

## Article

# Fractional Calculus for Convex Functions in Interval-Valued Settings and Inequalities

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**Abstract:** In this paper, we discuss the Riemann–Liouville fractional integral operator for left and right convex interval-valued functions (left and right convex  $I \cdot V \cdot F$ ), as well as various related notions and concepts. First, the authors used the Riemann–Liouville fractional integral to prove Hermite–Hadamard type ( $\mathcal{H}$ – $\mathcal{H}$  type) inequality. Furthermore,  $\mathcal{H}$ – $\mathcal{H}$  type inequalities for the product of two left and right convex  $I \cdot V \cdot F$ s have been established. Finally, for left and right convex  $I \cdot V \cdot F$ s, we found the Riemann–Liouville fractional integral Hermite–Hadamard type inequality ( $\mathcal{H}$ – $\mathcal{H}$  Fejér type inequality). The findings of this research show that this methodology may be applied directly and is computationally simple and precise.

**Keywords:** left and right convex interval-valued function; fractional integral operator; Hermite–Hadamard type inequality; Hermite–Hadamard Fejér type inequality



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## 1. Introduction

Mathematical inequality, finance, engineering, statistics, and probability all use convex functions in some way. Convex and symmetric convex functions have strong relationships with inequalities. Because of their intriguing features in the mathematical sciences, there are expansive properties and strong links between the symmetric function and different fields of convexity, including convex functions, probability theory, and convex geometry on convex sets. Convex functions have a long and illustrious history in science, and they have been a hot focus of study for more than a century. Several researchers have proposed different convex function guesses, expansions, and variants. Many inequalities or equalities, such as the Ostrowski-type inequality, Hardy-type inequality, Opial-type inequality, Simpson inequality, Fejér-type inequality, and Cebaysev-type inequalities, have been established using convex functions. Among these inequalities, the  $\mathcal{H}$ – $\mathcal{H}$  inequality [1,2], on which many publications have been published, is likely the one that attracts the most attention from scholars.  $\mathcal{H}$ – $\mathcal{H}$  inequality has been regarded as the most useful inequality in mathematical analysis since its discovery in 1883. It is also known as the conventional  $\mathcal{H}$ – $\mathcal{H}$  Inequality equation. The expansions and generalizations of the  $\mathcal{H}$ – $\mathcal{H}$  inequality have piqued the curiosity of a number of mathematicians. For various classes of convex functions and mappings, a number of

mathematicians in the fields of pure and applied mathematics have worked to expand, generalize, counterpart, and enhance the  $\mathcal{H}$ – $\mathcal{H}$  inequality (references [3–13] are a good place to start for interested readers).

Historically, Leibnitz and L'Hospital (1695) are credited with the invention of fractional calculus; however, Riemann, Liouville, and Grunwald–Letnikov, among others, made significant contributions to the field later on. The way that fractional operator speculation deciphers nature's existence in a grand and intentional fashion [14–19] has piqued the curiosity of researchers. By offering an enhanced form of an integral representation for the Appell k-series, Mubeen and Iqbal [20] have contributed to the present research.

Moreover, Khan et al. [21] exploited fuzzy order relations to introduce a new class of convex fuzzy-interval-valued functions (convex  $F$ – $I$ – $V$ – $F$ s), known as  $(h_1, h_2)$ -convex  $F$ – $I$ – $V$ – $F$ s, as well as a novel version of the  $\mathcal{H}$ – $\mathcal{H}$  type inequality for  $(h_1, h_2)$ -convex  $F$ – $I$ – $V$ – $F$ s that incorporates the fuzzy interval Riemann integral. Khan et al. went a step further by providing new convex and extended convex  $I$ – $V$ – $F$  classes, as well as new fractional  $\mathcal{H}$ – $\mathcal{H}$  type and  $\mathcal{H}$ – $\mathcal{H}$  type inequalities for left and right  $(h_1, h_2)$ -preinvex  $I$ – $V$ – $F$  [22], left and right  $p$ -convex  $I$ – $V$ – $F$ s [23], left and right log- $h$ -convex  $I$ – $V$ – $F$ s [24], and the references therein. For further analysis of the literature on the applications and properties of fuzzy Riemannian integrals, inequalities, and generalized convex fuzzy mappings, we refer the readers to cited works [25–56] and the references therein.

Motivated and inspired by the fascinating features of symmetry, convexity, and the fractional operator, we study the new  $\mathcal{H}$ – $\mathcal{H}$  and related  $\mathcal{H}$ – $\mathcal{H}$  type inequalities for left and right convex  $I$ – $V$ – $F$ s, based upon the pseudo order relation and the Riemann–Liouville fractional integral operator.

## 2. Preliminaries

First, we offer some background information on interval-valued functions, the theory of convexity, interval-valued integration, and interval-valued fractional integration, which will be utilized throughout the article.

We offer some fundamental arithmetic regarding interval analysis in this paragraph, which will be quite useful throughout the article.

$$Y = [Y_*, Y^*], Q = [Q_*, Q^*] \quad (Y_* \leq \omega \leq Y^* \text{ and } Q_* \leq z \leq Q^* \quad \omega, z \in \mathbb{R})$$

$$\begin{aligned} Y + Q &= [Y_*, Y^*] + [Q_*, Q^*] = [Y_* + Q_*, Y^* + Q^*], \\ Y - Q &= [Y_*, Y^*] - [Q_*, Q^*] = [Y_* - Q_*, Y^* - Q^*], \\ Y \times Q &= [Y_*, Y^*] \times [Q_*, Q^*] = [\min \mathcal{K}, \max \mathcal{K}] \\ \min \mathcal{K} &= \min\{Y_* Q_*, Y_* Q^*, Y^* Q_*, Y^* Q^*\}, \max \mathcal{K} = \max\{Y_* Q_*, Y_* Q^*, Y^* Q_*, Y^* Q^*\} \\ \nu \cdot [Y_*, Y^*] &= \begin{cases} [\nu Y_*, \nu Y^*] & \text{if } \nu > 0, \\ \{0\} & \text{if } \nu = 0, \\ [\nu Y^*, \nu Y_*] & \text{if } \nu < 0. \end{cases} \end{aligned}$$

Let  $\mathcal{X}_I$ ,  $\mathcal{X}_I^+$ ,  $\mathcal{X}_I^-$  be the set of all closed intervals of  $\mathbb{R}$ , the set of all closed positive intervals of  $\mathbb{R}$ , and the set of all closed negative intervals of  $\mathbb{R}$ , respectively.

For  $[Y_*, Y^*], [Q_*, Q^*] \in \mathcal{X}_I$ , the inclusion " $\subseteq$ " is defined by  $[Y_*, Y^*] \subseteq [Q_*, Q^*]$ , if and only if,  $Q_* \leq Y_*, Y^* \leq Q^*$ .

**Remark 1.** [21] The left and right relation " $\leq_p$ ", defined on  $\mathcal{X}_I$  by  $[Y_*, Y^*] \leq_p [Q_*, Q^*]$ , if and only if,  $Y_* \leq Q_*, Y^* \leq Q^*$ , for all  $[Y_*, Y^*], [Q_*, Q^*] \in \mathcal{X}_I$ , it is a pseudo order relation. For a given  $[Y_*, Y^*], [Q_*, Q^*] \in \mathcal{X}_I$ , we say that  $[Y_*, Y^*] \leq_p [Q_*, Q^*]$ , if and only if,  $Y_* \leq Q_*, Y^* \leq Q^*$  or  $Y_* \leq Q_*, Y^* < Q^*$ .

**Theorem 1.** [33] If  $Y : [t, s] \subset \mathbb{R} \rightarrow \mathcal{X}_I$  is an  $I$ – $V$ – $F$  on such that  $Y(\omega) = [Y_*(\omega), Y^*(\omega)]$ , then  $Y$  is Riemann integrable over  $[t, s]$ , if and only if,  $Y_*$  and  $Y^*$  are both Riemann integrable over  $[t, s]$ , such that

$$(IR) \int_t^s Y(\omega) d\omega = \left[ (R) \int_t^s Y_*(\omega) d\omega, (R) \int_t^s Y^*(\omega) d\omega \right].$$

**Definition 1.** [28,30] Let  $Y \in L([t, s], \mathcal{X}_I^+)$ . Then, interval fractional integrals,  $\mathcal{I}_{t+}^\alpha$  and  $\mathcal{I}_{s-}^\alpha$ , of order  $\alpha > 0$  are defined by

$$\mathcal{I}_{t+}^\alpha Y(\omega) = \frac{1}{\Gamma(\alpha)} \int_t^\omega (\omega - v)^{\alpha-1} Y(v) dv, \quad (\omega > t), \quad (1)$$

and

$$\mathcal{I}_{s-}^\alpha Y(\omega) = \frac{1}{\Gamma(\alpha)} \int_\omega^s (v - \omega)^{\alpha-1} Y(v) dv, \quad (\omega < s), \quad (2)$$

respectively, where  $\Gamma(\omega) = \int_0^\infty v^{\omega-1} e^{-v} dv$  is the Euler gamma function.

