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Bounds on the General Eccentric Connectivity Index

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Abstract: The general eccentric connectivity index of a graph R is defined as $\xi^{ec}(R) = \sum_{u \in V(G)} d(u)ec(u)^{\alpha}$, where α is any real number, $ec(u)$ and $d(u)$ represent the eccentricity and the degree of the vertex u in R, respectively. In this paper, some bounds on the general eccentric connectivity index are proposed in terms of graph-theoretic parameters, namely, order, radius, independence number, eccentricity, pendent vertices and cut edges. Moreover, extremal graphs are characterized by these bounds.

Keywords: eccentricity of vertex; eccentric connectivity index; general eccentric connectivity index; extremal graphs

MSC: 05C07; 05C35; 05C69

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1. Introduction

For terminologies and notations in graph theory that are not defined here, we refer to [1]. All considered graphs in this paper are finite, connected, and simple. A graph R is an ordered pair $R = (F, M)$. Here, F and M constitute the set of vertices and the set of edges, respectively. Cardinalities of these sets $|F|$ and $|M|$ are called the *order* and the *size* of the graph R. The *neighborhood* of a vertex u is the set of its adjacent vertices, and the cardinality of this set is said to be the *degree* of u. The degree of u in R is denoted as $d_R(u)$. A vertex u is called a pendant vertex if $d(u) = 1$. The length of any shortest path between the vertices u and v is called the *distance* between them and is denoted as $d_R(u, v)$. The maximum distance between the vertex u and any other vertex of R is called the *eccentricity*, $ec_R(u)$, of u in R. The maximum and the minimum eccentricity of any vertex among all vertices of the graph R is known as the *diameter* and the *radius* of R. The cardinality of the largest independent set of R is called the *independence number* of R and the minimum number of vertices whose removal from R generates a disconnected graph is called the *connectivity* of R. The *complement* of a graph R is a graph $\overline{R}$ with the same vertex set and two vertices are adjacent if and only if they are non-adjacent in R. For two graphs G and H, $G + H$ is the join graph of G and H whose vertex set is the vertices of G and H and the edge set consist of all the edges of G and H with all edges connecting each vertex of G with each vertex of H. Path, Star and complete graphs of order n are denoted by P_n, S_n and K_n.

Chemical graph theory is a branch of mathematical chemistry that deals with the nontrivial applications of graph theory to clear up molecular issues. In standard, a graph is used to represent a molecule with the aid of thinking about the atoms of the vertices of the graph and the molecular bonds as the edges. Then, the principal goal is to use algebraic invariants to reduce the topological structure of a molecule to a single quantity that could generate the same residences of the molecule. This single number is known as a *topological index*.

The *eccentric connectivity index* of R [2], is defined as

$$\xi^{ec}(R) = \sum_{u \in V(R)} ec_R(u) d_R(u).$$

This index has been considered for some classes of graphs. Zhou and Du [3] and Morgan et al. [4] independently investigated the sharp lower bound for the eccentric connectivity index of trees with given order and diameter. Morgan et al. [5] inveterate the sharp lower bound for the eccentric connectivity index of a connected graph with a given diameter. Recent results on the eccentric connectivity index of graphs can be seen in [2,4–21] and the references therein.

The authors in [22] introduced the general version of the eccentric connectivity index, namely the general eccentric connectivity index (GECI). For any real number α, GECI of graph R is defined as

$$\xi^{ec}_{\alpha}(R) = \sum_{u \in V(R)} d_R(u) ec_R(u)^{\alpha}.$$

For $\alpha = 1$ we obtain the classic eccentric connectivity index. The authors in [22] worked for trees and unicyclic graphs and put some bounds on these classes of graphs in terms of the order, diameter and pendant vertices for $\alpha > 0$.

2. Main Results

In this section, some lemmas and the main results of the paper are presented. In a graph, R adding an edge by joining two non-adjacent vertices increases the degrees and may decrease the eccentricity of the vertices. The following lemma tells that adding an edge in R increases the general eccentric connectivity index (GECI).

