symmetry

Article

A New Method for Blow-Up to Scale-Invariant Damped Wave Equations with Derivatives and Combined Nonlinear Terms

Yuanming Chen [1,2]

1 The School of Economics, Shanghai University of Finance and Economics, Shanghai 200433, China; chenyuanming98@163.sufe.edu.cn
2 Department of Mathematics, Lishui University, Zhejiang 323000, China

Abstract: The Cauchy problems of scale-invariant damped wave equations with derivative nonlinear terms and with combined nonlinear terms are studied. A new method is provided to show that the solutions will blow up in a finite time, if the nonlinear powers satisfy some conditions. The method is based on constructing appropriate test functions, by using the solution of an ordinary differential equation. It may be useful to prove the nonexistence for global solutions for other nonlinear evolution equations.

Keywords: blow-up; lifespan; damped wave equation; scale invariant; test function

check for updates

Citation: Chen, Y. A New Method for Blow-Up to Scale-Invariant Damped Wave Equations with Derivatives and Combined Nonlinear Terms. *Symmetry* **2022**, *14*, 198. https://doi.org/10.3390/sym14020198

Academic Editors: Manwai Yuen, Imre Ferenc Barna, Baofeng Feng, Biao Li and Li Jun Zhang

Received: 22 December 2021
Accepted: 17 January 2022
Published: 20 January 2022

1. Introduction

Many researchers have studied the damped wave equation, such as Usamah [1], who performed symmetry analysis and exhibited exact solutions for various forms of diffusivity and viscosity, but in the present, work we study the model:

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu}{1+t} u_t = |u_t|^p, & \text{in } [0,T) \times \mathbb{R}^n, \\ u(x,0) = \varepsilon f(x), \quad u_t(x,0) = \varepsilon g(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

and:

$$\begin{cases} u_{tt} - \Delta u + \dfrac{\mu}{1+t} u_t = |u_t|^p + |u|^q, & \text{in } [0,T) \times \mathbb{R}^n, \\ u(x,0) = \varepsilon f(x), \quad u_t(x,0) = \varepsilon g(x), \end{cases} \tag{2}$$

where $\mu > 0$ is a constant and $f(x), g(x)$ are the initial data with compact support, which satisfy:

$$f(x) \in H^1(\mathbb{R}^n), \ \ g(x) \in L^2(\mathbb{R}^n),$$

and

$$supp\, f(x), g(x) \in \{x : |x| \le 1\}. \tag{3}$$

The semilinear wave equation with scale-invariant damping has attracted more and more attention recently: on one the hand, it is the border of the "wave-like" and "heat-like" phenomena of the damped wave equation; on the other hand, it has a close relation to the Tricomi equation, which is used to describe gas dynamics. There are many literature works that have studied the semilinear wave equations with scale-invariant damping; see [2–14] and the references therein. For Problem (1), Lai and Takamura [15] showed the blow-up for $1 < p \le p_G(n+2\mu)$, which seems not to be the sharp blow-up power, since Palmieri and Tu [16] proved a blow-up result in the range $1 < p \le p_G(n+\sigma)$ for:

$$\sigma = \begin{cases} 2\mu, & for\ \mu \in [0,1), \\ 2, & for\ \mu \in [1,2), \\ \mu, & for\ \mu \in [2,\infty). \end{cases}$$

Obviously, when $\mu \in [0,1]$, the former result coincides with that in [15], and there is some improvement for $\mu \in (1,2)$, but still some gap for $p_G(n+\mu)$, while for $[2,\infty)$, they improved the blow-up power to the expected $p_G(n+\mu)$. Recently, Hamouda and Hamza [17] showed blow-up results for (1) when $1 < p \leq p_G(n+\mu)$ and for (2) when:

$$\gamma(p,q,n+\mu) < 4,$$

with:

$$\gamma(p,q,n) = (q-1)((n-1)p-2), \tag{4}$$

which improved the results in [15,16,18], by using a similar method in [11]. We should mention that there are many blow-up results for other nonlinear evolution equations; see [19–21] and the references therein.

In this work, we aim to show the blow-up results and lifespan estimate in [17] by using a new method, based on the works [22,23]. We constructed a special test function, the key ingredient of which is the solution of a ordinary differential equation. Inspired by [24], we can obtain the explicit solution of the ODE and furthermore obtain the asymptotic behavior.