**Definition 2.** [31] The I-V-F  $Y : K \rightarrow \mathcal{X}_I^+$  is named as the left and right convex-I-V-F on convex set  $K$  if the coming inequality,

$$Y(v\omega + (1-v)z) \leq_p vY(\omega) + (1-v)Y(z), \quad (3)$$

holds for all  $\omega, z \in K$  and  $v \in [0, 1]$  we have. If inequality (3) is reversed, then  $Y$  is named as the left and right concave on  $K$ .  $Y$  is affine, if and only if, it is both left and right convex and left and right concave.

**Theorem 2.** [31] Let  $Y : K \rightarrow \mathcal{X}_I^+$  be an I-V-F, such that

$$Y(\omega) = [Y_*(\omega), Y^*(\omega)], \quad \forall \omega \in K \quad (4)$$

for all  $\omega \in K$ . Then,  $Y$  is a left and right convex I-V-F on  $K$ , if and only if,  $Y_*(\omega)$  and  $Y^*(\omega)$  both are convex functions.

### 3. Interval Fractional Hermite–Hadamard Inequalities

The major goal, and the main purpose of this section, is to develop a novel version of the  $\mathcal{H}$ – $\mathcal{H}$  inequalities in the mode of interval-valued left and right convex functions.

**Theorem 3.** Let  $Y : [s, t] \rightarrow \mathcal{X}_I^+$  be a left and right convex I-V-F on  $[s, t]$  and provided by  $Y(\omega) = [Y_*(\omega), Y^*(\omega)]$  for all  $\omega \in [s, t]$ . If  $Y \in L([s, t], \mathcal{X}_I^+)$ , then

$$Y\left(\frac{s+t}{2}\right) \leq_p \frac{\Gamma(\alpha+1)}{2(t-s)^\alpha} [\mathcal{I}_{s+}^\alpha Y(t) + \mathcal{I}_{t-}^\alpha Y(s)] \leq_p \frac{Y(s) + Y(t)}{2}. \quad (5)$$

If  $Y(\omega)$  is a left and right concave I-V-F, then

$$Y\left(\frac{s+t}{2}\right) \geq_p \frac{\Gamma(\alpha+1)}{2(t-s)^\alpha} [\mathcal{I}_{s+}^\alpha Y(t) + \mathcal{I}_{t-}^\alpha Y(s)] \geq_p \frac{Y(s) + Y(t)}{2}. \quad (6)$$

**Proof.** Let  $Y : [s, t] \rightarrow \mathcal{X}_I^+$  be a left and right convex I-V-F. Then, by hypothesis, we have:

$$2Y\left(\frac{s+t}{2}\right) \leq_p Y(vs + (1-v)t) + Y((1-v)s + vt).$$

Therefore, we have

$$\begin{aligned} 2Y_*\left(\frac{s+t}{2}\right) &\leq Y_*(vs + (1-v)t) + Y_*((1-v)s + vt), \\ 2Y^*\left(\frac{s+t}{2}\right) &\leq Y^*(vs + (1-v)t) + Y^*((1-v)s + vt). \end{aligned}$$

Multiplying both sides by  $\nu^{\alpha-1}$  and integrating the obtained result, with respect to  $\nu$  over  $(0, 1)$ , we have

$$\begin{aligned} & 2 \int_0^1 \nu^{\alpha-1} Y_*\left(\frac{s+t}{2}\right) d\nu \\ & \leq \int_0^1 \nu^{\alpha-1} Y_*(\nu s + (1-\nu)t) d\nu + \int_0^1 \nu^{\alpha-1} Y_*((1-\nu)s + \nu t) d\nu, \\ & 2 \int_0^1 \nu^{\alpha-1} Y^*\left(\frac{s+t}{2}\right) d\nu \\ & \leq \int_0^1 \nu^{\alpha-1} Y^*(\nu s + (1-\nu)t) d\nu + \int_0^1 \nu^{\alpha-1} Y^*((1-\nu)s + \nu t) d\nu. \end{aligned}$$

Let  $\omega = \nu s + (1-\nu)t$  and  $z = (1-\nu)s + \nu t$ . Then, we have

$$\begin{aligned} \frac{2}{\alpha} Y_*\left(\frac{s+t}{2}\right) & \leq \frac{1}{(t-s)^\alpha} \int_s^t (t-z)^{\alpha-1} Y_*(z) dz + \frac{1}{(t-s)^\alpha} \int_s^t (\omega-s)^{\alpha-1} Y_*(\omega) d\omega \\ \frac{2}{\alpha} Y_*\left(\frac{s+t}{2}\right) & \leq \frac{1}{(t-s)^\alpha} \int_s^t (t-z)^{\alpha-1} Y^*(z) dz + \frac{1}{(t-s)^\alpha} \int_s^t (\omega-s)^{\alpha-1} Y^*(\omega) d\omega, \\ & \leq \frac{\Gamma(\alpha)}{(t-s)^\alpha} [\mathcal{I}_{s^+}^\alpha Y_*(t) + \mathcal{I}_{t^-}^\alpha Y_*(s)] \\ & \leq \frac{\Gamma(\alpha)}{(t-s)^\alpha} [\mathcal{I}_{s^+}^\alpha Y^*(t) + \mathcal{I}_{t^-}^\alpha Y^*(s)], \end{aligned}$$

That is,

$$\frac{2}{\alpha} \left[ Y_*\left(\frac{s+t}{2}\right), Y^*\left(\frac{s+t}{2}\right) \right] \leq_p \frac{\Gamma(\alpha)}{(t-s)^\alpha} [[\mathcal{I}_{s^+}^\alpha Y_*(t) + \mathcal{I}_{t^-}^\alpha Y_*(s)], [\mathcal{I}_{s^+}^\alpha Y^*(t) + \mathcal{I}_{t^-}^\alpha Y^*(s)]]$$

Thus:

$$\frac{2}{\alpha} Y\left(\frac{s+t}{2}\right) \leq_p \frac{\Gamma(\alpha)}{(t-s)^\alpha} [\mathcal{I}_{s^+}^\alpha Y(t) + \mathcal{I}_{t^-}^\alpha Y(s)] \quad (7)$$

Similar to the above, we have

$$\frac{\Gamma(\alpha)}{(t-s)^\alpha} [\mathcal{I}_{s^+}^\alpha Y(t) + \mathcal{I}_{t^-}^\alpha Y(s)] \leq_p \frac{Y(s) + Y(t)}{2} \quad (8)$$

Combining (7) and (8), we have

$$Y\left(\frac{s+t}{2}\right) \leq_p \frac{\Gamma(\alpha+1)}{2(t-s)^\alpha} [\mathcal{I}_{s^+}^\alpha Y(t) + \mathcal{I}_{t^-}^\alpha Y(s)] \leq_p \frac{Y(s) + Y(t)}{2}$$

Hence, we achieve the required result.  $\square$

**Remark 2.** We may observe from Theorem 3 that:

Let one take  $\alpha = 1$ . Then, from Theorem 1 and (5), we achieve the coming inequality (see [23]):

$$Y\left(\frac{s+t}{2}\right) \leq_p \frac{1}{t-s} \int_s^t Y(\omega) d\omega \leq_p \frac{Y(s) + Y(t)}{2}$$

If we take  $Y_*(\omega) = Y^*(\omega)$ , then from Theorem 3 and (5), we acquire the coming inequality (see [32]):

$$Y\left(\frac{s+t}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(t-s)^\alpha} [\mathcal{I}_{s^+}^\alpha Y(t) + \mathcal{I}_{t^-}^\alpha Y(s)] \leq \frac{Y(s) + Y(t)}{2}$$

Let one take  $\alpha = 1$  and  $Y_*(\omega) = Y^*(\omega)$ . Then, from Theorem 1 and (5), we achieve classical  $\mathcal{H}$ - $\mathcal{H}$  type inequality.

