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Type I Shapovalov Wave Spacetimes in the Brans–Dicke Scalar-Tensor Theory of Gravity

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Abstract: Exact solutions for Shapovalov wave spacetimes of type I in Brans–Dicke's scalar-tensor theory of gravity are constructed. Shapovalov wave spacetimes describe gravitational wave models that allow for the the separation of wave variables in privileged coordinate systems. In contrast to general relativity, the vacuum field equations of the Brans–Dicke scalar-tensor theory of gravity lead to exact solutions for type I Shapovalov spaces, allowing for the the construction of observational tests to detect such wave disturbances. Furthermore, the equations for the trajectories of the test particles are obtained for the models considered.

Keywords: Brance–Dicke theory of gravity; gravitational waves; Hamilton–Jacobi equation; Killing fields; Shapovalov spacetimes; test particle trajectories

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1. Introduction

The recently proposed classification of Shapovalov wave spacetimes [1] provides additional mathematical tools for the construction of exact integrable models of gravitational waves, including primordial gravitational waves in Bianchi universes [2,3] and other exact models of plane gravitational waves [4,5]. Recent advances in gravitational wave astronomy in the detection of gravitational waves and the resulting astrophysical information [6–8] have also increased interest in the mathematical aspects of gravitational wave research.

The relevance of these lines of research is also related to the importance of the discovery of primordial gravitational waves for constructing a theory of the early universe and the possible confirmation of the stage of inflation predicted by some theories. The interest in gravitational wave research is also due to possible secondary physical effects in a gravitational wave, such as the formation of black holes [9,10], the capture of astrophysical objects by a gravitational wave [11], gravitational wave lensing, and other effects.

The Shapovalov wave spacetimes allow for the presence of special symmetries of the Hamilton–Jacobi equation, leading to the existence of privileged coordinate systems, where the equations of motion of test particles and the eikonal equation for radiation in the Hamilton–Jacobi formalism allow for the exact integration by the method of separation of variables. Moreover, among the separable variables on which the spacetime metric depends (non-ignored variables), there are wave variables along which the spacetime interval vanishes. The gravitational wave velocity, according to the gravitational wave detection data from neutron star mergers [12], equals the velocity of light. Therefore, the existence of the possibility of separating null variables indicates the wave nature of such spacetime models, and the use of wave variables in describing gravitational waves is experimentally justified.

Shapovalov wave spaces in the four-dimensional cases include three main types according to the number of commuting killing vectors they allow and, accordingly, in privileged coordinate systems, the number of killing vectors determines the number of non-ignorable variables on which the spacetime metric depends. For privileged coordinate systems, Vladimir Shapovalov obtained the structure of the space metric [13–15], which leads to the separation of variables in the Hamilton–Jacobi equation, which allows one to construct exact integrable models of gravitational waves in various theories of gravity in which test particles move along geodesic curves of spacetime.

Currently, there is interest in studying the properties of modified theories of gravity that could provide corrections to Einstein's theory of gravity in the early stages of the universe (quantum corrections, quadratic theories, $f(R)$-theories of gravity, theories with a scalar field, etc.) and could provide a theoretical description of the phenomena of "inflation", "dark matter", and "dark energy" [16–19]. Therefore, the use of additional mathematical tools for the comparative analysis of exact models of gravitational waves in different theories of gravity also enables the selection of the most realistic theories and models.

In this direction, one of the first models of modified theories of this type was the Brans–Dicke scalar-tensor theory of gravity [20]. Scalar-tensor theories have taken a firm place in theoretical research on gravity and cosmology [21], determined by attempts to use an additional scalar field to describe various possible scenarios for the dynamics of the universe and other theoretical constructions [22–25].

In this work, Shapovalov wave spaces of type I are considered in the scalar-tensor theory of Brans–Dicke gravity. An additional scalar field in the theory complicates the field equations but also provides additional possibilities. The resulting equation for a Klein–Gordon equation-type scalar field also requires further investigation. As Shapovalov [13,14] showed, the scalar equation can be integrated by separating the variables in the same privileged coordinate systems as the Hamilton–Jacobi equation. Therefore, the use of Shapovalov wave spaces also provides additional opportunities for the exact solution of the scalar equation [26–29]. Recently, new results have also been obtained in the study of the symmetries of the Klein–Gordon–Fock equations. [30–33].