Lemma 1. *Let u and v be non-adjacent vertices in R, then for $\alpha < 0$*

$$\xi^{ec}_{\alpha}(R) < \xi^{ec}_{\alpha}(R + uv).$$

Proof. From the definition of the general eccentric connectivity index for $\alpha < 0$, we have, $\xi^{ec}_{\alpha}(R) - \xi^{ec}_{\alpha}(R + uv) \leq ec(u)^{\alpha}(d(u) - d(u) - 1) + ec(v)^{\alpha}(d(v) - d(v) - 1) < 0.$ □

The following result gives the upper bound on GECI for R in terms of the order of R.

Theorem 1. *Let R be a connected simple graph on r vertices and m edges, then for $\alpha < 0$*

$$\xi^{ec}_{\alpha}(R) \leq 2m \leq r(r-1)$$

equality holds if and only if $R \cong K_r$.

Proof. Since $ec(u) \geq 1$ for all $u \in V(R)$, so

$$\xi^{ec}_{\alpha}(R) = \sum_{u \in V(R)} d(u) ec(u)^{\alpha} \leq \sum_{u \in V(R)} d(u) = 2m \leq r(r-1)$$

equality holds if and only if every vertex of R has eccentricity one, i.e., $R = K_n$. □

Let r_1 represent the number of vertices with the eccentricity one in R. Let $K_n - qe$ be the graph achieved from K_n by removing q independent edges for $0 \leq q \leq \lfloor \frac{r}{2} \rfloor$.

Theorem 2. *Let R be a connected graph of order r with $r_1 \geqslant 1$ number of vertices with eccentricity one and for $\alpha < 0$, we have*

$$\xi^{ec}_{\alpha}(R) \leq r2^{\alpha}(r-2) + r_1(r - 1 - (r-2)2^{\alpha})$$

and the equality holds if and only if $R = K_{r_1} + (K_{r-nr_1} - \frac{r-r_1}{2}e)$, where $r - r_1$ is even.

Proof. Let $E_1 = \{u; ec(u) = 1, u \in V(G)\}$, then for $w \in V(R)\backslash E_1$, we have $ec(w) \geq 2$ it follows that $d(w) \leq r - 2$. Now

$$\begin{aligned}
\xi_\alpha^{ec}(R) &= \sum_{u \in E_1} d(u)ec(u)^\alpha + \sum_{w \in V(R \backslash E_1)} d(w)ec(w)^\alpha \\
&\leq r_1(r-1) + \sum_{w \in V(R \backslash E_1)} (r-2)2^\alpha \\
&= r_1(r-1) + (r - r_1)(r-2)2^\alpha \\
&= r_1(r - 1 - (r-2)2^\alpha) + r(r-2)2^\alpha
\end{aligned}$$

and the equality holds if and only if $ec(w) = 2$ and $d(w) = r - 2$ for any $w \in V(R)\backslash E_1$. Hence, the required result. □

Theorem 3. *Let R be a connected graph of order r and size m. Let* $a = \left\lfloor \frac{2r-1-\sqrt{(2r-1)^2-8m}}{2} \right\rfloor$ *be the greatest integer satisfying that* $y^2 + (1-2r)y + 2m \geq 0$. *Then, for* $\alpha < 0$, *we have*

$$\xi_\alpha^{ec}(R) \leq \frac{r(r-2)}{2} + \frac{r}{2}a$$

and the equality holds if and only if $R = K_a + (K_{r-a} - \frac{r-a}{2}e)$, *where* $r - a$ *is even.*

Proof. From Theorem 2, we know that $\xi_\alpha^{ec}(R) \leq r_1(r - 1 - (r-2)2^\alpha) + r(r-2)2^\alpha$. Moreover, $2m = \sum_{w \in V(R)} \geq r_1(r-1) + r_1(r - r_1)$ this implies that $r_1 \leq a$. So, $\xi_\alpha^{ec}(R) \leq r_1(r - 1 - (r-2)2^\alpha) + r(r-2)2^\alpha \leq a(r - 1 - (r-2)2^\alpha) + r(r-2)2^\alpha$ and the equality holds if and only if R has exactly a vertices with eccentricity one and rest of vertices having degree two. □

