive solution. □

In the case that $Q(t)$ is noncompact for $t > 0$, we impose the following hypothesis.

H3. *there exists a constant $K > 0$ such that for any bounded $D \subseteq X$,*

$$
\chi(G(t,D)) \leq Kt^{1-\mu+\beta\mu-\beta k}\chi(D), \quad \text{for a.e. } t \in [0,\infty),
$$

where χ is Kuratowski's measure of noncompactness.

Theorem 2. *Suppose that $Q(t)$ is noncompact for $t > 0$. Further assume that H1, H2 and H3 hold. Then (i) there is at least one mild solution for (1); (ii) all mild solutions of (1) are attractive.*

Proof. (i) Let $u_0(t) = t^{1-\mu+\beta\mu-\beta k}\mathcal{K}_{\mu,\beta}(t)y_0$ for all $t \in [0,\infty)$ and $u_{m+1} = \Phi u_m$, $m = 0,1,2,\cdots$. Consider set $\mathcal{V} = \{v_m : v_m(t) = (\Phi u_m)(t)/(1+t),\ u_m \in \Omega_1\}_{m=0}^\infty$, and we will prove set $\mathcal{V}$ is relatively compact.

In view of Lemmas 8 and 9, the set $\mathcal{V}$ is equicontinuous and $\lim_{t\to\infty}|(\Phi u_m)(t)|/(1+t) = 0$ uniformly for $u_m \in \Omega_1$. According to Lemma 7, we only need to prove $\mathcal{V}(t) = \{v_m(t) : v_m(t) = (\Phi u_m)(t)/(1+t),\ u_m \in \Omega_1\}_{m=0}^\infty$ is relatively compact in X for $t \in [0,\infty)$.

Under the condition H3, by the properties of measure of noncompactness and Lemma 2, for any $t \in [0,\infty)$, we have

$$
\chi\Big(\Big\{\frac{u_m(t)}{1+t}\Big\}_{m=0}^\infty\Big) = \chi\Big(\Big\{\frac{u_0(t)}{1+t}\Big\} \cup \Big\{\frac{u_m(t)}{1+t}\Big\}_{m=1}^\infty\Big) = \chi\Big(\Big\{\frac{u_m(t)}{1+t}\Big\}_{m=1}^\infty\Big)
$$

and

$$\begin{aligned}
&\chi\Big(\Big\{\frac{u_m(t)}{1+t}\Big\}_{m=1}^{\infty}\Big)\\
=&\chi\Big(\Big\{\frac{u_m(t)}{1+t}\Big\}_{m=0}^{\infty}\Big)\\
=&\chi\Big(\Big\{\frac{t^{1-\mu+\beta\mu-\beta k}}{1+t}\mathcal{K}_{\mu,\beta}(t)y_0+\frac{t^{1-\mu+\beta\mu-\beta k}}{1+t}\int_0^t\mathcal{Q}_\beta(t-s)G(s,y_m(s))ds\Big\}_{m=0}^{\infty}\Big)\\
=&\chi\Big(\Big\{\frac{t^{1-\mu+\beta\mu-\beta k}}{1+t}\int_0^t\mathcal{Q}_\beta(t-s)G(s,y_m(s))ds\Big\}_{m=0}^{\infty}\Big)\\
\leq&2L_1\frac{t^{1-\mu+\beta\mu-\beta k}}{1+t}\int_0^t(t-s)^{\beta k-1}\chi\Big(G(s,\{s^{-1+\mu-\beta\mu+\beta k}u_m(s)\}_{m=0}^{\infty})\Big)ds\\
\leq&2L_1KM^*\int_0^t(t-s)^{\beta k-1}s^{1-\mu+\beta\mu-\beta k}\chi\Big(\{s^{-1+\mu-\beta\mu+\beta k}u_m(s)\}_{m=0}^{\infty}\Big)ds\\
\leq&2L_1KM^*\int_0^t(t-s)^{\beta k-1}(1+s)\chi\Big(\Big\{\frac{u_m(s)}{1+s}\Big\}_{m=0}^{\infty}\Big)ds,
\end{aligned}$$

then

$$\chi(\mathcal{V}(t))\leq 2L_1KM^*\int_0^t(t-s)^{\beta k-1}(1+s)\chi(\mathcal{V}(s))ds, \tag{18}$$

where

$$M^*=\max_{t\in[0,\infty)}\Big\{\frac{t^{1-\mu+\beta\mu-\beta k}}{1+t}\Big\}.$$

From (18), we know that

$$\chi(\mathcal{V}(t))\leq 4L_1KM^*\int_0^t(t-s)^{\beta k-1}\chi(\mathcal{V}(s))ds,$$

or

$$\chi(\mathcal{V}(t))\leq 4L_1KM^*\int_0^t(t-s)^{\beta k-1}s\chi(\mathcal{V}(s))ds,$$

holds. Therefore, by the inequality in [25] (p. 188), we obtain that $\chi(\mathcal{V}(t))=0$, and hence $\mathcal{V}(t)$ is relatively compact. Consequently, it follows from Lemma 7 that set $\mathcal{V}$ is relatively compact, that is, there exists a convergent subsequence of $\{u_m\}_{m=0}^{\infty}$. Without any confusion, let $\lim_{m\to\infty}u_m=u^*\in\Omega_1$.

Thus, by continuity of the operator Φ, we have

$$u^*=\lim_{m\to\infty}u_m=\lim_{m\to\infty}\Phi u_{m-1}=\Phi\Big(\lim_{m\to\infty}u_{m-1}\Big)=\Phi u^*,$$

Let $y^*(t)=t^{-1+\mu-\beta\mu+\beta k}u^*(t)$. Thus, $y^*(t)$ is a mild solution of (1).

(ii) This proof is similar to (ii) of Theorem 1, so we omit it. □

5. Conclusions

In this paper, by using the generalized Ascoli–Arzela theorem, we investigated the existence of attractive solutions for Hilfer fractional evolution equations with an almost sectorial operator. We have obtained the global existence and attractivity results when the semigroup is compact as well as noncompact. In particular, we do not need to assume that the $G(t,\cdot)$ satisfies the Lipschitz condition. It is worth mentioning that we have developed some new techniques, for example, structuring the space $C_\beta([0,\infty),X)$ which is the key method concerning the existence of global solutions for fractional evolution equations on infinite intervals. The method employed in this paper can be applied to infinite intervals problems for fractional evolution equations with instantaneous/non-instantaneous impulses, fractional stochastic evolution equations.

Author Contributions: Conceptualization, M.Z. and Y.Z.; formal analysis, M.Z. and B.A.; investigation, M.Z. and Y.Z.; writing—review and editing, M.Z. and B.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Fundo para o Desenvolvimento das Ciências e da Tecnologia of Macau grant number 0074/2019/A2.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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