Type I Shapovalov metrics are the most general gravitational wave models of Shapovalov spacetimes since they depend on three variables in the privileged coordinate system, including the wave variable. Note that metrics of this type lead to degeneracy when using Einstein's vacuum equations since the number of non-ignorable variables in the metric decreases as the vacuum field equations are solved, and the space becomes either a Shapovalov wave space of type II with two non-ignorable variables in the metric, or a space of type III with one wave variable in the metric. The preservation of type I metrics is possible if additional fields and matter exist in the models under consideration. The existence of non-degenerate type I Shapovalov models in the Brans–Dicke theory of gravity and the possibility of detecting such gravitational waves in observations would therefore be additional evidence for the scalar-tensor theories of gravity.

The authors investigate exact wave models of spacetime to find out the differences that may arise in these gravitational wave models for Einstein's theory and for modified theories of gravity. Shapovalov type I wave spaces have internal symmetries that relate them to mathematical models of gravitational waves, but still, these models of gravitational waves do not lead to the appearance of exact solutions of Einstein's vacuum equations, unlike the type II and III models. Additionally, in the modified Brans–Dicke theory of gravity, exact wave solutions occur for certain spaces, as we show below. This allows us to observe differences in the detection of gravitational waves by existing and future detectors. This is also important for analyzing the stochastic gravitational wave noise detected by existing gravitational wave detectors. In addition, primordial gravitational waves of this type could leave an imprint on the cosmic microwave background. Detecting "traces" of such gravitational waves would argue that the scalar-tensor gravity theories are more realistic than Einstein's theory. The possibility of the exact integration of the equations of motion of the test particles and the equations for geodesic deviation in these models of gravitational waves allows gravitational wave detectors to more efficiently analyze and select such disturbances in signals with high noise.

2. Gravitational-Wave Spacetimes of Shapovalov

Shapovalov wave spaces allow for the exact integration of the equations of motion of test particles in the Hamilton–Jacobi formalism

$$g^{\alpha\beta}\frac{\partial S}{\partial x^{\alpha}}\frac{\partial S}{\partial x^{\beta}}=m^2c^2, \qquad \alpha,\beta,\gamma=0,1,2,3. \tag{1}$$

with the separation of non-ignorable wave variables (the space metric depends on these variables) along which the spacetime interval vanishes [1]. Here, S is the action function of the test particle, m is the mass of the particle, and c is the speed of light. Let us also choose a system of units in which the speed of light is equal to one.

Type I Shapovalov spacetimes admit a killing vector contained in the so-called "complete set" of vectors and second-rank killing tensors that determine integrals of motion linearly and quadratically in the momenta. In a privileged coordinate system where the Equation (1) allows for the complete separation of variables, the type I Shapovalov spacetime metric depends on three variables, including the null wave variable denoted by x^0. Another null variable x^1 is ignored (cyclic) and is not included in the metric.

Consider a model with a type I Shapovalov wave space metric, which can be written in a privileged coordinate system in the following form:

$$g^{\alpha\beta}=\frac{1}{f_0}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{W} & 0 \\ 0 & 0 & 0 & \frac{1}{W} \end{pmatrix}, \tag{2}$$

where

$$f_0=f_0(x^0), \quad W(x^2,x^3)=t_3(x^3)-t_2(x^2). \tag{3}$$

The scalar curvature R takes the following form:

$$R(x^0,x^2,x^3)=\frac{f_0}{(t_2-t_3)^3}\Big((t_2-t_3)\big(t_2{}''-t_3{}''\big)-t_2{}'^2-t_3{}'^2\Big), \tag{4}$$

where the top prime means the ordinary derivative with respect to the variable on which the function depends.

The nonzero components of the Riemann curvature tensor have the following form:

$$R_{0202}=R_{0303}=\frac{\big(f_0{}'^2-2f_0f_0{}''\big)(t_2-t_3)}{4f_0{}^3}, \tag{5}$$

$$R_{2323}=\frac{(t_2-t_3)(t_2{}''-t_3{}'')-t_2{}'^2-t_3{}'^2}{2f_0(t_2-t_3)}. \tag{6}$$

Moreover, all nonzero components of Weyl's conformal curvature tensor $C_{\alpha\beta\gamma\delta}$ are proportional to the scalar curvature R, and when the scalar curvature vanishes, they also vanish, leading to a conformally flat spacetime.

