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Editorial

Recent Progress in Studies of Stability of Numerical Schemes

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Abstract: Applications and modeling of various phenomena in all areas of scientific research require finding numerical solutions for differential, partial differential, integral, or integro-differential equations. In addition to proving theoretical convergence and giving error estimates, stability of numerical methods for such operator equations is a fundamental property that it is necessary for the method to produce a valid solution. This Special Issue focuses on new theoretical and numerical studies concerning the techniques used for proving stability or instability of numerical schemes, which extend or improve known results. It also includes applications to non-linear physical, chemical, and engineering systems, arising in dynamics of waves, diffusion, or transport problems.

Keywords: numerical stability; Ulam–Hyers stability; split–step exponential scheme; numerical schemes for stochastic differential equations; difference schemes with interpolation; KPZ equation; fractional nonlinear Schrödinger equation; method of fundamental solutions

1. Introduction

In many applications, the model leads to operator equations with initial, boundary, bilocal or mixed conditions. Solving such equations analytically is, most of the time, nearly impossible. Instead, numerical methods are employed, which produce an approximate solution at a discrete set of points. In deriving an approximate method, one of the most important tasks is the study of *stability* of the scheme, which directly affects its applicability. It is necessary to know under what conditions the method is stable, what type of stability is involved and for what kind of equations or parameters some stability property holds.

There are various concepts of numerical stability for different types of equations.

For ODE's, A-stability can be considered, which is related to some concept of stability in the dynamical systems sense. Approaches based on continuous dependence on data and on Grönwall-type inequalities, in differential or integral form, can also be employed to establish stability results.

The von Neumann analysis is the standard tool for establishing the stability of a numerical scheme for PDE's. However, recently, a number of studies have addressed the stability of numerical methods (e.g., the split-step method for dispersive/parabolic equations, or the method of characteristics for hyperbolic equations) with approaches that go beyond the von Neumann analysis.

For functional equations, Ulam–Hyers–Rassias stability (possibly in conjunction with Mittag–Leffler's theorem) is analyzed. Well-known results in Banach spaces have recently been extended to b-metric spaces.

The purpose of this Special Issue was to bring together new theoretical and numerical studies concerning techniques used for proving the stability or instability of numerical schemes, which extend or improve known results.

In response to our call, we had 11 submissions, of which 6 were accepted. The accepted articles were co-authored by 18 researchers from 8 countries (China, France, Hungary, India, Iran, Romania, Slovakia, and USA). This Special Issue consists of six original articles and one editorial. The articles cover stability analysis for various types of equations,

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Citation: Lakoba, T.I.; Micula, S. Recent Progress in Studies of Stability of Numerical Schemes. *Symmetry* **2022**, *14*, 2692. https://doi.org/10.3390/sym14122692

Received: 13 December 2022
Accepted: 15 December 2022
Published: 19 December 2022

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such as stochastic differential equations, transport equations with delay and advanced arguments, KPZ equations, fractional non-linear Schrödinger equations, or wave–current interaction equations. In what follows, we give a brief description of the contributions to this Special Issue.

2. Overview

In [1], the authors present fixed-point results for Subrahmanyan contraction in the setting of a b-metric space (also called a quasi-metric space). The b-metric is a generalization of the notion of a metric; therefore, the results obtained in the paper generalize classical fixed-point results for metric spaces. The contribution of this research is not only in that the space is considered as non-classical, but also in that the authors consider multi-valued operators (with the example in point being the Subrahmanyan contraction). The results obtained are used to derive a local Ulam–Hyers stability result for the fixed point inclusion. Explicit conditions for the existence of a strict fixed point of a multi-valued operator are also given.

In [2], the authors introduce two split-step schemes, based on the explicit exponential Milstein scheme, for solving stiff stochastic differential equations. Under the Lipschitz condition and linear-growth bounds, it is shown theoretically and confirmed numerically that the proposed explicit schemes have the convergence order 1.0 in the mean-square sense. Additionally, the new schemes proposed are proven to be stable in the asymptotic mean-square sense for a certain class of two-dimensional systems driven by two commuting noise terms.

In [3], the authors prove a maximum principle for a delay-differential equation (DDE) representing one-dimensional transport system with delayed and advanced arguments. DDEs arise in various fields of engineering and science, such as control theory, mathematical biology, and climate modeling. As an application of the maximum principle, the stability of the solution of the DDE in question is established. It is also shown that a discontinuity in the initial condition will propagate in such a system. This system is further analyzed numerically with finite-difference schemes based on linear interpolation; both conditionally and unconditionally stable versions of such schemes are presented. An unconditionally stable numerical scheme is applied, as an illustration, to a DDE with symmetric delay arguments and variable delays.

In [4], the authors study numerically the Kardar–Parisi–Zhang (KPZ) equation in 1-space and 1-time dimensions and with an additive stochastic term. Specifically, they compare the performance of several schemes: a recently proposed version of the leapfrog–hopscotch (LH) method, a standard forward-time centered-space scheme, and the Heun method. Due to a special symmetry of the time–space discretization, the new LH method is shown to clearly outperform the other two numerical methods.

In [5], the authors investigate stability and dynamics of the plane wave solutions of the fractional non-linear Schrödinger (fNLS) equation, where the long-range dispersion is described by the fractional Laplacian $(-\Delta)^{\alpha/2}$. The linear stability analysis is used to establish stable and unstable regimes depending on the signs of non-linearity and the value of α. The split-step Fourier spectral (SSFS) method is then used to simulate the non-linear stage of the plane waves dynamics. In agreement with earlier studies, a decrease of α from 2 (the "classical" Laplacian) to 1 leads to the solution evolving towards an increasingly localized pulse, existing on the background of a "sea" of small-amplitude dispersive waves. For the focusing fNLS with $\alpha \leq 1$, the solution undergoes collapse. (There is no collapse for the defocusing fNLS.) It is also shown that for initial conditions that are traveling, as opposed to standing, plane waves, the onset of collapse is delayed with the increase in the wave's initial "speed" parameter. Additionally, a stability condition on the time step of the SSFS is derived.

In [6], the authors discuss stability of the numerical method of fundamental solutions (MFS), where boundary conditions are treated by the generating–absorbing boundary conditions, for a system that describes wave-current interactions. They perform numerical simulations that cover a wide range of system parameters: types of currents (coplanar current, no current, and opposing current), and water depths. The accuracy and stability of the MFS is evaluated for different locations and numbers of source points.

Author Contributions: Conceptualization, T.I.L. and S.M.; methodology, T.I.L. and S.M.; formal analysis, T.I.L. and S.M.; writing—original draft preparation, T.I.L. and S.M.; writing—review and editing, T.I.L. and S.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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