2. Main Result

Definition 1. *We define the upper bound of the lifespan for* (1) *and* (2) *as:*

$$T_\varepsilon = \sup\{T > 0;\ \textit{there exists an energy solution to } (1) \textit{ and } (2) \textit{ in } [0,T)\}.$$

Then, our main results read as follows:

Theorem 1. *Let $1 < p \leq p_G(n+\mu)$. Assume that the initial data f, g are non-negative and do not vanish identically. Furthermore, the compact support assumption* (3) *holds. If we further assume the energy solution satisfies:*

$$supp\ u(t,x) \subset \{x\big| : |x| \leq t+1\}, \tag{5}$$

then the solution of (1) *will blow up in a finite time, and the upper bound of the lifespan will satisfy:*

$$T \leq \begin{cases} C\varepsilon^{-(p-1)/\{1-(n+\mu-1)(p-1)/2\}} & \textit{for } 1 < p < p_G(n+\mu), \\ \exp\Big(C\varepsilon^{-(p-1)}\Big) & \textit{for } p = p_G(n+\mu), \end{cases} \tag{6}$$

where C denotes a positive constant, which may have a different value from line to line and is independent of ε.

Theorem 2. *Let $\gamma(p,q,n+\mu) < 4$. Assume that the initial data f, g are non-negative and do not vanish identically. Furthermore, the compact support assumption* (3) *holds. If we further assume the energy solution satisfies:*

$$supp\ u(t,x) \subset \{x\big| : |x| \leq t+1\}, \tag{7}$$

then the solution of (2) *will blow up in a finite time, and the upper bound of the lifespan will satisfy:*

$$T \leq C\varepsilon^{-\frac{2p(q-1)}{4-\gamma(p,q,n+\mu)}}, \tag{8}$$

where C denotes a positive constant, which may have a different value from line to line and is independent of ε.

3. Test Function

As mentioned above, the key ingredient of the test function is one of the solutions of the following ODEs:

$$\lambda''(t) - \frac{\mu}{1+t}\lambda'(t) - \lambda(t) = 0. \tag{9}$$

Lemma 1. *The ODE (9) admits one solution:*

$$\lambda(t) = (1+t)^{\frac{\mu+1}{2}} K_{\frac{\mu+1}{2}}(1+t),$$

where $K_\nu(z)$ is the modified Bessel functions of the second kind. In particular, λ is a real and positive function satisfying:

$$\lambda(0) = K_{\frac{\mu+1}{2}}(1) > 0, \qquad \lambda'(0) = -K_{\frac{\mu-1}{2}}(1) < 0, \qquad \lambda'(t) < 0, \tag{10}$$

and, for large t,

$$\lambda(t) = \frac{1}{e}\sqrt{\frac{\pi}{2}}(1+t)^{\frac{\mu}{2}} e^{-t} \times \left(1 + O\left(\frac{1}{1+t}\right)\right) = -\lambda'(t). \tag{11}$$

Proof. We first collect some useful relations, which can be found in [25].

$$K_\nu(z) = K_{-\nu}(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}, \tag{12}$$

where $I_\nu(z)$ is the modified Bessel functions of the first kind, and when ν is an integer, the right hand-side of this equation is replaced by its limiting value.

$$K'_\nu(z) = -K_{\nu-1}(z) - \frac{\nu}{z}K_\nu(z), \;\; K'_\nu(z) = -K_{\nu+1}(z) + \frac{\nu}{z}K_\nu(z), \tag{13}$$

$$K_\nu(z) = \sqrt{\frac{\pi}{2}} z^{-1/2} e^{-z} \times (1 + O(z^{-1})), \; \textit{for } |z| \textit{ large and } |\arg z| < \frac{3}{2}\pi. \tag{14}$$