Note that the ability to exactly integrate the Hamilton–Jacobi equations for Shapovalov wave spaces makes it possible to obtain the trajectories of test particles, as well as to find exact solutions for the geodesic deviation equation and the exact form of the tidal accelerations in these spaces. These possibilities allow one to determine all of the physical effects in gravitational waves.

3. Brans–Dicke Scalar-Tensor Theory of Gravity

The Lagrange function for the Brans–Dicke theory of gravity (BDT) can be written in the following form [21]:

$$L = \phi\,(R + 2\Lambda) - \frac{\omega}{\phi}\,g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + 16\pi\kappa L_{matter}, \tag{7}$$

where ϕ is a scalar field, ω is a constant parameter of the theory, and Λ is a cosmological constant.

The field equations of the Brans–Dicke scalar-tensor gravity theory with cosmological constant Λ can be written as follows:

$$G_{\alpha\beta} = \frac{8\pi}{\Phi}T_{\alpha\beta} + \Lambda g_{\alpha\beta} + \frac{\omega}{\Phi^2}(\Phi_{,\alpha}\Phi_{,\beta} - \frac{1}{2}g_{\alpha\beta}g^{\gamma\delta}\Phi_{,\gamma}\Phi_{,\delta}) + \frac{1}{\Phi}(\Phi_{;\alpha\beta} - g_{\alpha\beta}\Phi^{;\gamma}{}_{;\gamma}), \tag{8}$$

$$\frac{3+2\omega}{\Phi}\,\Phi^{;\gamma}{}_{;\gamma} = \frac{8\pi}{\Phi}T^{\gamma}{}_{\gamma} + 2\Lambda. \tag{9}$$

Here, the comma denotes the ordinary partial derivative, and the semicolon denotes the covariant derivative.

To obtain exact gravitational wave solutions, consider the vacuum model $T_{\alpha\beta} = 0$. It is convenient to pass to the form of the scalar field $\Phi = e^{\phi}$; then, the BDT equations will take the following more compact form:

$$G_{\alpha\beta} + g_{\alpha\beta}(\phi^{;\gamma}{}_{;\gamma} + (1+\omega/2)\phi^{,\gamma}\phi_{,\gamma} - \Lambda) - \phi_{;\alpha\beta} - (\omega+1)\phi_{,\alpha}\phi_{,\beta} = 0, \tag{10}$$

$$(\omega + 3/2)(\phi^{;\gamma}{}_{;\gamma} + \phi^{,\gamma}\phi_{,\gamma}) - \Lambda = 0. \tag{11}$$

Note that the higher derivatives of the scalar field ϕ can be found in Equation (10) and substituted into the scalar Equation (11) to reduce the order of this equation.

The equations of the scalar-tensor theory of gravity of Brans–Dicke for the considered metric lead to the following form of field equations:

$$\phi_{,00} = \frac{f_0{}'}{f_0}\phi_{,0} + \frac{3}{2}\frac{f_0{}'^2}{f_0{}^2} - (\omega+1)\phi_{,0}{}^2 + \frac{f_0{}''}{f_0}, \tag{12}$$

$$\phi_{,01} = (\omega+1)\left(\frac{2\Lambda f_0}{3+2\omega} - \phi_{,0}\phi_{,1}\right), \tag{13}$$

$$\phi_{,11} = -(\omega+1)\phi_{,1}{}^2, \tag{14}$$

$$\phi_{,1\mu} = -(\omega+1)\phi_{,1}\phi_{,\mu}, \qquad \mu,\nu = 2,3, \tag{15}$$

$$\phi_{,0\mu} = \phi_{,\mu}\left(\frac{1}{2}\frac{f_0{}'}{f_0} - (\omega+1)\phi_{,0}\right), \tag{16}$$

$$\begin{aligned}\phi_{,\mu\mu} = {} & W\left(\frac{1}{3}(R+\Lambda)f_0 - \frac{1}{2}\frac{f_0{}'}{f_0}\phi_{,1} + \frac{1}{3}\omega\phi_{,0}\phi_{,1}\right) - \frac{5\omega+6}{6}\phi_{,2}{}^2 \\ & + \omega\phi_{,3}{}^2 - (-1)^{\mu}\frac{\phi_{,2}t_2{}' + \phi_{,3}t_3{}'}{2W},\end{aligned} \tag{17}$$

$$\phi_{,23} = \frac{\phi_{,2}t_3{}' + \phi_{,3}t_2{}'}{2W} - (\omega+1)\phi_{,2}\phi_{,3}, \tag{18}$$

$$W((2\omega+3)R + 2(4\omega+3)\Lambda)f_0 = \omega(2\omega+3)(2W\phi_{,0}\phi_{,1} + \phi_{,2}{}^2 + \phi_{,3}{}^2). \tag{19}$$

Equation (19) arises as a consequence of scalar Equation (9) when the higher derivatives of the scalar field ϕ from the other field equations of the theory are substituted into it.