Theorem 4. *Let R be a graph of order r and radius h. Then, for* $\alpha < 0$

$$\xi_\alpha^{ec}(R) \leq r(r-h)h^\alpha$$

and the equality holds if and only if either $R = K_r$ *or* $R = K_r - \frac{r}{2}e$ *for even n.*

Proof. For any $u \in V(R)$, we know that $d(u) \leq r - ec(u)$. Then, for $\alpha < 0$, we have $\xi_\alpha^{ec}(R) = \sum_{u \in V(R)} d(u)ec(u)^\alpha \leq \sum_{u \in V(R)} (r - ec(u))ec(u)^\alpha \leq r(r-h)h^\alpha$. Function $f(x) = (r-x)x^\alpha$ is a decreasing function as $f'(x) < 0$ for $\alpha < 0$ and $r > x$. The equalities in the above hold when $ec(u) = r - d(u)$ and $ec(u) = h$ for every vertex of R, i.e., G is an $r - h$ regular graph with $ec(u) = h$ for any vertex u of R. This follows that either $R = K_r$ or $R = K_r - \frac{r}{2}e$ for even r. □

Let R_1 and R_2 be two graphs obtained from $K_{r-\gamma} + \overline{K_\gamma}$ by removing an edge joining two vertices in $K_{r-\gamma}$ and $\overline{K_\gamma}$ and removing an edge incident two vertices in $K_{r-\gamma}$, respectively.

Theorem 5. *Let R be a graph with order* $r > 5$ *having independence number* γ. *Then, for* $\alpha < 0$, *we have*

i. $\xi_\alpha^{ec}(R) \leq (r-\gamma)(r-1) + \gamma(r-\gamma)2^\alpha$ *and the equality holds if and only if* $R = K_{r-\gamma} + \overline{K_\gamma}$,

ii. $\xi_\alpha^{ec}(R) \leq (r+\gamma-1)(r-1) + (r-2)2^\alpha + (r-\gamma-1)2^\alpha + (\gamma-1)(r-\gamma)2^\alpha$ *and the equality holds if and only if* $R = R_1$.

Proof. From Lemma 1, we know that for $\alpha < 0$, adding an edge between two non-adjacent vertices of G increases the GECI. $K_{r-\gamma} + \overline{K_\gamma}$ has the maximum number of edges with independent number γ, this implies that (i) $\xi_\alpha^{ec}(R) \leq (r-\gamma)(r-1) + \gamma(r-\gamma)2^\alpha$.

For (ii), whenever we remove an edge e from $K_{r-\gamma} + \overline{K_\gamma}$ we always obtain either R_1 or R_2. Moreover, $\xi_\alpha^{ec}(R_1) = (r+\gamma-1)(r-1) + (r-2)2^\alpha + (r-\gamma-1)2^\alpha + (\gamma-1)(r-\gamma)2^\alpha$

and $\xi_\alpha^{ec}(R_2) = (r-\gamma-2)(r-1) + 2(r-2)2^\alpha + \gamma(n-\gamma)2^\alpha$. Lemma 1 implies that in order to obtain the result we only need to compare the GECI of R_1 and R_2 for $\alpha < 0$, which is $\xi_\alpha^{ec}(R_1) - \xi_\alpha^{ec}(R_2) = (r-1)(1-2^\alpha) > 0$, which is the required result. □

Let λ denote the covering number of R, since $\gamma(R) + \lambda(R) = r$ then we have the following corollary.

Corollary 1. *Let R be a graph of order r and covering number λ, then for $\alpha < 0$, we have*

$$\xi_\alpha^{ec}(R) \leq \lambda\Big(r - 1 + \frac{r-\lambda}{2}\Big)^\alpha$$

and the equality holds if and only if $R = K_\lambda + \overline{K_{r-\lambda}}$.