Then, it is easy to check from (13):

$$\begin{aligned}
\lambda'(t) &= \frac{\mu+1}{2}(1+t)^{\frac{\mu-1}{2}} K_{\frac{\mu+1}{2}}(1+t) + (1+t)^{\frac{\mu+1}{2}} K'_{\frac{\mu+1}{2}}(1+t) \\
&= \frac{\mu+1}{2}(1+t)^{\frac{\mu-1}{2}} K_{\frac{\mu+1}{2}}(1+t) + (1+t)^{\frac{\mu+1}{2}} [-K_{\frac{\mu-1}{2}}(1+t) \\
&- \frac{\mu+1}{2(1+t)} K_{\frac{\mu+1}{2}}(1+t)] \\
&= -(1+t)^{\frac{\mu+1}{2}} K_{\frac{\mu-1}{2}}(1+t), \\
\lambda''(t) &= -\frac{\mu+1}{2}(1+t)^{\frac{\mu-1}{2}} K_{\frac{\mu-1}{2}}(1+t) - (1+t)^{\frac{\mu+1}{2}} K'_{\frac{\mu-1}{2}}(1+t) \\
&= -\frac{\mu+1}{2}(1+t)^{\frac{\mu-1}{2}} K_{\frac{\mu-1}{2}}(1+t) + (1+t)^{\frac{\mu+1}{2}} K_{\frac{\mu+1}{2}}(1+t) \\
&- \frac{\mu-1}{2}(1+t)^{\frac{\mu-1}{2}} K_{\frac{\mu-1}{2}}(1+t) \\
&= -\mu(1+t)^{\frac{\mu-1}{2}} K_{\frac{\mu-1}{2}}(1+t) + (1+t)^{\frac{\mu+1}{2}} K_{\frac{\mu+1}{2}}(1+t) \\
&= \frac{\mu}{1+t}\lambda'(t) + \lambda(t).
\end{aligned}$$

Hence, λ solves (9). Thanks to the identity (12), we know that $K_\nu(z) = K_{|\nu|}(z)$ for every $\nu \in \mathbb{R}$. Since $K_\nu(z)$ is real and positive for $z > 0$, also λ is real and positive, whereas λ' is negative. Then, we achieve (10), while exploiting Formula (14):

$$K_{\frac{\mu+1}{2}}(1+t) = K_{\frac{\mu-1}{2}}(1+t) = \sqrt{\frac{\pi}{2}}(1+t)^{-1/2}e^{-(1+t)} \times \left(1 + O\left(\frac{1}{1+t}\right)\right),$$

$$\begin{aligned}\lambda(t) &= (1+t)^{\frac{\mu+1}{2}}\sqrt{\frac{\pi}{2}}(1+t)^{-1/2}e^{-(1+t)} \times \left(1 + O\left(\frac{1}{1+t}\right)\right) \\ &= \sqrt{\frac{\pi}{2}}\,(1+t)^{\frac{\mu}{2}}e^{-(1+t)} \times \left(1 + O\left(\frac{1}{1+t}\right)\right) \\ &= -\lambda'(t).\end{aligned}$$

We obtain the relations (11). $\square$

4. Proof of the Theorem 1

As in [22], we introduce two cut-off functions:

$$\eta(t) = \begin{cases} 1 & \text{for } t \le \frac{1}{2}, \\ decreasing & \text{for } \frac{1}{2} < t < 1, \\ 0 & \text{for } t \ge 1, \end{cases} \quad \theta(t) = \begin{cases} 0 & \text{for } t < \frac{1}{2}, \\ \eta(t) & \text{for } t \ge \frac{1}{2}, \end{cases}$$

with:

$$|\eta'(t)| \le C, \; |\eta''(t)| \le C,$$

and

$$\eta_M(t) = \eta\left(\frac{t}{M}\right).$$

Then, we construct our test function as:

$$\Phi(t,x) = -\partial_t\left(\eta_M^{2p'}(t)\lambda(t)\phi(x)\right),$$

where $M \in (1, T]$ for any $T \in [1, T(\varepsilon)]$ and:

$$\phi(x) = \int_{S^{n-1}} e^{x\cdot\omega}d\sigma,$$

which satisfies:

$$0 < \phi(x) \le C(1+|x|)^{-\frac{n-1}{2}}e^{|x|}. \tag{15}$$

Then, we have:

$$\begin{aligned}\Phi(t,x) &= -\left(\partial_t\eta_M^{2p'}(t)\lambda(t)\phi(x) + \eta_M^{2p'}(t)\lambda'(t)\phi(x)\right) \\ &\ge \eta_M^{2p'}(t)|\lambda'(t)|\phi(x) \\ &\ge 0,\end{aligned} \tag{16}$$

where we used the fact that both $\eta_M(t)$ and $\lambda(t)$ are non-increasing functions.