**Example 1.** Let  $\alpha = \frac{1}{2}$ ,  $\omega \in [2, 3]$ , and the I-V-F  $Y: [s, t] = [2, 3] \rightarrow \mathcal{X}_I^+$ , provided by  $Y(\omega) = [1, 2] \left(2 - \omega^{\frac{1}{2}}\right)$ . Since the left and right endpoint functions,  $Y_*(\omega) = 2 - \omega^{\frac{1}{2}}$ ,  $Y^*(\omega) =$

$2\left(2 - \omega^{\frac{1}{2}}\right)$ , are left and right convex functions, then  $Y(\omega)$  is a left and right convex I·V-F. We clearly see that  $Y \in L([s, t], \mathcal{X}_I^+)$ , and

$$\begin{aligned} Y_*\left(\frac{s+t}{2}\right) &= Y_*\left(\frac{5}{2}\right) = \frac{4-\sqrt{10}}{2} \\ Y^*\left(\frac{s+t}{2}\right) &= Y^*\left(\frac{5}{2}\right) = 4 - \sqrt{10} \\ \frac{Y_*(s)+Y_*(t)}{2} &= \frac{4-\sqrt{2}-\sqrt{3}}{2} \\ \frac{Y^*(s)+Y^*(t)}{2} &= 4 - \sqrt{2} - \sqrt{3} \end{aligned}$$

Note that

$$\begin{aligned} &\frac{\Gamma(a+1)}{2(t-s)^a} [\mathcal{I}_{s^+}^a Y_*(t) + \mathcal{I}_{t^-}^a Y_*(s)] \\ &= \frac{\Gamma(\frac{3}{2})}{2} \frac{1}{\sqrt{\pi}} \int_{\frac{2}{2}}^3 (3-\omega)^{-\frac{1}{2}} \cdot \left(2 - \omega^{\frac{1}{2}}\right) d\omega \\ &\quad + \frac{\Gamma(\frac{3}{2})}{2} \frac{1}{\sqrt{\pi}} \int_{\frac{2}{2}}^3 (\omega-2)^{-\frac{1}{2}} \cdot \left(2 - \omega^{\frac{1}{2}}\right) d\omega \\ &= \frac{1}{4} \left[ \frac{7393}{10,000} + \frac{9501}{10,000} \right] \\ &= \frac{8447}{20,000}. \\ &\frac{\Gamma(a+1)}{2(t-s)^a} [\mathcal{I}_{s^+}^a Y^*(t) + \mathcal{I}_{t^-}^a Y^*(s)] \\ &= \frac{\Gamma(\frac{3}{2})}{2} \frac{1}{\sqrt{\pi}} \int_{\frac{2}{2}}^3 (3-\omega)^{-\frac{1}{2}} \cdot 2\left(2 - \omega^{\frac{1}{2}}\right) d\omega \\ &\quad + \frac{\Gamma(\frac{3}{2})}{2} \frac{1}{\sqrt{\pi}} \int_{\frac{2}{2}}^3 (\omega-2)^{-\frac{1}{2}} \cdot 2\left(2 - \omega^{\frac{1}{2}}\right) d\omega \\ &= \frac{1}{2} \left[ \frac{7393}{10,000} + \frac{9501}{10,000} \right] \\ &= \frac{8447}{10,000} \end{aligned}$$

Therefore,

$$\left[ \frac{4 - \sqrt{10}}{2}, 4 - \sqrt{10} \right] \leq_p \left[ \frac{8447}{20,000}, \frac{8447}{10,000} \right] \leq_p \left[ \frac{4 - \sqrt{2} - \sqrt{3}}{2}, 4 - \sqrt{2} - \sqrt{3} \right]$$

and Theorem 3 is verified.

The upcoming two results acquire the fractional inequalities for the product of left and right convex I·V-Fs.

**Theorem 4.** Let  $Y, \mathfrak{G} : [s, t] \rightarrow \mathcal{X}_I^+$  be two left and right convex I·V-Fs on  $[s, t]$ , provided by  $Y(\omega) = [Y_*(\omega), Y^*(\omega)]$  and  $\mathfrak{G}(\omega) = [\mathfrak{G}_*(\omega), \mathfrak{G}^*(\omega)]$  for all  $\omega \in [s, t]$ . If  $Y \times \mathfrak{G} \in L([s, t], \mathcal{X}_I^+)$ , then

$$\begin{aligned} &\frac{\Gamma(a)}{2(t-s)^a} [\mathcal{I}_{s^+}^a Y(t) \times \mathfrak{G}(t) + \mathcal{I}_{t^-}^a Y(s) \times \mathfrak{G}(s)] \\ &\leq_p \left( \frac{1}{2} - \frac{a}{(a+1)(a+2)} \right) \varphi(s, t) + \left( \frac{a}{(a+1)(a+2)} \right) \nabla(s, t) \end{aligned}$$

where  $\varphi(s, t) = Y(s) \times \mathfrak{G}(s) + Y(t) \times \mathfrak{G}(t)$ ,  $\nabla(s, t) = Y(s) \times \mathfrak{G}(t) + Y(t) \times \mathfrak{G}(s)$ , and  $\varphi(s, t) = [\varphi_*(s, t), \varphi^*(s, t)]$  and  $\nabla(s, t) = [\nabla_*(s, t), \nabla^*(s, t)]$ .

**Proof.** Since  $Y, \mathfrak{G}$  are both left and right convex I·V-Fs, then we have

$$Y_*(vs + (1-v)t) \leq vY_*(s) + (1-v)Y_*(t),$$

$$Y^*(vs + (1-v)t) \leq vY^*(s) + (1-v)Y^*(t).$$

and

$$\mathfrak{G}_*(\nu s + (1-\nu)t) \leq \nu \mathfrak{G}_*(s) + (1-\nu)\mathfrak{G}_*(t),$$

$$\mathfrak{G}^*(\nu s + (1-\nu)t) \leq \nu \mathfrak{G}^*(s) + (1-\nu)\mathfrak{G}^*(t).$$

From the definition of left and right convex  $I$ - $V$ -Fs, it follows that  $0 \leq_p Y(\omega)$  and  $0 \leq_p \mathfrak{G}(\omega)$ , so

$$\begin{aligned} Y_*(\nu s + (1-\nu)t) \times \mathfrak{G}_*(\nu s + (1-\nu)t) &\leq (\nu Y_*(s) + (1-\nu)Y_*(t))(\nu \mathfrak{G}_*(s) + (1-\nu)\mathfrak{G}_*(t)) \\ &= \nu^2 Y_*(s) \times \mathfrak{G}_*(s) + (1-\nu)^2 Y_*(t) \times \mathfrak{G}_*(t) \\ &\quad + \nu(1-\nu)Y_*(s) \times \mathfrak{G}_*(t) + \nu(1-\nu)Y_*(t) \times \mathfrak{G}_*(s) \\ Y^*(\nu s + (1-\nu)t) \times \mathfrak{G}^*(\nu s + (1-\nu)t) &\leq (\nu Y^*(s) + (1-\nu)Y^*(t))(\nu \mathfrak{G}^*(s) + (1-\nu)\mathfrak{G}^*(t)) \\ &= \nu^2 Y^*(s) \times \mathfrak{G}^*(s) + (1-\nu)^2 Y^*(t) \times \mathfrak{G}^*(t) \\ &\quad + \nu(1-\nu)Y^*(s) \times \mathfrak{G}^*(t) + \nu(1-\nu)Y^*(t) \times \mathfrak{G}^*(s), \end{aligned} \quad (9)$$

Analogously, we have

$$\begin{aligned} Y_*((1-\nu)s + \nu t) \times \mathfrak{G}_*((1-\nu)s + \nu t) &\leq (1-\nu)^2 Y_*(s) \times \mathfrak{G}_*(s) + \nu^2 Y_*(t) \times \mathfrak{G}_*(t) \\ &\quad + \nu(1-\nu)Y_*(s) \times \mathfrak{G}_*(t) + \nu(1-\nu)Y_*(t) \times \mathfrak{G}_*(s) \\ Y^*((1-\nu)s + \nu t) \times \mathfrak{G}^*((1-\nu)s + \nu t) &\leq (1-\nu)^2 Y^*(s) \times \mathfrak{G}^*(s) + \nu^2 Y^*(t) \times \mathfrak{G}^*(t) \\ &\quad + \nu(1-\nu)Y^*(s) \times \mathfrak{G}^*(t) + \nu(1-\nu)Y^*(t) \times \mathfrak{G}^*(s). \end{aligned} \quad (10)$$