Theorem 6. *Let R be graph of order n and connectivity κ, then for $\alpha < 0$, we have*

$$\xi_\alpha^{ec}(R) \leq 2^\alpha\big(r^2 - r(3+\kappa) + 3\kappa + 2\big) + \kappa(r-1)$$

and the equality holds if and only if $R = (K_1 \bigcup K_{r-\kappa-1}) + K_\kappa$.

Proof. For $\alpha < 0$, let R' be a graph with maximum ξ_α^{ec} among all graphs of order n and connectivity κ. Then, there is a set of vertices C with cardinality κ such that $R' - C = \bigcup_i^p R_i$, where $p \geq 2$ and R_i's are the connected components of $R - C$. By Lemma 1 adding edges increases the ξ_α^{ec} for $\alpha < 0$, so R must be like $R' = (K_{r_1} \bigcup K_{r_2}) + K_\kappa$ such that $r_1 + r_2 = r - \kappa$ and without loss of generality suppose $r_1 \leq r_2$. Now,

$$\begin{aligned}
\xi_\alpha^{ec}(R') &= \sum_{u \in V(K_{r_1})} d(u)ec(u)^\alpha + \sum_{u \in V(K_{r_2})} d(u)ec(u)^\alpha + \sum_{u \in V(K_\kappa)} d(u)ec(u)^\alpha \\
&= r_1(r_1 + \kappa - 1)2^\alpha + r_2(r_2 + \kappa - 1)2^\alpha + k(r-1) \\
&= 2^\alpha(r-\kappa)\big(r-1\big) - 2^{\alpha+1}r_1r_2 + \kappa(r-1) \\
&\leq 2^\alpha(r-\kappa)\big(r-1\big) - 2^{\alpha+1}(r-\kappa-1) + \kappa(r-1) \\
&= 2^\alpha\big(r^2 - r(3+\kappa) + 3\kappa + 2\big) + \kappa(r-1)
\end{aligned}$$

and equality holds if and only if $r_1 = 1$ and $r_2 = r - \kappa - 1$. □

For edge connectivity and minimum degree, we have a famous relation $\kappa(R) \leq \kappa'(R) \leq \delta(R)$. Let $\phi(x) = 2^\alpha\big(r^2 - r(3+x) + 3x + 2\big) + x(r-1)$, then $\phi'(x) > 0$ for $\alpha < 0$ this implies that $\phi(x)$ is an increasing function for $r > x > 0$ so we have $\phi(\kappa) \leq \phi(\kappa') \leq \phi(\delta)$. This gives the following given results.

Corollary 2. *Let R be a graph of order n and edge-connectivity κ', then for $\alpha < 0$ we have*

$$\xi_\alpha^{ec}(R) \leq 2^\alpha\big(r^2 - r(3+\kappa') + 3\kappa' + 2\big) + \kappa'(r-1)$$

and the equality holds if and only if $R = (K_1 \bigcup K_{r-\kappa'-1}) + K_{\kappa'}$.

Corollary 3. *Let R be a graph of order n with minimum degree δ, then for $\alpha < 0$, we have*

$$r\delta(r-\delta)^\alpha \leq \xi_\alpha^{ec}(R) \leq 2^\alpha\big(r^2 - r(3+\delta) + 3\delta + 2\big) + \delta(r-1)$$

the left equality holds if and only if either $R = K_r$ or $R = K_r - \frac{r}{2}e$ for even n and the right equality holds if and only if $R = (K_1 \bigcup K_{r-\delta-1}) + K_\delta$.