Remark 1. *Note that the test function $\phi(x)$ admits some good properties. First, it is non-negative, and it satisfies:*

$$\Delta\phi = \phi.$$

Finally, it has the asymptotic behavior (15).

Multiplying the equation in (1) with $\Phi(t,x)$ and integrating over $[0,T]\times\mathbb{R}^n$, then by integration by parts, we obtain:

$$\begin{aligned}
&-\varepsilon\int_{\mathbb{R}^n}\lambda'(0)g(x)\phi(x)dx+\varepsilon\int_{\mathbb{R}^n}\lambda(0)f(x)\phi(x)dx\\
&+\int_0^T\int_{\mathbb{R}^n}|u_t|^p\eta_M^{2p'}(t)|\lambda'(t)|\phi(x)dxdt\\
\le&\int_0^T\int_{\mathbb{R}^n}u_t\eta_M^{2p'}(t)\left(\lambda''(t)-\frac{\mu}{1+t}\lambda'(t)-\lambda(t)\right)dxdt\\
&+\int_0^T\int_{\mathbb{R}^n}u_t\partial_t^2\eta_M^{2p'}(t)\lambda(t)\phi(x)dxdt\\
&+2\int_0^T\int_{\mathbb{R}^n}u_t\partial_t\eta_M^{2p'}(t)\lambda'(t)\phi(x)dxdt\\
&-\int_0^T\int_{\mathbb{R}^n}\frac{\mu}{1+t}u_t\partial_t\eta_M^{2p'}(t)\lambda(t)\phi(x)dxdt,
\end{aligned}\tag{17}$$

which yields for some positive constant $C_1=C(f,g,\mu)$ by combining (9) and (10):

$$\begin{aligned}
&C_1\varepsilon+\int_0^T\int_{\mathbb{R}^n}|u_t|^p\eta_M^{2p'}(t)|\lambda'(t)|\phi(x)dxdt\\
\le&\int_0^T\int_{\mathbb{R}^n}u_t\partial_t^2\eta_M^{2p'}(t)\lambda(t)\phi(x)dxdt\\
&+2\int_0^T\int_{\mathbb{R}^n}u_t\partial_t\eta_M^{2p'}(t)\lambda'(t)\phi(x)dxdt\\
&-\int_0^T\int_{\mathbb{R}^n}\frac{\mu}{1+t}u_t\partial_t\eta_M^{2p'}(t)\lambda(t)\phi(x)dxdt,\\
\triangleq& I+II+III.
\end{aligned}\tag{18}$$

We estimate the three terms I,II,III by the nonlinear term by using the Hölder inequality. For I, it follows from (11) and (15) that:

$$\begin{aligned}
I\le&CM^{-2}\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p\theta_M^{2p'}(t)|\lambda'(t)|\phi(x)dxdt\right)^{\frac{1}{p}}\\
&\times\left(\int_{\frac{M}{2}}^M\int_{|x|\le1+t}|\lambda'(t)|^{-\frac{1}{p-1}}|\lambda(t)|^{\frac{p}{p-1}}\phi(x)dxdt\right)^{\frac{1}{p'}}\\
\le&CM^{-2+\frac{n+\mu+1}{2}\frac{1}{p'}}\times\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p\eta_M^{2p'}(t)|\lambda'(t)|\phi(x)dxdt\right)^{\frac{1}{p}}.
\end{aligned}\tag{19}$$

In the same way for II and III, we have:

$$II:\le CM^{-1+\frac{n+\mu+1}{2}\frac{1}{p'}}\times\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p\eta_M^{2p'}(t)|\lambda'(t)|\phi(x)dxdt\right)^{\frac{1}{p}},\tag{20}$$

and:

$$III:\le CM^{-2+\frac{n+\mu+1}{2}\frac{1}{p'}}\times\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p\eta_M^{2p'}(t)|\lambda'(t)|\phi(x)dxdt\right)^{\frac{1}{p}}.\tag{21}$$