Adding (9) and (10), we have

$$\begin{aligned} &Y_*(\nu s + (1-\nu)t) \times \mathfrak{G}_*(\nu s + (1-\nu)t) \\ &\quad + Y_*((1-\nu)s + \nu t) \times \mathfrak{G}_*((1-\nu)s + \nu t) \\ &\leq [\nu^2 + (1-\nu)^2][Y_*(s) \times \mathfrak{G}_*(s) + Y_*(t) \times \mathfrak{G}_*(t)] \\ &\quad + 2\nu(1-\nu)[Y_*(t) \times \mathfrak{G}_*(s) + Y_*(s) \times \mathfrak{G}_*(t)] \\ &Y^*(\nu s + (1-\nu)t) \times \mathfrak{G}^*(\nu s + (1-\nu)t) \\ &\quad + Y^*((1-\nu)s + \nu t) \times \mathfrak{G}^*((1-\nu)s + \nu t) \\ &\leq [\nu^2 + (1-\nu)^2][Y^*(s) \times \mathfrak{G}^*(s) + Y^*(t) \times \mathfrak{G}^*(t)] \\ &\quad + 2\nu(1-\nu)[Y^*(t) \times \mathfrak{G}^*(s) + Y^*(s) \times \mathfrak{G}^*(t)]. \end{aligned} \quad (11)$$

Taking the multiplication of (11) by  $\nu^{a-1}$  and integrating the obtained result, with respect to  $\nu$  over  $(0, 1)$ , we have

$$\begin{aligned} &\int_0^1 \nu^{a-1} Y_*(\nu s + (1-\nu)t) \times \mathfrak{G}_*(\nu s + (1-\nu)t) \\ &\quad + \nu^{a-1} Y_*((1-\nu)s + \nu t) \times \mathfrak{G}_*((1-\nu)s + \nu t) d\nu \\ &\leq \varphi_*(s, t) \int_0^1 \nu^{a-1} [\nu^2 + (1-\nu)^2] d\nu + 2\nabla_*(s, t) \int_0^1 \nu^{a-1} \nu(1-\nu) d\nu \\ &\int_0^1 \nu^{a-1} Y^*(\nu s + (1-\nu)t) \times \mathfrak{G}^*(\nu s + (1-\nu)t) \\ &\quad + \nu^{a-1} Y^*((1-\nu)s + \nu t) \times \mathfrak{G}^*((1-\nu)s + \nu t) d\nu \\ &\leq \varphi^*(s, t) \int_0^1 \nu^{a-1} [\nu^2 + (1-\nu)^2] d\nu + 2\nabla^*(s, t) \int_0^1 \nu^{a-1} \nu(1-\nu) d\nu. \end{aligned}$$

It follows that

$$\begin{aligned} & \frac{\Gamma(\alpha)}{(t-s)^\alpha} [\mathcal{I}_{s+}^\alpha Y_*(t) \times \mathfrak{G}_*(t) + \mathcal{I}_{t-}^\alpha Y_*(s) \times \mathfrak{G}_*(s)] \\ & \leq \frac{2}{\alpha} \left( \frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \varphi_*(s, t) + \frac{2}{\alpha} \left( \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \nabla_*(s, t) \\ & \frac{\Gamma(\alpha)}{(t-s)^\alpha} [\mathcal{I}_{s+}^\alpha Y^*(t) \times \mathfrak{G}^*(t) + \mathcal{I}_{t-}^\alpha Y^*(s) \times \mathfrak{G}^*(s)] \\ & \leq \frac{2}{\alpha} \left( \frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \varphi^*(s, t) + \frac{2}{\alpha} \left( \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \nabla^*(s, t), \\ & \frac{\Gamma(\alpha)}{(t-s)^\alpha} [\mathcal{I}_{s+}^\alpha Y_*(t) \times \mathfrak{G}_*(t) + \mathcal{I}_{t-}^\alpha Y_*(s) \times \mathfrak{G}_*(s)] \\ & \leq \frac{2}{\alpha} \left( \frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \varphi_*(s, t) + \frac{2}{\alpha} \left( \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \nabla_*(s, t) \\ & \frac{\Gamma(\alpha)}{(t-s)^\alpha} [\mathcal{I}_{s+}^\alpha Y^*(t) \times \mathfrak{G}^*(t) + \mathcal{I}_{t-}^\alpha Y^*(s) \times \mathfrak{G}^*(s)] \\ & \leq \frac{2}{\alpha} \left( \frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \varphi^*(s, t) + \frac{2}{\alpha} \left( \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \nabla^*(s, t), \end{aligned}$$

That is,

$$\begin{aligned} & \frac{\Gamma(\alpha)}{(t-s)^\alpha} [\mathcal{I}_{s+}^\alpha Y_*(t) \times \mathfrak{G}_*(t) + \mathcal{I}_{t-}^\alpha Y_*(s) \times \mathfrak{G}_*(s), \mathcal{I}_{s+}^\alpha Y^*(t) \times \mathfrak{G}^*(t) + \mathcal{I}_{t-}^\alpha Y^*(s) \times \mathfrak{G}^*(s)] \\ & \leq_p \frac{2}{\alpha} \left( \frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) [\varphi_*(s, t), \varphi^*(s, t)] + \frac{2}{\alpha} \left( \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) [\nabla_*(s, t), \nabla^*(s, t)] \end{aligned}$$

Thus,

$$\begin{aligned} & \frac{\Gamma(\alpha)}{2(t-s)^\alpha} [\mathcal{I}_{s+}^\alpha Y(t) \times \mathfrak{G}(t) + \mathcal{I}_{t-}^\alpha Y(s) \times \mathfrak{G}(s)] \\ & \leq_p \left( \frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \varphi(s, t) + \left( \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \nabla(s, t) \end{aligned}$$

and the theorem has been established.  $\square$

**Example 2.** Let  $[s, t] = [0, 2]$ ,  $\alpha = \frac{1}{2}$ ,  $Y(\omega) = [\frac{\omega}{2}, \frac{3\omega}{2}]$ , and  $\mathfrak{G}(\omega) = [\omega, 3\omega]$ . Since the left and right endpoint functions,  $Y_*(\omega) = \frac{\omega}{2}$ ,  $Y^*(\omega) = \frac{3\omega}{2}$ ,  $\mathfrak{G}_*(\omega) = \omega$  and  $\mathfrak{G}^*(\omega) = 3\omega$ , are left and right convex functions, then  $Y(\omega)$  and  $\mathfrak{G}(\omega)$  are both left and right convex I-V-Fs. We clearly see that  $Y(\omega) \times \mathfrak{G}(\omega) \in L([s, t], \mathcal{X}_I^+)$ , and

$$\begin{aligned} & \frac{\Gamma(1+\alpha)}{2(t-s)^\alpha} [\mathcal{I}_{s+}^\alpha Y_*(t) \times \mathfrak{G}_*(t) + \mathcal{I}_{t-}^\alpha Y_*(s) \times \mathfrak{G}_*(s)] \\ & = \frac{\Gamma(\frac{3}{2})}{2\sqrt{2}} \frac{1}{\sqrt{\pi}} \int_0^2 (2-\omega)^{-\frac{1}{2}} \left( \frac{1}{2} \omega^2 \right) d\omega + \frac{\Gamma(\frac{3}{2})}{2\sqrt{2}} \frac{1}{\sqrt{\pi}} \int_0^2 (\omega)^{-\frac{1}{2}} \left( \frac{1}{2} \omega^2 \right) d\omega \approx 0.7333, \\ & \frac{\Gamma(1+\alpha)}{2(t-s)^\alpha} [\mathcal{I}_{s+}^\alpha Y^*(t) \times \mathfrak{G}^*(t) + \mathcal{I}_{t-}^\alpha Y^*(s) \times \mathfrak{G}^*(s)] \\ & = \frac{\Gamma(\frac{3}{2})}{2\sqrt{2}} \frac{1}{\sqrt{\pi}} \int_0^2 (2-\omega)^{-\frac{1}{2}} \cdot \frac{9}{2} \omega^2 d\omega + \frac{\Gamma(\frac{3}{2})}{2\sqrt{2}} \frac{1}{\sqrt{\pi}} \int_0^2 (\omega)^{-\frac{1}{2}} \cdot \frac{9}{2} \omega^2 d\omega \approx 6.5997, \end{aligned}$$

Note that

$$\begin{aligned} & \left( \frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \varphi_*(s, t) = [Y_*(s) \times \mathfrak{G}_*(s) + Y_*(t) \times \mathfrak{G}_*(t)] = \frac{11}{15}, \\ & \left( \frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \varphi^*(s, t) = [Y^*(s) \times \mathfrak{G}^*(s) + Y^*(t) \times \mathfrak{G}^*(t)] = \frac{33}{5}, \\ & \left( \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \nabla_*(s, t) = [Y_*(s) \times \mathfrak{G}_*(t) + Y_*(t) \times \mathfrak{G}_*(s)] = \frac{2}{15}(0), \\ & \left( \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \nabla^*(s, t) = [Y^*(s) \times \mathfrak{G}^*(t) + Y^*(t) \times \mathfrak{G}^*(s)] = \frac{2}{15}(0). \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \left( \frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \varphi(s, t) + \left( \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \nabla(s, t) \\ & = \left[ \frac{11}{15}, \frac{33}{5} \right] + \frac{2}{15} [0, 0] = \left[ \frac{11}{15}, \frac{33}{5} \right] \end{aligned}$$

It follows that

$$[0.7333, 6.5997] \leq_p \left[ \frac{11}{15}, \frac{33}{5} \right]$$

and Theorem 4 has been demonstrated.