3.1. Scalar Field of the Form $\phi = \phi(x^0, x^1, x^2, x^3)$ and $\phi_{,1} \neq 0$.

Assume that the scalar field ϕ depends on all variables, including the null variable x^1, on which the spacetime metric does not depend.

In this case, the system of BDT equations is quite cumbersome, but it allows one to find the conditions for the compatibility of field equations, which leads to relations of the following form:

$$\phi_{,2}(2f_0f_0'' - 3f_0'^2) = \phi_{,3}(2f_0f_0'' - 3f_0'^2) = 0. \tag{20}$$

The requirement $\phi_{,1} \neq 0$ imposes strict conditions on the dependence of the scalar field on the variables x^2 and x^3, so the case of $\phi_{,2}{}^2 + \phi_{,3}{}^2 \neq 0$ in (20) leads to the degeneration of the space into a flat one for any ω.

Thus, the scalar field for $\phi_{,1} \neq 0$ can only have the following form:

$$\phi = \phi(x^0, x^1).$$

In this case, the system of BDT equations takes the following form:

$$\phi_{,00} = \frac{f_0'}{f_0}\phi_{,0} + \frac{3}{2}\frac{f_0'^2}{f_0^2} - (\omega+1)\phi_{,0}{}^2 + \frac{f_0''}{f_0} \tag{21}$$

$$\phi_{,01} = \frac{\Lambda}{2}f_0 - \frac{\omega+2}{2}\phi_{,0}\phi_{,1} - \frac{1}{4}\frac{f_0'}{f_0}\phi_{,1} \tag{22}$$

$$\phi_{,11} = -(\omega+1)\phi_{,1}{}^2 \tag{23}$$

$$(2\omega f_0\phi_{,0} - f_0')(2\omega+3)\phi_{,1} - 2(2\omega+1)\Lambda f_0^2 = 0 \tag{24}$$

$$(R+\Lambda)f_0 + \omega\phi_{,0}\phi_{,1} - \frac{3}{2}\frac{f_0'}{f_0}\phi_{,0} = 0 \tag{25}$$

Consider Equation (23). For $\omega \neq -1$, the scalar field is in the form

$$\phi(x^0, x^1) = \frac{1}{\omega+1}\ln\Big(a(x^0)x^1 + b(x^0)\Big), \tag{26}$$

however, as a result of substitution into other equations of the system and the separation of the coefficients of x^1, it turns out that $a(x^0) = 0$, and the scalar field does not depend on x^1. Consequently, a solution that satisfies the stated requirements is possible only for $\omega = -1$.

The integration of the Equation (23) and the separation of the coefficients for x^1 for $\omega = -1$ leads to the scalar field $\phi = \alpha x^1 + b(x^0)$, $\alpha - const$. Substitution into other equations of the system allows one to find all of the unknown functions.

The final solution for the case $\phi = \phi(x^0, x^1)$ is as follows:

$$f_0 = e^{\beta x^0}, \tag{27}$$

$$(t_2')^2 = 2\alpha\beta\, t_2{}^3 + \lambda\, t_2{}^2 + \gamma\, t_2 + \delta, \tag{28}$$

$$(t_3')^2 = -\Big(2\alpha\beta\, t_3{}^3 + \lambda\, t_3{}^2 + \gamma\, t_3 + \delta\Big) \tag{29}$$

$$\omega = -1, \qquad \phi(x^0, x^1) = \frac{\Lambda}{\alpha\beta}e^{\beta x^0} - \frac{\beta}{2}x^0 + \alpha x^1, \tag{30}$$

where α, β, γ, δ, and λ are constants.