By combining (18)–(21), we obtain:

$$\begin{aligned}
&C_1\varepsilon+\int_0^T\int_{\mathbb{R}^n}|u_t|^p\eta_M^{2p'}(t)|\lambda'(t)|\phi(x)dxdt\\
\le&CM^{-1+\frac{n+\mu+1}{2}\frac{1}{p'}}\times\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p\theta_M^{2p'}(t)|\lambda'(t)|\phi(x)dxdt\right)^{\frac{1}{p}}.
\end{aligned}\tag{22}$$

If for a function $w(t,x)$, we set:

$$Y[w](M) = \int_1^M \left(\int_0^T \int_{\mathbb{R}^n} w(t,x)\theta_\sigma^{2p'}(t)dxdt \right)\sigma^{-1}d\sigma,$$

then as in [24], we have:

$$\begin{aligned} &Y\big[|u_t|^p|\lambda'(t)|\phi(x)\big](M) \\ =&\int_1^M \left(\int_0^T \int_{\mathbb{R}^n} |u_t|^p|\lambda'(t)|\phi(x)\theta_\sigma^{2p'}(t)dxdt \right)\sigma^{-1}d\sigma \\ \leq& C\log 2\int_0^T \int_{\mathbb{R}^n} \eta_M^{2p'}|u_t|^p|\lambda'(t)|\phi(x)dxdt. \end{aligned} \tag{23}$$

For simplicity, we denote $Y(M)$ for $Y[|u_t|^p|\lambda'(t)|\phi(x)](M)$, then by (23), we have:

$$Y(M) \leq C\log 2\int_0^T \int_{\mathbb{R}^n} \eta_M^{2p'}|u_t|^p|\lambda'(t)|\phi(x)dxdt, \tag{24}$$

$$\frac{d}{dM}Y(M) = M^{-1}\int_0^T \int_{\mathbb{R}^n} |u_t|^p|\lambda'(t)|\phi(x)\theta_M^{2p'}(t)dxdt. \tag{25}$$

Hence, by combining (22), (24), and (25), we know there exist positive constants C_2, C_3 such that:

$$MY'(M) \geq CM^{p-\frac{(n+\mu+1)(p-1)}{2}}(C_2\varepsilon + C_3Y(M))^p, \tag{26}$$

which leads to the lifespan estimate (6).

5. Proof for Theorem 2

For the problem with combined nonlinearity, we have to introduce another cut-off function:

$$\zeta(t) = \begin{cases} 0 & \text{for } t \leq \frac{1}{4}, \\ increasing & \text{for } \frac{1}{4} < t < \frac{1}{2}, \\ \theta(t) & \text{for } t \geq \frac{1}{2}. \end{cases}$$

Let:

$$\zeta_M(t) = \zeta\left(\frac{t}{M}\right), \quad \psi_M(t) = \zeta_M^k(t),$$

with $k > 0$, which will be determined later, and $M \in (1,T)$. It is easy to obtain:

$$\begin{aligned} &|\partial_t^2\psi_M(t)| \leq CM^{-2}\psi_M^{1-\frac{2}{k}}(t), \\ &|\partial_t\psi_M(t)| = CM^{-1}\psi_M^{1-\frac{1}{k}}(t). \end{aligned} \tag{27}$$

Multiplying the equation in (2) with $\psi_M(t,x)$ and integrating over $[0,T] \times \mathbb{R}^n$, then by integration by parts, we obtain:

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^n} (|u_t|^p + |u|^q)\psi_M(t)dxdt \\ =&\int_0^T \int_{\mathbb{R}^n} u\partial_t^2\psi_M dxdt + \int_0^T \int_{\mathbb{R}^n} \frac{\mu}{(1+t)^2}u\psi_M dxdt - \int_0^T \int_{\mathbb{R}^n} \frac{\mu}{1+t}u\partial_t\psi_M dxdt \\ \triangleq& IV + V + VI. \end{aligned} \tag{28}$$