**Theorem 5.** Let  $Y, \mathfrak{G} : [s, t] \rightarrow \mathcal{X}_I^+$  be two left and right convex  $I$ -V-Fs, provided by  $Y(\omega) = [Y_*(\omega), Y^*(\omega)]$  and  $\mathfrak{G}(\omega) = [\mathfrak{G}_*(\omega), \mathfrak{G}^*(\omega)]$  for all  $\omega \in [s, t]$ . If  $Y \times \mathfrak{G} \in L([s, t], \mathcal{X}_I^+)$ , then

$$\begin{aligned} \frac{1}{a} Y\left(\frac{s+t}{2}\right) \times \mathfrak{G}\left(\frac{s+t}{2}\right) &\leq_p \frac{\Gamma(a+1)}{4(t-s)^a} [\mathcal{I}_{s^+}^a Y(t) \times \mathfrak{G}(t) + \mathcal{I}_{t^-}^a Y(s) \times \mathfrak{G}(s)] \\ &\quad + \frac{1}{2a} \left( \frac{1}{2} - \frac{a}{(a+1)(a+2)} \right) \nabla(s, t) + \frac{1}{2a} \left( \frac{a}{(a+1)(a+2)} \right) \varphi(s, t) \end{aligned}$$

where  $\varphi(s, t) = Y(s) \times \mathfrak{G}(s) + Y(t) \times \mathfrak{G}(t)$ ,  $\nabla(s, t) = Y(s) \times \mathfrak{G}(t) + Y(t) \times \mathfrak{G}(s)$ ,  $\varphi(s, t) = [\varphi_*(s, t), \varphi^*(s, t)]$ , and  $\nabla(s, t) = [\nabla_*(s, t), \nabla^*(s, t)]$ .

**Proof.** Consider that  $Y, \mathfrak{G} : [s, t] \rightarrow \mathcal{X}_I^+$  are left and right convex  $I$ -V-Fs. Then, by hypothesis, we have

$$\begin{aligned} &Y_*\left(\frac{s+t}{2}\right) \times \mathfrak{G}_*\left(\frac{s+t}{2}\right) \\ &Y^*\left(\frac{s+t}{2}\right) \times \mathfrak{G}^*\left(\frac{s+t}{2}\right) \\ &\leq \frac{1}{4} \left[ \begin{array}{l} Y_*(vs + (1-v)t) \times \mathfrak{G}_*(vs + (1-v)t) \\ + Y_*(vs + (1-v)t) \times \mathfrak{G}_*((1-v)s + vt) \end{array} \right] \\ &+ \frac{1}{4} \left[ \begin{array}{l} Y_*((1-v)s + vt) \times \mathfrak{G}_*(vs + (1-v)t) \\ + Y_*((1-v)s + vt) \times \mathfrak{G}_*((1-v)s + vt) \end{array} \right] \\ &\leq \frac{1}{4} \left[ \begin{array}{l} Y^*(vs + (1-v)t) \times \mathfrak{G}^*(vs + (1-v)t) \\ + Y^*(vs + (1-v)t) \times \mathfrak{G}^*((1-v)s + vt) \end{array} \right] \\ &+ \frac{1}{4} \left[ \begin{array}{l} Y^*((1-v)s + vt) \times \mathfrak{G}^*(vs + (1-v)t) \\ + Y^*((1-v)s + vt) \times \mathfrak{G}^*((1-v)s + vt) \end{array} \right], \\ &\leq \frac{1}{4} \left[ \begin{array}{l} Y_*(vs + (1-v)t) \times \mathfrak{G}_*(vs + (1-v)t) \\ + Y_*((1-v)s + vt) \times \mathfrak{G}_*((1-v)s + vt) \end{array} \right] \\ &\quad + \frac{1}{4} \left[ \begin{array}{l} (vY_*(s) + (1-v)Y_*(t)) \\ \times ((1-v)\mathfrak{G}_*(s) + v\mathfrak{G}_*(t)) \\ + ((1-v)Y_*(s) + vY_*(t)) \\ \times (v\mathfrak{G}_*(s) + (1-v)\mathfrak{G}_*(t)) \end{array} \right] \\ &\leq \frac{1}{4} \left[ \begin{array}{l} Y^*(vs + (1-v)t) \times \mathfrak{G}^*(vs + (1-v)t) \\ + Y^*((1-v)s + vt) \times \mathfrak{G}^*((1-v)s + vt) \end{array} \right] \\ &\quad + \frac{1}{4} \left[ \begin{array}{l} (vY^*(s) + (1-v)Y^*(t)) \\ \times ((1-v)\mathfrak{G}^*(s) + v\mathfrak{G}^*(t)) \\ + ((1-v)Y^*(s) + vY^*(t)) \\ \times (v\mathfrak{G}^*(s) + (1-v)\mathfrak{G}^*(t)) \end{array} \right], \\ &= \frac{1}{4} \left[ \begin{array}{l} Y_*(vs + (1-v)t) \times \mathfrak{G}_*(vs + (1-v)t) \\ + Y_*((1-v)s + vt) \times \mathfrak{G}_*((1-v)s + vt) \end{array} \right] \\ &\quad + \frac{1}{4} \left[ \begin{array}{l} \{v^2 + (1-v)^2\} \nabla_*(s, t) \\ + \{v(1-v) + (1-v)v\} \varphi_*(s, t) \end{array} \right] \\ &= \frac{1}{4} \left[ \begin{array}{l} Y^*(vs + (1-v)t) \times \mathfrak{G}^*(vs + (1-v)t) \\ + Y^*((1-v)s + vt) \times \mathfrak{G}^*((1-v)s + vt) \end{array} \right] \\ &\quad + \frac{1}{4} \left[ \begin{array}{l} \{v^2 + (1-v)^2\} \nabla^*(s, t) \\ + \{v(1-v) + (1-v)v\} \varphi^*(s, t) \end{array} \right]. \end{aligned} \tag{12}$$

Taking the multiplication of (12) with  $\nu^{a-1}$  and integrating over  $(0, 1)$ , we get

$$\begin{aligned}
 & \frac{1}{a} Y_*\left(\frac{s+t}{2}\right) \times \mathfrak{G}_*\left(\frac{s+t}{2}\right) \\
 & \leq \frac{1}{4(t-s)^a} \left[ \int_s^t (t-\omega)^{a-1} Y_*(\omega) \times \mathfrak{G}_*(\omega) d\omega \right. \\
 & \quad \left. + \int_s^t (z-s)^{a-1} Y_*(z) \times \mathfrak{G}_*(z) dz \right] \\
 & + \frac{1}{2a} \left( \frac{1}{2} - \frac{a}{(a+1)(a+2)} \right) \nabla_*(s, t) + \frac{1}{2a} \left( \frac{a}{(a+1)(a+2)} \right) \varphi_*(s, t) \\
 & = \frac{\Gamma(a+1)}{4(t-s)^a} [\mathcal{I}_{s+}^a Y_*(t) \times \mathfrak{G}_*(t) + \mathcal{I}_{t-}^a Y_*(s) \times \mathfrak{G}_*(s)] \\
 & + \frac{1}{2a} \left( \frac{1}{2} - \frac{a}{(a+1)(a+2)} \right) \nabla_*(s, t) + \frac{1}{2a} \left( \frac{a}{(a+1)(a+2)} \right) \varphi_*(s, t) \\
 & \frac{1}{a} Y^*\left(\frac{s+t}{2}\right) \times \mathfrak{G}^*\left(\frac{s+t}{2}\right) \\
 & \leq \frac{1}{4(t-s)^a} \left[ \int_s^t (t-\omega)^{a-1} Y^*(\omega) \times \mathfrak{G}^*(\omega) d\omega \right. \\
 & \quad \left. + \int_s^t (z-s)^{a-1} Y^*(z) \times \mathfrak{G}^*(z) dz \right] \\
 & + \frac{1}{2a} \left( \frac{1}{2} - \frac{a}{(a+1)(a+2)} \right) \nabla^*(s, t) + \frac{1}{2a} \left( \frac{a}{(a+1)(a+2)} \right) \varphi^*(s, t) \\
 & = \frac{\Gamma(a+1)}{4(t-s)^a} [\mathcal{I}_{s+}^a Y^*(t) \times \mathfrak{G}^*(t) + \mathcal{I}_{t-}^a Y^*(s) \times \mathfrak{G}^*(s)] \\
 & + \frac{1}{2a} \left( \frac{1}{2} - \frac{a}{(a+1)(a+2)} \right) \nabla^*(s, t) + \frac{1}{2a} \left( \frac{a}{(a+1)(a+2)} \right) \varphi^*(s, t),
 \end{aligned}$$