Proof. Since $d(u) \geq \delta$ for every $u \in V(R)$ from Theorem 4 we know that $f(x)$ is a decreasing function, we have

$$\zeta_\alpha^{ec}(R) \geq \sum_{u \in V(R)} d(u)(r-d(u))^\alpha \geq \sum_{u \in V(R)} \delta(r-\delta)^\alpha = r\delta(r-\delta)^\alpha.$$

□

Let K_r^t be a graph obtained from K_{r-t} by joining t pendant vertices to one vertex of K_{r-t}. Let $\top_r^t$ be the set of all trees achieved by identifying both end vertices of P_{r-t} with the centers of S_{p+1} and S_{q+1}, respectively, where $p, q \geq 0$ and $p+q=k$. One can notice that $\zeta_\alpha^{ec}(R) = \zeta_\alpha^{ec}(H)$ for every pair of graphs $R, H \in \top_r^t$. Let $A(k)$ be the k-th harmonic number, i.e., $A(k) = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$.

Theorem 7. *Let R be a graph of order r with t pendant vertices, for $\alpha < 0$, we have*

$$\lambda \leq \zeta_\alpha^{ec}(R) \leq 2^\alpha\Big((r-t-1)^2 + t\Big) + r - 1$$

where

$$\lambda = \begin{cases} t(r-t+1)^\alpha + (t+2)(r-t)^\alpha + 4A(r-t-1) - 4A\left(\frac{r-t}{2}\right); \\ \text{when } r-t \text{ is even} \\ \\ \left(t+2^{1-\alpha}\right)(r-t+1)^\alpha + (t+2)(r-t)^\alpha + 4A(r-t-1) - 4A\left(\frac{r-t+1}{2}\right); \\ \text{when } r-t \text{ is odd} \end{cases}$$

and the upper bound achieved if and only if $R = K_r^t$ and lower bound is achieved if and only if $R \in \top_r^t$.

Proof. Let R have the maximum GECI among all graphs of order r with t pendant vertices. Let $\{u_1, u_2, \cdots, u_t\}$ be pendant vertices of R, then by Lemma 1 $V(R)\backslash\{u_1, u_2, \cdots, u_t\}$ induces a complete graph H. Now, all the pendant vertices are adjacent to some vertices of H. Now our next goal is to show that $R = K_r^t$.

Suppose that u and v are the only vertices of R such that $d(u) \geq d(v) > r-t-1$. A new graph R' is achieved from R by shifting all pendant vertices from v to u. It is easy to see that for any $w \in V(R)\backslash\{u\}$, we have $ec_R(w) = ec_{R'}(w)$, $ec_R(u) > ec_{R'}(u)$ and $ec_{R'}(w) = ec_R(w) - 1$ for any pendant vertex w. The degrees of vertices in $V(R)\backslash\{u, v\}$ remain the same, and the degree of u increases by $d_R(v) - r + t + 1$ and the degree of v decreases by $d_R(v) - r - t + 1$. Now, we compare the GECI values of R and R' for $\alpha < 0$ as which is a contradiction against the supposition that R has maximum GECI.

Now, consider that H contains at least two vertices of degrees at least $r-t$. By repeating the above procedure we can obtain a graph with a larger GECI, which is again a contradiction. Hence, $R = K_r^t$.

Now, for the lower bound, suppose that K is the graph with minimum GECI among all graphs of order n and t pendant vertices. Lemma 1 tells that K must be a tree. Further, we need to prove that $K \in \top_r^t$. □

Claim 1. *K must be a caterpillar.*

Proof of Claim 1. Let $P_{s+1} = u_0u_1u_2\cdots u_s$ be a longest path in K. We consider $s \geq 4$ because $s = 3$ is a caterpillar. Suppose that $j \in \{2, 3, \cdots, \lfloor\frac{t+1}{2}\rfloor\}$ is the smallest integer such that there is a vertex w different from u_{i-1} and u_{i+1} which is adjacent to u_i. Let $N_R(w) = \{u_i, w_1, w_2, \cdots, w_q\}$, where $q \geq 2$. Let T_1 be the subtree of $K - u_iu_{i+1} - u_iw$ having u_i and T_2 and T_3 are subtrees of $K - u_iu_{i+1} - u_iw$ having u_{i+1} and w, respectively.