We estimate IV as:

$$\begin{aligned} IV =& \int_0^T \int_{\mathbb{R}^n} u\partial_t^2 \psi_M dxdt \\ \leq & CM^{-2} \int_0^T \int_{\mathbb{R}^n} |u| \psi_M^{1-\frac{2}{k}-\frac{1}{2q'}} \psi_M^{\frac{1}{2q'}} dxdt \\ \leq & CM^{-2} \left(\int_0^T \int_{\mathbb{R}^n} |u|^q \psi_M^{q(1-\frac{2}{k}-\frac{1}{2q'})} dxdt \right)^{\frac{1}{q}} \left(\int_0^T \int_{|x|\leq 1+t} \psi_M^{\frac{1}{2}} dxdt \right)^{\frac{1}{q'}} \\ \leq & CR^{-2+\frac{n+1}{q'}} \left(\int_0^T \int_{\mathbb{R}^n} |u|^q \psi_M dxdt \right)^{\frac{1}{q}}, \end{aligned} \tag{29}$$

if we choose k large enough such that:

$$q\left(1 - \frac{2}{k} - \frac{1}{2q'}\right) \geq 1. \tag{30}$$

For V, although there is no derivative on the cut-off function ψ_M, note that:

$$supp\ \psi_M \subset \left[\frac{R}{4}, R\right],$$

We can obtain in a similar way as for IV:

$$V, VI \leq CR^{-2+\frac{n+1}{q'}} \left(\int_0^T \int_{\mathbb{R}^n} |u|^q \psi_M dxdt \right)^{\frac{1}{q}}, \tag{31}$$

where we need to choose $k > 0$ satisfying for VI:

$$q\left(1 - \frac{1}{k} - \frac{1}{2q'}\right) \geq 1. \tag{32}$$

By combining (28), (29) and (31), we have:

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^n} (|u_t|^p + |u|^q) \psi_M(t) dxdt \\ \leq & CR^{-2+\frac{n+1}{q'}} \left(\int_0^T \int_{\mathbb{R}^n} |u|^q \psi_M dxdt \right)^{\frac{1}{q}} \\ \leq & CM^{n-\frac{q+1}{q-1}} + \frac{1}{2} \int_0^T \int_{\mathbb{R}^n} |u|^q \psi_M dxdt, \end{aligned} \tag{33}$$

which yields:

$$\int_0^T \int_{\mathbb{R}^n} |u_t|^p \psi_M(t) dxdt \leq CM^{n-\frac{q+1}{q-1}}. \tag{34}$$

The next step is to use the test function:

$$\Psi(t, x) = -\partial_t \left(\eta_M^k(t) \lambda(t) \phi(x) \right) \geq 0$$

to obtain the lower bound of the nonlinear term. Multiplying the equation in (2) with $\Psi(t,x)$ and integrating over $[0,T]\times\mathbb{R}^n$, then by integration by parts, we obtain:

$$\begin{aligned}
&-\varepsilon\int_{\mathbb{R}^n}\lambda'(0)g(x)\phi(x)dx+\varepsilon\int_{\mathbb{R}^n}\lambda(0)f(x)\phi(x)dx\\
&+\int_0^T\int_{\mathbb{R}^n}|u_t|^p\eta_M^k(t)|\lambda'(t)|\phi(x)dxdt\\
=&\int_0^T\int_{\mathbb{R}^n}u_t\eta_M^k(t)\Big(\lambda''(t)-\frac{\mu}{1+t}\lambda'(t)-\lambda(t)\Big)dxdt\\
&+\int_0^T\int_{\mathbb{R}^n}u_t\partial_t^2\eta_M^k(t)\lambda(t)\phi(x)dxdt\\
&+2\int_0^T\int_{\mathbb{R}^n}u_t\partial_t\eta_M^k(t)\lambda'(t)\phi(x)dxdt\\
&-\int_0^T\int_{\mathbb{R}^n}\frac{\mu}{1+t}u_t\partial_t\eta_M^k(t)\lambda(t)\phi(x)dxdt,
\end{aligned}\tag{35}$$

which yields for some positive constant $C_2=C(f,g,\mu)$ by combining (9) and (10):

$$\begin{aligned}
&C_2\varepsilon+\int_0^T\int_{\mathbb{R}^n}|u_t|^p\eta_M^k(t)|\lambda'(t)|\phi(x)dxdt\\
\le&\int_0^T\int_{\mathbb{R}^n}|u_t|\partial_t^2\eta_M^k(t)\lambda(t)\phi(x)dxdt\\
&+2\int_0^T\int_{\mathbb{R}^n}|u_t|\partial_t\eta_M^k(t)|\lambda'(t)|\phi(x)dxdt\\
&+\int_0^T\int_{\mathbb{R}^n}\frac{\mu}{1+t}|u_t|\partial_t\eta_M^k(t)\lambda(t)\phi(x)dxdt,\\
\triangleq& I_c+II_c+III_c.
\end{aligned}\tag{36}$$