In the resulting solution, we can set $\alpha = \beta = 1$ by scaling transformations; then,

$$f_0 = \exp x^0, \qquad (t_\mu')^2 = (-1)^\mu\Big(2t_\mu{}^3 + \lambda t_\mu{}^2 + \gamma t_\mu + \delta\Big), \tag{31}$$

$$\omega = -1, \qquad \phi(x^0, x^1) = \Lambda e^{x^0} - x^0/2 + x^1. \tag{32}$$

The cosmological constant Λ remains arbitrary, the Riemann curvature tensor R_{ijkl} and the Weyl conformal curvature tensor C_{ijkl} do not vanish, and for the scalar curvature R we obtain:

$$R = e^{-x^0}, \qquad C_{ijkl} \neq 0. \tag{33}$$

Thus, a nontrivial solution for Shapovalov type I wave spaces in the Brans–Dicke theory is obtained with a scalar field depending on null variables, with a cosmological constant and an exponential conformal factor depending on the wave variable. Note that a vacuum solution of this type does not arise in general relativity [1].

3.2. Scalar Field of the Form $\phi = \phi(x^0, x^2, x^3)$.

Consider the case where the scalar field ϕ does not depend on the "ignored" null variable x^1, on which the metric also does not depend. The system of BDT Equations (10) and (11), resolved in terms of higher derivatives, has the following form:

$$\phi_{,00} = \frac{f_0{}'}{f_0}\phi_{,0} + \frac{3}{2}\frac{f_0{}'^2}{f_0{}^2} - \frac{f_0{}''}{f_0} - (\omega+1)\phi_{,0}{}^2, \tag{34}$$

$$\phi_{,02} = \phi_{,2}\left(\frac{1}{2}\frac{f_0{}'}{f_0} - (\omega+1)\phi_0\right), \tag{35}$$

$$\phi_{,03} = \phi_{,3}\left(\frac{1}{2}\frac{f_0{}'}{f_0} - (\omega+1)\phi_0\right), \tag{36}$$

$$\phi_{,22} = \Lambda W f_0 + \frac{\omega\phi_{,3}{}^2 - (\omega+2)\phi_{,2}{}^2}{2} - \frac{\phi_{,2}t_2{}' + \phi_{,3}t_3{}'}{2W}, \tag{37}$$

$$\phi_{,22} + \phi_{,33} = 2\Lambda W f_0 - (\phi_{,2}{}^2 + \phi_{,3}{}^2) \tag{38}$$

$$\phi_{,23} = -(\omega+1)\phi_{,2}\phi_{,3} + \frac{\phi_{,2}t_3{}' - \phi_{,3}t_2{}'}{2W}, \tag{39}$$

$$(2\Lambda - R)W f_0 + \omega(\phi_{,2}{}^2 + \phi_{,3}{}^2) = 0, \tag{40}$$

$$R = \frac{t_2{}'^2 + t_3{}'^2 + W(t_2{}'' - t_3{}'')}{W^3 f_0}, \tag{41}$$

$$\Lambda(\omega+1) = 0. \tag{42}$$

The study of the compatibility of the subsystem of Equations (34)–(36) leads to relations that are further used to classify possible solutions:

$$\phi_{,2}(2f_0 f_0{}'' - 3f_0{}'^2) = \phi_{,3}(2f_0 f_0{}'' - 3f_0{}'^2) = 0. \tag{43}$$

The compatibility Equation (43) lead to the following two possible cases:

(1) $\phi_{,2}\,\phi_{,3} \neq 0, \quad 2f_0 f_0{}'' - 3f_0{}'^2 = 0;$
(2) $\phi_{,2} = \phi_{,3} = 0, \quad \phi = \phi(x^0).$

In the following, we will consider these cases separately.

3.2.1. Scalar Field of the Form $\phi = \phi(x^0, x^2, x^3)$ and $\phi_{,2}\,\phi_{,3} \neq 0$.

From the compatibility conditions, we obtain $f_0 = 1/\left(x^0\right)^2$, and now we can integrate Equations (34)–(36). In this case, two cases are possible depending on the values of the constant ω:

(A) $\omega = -1$, $\phi = \xi(x^2, x^3)/x^0$. Substitution into Equation (41) allows us to separate the coefficients at x^0. As a result, we obtain $R = 0$, which, in view of (6) and (43), means the degenerate case since spacetime becomes flat in this case.