That is,

$$\begin{aligned}
 \frac{1}{a} Y\left(\frac{s+t}{2}\right) \times \mathfrak{G}\left(\frac{s+t}{2}\right) & \leq_p \frac{\Gamma(a+1)}{4(t-s)^a} [\mathcal{I}_{s+}^a Y(t) \times \mathfrak{G}(t) + \mathcal{I}_{t-}^a Y(s) \times \mathfrak{G}(s)] \\
 & + \frac{1}{2a} \left( \frac{1}{2} - \frac{a}{(a+1)(a+2)} \right) \nabla(s, t) + \frac{1}{2a} \left( \frac{a}{(a+1)(a+2)} \right) \varphi(s, t).
 \end{aligned}$$

Hence, the required result is achieved.  $\square$

The upcoming results discuss the  $\mathcal{H}$ - $\mathcal{H}$  Fejér type inequality left and right convex  $I$ - $V$ - $F$ . Firstly, we achieve second  $\mathcal{H}$ - $\mathcal{H}$  Fejér type inequality.

**Theorem 6.** Let  $Y : [s, t] \rightarrow \mathcal{X}_I^+$  be a left and right convex  $I$ - $V$ - $F$ , with  $s < t$ , provided by  $Y(\omega) = [Y_*(\omega), Y^*(\omega)]$  for all  $\omega \in [s, t]$ . Let  $Y \in L([s, t], \mathcal{X}_I^+)$  and  $\mathfrak{C} : [s, t] \rightarrow \mathbb{R}$ ,  $\mathfrak{C}(\omega) \geq 0$ , be symmetric with respect to  $\frac{s+t}{2}$ . Then,

$$[\mathcal{I}_{s+}^a Y\mathfrak{C}(t) + \mathcal{I}_{t-}^a Y\mathfrak{C}(s)] \leq_p \frac{Y(s) + Y(t)}{2} [\mathcal{I}_{s+}^a \mathfrak{C}(t) + \mathcal{I}_{t-}^a \mathfrak{C}(s)] \quad (13)$$

If  $Y$  is a concave  $I$ - $V$ - $F$ , then inequality (13) is reversed.

**Proof.** Let  $Y$  be a left and right convex  $I$ - $V$ - $F$  and  $\nu^{a-1}\mathfrak{C}(\nu s + (1-\nu)t) \geq 0$ . Then, we have

$$\begin{aligned}
 & \nu^{a-1} Y_*(\nu s + (1-\nu)t) \mathfrak{C}(\nu s + (1-\nu)t) \\
 & \leq \nu^{a-1} (\nu Y_*(s) + (1-\nu) Y_*(t)) \mathfrak{C}(\nu s + (1-\nu)t) \\
 & \nu^{a-1} Y^*(\nu s + (1-\nu)t) \mathfrak{C}(\nu s + (1-\nu)t) \\
 & \leq \nu^{a-1} (\nu Y^*(s) + (1-\nu) Y^*(t)) \mathfrak{C}(\nu s + (1-\nu)t).
 \end{aligned} \quad (14)$$

and

$$\begin{aligned}
 & \nu^{a-1} Y_*((1-\nu)s + \nu t) \mathfrak{C}((1-\nu)s + \nu t) \\
 & \leq \nu^{a-1} ((1-\nu) Y_*(s) + \nu Y_*(t)) \mathfrak{C}((1-\nu)s + \nu t) \\
 & \nu^{a-1} Y^*((1-\nu)s + \nu t) \mathfrak{C}((1-\nu)s + \nu t) \\
 & \leq \nu^{a-1} ((1-\nu) Y^*(s) + \nu Y^*(t)) \mathfrak{C}((1-\nu)s + \nu t).
 \end{aligned} \quad (15)$$

After adding (14) and (15), and integrating over  $[0, 1]$ , we get

$$\begin{aligned}
 & \int_0^1 \nu^{\alpha-1} Y_*(\nu s + (1-\nu)t) \mathfrak{C}(\nu s + (1-\nu)t) d\nu \\
 & \quad + \int_0^1 \nu^{\alpha-1} Y_*((1-\nu)s + \nu t) \mathfrak{C}((1-\nu)s + \nu t) d\nu \\
 & \leq \int_0^1 \left[ \begin{aligned} & \nu^{\alpha-1} Y_*(s) \{ \nu \mathfrak{C}(\nu s + (1-\nu)t) + (1-\nu) \mathfrak{C}((1-\nu)s + \nu t) \} \\ & + \nu^{\alpha-1} Y_*(t) \{ (1-\nu) \mathfrak{C}(\nu s + (1-\nu)t) + \nu \mathfrak{C}((1-\nu)s + \nu t) \} \end{aligned} \right] d\nu, \\
 & \int_0^1 \nu^{\alpha-1} Y^*((1-\nu)s + \nu t) \mathfrak{C}((1-\nu)s + \nu t) d\nu \\
 & \quad + \int_0^1 \nu^{\alpha-1} Y^*(\nu s + (1-\nu)t) \mathfrak{C}(\nu s + (1-\nu)t) d\nu \\
 & \leq \int_0^1 \left[ \begin{aligned} & \nu^{\alpha-1} Y^*(s) \{ \nu \mathfrak{C}(\nu s + (1-\nu)t) + (1-\nu) \mathfrak{C}((1-\nu)s + \nu t) \} \\ & + \nu^{\alpha-1} Y^*(t) \{ (1-\nu) \mathfrak{C}(\nu s + (1-\nu)t) + \nu \mathfrak{C}((1-\nu)s + \nu t) \} \end{aligned} \right] d\nu, \\
 & = Y_*(s) \int_0^1 \nu^{\alpha-1} C(\nu s + (1-\nu)t) d\nu + Y_*(t) \int_0^1 \nu^{\alpha-1} C((1-\nu)s + \nu t) d\nu, \\
 & = Y^*(s) \int_0^1 \nu^{\alpha-1} C(\nu s + (1-\nu)t) d\nu + Y^*(t) \int_0^1 \nu^{\alpha-1} C((1-\nu)s + \nu t) d\nu.
 \end{aligned}$$

Since  $\mathfrak{C}$  is symmetric, then

$$\begin{aligned}
 & = [Y_*(s) + Y_*(t)] \int_0^1 \nu^{\alpha-1} C((1-\nu)s + \nu t) d\nu \\
 & = [Y^*(s) + Y^*(t)] \int_0^1 \nu^{\alpha-1} C((1-\nu)s + \nu t) d\nu. \\
 & = \frac{Y_*(s) + Y_*(t)}{2} \frac{\Gamma(\alpha)}{(t-s)^\alpha} [\mathcal{I}_{s^+}^\alpha \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha \mathfrak{C}(s)], \\
 & = \frac{Y^*(s) + Y^*(t)}{2} \frac{\Gamma(\alpha)}{(t-s)^\alpha} [\mathcal{I}_{s^+}^\alpha \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha \mathfrak{C}(s)].
 \end{aligned} \tag{16}$$