Furthermore, it is easy to obtain:

$$\begin{aligned}
|\partial_t^2\eta_M^k(t)|&\le CM^{-2}\theta_M^{k-2}(t),\\
|\partial_t\eta_M(t)|&=CM^{-1}\theta_M^{k-1}(t).
\end{aligned}\tag{37}$$

As above, the term I_c can be estimated as:

$$\begin{aligned}
I_c\le&CM^{-2}\Big(\int_0^T\int_{\mathbb{R}^n}|u_t|^p\theta_M^k(t)dxdt\Big)^{\frac{1}{p}}\\
&\times\Big(\int_{\frac{M}{2}}^M\int_{|x|\le1+t}\theta_M^{k-2p'}|\lambda(t)\phi(x)|^{p'}dxdt\Big)^{\frac{1}{p'}}\\
\le&CM^{-2}\Big(\int_0^T\int_{\mathbb{R}^n}|u_t|^p\theta_M^k(t)dxdt\Big)^{\frac{1}{p}}\\
&\times\Big(\int_{\frac{M}{2}}^M\int_{|x|\le1+t}(1+t)^{\frac{\mu p'}{2}}e^{-tp'}(1+r)^{n-1-\frac{n-1}{2}p'}e^{p'r}dxdt\Big)^{\frac{1}{p'}}\\
\le&CM^{-2+\frac{\mu}{2}+\frac{n}{p'}-\frac{n-1}{2}}\Big(\int_0^T\int_{\mathbb{R}^n}|u_t|^p\theta_M^k(t)dxdt\Big)^{\frac{1}{p}}.
\end{aligned}\tag{38}$$

In the same way, we have:

$$II_c,III_c\le CM^{-1+\frac{\mu}{2}+\frac{n}{p'}-\frac{n-1}{2}}\Big(\int_0^T\int_{\mathbb{R}^n}|u_t|^p\theta_M^k(t)dxdt\Big)^{\frac{1}{p}}.\tag{39}$$

Then, (36), (38) and (39), yield:

$$C\varepsilon \le CM^{-1+\frac{\mu}{2}+\frac{n}{p'}-\frac{n-1}{2}}\left(\int_0^T\int_{\mathbb{R}^n}|u_t|^p\theta_M^k(t)dxdt\right)^{\frac{1}{p}}, \tag{40}$$

which in turn yields:

$$C\varepsilon^p M^{n-\frac{(n+\mu-1)p}{2}} \le \int_0^T\int_{\mathbb{R}^n}|u_t|^p\theta_M^k(t)dxdt. \tag{41}$$

Since:

$$\theta_M^k(t) \le \psi_M(t) = \zeta_M^k(t),$$

we then conclude the lifespan (8) by combining (34) and (41).

Funding: This research received no external funding.