(B) $\omega \neq -1$, $\Lambda = 0$, $\phi = \frac{1}{\omega+1} \ln \frac{\xi(x^2,x^3)}{x^0}$, where the function $\xi(x^2, x^3)$ must satisfy the following equations:

$$\xi_{,22} = \frac{\omega}{2(\omega+1)} \frac{\xi_{,2}{}^2 + \xi_{,3}{}^2}{\xi} - \frac{\xi_{,2} t_2{}' + \xi_{,3} t_3{}'}{2W}, \tag{44}$$

$$\xi_{,33} = \frac{\omega}{2(\omega+1)} \frac{\xi_{,2}{}^2 + \xi_{,3}{}^2}{\xi} + \frac{\xi_{,2} t_2{}' + \xi_{,3} t_3{}'}{2W}, \tag{45}$$

$$\xi_{,23} = \frac{\xi_{,2} t_3{}' - \xi_{,3} t_2{}'}{2W}, \tag{46}$$

$$\frac{\omega}{(\omega+1)^2} \frac{\xi_{,2}{}^2 + \xi_{,3}{}^2}{\xi^2} = \frac{t_2{}'^2 + t_3{}'^2 + W(t_2{}'' - t_3{}'')}{W^2}. \tag{47}$$

Let us consider the solution when the scalar field ϕ admits the separation of variables:

$$\xi(x^2, x^3) = \xi_2(x^2)\, \xi_3(x^3), \qquad \xi_2{}'\, \xi_3{}' \neq 0. \tag{48}$$

From here, we obtain the relations:

$$\xi_{,2} = \xi \frac{d}{dx^2} \log \xi_2, \qquad \xi_{,3} = \xi \frac{d}{dx^3} \log \xi_3. \tag{49}$$

Substituting the relations (49) into Equation (46) and separating the variables, we obtain

$$\xi_2 t_2{}' / \xi_2{}' - 2t_2 = \xi_3 t_3{}' / \xi_3{}' - 2t_3 = \text{const} = 2c. \tag{50}$$

In this way,

$$2 \frac{d \log \xi_2}{dx^2} = \frac{d \log (t_2 + c)}{dx^2}, \qquad 2 \frac{d \log \xi_3}{dx^3} = \frac{d \log (t_3 + c)}{dx^3}. \tag{51}$$

From this, we obtain relations relating the functions t_μ and ξ_μ of the following form:

$$(\xi_2)^2 = a(t_2 + c), \qquad (\xi_3)^2 = b(t_3 + c), \tag{52}$$

where a, b, and c are constants.

Since the functions t_2, t_3 enter the metric only as a difference, and the constants a, b add only a constant term to the scalar field ϕ, then the constants a, b, c can be chosen in the simplest way. As a result, we have an intermediate result for the scalar field ϕ:

$$\xi = \sqrt{t_2 t_3}, \quad \phi = \frac{1}{2(\omega+1)} \ln \frac{t_2 t_3}{x^{0\,2}}. \tag{53}$$

Having this, excluding the second derivatives from the Equations (44), (45), and (47), we obtain the following relation:

$$\omega(\omega + 2) = 0. \tag{54}$$

The case $\omega = 0$ leads to a flat solution, and it is easy to see this, taking into account the relations (6) and (41); thus,

$$\omega = -2. \tag{55}$$

The sum of Equations (44) and (45) after substituting (53) and (55) allows us to separate variables, and we obtain

$$\frac{t_2{}''}{t_2} - \frac{3}{2} \frac{t_2{}'^2}{t_2{}^2} = -\left(\frac{t_3{}''}{t_3} - \frac{3}{2} \frac{t_3{}'^2}{t_3{}^2} \right) = \text{const} = \lambda, \tag{56}$$

the solution is the following:

$$t_2 = \alpha / \cos^2 \sqrt{\lambda} x^2, \quad t_3 = \beta / \cosh^2 \sqrt{\lambda} x^3. \tag{57}$$

Substituting this solution into Equations (44) and (45) leads to a simple condition $\alpha = \beta$; furthermore, the scaling constants α, λ can be converted to the simplest form, and as a result, the final solution has the following form:

$$f_0 = 1/\left(x^0\right)^2, \qquad t_2 = \epsilon / \cos^2 x^2, \qquad t_3 = \epsilon / \cosh^2 x^3, \qquad \epsilon = \pm 1, \tag{58}$$

$$\phi = \ln(x^0 \cos x^2 \cosh x^3), \qquad \Lambda = 0. \tag{59}$$

For the resulting exact solution (59) of the equations of the Brans–Dicke theory, the Riemann curvature tensor, the scalar curvature, and the Weyl conformal curvature tensor do not vanish:

$$R = 2\epsilon {x^0}^2, \quad R_{2323} \neq 0, \quad C_{ijkl} \neq 0. \tag{60}$$