Let $T^* = K - \{ww_1, ww_2, \cdots, ww_q\} + \{u_sw_1, u_sw_2, \cdots, u_sw_q\}$. Clearly, T^* has t pendant vertices and

i. $ec_K(u) < ec_{T^*}(u)$ and $d_K(u) = d_{T^*}(u)$ for all $u \in (T_1 \cup (T_3 - w))$,
ii. $ec_K(w) < ec_{T^*}(w), d_K(w) = q+1, d_{T^*}(w) = 1$,

iii. $ec_K(u) \leq ec_{T^*}(u)$ and $d_K(u) = d_{T^*}(u)$ for all $u \in (T_2 - u_s)$,
iv. $ec_K(u_s) = ec_{T^*}(u_s), d_K(u_s) = 1, d_{T^*}(u_s) = q + 1$.

From the above, we have

$$\begin{aligned}\xi_\alpha^{ec}(T^*) - \xi_\alpha^{ec}(K) > & d_{T^*}(w)ec_{T^*}(w)^\alpha + d_{T^*}(u_s)ec_{T^*}(u_s)^\alpha - d_K(w)ec_K(w)^\alpha - \\ & d_K(u_s)ec_K(u_s)^\alpha \\ =& ec_{T^*}(w)^\alpha + (q+1)ec_{T^*}(u_s)^\alpha - (q+1)ec_K(w)^\alpha - ec_K(u_s)^\alpha \\ <& q(ec_{T^*}(u_s)^\alpha - ec_K(w)^\alpha) \leq 0\end{aligned}$$

the second last inequality is due to $ec_K(w) < ec_{T^*}(w)$ while the last inequality is due to $ec_{T^*}(u_s)^\alpha - ec_K(w)^\alpha \leq 0$, and the result gives a contradiction that K has the minimum GECI for $\alpha < 0$.

Suppose that $j \in \{\lceil \frac{t+1}{2} \rceil, \cdots, s-2\}$ be a largest integer such that there is a vertex x adjacent with u_j and $u_{j-1} \neq x \neq u_{j+1}$. Let $N_K(x) = \{u_j, w'_1, w'_2, \cdots, w'_p\}$, where $p \geq 2$. Similar to the above discussion we can construct a new tree by removing $w'_k s$ from x and joining these to u_0, and the new graph has less GECI for $\alpha < 0$, which leads to a contradiction. Hence, K is a caterpillar. □

Claim 2. *K is an element of $\top_r^t$.*

Proof of Claim 2. From the above claim, we conclude that K has the diameter $r - t + 1$. Let $P_{r-t+2} = u_0u_1 \cdots u_{r-t+1}$ be the longest path of K. Suppose that $i \in \{2, 3, \cdots, \lceil \frac{t+1}{2} \rceil\}$ be the smallest integer such that $d(u_i) > 0$ and $N_K(u_i) = \{u_{i-1}, u_{i+1}, w_1, \cdots, w_q\}$. Now, we construct a new graph $T^{**} = K - \{u_iw_1, u_iw_2, \cdots, u_iw_q\} + \{u_1w_1, u_1w_2, \cdots, u_1w_q\}$.

Clearly, $ec_K(w_i) < ec_{T^{**}}(w_i)$ for every w_i's and for other vertices $ec_K(x) = ec_{T^{**}}(x)$. Moreover, $d_{T^{**}}(v_i) = d_K(v_i) - s$, $d_{T^{**}}(v_1) = d_K(v_1) + s$ and for any other vertices we have $d_{T^{**}}(x) = d_K(x)$. This implies that

$$\begin{aligned}\xi_\alpha^{ec}(K) - \xi_\alpha^{ec}(T^{**}) > & d_k(u_1)ec_K(u_1)^\alpha + d(u_i)ec_K(u_i)^\alpha - d_{T^{**}}(u_1)ec_{T^{**}}(u_1)^\alpha - \\ & d_{T^{**}}(u_i)ec_{T^{**}}(u_i)^\alpha \\ =& q(-ec_K(u_1)^\alpha + ec_K(u_i)^\alpha) > 0\end{aligned}$$

the last inequality is due to $ec_K(u_i) < ec_K(u_1)$ and $\alpha < 0$. This gives a contradiction that K has the minimum generalized GECI.