Acknowledgments: The author thanks Ning-An Lai for recommending the study of the topic of this work. He would also like to express his sincere thanks to the anonymous Reviewers for the helpful suggestions and comments.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Usamh, S.A.-A.; Borhari, A.H.; Kara, H.; Zaman, F.D. Symmetry analysis and exact solutions of the damped wave equation on the surface of the sphere. *Adv. Differ. Equ. Control Process.* **2017**, *17*, 321–333.
2. D'Abbicco, M. The threshold of effective damping for semilinear wave equations. *Math. Methods Appl. Sci.* **2015**, *38*, 1032–1045. [CrossRef]
3. D'Abbicco, M. Small data solutions for the Euler-Poisson-Darboux equation with a power nonlinearity. *J. Differ. Equ.* **2021**, *286*, 531–556. [CrossRef]
4. D'Abbicco, M.; Lucente, S. A modified test function method for damped wave equations. *Adv. Nonlinear Stud.* **2013**, *13*, 867–892. [CrossRef]
5. D'Abbicco, M.; Lucente, S. NLWE with a special scale invariant damping in odd space dimension. In Proceedings of the 10th AIMS Conference on Dynamical Systems, Differential Equations and Applications AIMS Proceedings, Madrid, Spain, 7–11 July 2014; pp. 312–319.
6. D'Abbicco, M.; Lucente, S.; Reissig, M. A shift in the Strauss exponent for semilinear wave equations with a not effective damping. *J. Differ. Equ.* **2015**, *259*, 5040–5073. [CrossRef]
7. Ikeda, M.; Sobajima, M. Life-span of solutions to semilinear wave equation with time-dependent critical damping for specially localized initial data. *Math. Ann.* **2018**, *372*, 1017–1040. [CrossRef]
8. Lai, N. Weighted $L^2 - L^2$ estimate for wave equation and its applications. *Adv. Stud. Pure Math.* **2020**, *85*, 269–279.
9. Lai, N.; Schiavone, N.; Takamura, H. Heat-like and wave-like lifespan estimates for solutions of semilinear damped wave equations via a Kato's type lemma. *J. Differ. Equ.* **2020**, *269*, 11575–11620. [CrossRef]
10. Kato, M.; Sakuraba, M. Global existence and blow-up for semilinear damped wave equations in three space dimensions. *Nonlinear Anal.* **2019**, *182*, 209–225. [CrossRef]
11. Lai, N.; Takamura, H.; Wakasa, K. Blow-up for semilinear wave equations with the scale invariant damping and super-Fujita exponent. *J. Differ. Equ.* **2017**, *263*, 5377–5394. [CrossRef]
12. Lai, N.; Zhou, Y. Global existence for semilinear wave equations with scaling invariant damping in 3-D. *Nonlinear Anal.* **2021**, *210*, 112392. [CrossRef]
13. Palmieri, A. A global existence result for a semilinear scale-invariant wave equation in even dimension. *Math. Methods Appl. Sci.* **2019**, *42*, 2680–2706. [CrossRef]
14. Wakasa, K. The lifespan of solutions to semilinear damped wave equations in one space dimension. *Commun. Pure Appl. Anal.* **2016**, *15*, 1265–1283. [CrossRef]
15. Lai, N.; Takamura, H. Nonexistence of global solutions of nonlinear wave equations with weak time dependent damping related to Glasseys conjecture. *Differ. Integral Equ.* **2019**, *32*, 37–48.
16. Palmieri, A.; Tu, Z. A blow-up result for a semilinear wave equation with scale-invariant damping and mass and nonlinearity of derivative type. *Calc. Var. Partial. Differ. Equ.* **2021**, *60*, 72. [CrossRef]
17. Hamouda, M.; Hamza, M.A. Improvement on the blow-up of the wave equation with the scale-invariant damping and combined nonlinearities. *Nonlinear Anal. Real World Appl.* **2021**, *59*, 103275. [CrossRef]
18. Hamouda, M.; Hamza, M.A. Blow-up for wave equation with the scale-invariant damping and combined nonlinearities. *Math. Meth. Appl. Sci.* **2021**, *44*, 1127–1136. [CrossRef]

19. Mitidieri, E.; Pokhozhaev, S.I. A priori estimates and blow-up of solutions to nonlinear partial differential equations and inequalities. *Tr. Mat. Instituta Im. VA Steklova* **2001**, *234*, 3–383.
20. Samarsky, A.A.; Galaktionov, V.A.; Kurdyumov, S.P. *Regimes with Peaking in Problems for Quasilinear Parabolic Equations*; Nauka: Moskow, Russia, 1987; 476p.
21. Sidorov, D.N. Existence and blow-up of Kantorovich principal continuous solutions of nonlinear integral equations. *Differ. Equ.* **2014**, *50*, 1217–1224. [CrossRef]
22. Ikeda, M.; Sobajima, M.; Wakasa, K. Blow-up phenomena of semilinear wave equations and their weakly coupled systems. *J. Differ. Equ.* **2019**, *267*, 5165–5201. [CrossRef]
23. Lai, N.; Tu, Z. Strauss exponent for semilinear wave equations with scattering space dependent damping. *J. Math. Anal. Appl.* **2020**, *489*, 124189. [CrossRef]
24. Lai, N.; Schiavone, M. Lifespan estimate for semilinear generalized Tricomi equations. *arXiv* **2007**, arXiv:16003v2.
25. Abramowitz, M.; Stegun, I.A. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Table*; Dover: New York, NY, USA, 1965; Volume 2172.