3.2.2. Scalar Field of the Form $\phi = \phi(x^0)$.

Let us consider the case where the scalar field ϕ depends only on the non-ignorable wave variable x^0 ($\phi_{,2} = \phi_{,3} = 0$), on which the wave metric itself depends. Equations (37) and (41) imply that the cosmological constant Λ and the scalar curvature R vanish in this case:

$$\Lambda = R = 0. \tag{61}$$

Then, one can immediately separate the variables in the Equation (41), which allows you to find differential equations for the functions $t_\mu(x^\mu)$:

$$\left({t_2}'\right)^2 = a{t_2}^2 + 2bt_2 + c, \tag{62}$$

$$\left({t_3}'\right)^2 = -\left(a{t_3}^2 + 2bt_3 + c\right), \tag{63}$$

where a, b, and c are constants.

The system of field equations is left with only one Equation (34) connecting the function f_0 (scale factor of the metric) and the scalar field $\phi(x^0)$:

$$\phi'' = \phi' \left(\frac{{f_0}'}{f_0} - (\omega + 1)\phi' \right) - \frac{2f_0{f_0}'' - 3{f_0}'^2}{2{f_0}^2}. \tag{64}$$

Equation (64) obviously admits the solutions for the scalar field $\phi = \phi_0(x^0)$ for a given conformal factor $f_0(x^0)$ of the metric.

In the case under consideration, the Weil tensor C_{ijkl} vanishes, and only two nonzero components of the Riemann curvature tensor remain R_{ijkl}:

$$R_{0202} = R_{0303} = \frac{W}{4f_0}(3{f_0}'^2 - 2f_0{f_0}''), \tag{65}$$

$$C_{ijkl} = R_{2323} = 0. \tag{66}$$

Thus, this solution is the case of conformally flat space, and the metric and components of the Riemann curvature tensor depend on the wave variable x^0. The scalar field ϕ and the conformal factor of the metric f_0 remain arbitrary functions of the wave variable x^0 but are related by the Equation (64).

Note that Equation (64) admits a de Sitter-type solution. Consider the scale factor of the metric in the following form:

$$f_0(x^0) = ke^{\beta x^0}, \qquad k, \beta - \text{const}. \tag{67}$$

Then, Equation (64) takes the following form:

$$\phi'' = \phi'(\beta - (\omega+1)\phi') + \beta^2/2. \tag{68}$$

Equation (68) for scalar field ϕ for metric (2) with scale factor (67) has three solutions depending on the values of the constant ω:

(1) $\omega = -1$. Then,

$$\phi = \beta e^{\beta x^0} - \frac{\beta}{2}x^0, \tag{69}$$

(2) $\omega < -3/2$. Then,

$$\phi = \frac{\beta}{2(\omega+1)}x^0 + \frac{1}{\omega+1}\ln\cos(\beta\sqrt{-(2\omega+3)}\,x^0/2), \tag{70}$$

(3) $\omega > -3/2,\ \omega \neq -1$. Then,

$$\phi = \frac{\beta}{2(\omega+1)}x^0 - \frac{1}{\omega+1}\ln\cosh(\beta\sqrt{2\omega+3}\,x^0/2), \tag{71}$$

where β is a constant parameter.

Thus, we have obtained a set of exact solutions to the equations of the scalar-tensor Brans–Dicke theory in vacuum for the type I Shapovalov wave space. Recall that there are no exact solutions of this type for the case of the vacuum Einstein equations [1] since Shapovalov spaces of type I degenerate in the case of vacuum Einstein equations.

Therefore, scalar-tensor theories of gravity can provide such exact models of gravitational waves that do not appear in general relativity.

4. Equations of Motion and Trajectories of Test Particles

Let us consider the equation for the motion of test particles in a gravitational field in the Hamilton–Jacobi formalism (the speed of light is chosen to be unity):

$$g^{\alpha\beta}\frac{\partial S}{\partial x^\alpha}\frac{\partial S}{\partial x^\beta} = m^2.$$

The complete integral of the Hamilton–Jacobi Equation (1) for the Shapovalov spacetimes in the privileged coordinate system can be represented in a separated form:

$$S = \theta_0(x^0) + \theta_1(x^1) + \theta_2(x^2) + \theta_3(x^3), \tag{72}$$

moreover, for the ignored variable x^1, on which the metric in the privileged coordinate system does not depend, the function $\theta_1(x^1)$ can be reduced by admissible coordinate transformations to the form kx^1, where k is a constant.