Since

$$\begin{aligned}
 & \int_0^1 \nu^{\alpha-1} Y_*(\nu s + (1-\nu)t) \mathfrak{C}((1-\nu)s + \nu t) d\nu \\
 & \quad + \int_0^1 \nu^{\alpha-1} Y_*((1-\nu)s + \nu t) \mathfrak{C}((1-\nu)s + \nu t) d\nu \\
 & = \frac{1}{(t-s)^\alpha} \int_s^t (\omega - s)^{\alpha-1} Y_*(s + t - \omega) \mathfrak{C}(\omega) d\omega \\
 & \quad + \frac{1}{(t-s)^\alpha} \int_s^t (\omega - s)^{\alpha-1} Y_*(\omega) \mathfrak{C}(\omega) d\omega \\
 & = \frac{1}{(t-s)^\alpha} \int_s^t (\omega - s)^{\alpha-1} Y_*(\omega) \mathfrak{C}(s + t - \omega) d\omega \\
 & \quad + \frac{1}{(t-s)^\alpha} \int_s^t (\omega - s)^{\alpha-1} Y_*(\omega) \mathfrak{C}(\omega) d\omega \\
 & = \frac{\Gamma(\alpha)}{(t-s)^\alpha} [\mathcal{I}_{s^+}^\alpha Y_* \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha Y_* \mathfrak{C}(s)], \\
 & \int_0^1 \nu^{\alpha-1} Y^*(\nu s + (1-\nu)t) \mathfrak{C}((1-\nu)s + \nu t) d\nu \\
 & \quad + \int_0^1 \nu^{\alpha-1} Y^*((1-\nu)s + \nu t) \mathfrak{C}((1-\nu)s + \nu t) d\nu \\
 & = \frac{\Gamma(\alpha)}{(t-s)^\alpha} [\mathcal{I}_{s^+}^\alpha Y^* \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha Y^* \mathfrak{C}(s)].
 \end{aligned} \tag{17}$$

then, from (16), we have

$$\begin{aligned}
 & \frac{\Gamma(\alpha)}{(t-s)^\alpha} [\mathcal{I}_{s^+}^\alpha Y^* \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha Y^* \mathfrak{C}(s)] \\
 & \leq \frac{Y^*(s) + Y^*(t)}{2} \frac{\Gamma(\alpha)}{(t-s)^\alpha} [\mathcal{I}_{s^+}^\alpha \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha \mathfrak{C}(s)] \\
 & \leq \frac{Y^*(s) + Y^*(t)}{2} \frac{\Gamma(\alpha)}{(t-s)^\alpha} [\mathcal{I}_{s^+}^\alpha \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha \mathfrak{C}(s)], \\
 & \frac{\Gamma(\alpha)}{(t-s)^\alpha} [\mathcal{I}_{s^+}^\alpha Y_* \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha Y_* \mathfrak{C}(s)] \\
 & \leq \frac{Y^*(s) + Y^*(t)}{2} \frac{\Gamma(\alpha)}{(t-s)^\alpha} [\mathcal{I}_{s^+}^\alpha \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha \mathfrak{C}(s)] \\
 & \leq \frac{Y^*(s) + Y^*(t)}{2} \frac{\Gamma(\alpha)}{(t-s)^\alpha} [\mathcal{I}_{s^+}^\alpha \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha \mathfrak{C}(s)],
 \end{aligned}$$

That is,

$$\frac{\Gamma(\alpha)}{(t-s)^\alpha} [[\mathcal{I}_{s^+}^\alpha Y_* \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha Y_* \mathfrak{C}(s)], \mathcal{I}_{s^+}^\alpha Y^* \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha Y^* \mathfrak{C}(s)]$$

$$\begin{aligned} &\leq_p \frac{\Gamma(\alpha)}{(t-s)^\alpha} \left[ \frac{Y_*(s) + Y_*(t)}{2}, \frac{Y^*(s) + Y^*(t)}{2} \right] [\mathcal{I}_{s^+}^\alpha \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha \mathfrak{C}(s)] \\ &\leq_p \frac{\Gamma(\alpha)}{(t-s)^\alpha} \left[ \frac{Y_*(s) + Y_*(t)}{2}, \frac{Y^*(s) + Y^*(t)}{2} \right] [\mathcal{I}_{s^+}^\alpha \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha \mathfrak{C}(s)], \end{aligned}$$

Hence,

$$\begin{aligned} [\mathcal{I}_{s^+}^\alpha Y\mathfrak{C}(t) \leq_p \mathcal{I}_{t^-}^\alpha Y\mathfrak{C}(s)] &\leq_p \frac{Y(s) + Y(t)}{2} [\mathcal{I}_{s^+}^\alpha \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha \mathfrak{C}(s)] \\ &\leq_p \frac{Y(s) + Y(t)}{2} [\mathcal{I}_{s^+}^\alpha \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha \mathfrak{C}(s)]. \end{aligned}$$

Now, we first obtain the  $\mathcal{H}$ - $\mathcal{H}$  Fejér type inequality for the left and right convex  $I$ - $V$ - $F$ .  $\square$

**Theorem 7.** Let  $Y : [s, t] \rightarrow \mathcal{X}_I^+$  be a left and right convex  $I$ - $V$ - $F$ , with  $s < t$ , and defined by  $Y(\omega) = [Y_*(\omega), Y^*(\omega)]$  for all  $\omega \in [s, t]$ . If  $Y \in L([s, t], \mathcal{X}_I^+)$  and  $\mathfrak{C} : [s, t] \rightarrow \mathbb{R}$ ,  $\mathfrak{C}(\omega) \geq 0$  are symmetric with respect to  $\frac{s+t}{2}$ , then

$$Y\left(\frac{s+t}{2}\right) [\mathcal{I}_{s^+}^\alpha \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha \mathfrak{C}(s)] \leq_p [\mathcal{I}_{s^+}^\alpha Y\mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha Y\mathfrak{C}(s)] \quad (18)$$

If  $Y$  is a concave  $I$ - $V$ - $F$ , then inequality (18) is reversed.

**Proof.** Since  $Y$  is a left and right convex  $I$ - $V$ - $F$ , then we have

$$\begin{aligned} Y_*\left(\frac{s+t}{2}\right) &\leq \frac{1}{2}(Y_*(vs + (1-v)t) + Y_*((1-v)s + vt)) \\ Y^*\left(\frac{s+t}{2}\right) &\leq \frac{1}{2}(Y^*(vs + (1-v)t) + Y^*((1-v)s + vt)), \end{aligned} \quad (19)$$

Since  $\mathfrak{C}(vs + (1-v)t) = \mathfrak{C}((1-v)s + vt)$ , then by multiplying (19) by  $v^{\alpha-1}\mathfrak{C}((1-v)s + vt)$  and integrating it, with respect to  $v$  over  $[0, 1]$ , we obtain

$$\begin{aligned} &Y_*\left(\frac{s+t}{2}\right) \int_0^1 v^{\alpha-1} \mathfrak{C}((1-v)s + vt) dv \\ &\leq \frac{1}{2} \left( \int_0^1 v^{\alpha-1} Y_*(vs + (1-v)t) \mathfrak{C}((1-v)s + vt) dv \right. \\ &\quad \left. + \int_0^1 v^{\alpha-1} Y_*((1-v)s + vt) \mathfrak{C}((1-v)s + vt) dv \right), \\ &Y^*\left(\frac{s+t}{2}\right) \int_0^1 \mathfrak{C}((1-v)s + vt) dv \\ &\leq \frac{1}{2} \left( \int_0^1 v^{\alpha-1} Y^*(vs + (1-v)t) \mathfrak{C}((1-v)s + vt) dv \right. \\ &\quad \left. + \int_0^1 v^{\alpha-1} Y^*((1-v)s + vt) \mathfrak{C}((1-v)s + vt) dv \right) \end{aligned} \quad (20)$$

Let  $\omega = (1-v)s + vt$ . Then, we have

$$\begin{aligned} &\int_0^1 v^{\alpha-1} Y_*(vs + (1-v)t) \mathfrak{C}((1-v)s + vt) dv \\ &\quad + \int_0^1 v^{\alpha-1} Y_*((1-v)s + vt) \mathfrak{C}((1-v)s + vt) dv \\ &= \frac{1}{(t-s)^\alpha} \int_s^t (\omega - s)^{\alpha-1} Y_*(s + t - \omega) \mathfrak{C}(\omega) d\omega \\ &\quad + \frac{1}{(t-s)^\alpha} \int_s^t (\omega - s)^{\alpha-1} Y_*(\omega) \mathfrak{C}(\omega) d\omega \\ &= \frac{1}{(t-s)^\alpha} \int_s^t (\omega - s)^{\alpha-1} Y_*(\omega) \mathfrak{C}(s + t - \omega) d\omega \\ &\quad + \frac{1}{(t-s)^\alpha} \int_s^t (\omega - s)^{\alpha-1} Y_*(\omega) \mathfrak{C}(\omega) d\omega \\ &= \frac{\Gamma(\alpha)}{(t-s)^\alpha} [\mathcal{I}_{s^+}^\alpha Y_*\mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha Y_*\mathfrak{C}(s)], \\ &\int_0^1 v^{\alpha-1} Y^*(vs + (1-v)t) \mathfrak{C}((1-v)s + vt) dv \\ &\quad + \int_0^1 v^{\alpha-1} Y^*((1-v)s + vt) \mathfrak{C}((1-v)s + vt) dv \\ &= \frac{\Gamma(\alpha)}{(t-s)^\alpha} [\mathcal{I}_{s^+}^\alpha Y^*\mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha Y^*\mathfrak{C}(s)]. \end{aligned} \quad (21)$$

Then, from (21), we have

$$\begin{aligned} & \frac{\Gamma(\alpha)}{(t-s)^\alpha} Y_*\left(\frac{s+t}{2}\right) [\mathcal{I}_{s^+}^\alpha \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha \mathfrak{C}(s)] \\ & \leq \frac{\Gamma(\alpha)}{(t-s)^\alpha} [\mathcal{I}_{s^+}^\alpha Y_* \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha Y_* \mathfrak{C}(s)] \\ & \frac{\Gamma(\alpha)}{(t-s)^\alpha} Y^*\left(\frac{s+t}{2}\right) [\mathcal{I}_{s^+}^\alpha \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha \mathfrak{C}(s)] \\ & \leq \frac{\Gamma(\alpha)}{(t-s)^\alpha} [\mathcal{I}_{s^+}^\alpha Y^* \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha Y^* \mathfrak{C}(s)] \end{aligned}$$

from which, we have

$$\begin{aligned} & \frac{\Gamma(\alpha)}{(t-s)^\alpha} [Y_*\left(\frac{s+t}{2}\right), Y^*\left(\frac{s+t}{2}\right)] [\mathcal{I}_{s^+}^\alpha \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha \mathfrak{C}(s)] \\ & \leq {}_p \frac{\Gamma(\alpha)}{(t-s)^\alpha} [\mathcal{I}_{s^+}^\alpha Y_* \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha Y_* \mathfrak{C}(s), \mathcal{I}_{s^+}^\alpha Y^* \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha Y^* \mathfrak{C}(s)], \end{aligned}$$

That is,

$$\begin{aligned} & \frac{\Gamma(\alpha)}{(t-s)^\alpha} Y\left(\frac{s+t}{2}\right) [\mathcal{I}_{s^+}^\alpha \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha \mathfrak{C}(s)] \\ & \leq {}_p \frac{\Gamma(\alpha)}{(t-s)^\alpha} [\mathcal{I}_{s^+}^\alpha Y \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha Y \mathfrak{C}(s)]. \end{aligned}$$

This completes the proof.  $\square$

**Example 3.** We consider the I-V-F  $Y : [0, 2] \rightarrow \mathcal{X}_I^+$ , defined by  $Y(\omega) = [(2 - \sqrt{\omega}), 2(2 - \sqrt{\omega})]$ . Since endpoint functions  $Y_*(\omega)$ ,  $Y^*(\omega)$  are convex functions, then  $Y(\omega)$  is a left and right convex I-V-F. If

$$\mathfrak{C}(\omega) = \begin{cases} \sqrt{\omega}, & \omega \in [0, 1], \\ \sqrt{2-\omega}, & \omega \in (1, 2], \end{cases}$$

then  $\mathfrak{C}(2-\omega) = \mathfrak{C}(\omega) \geq 0$  for all  $\omega \in [0, 2]$ . Since  $Y_*(\omega) = 2 - \sqrt{\omega}$  and  $Y^*(\omega) = 2(2 - \sqrt{\omega})$ , if  $\alpha = \frac{1}{2}$ , then we compute the following:

$$\begin{aligned} [\mathcal{I}_{s^+}^\alpha Y \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha Y \mathfrak{C}(s)] & \leq {}_p \frac{Y(s) + Y(t)}{2} [\mathcal{I}_{s^+}^\alpha \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha \mathfrak{C}(s)] \\ \frac{Y_*(s) + Y_*(t)}{2} [\mathcal{I}_{s^+}^\alpha \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha \mathfrak{C}(s)] & = \frac{\pi}{\sqrt{2}} \left( \frac{4-\sqrt{2}}{2} \right) \\ \frac{Y^*(s) + Y^*(t)}{2} [\mathcal{I}_{s^+}^\alpha \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha \mathfrak{C}(s)] & = \frac{\pi}{\sqrt{2}} (4 - \sqrt{2}), \end{aligned} \quad (22)$$

$$\begin{aligned} \frac{Y_*(s) + Y_*(t)}{2} [\mathcal{I}_{s^+}^\alpha \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha \mathfrak{C}(s)] & = \frac{\pi}{\sqrt{2}} \left( \frac{4-\sqrt{2}}{2} \right) \\ \frac{Y^*(s) + Y^*(t)}{2} [\mathcal{I}_{s^+}^\alpha \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha \mathfrak{C}(s)] & = \frac{\pi}{\sqrt{2}} (4 - \sqrt{2}), \end{aligned} \quad (23)$$

$$\begin{aligned} [\mathcal{I}_{s^+}^\alpha Y_* \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha Y_* \mathfrak{C}(s)] & = \frac{1}{\sqrt{\pi}} \left( 2\pi + \frac{4-8\sqrt{2}}{3} \right), \\ [\mathcal{I}_{s^+}^\alpha Y^* \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha Y^* \mathfrak{C}(s)] & = \frac{2}{\sqrt{\pi}} \left( 2\pi + \frac{4-8\sqrt{2}}{3} \right). \end{aligned} \quad (24)$$

From (22)–(24), (13) we have

$$\frac{1}{\sqrt{\pi}} \left[ \left( 2\pi + \frac{4-8\sqrt{2}}{3} \right), 2 \left( 2\pi + \frac{4-8\sqrt{2}}{3} \right) \right] \leq {}_p \frac{\pi}{\sqrt{2}} \left[ \frac{4-\sqrt{2}}{2}, 4 - \sqrt{2} \right] = \frac{\pi}{\sqrt{2}} \left[ \frac{4-\sqrt{2}}{2}, 4 - \sqrt{2} \right].$$

Hence, Theorem 6 is verified.

For Theorem 7, we have

$$\begin{aligned} Y_*\left(\frac{s+t}{2}\right) [\mathcal{I}_{s^+}^\alpha \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha \mathfrak{C}(s)] & = \sqrt{\pi}, \\ Y^*\left(\frac{s+t}{2}\right) [\mathcal{I}_{s^+}^\alpha \mathfrak{C}(t) + \mathcal{I}_{t^-}^\alpha \mathfrak{C}(s)] & = 2\sqrt{\pi}. \end{aligned} \quad (25)$$

From (24) and (25), we have

$$\sqrt{\pi}[1, 2] \leq {}_p \frac{1}{\sqrt{\pi}} \left[ 2\pi + \frac{4-8\sqrt{2}}{3}, 2 \left( 2\pi + \frac{4-8\sqrt{2}}{3} \right) \right]$$

Hence, (18) has been verified.

**Remark 3.** If one takes  $\mathfrak{C}(\omega) = 1$ , then, from (13) and (18), we acquire (5).

Let us take  $\mathfrak{a} = 1$ . Then, we achieve the coming inequality (see [22]).

$$Y\left(\frac{s+t}{2}\right) \leq_p \frac{1}{\int_s^t \mathfrak{C}(\omega) d\omega} \int_s^t Y(\omega) \mathfrak{C}(\omega) d\omega \leq_p \frac{Y(s) + Y(t)}{2}$$

If we take  $Y_*(\omega) = Y^*(\omega)$ , then from (13) and (18), we acquire the coming inequality (see [33]).

$$\begin{aligned} Y\left(\frac{s+t}{2}\right) [\mathcal{I}_{s+}^{\mathfrak{a}} \mathfrak{C}(t) + \mathcal{I}_{t-}^{\mathfrak{a}} \mathfrak{C}(s)] &\leq_p [\mathcal{I}_{s+}^{\mathfrak{a}} Y \mathfrak{C}(t) + \mathcal{I}_{t-}^{\mathfrak{a}} Y \mathfrak{C}(s)] \\ &\leq_p \frac{Y(s) + Y(t)}{2} [\mathcal{I}_{s+}^{\mathfrak{a}} \mathfrak{C}(t) + \mathcal{I}_{t-}^{\mathfrak{a}} \mathfrak{C}(s)] \end{aligned}$$

If one takes  $Y_*(\omega) = Y^*(\omega)$  with  $\mathfrak{a} = 1$ , then from (13) and (18), we achieve the classical  $\mathcal{H}$ - $\mathcal{H}$  Fejér inequality (see [26]).

#### 4. Conclusions

In applied sciences, convex functions and fractional calculus are essential. The new interval-valued left and right convex functions are presented in this article. Some novel Riemann–Liouville fractional integral  $\mathcal{H}$ – $\mathcal{H}$  and Fejér-type inequalities are provided, utilizing the idea of interval-valued left and right convex functions and some supplementary interval analysis findings. Our results are a generalization of a number of previously published findings. In the future, we will use generalized interval and fuzzy Riemann–Liouville fractional operators to investigate this concept for generalized left and right convex  $I$ · $V$ -Fs and  $F$ · $I$ · $V$ -Fs by using interval Katugampola fractional integrals and fuzzy Katugampola fractional integrals. For applications, see [53–56].

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