From Equation (1), separating the variable x^0, we obtain:

$$\frac{m^2}{f_0(x^0)} - 2k\theta_0{}'(x^0) = \frac{1}{t_3(x^3) - t_2(x^2)}\left(\left(\theta_2{}'(x^2)\right)^2 + \left(\theta_3{}'(x^3)\right)^2\right) = \text{const} = p. \tag{73}$$

Finally, separating the variables x^2 and x^3 we obtain:

$$\left(\theta_2{}'(x^2)\right)^2 + pt_2(x^2) = -\left(\theta_3{}'(x^3)\right)^2 + pt_3(x^3) = \text{const} = q. \tag{74}$$

Thus, for the complete integral of the particle action function $S(x^\alpha, k, p, q)$ we obtain the following expression:

$$S = kx^1 - \frac{p}{2k}x^0 + \frac{m^2}{2k}\int\frac{dx^0}{f_0(x^0)} + \varepsilon_2\int\sqrt{q - pt_2(x^2)}\,dx^2$$

$$+\varepsilon_3 \int \sqrt{pt_3(x^3) - q}\, dx^2, \qquad \varepsilon_2, \varepsilon_3 = \pm 1, \tag{75}$$

where k, p, and q are constant parameters.

The equations of the trajectories of the test particles for the considered Shapovalov spaces of type I in the Hamilton–Jacobi formalism take the following form:

$$\frac{\partial S}{\partial k} = \text{const} = \sigma_1 \quad \rightarrow \quad x^1 + \frac{p}{2k^2}x^0 + \frac{m^2}{2k^2}\int \frac{dx^0}{f_0(x^0)} = \sigma_1, \qquad k \neq 0, \tag{76}$$

$$\frac{\partial S}{\partial p} = \text{const} = \sigma_2 \quad \rightarrow \quad -\frac{x^0}{2k} - \frac{\varepsilon_2}{2}\int \frac{t_2(x^2)\, dx^2}{\sqrt{q - pt_2(x^2)}} + \frac{\varepsilon_3}{2}\int \frac{t_3(x^3)\, dx^3}{\sqrt{pt_3(x^3) - q}} = \sigma_2, \tag{77}$$

$$\frac{\partial S}{\partial q} = \text{const} = \sigma_3 \quad \rightarrow \quad \frac{\varepsilon_2}{2}\int \frac{dx^2}{\sqrt{q - pt_2(x^2)}} - \frac{\varepsilon_3}{2}\int \frac{dx^3}{\sqrt{pt_3(x^3) - q}} = \sigma_3, \tag{78}$$

where σ_1, σ_2, and σ_3 are new independent constant parameters of the test particle motion determined by the initial conditions. Thus, the motion of test particles in the considered spacetime models is determined by a set of constant parameters k, p, q, σ_1, σ_2, and σ_3, given by the initial conditions.

Note that the proper time of the test particle τ using the obtained trajectory equations can be written in the following form in terms of the coordinate variables on the trajectory:

$$\tau = S\,\big|_{m=1} = 2kx^1 + \frac{p}{k}\, x^0. \tag{79}$$

Here, we set the additive constant equal to zero by choosing the origin. Thus, proper time, as expected, is a linear combination of the null variables x^0 and x^1.

If we substitute (79) into Equation (76), then on the trajectories of test particles, we obtain:

$$\tau = -\frac{1}{k}\int \frac{dx^0}{f_0(x^0)} + 2k\sigma_1, \tag{80}$$

Thus, Equations (79) and (80), together with Equations (77) and (78), define the coordinates of the test particle x^α on the trajectory as a function of the proper time of the particle τ.

5. Conclusions

The exact solutions of the vacuum equations of the Brans–Dicke theory of gravity for Shapovalov type I wave spaces are found. The solutions for gravitational waves are obtained depending on the maximum possible number of variables for wave metrics of four-dimensional spacetime (three variables, including the wave variables) in privileged coordinate systems where it is possible to separate the wave variables in the Hamilton–Jacobi equation for test particles and in the Eikonal equation for radiation. The situation is shown to be different from general relativity, where these gravitational-wave models based on Einstein's vacuum equations are degenerate [1]. Shapovalov's wave spaces thus provide an additional mathematical tool to obtain exact models of gravitational waves, allowing us to study differences in modified theories of gravity and to form tests to verify these differences by observations.

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