Now, assume that $i \in \{\lceil \frac{t+1}{2} \rceil, \cdots, r-t-1\}$ is the largest integer such that $d(v_i) > 2$ and $N_K(u_i) = \{u_{i-1}, u_{i+1}, w'_1 \cdots, w'_q\}$. We construct a new graph $T^{***} = K - u_iw'_1 - u_iw'_2 - \cdots - u_iw'_q + u_{n-t}w'_1 + u - r - tw'_2 + \cdots + u - r - tw'_q$. Similarly, as above, we obtain $\xi_\alpha^{ec}(K) - \xi_\alpha^{ec}(T^{***}) > 0$ and we have again a contradiction. Hence, we have $K \in \top_r^t$. □

Let R_1 and R_2 be two non-trivial graphs having $u \in V(R_1)$ and $v \in V(R_2)$. The graphs R^* and R^{**} are obtained from $R_1 \cup R_2$ by adding an edge uv and by identifying u and v to a new vertex say u and adding a pendant edge uv, respectively.

Lemma 2. *Let R^* and R^{**} be two above defined graphs, then for $\alpha < 0$, we have $\xi_\alpha^{ec}(R^*) < \xi_\alpha^{ec}(R^{**})$.*

Proof. Clearly, for any $w \in R_1 \cup R_2 - \{u, v\}$, we have $ec_{R^*}(w) \geq ec_{R^{**}}(w)$ and $d_{R^*}(w) = d_{R^{**}}(w)$. For vertices u and v, we have

- $ec_{R^*}(u) = max\{ec_{H_1(u)}, ec_{H_2}(v) + 1\}, d_{R^*}(u) = d_{H_1}(u) + 1$,
- $ec_{R^**}(u) = max\{ec_{H_1(u)}, ec_{H_2}(v)\}, d_{R^{**}}(u) = d_{H_1}(u) + d_{H_2}(v) + 1$,
- $ec_{R^*}(v) = max\{ec_{H_1(u)} + 1, ec_{H_2}(v)\}, d_{R^*}(v) = d_{H_2}(v) + 1$,
- $ec_{R^**}(v) = max\{ec_{H_1(u)} + 1, ec_{H_2}(v) + 1\}, d_{G^*}(v) = 1$.

Now, we have the following cases:

Case 1. $ec_{H_1}(u) \geq ec_{H_2}(v)$.

Here, we have $ec_{R^*}(u) = ec_{H_1}(u), ec_{R^{**}}(u) = ec_{H_1}(u), ec_{R^*}(v) = ec_{H_1}(u)+1$, and $ec_{R^{**}}(v) = ec_{H_1}(u)+1$. This gives

$$\begin{aligned}\xi^{ec}_\alpha(R^*) - \xi^{ec}_\alpha(R^{**}) &\leq d_{R^*}(u)ec_{H_1}(u)^\alpha - d_{R^{**}}(u)ec_{H_1}(u)^\alpha + d_{R^*}(v)\big(ec_{H_1}(u)+1\big)^\alpha \\ &\quad - d_{R^{**}}(v)\big(ec_{H_1}(u)+1\big)^\alpha \\ &= d_{H_2}(v)\big((ec_{H_1}+1)^\alpha - ec_{H_1}(u)^\alpha\big) < 0\end{aligned}$$

as $\alpha < 0$.

Case 2. $ec_{H_1}(u) \leq ec_{H_2}(v)$.

First consider $ec_{H_1}(u) = ec_{H_2}(v)$. Here, we have $ec_{R^*}(u) = ec_{H_1}(u)+1, ec_{R^**}(u) = ec_{H_1}(u), ec_{R^*}(v) = ec_{H_1}+1, ec_{R^{**}}(v) = ec_{H_1}(u)+1$.

This implies that

$$\begin{aligned}\xi^{ec}_\alpha(R^*) - \xi^{ec}_\alpha(R^{**}) &\leq \big(d_{H_1}(u)+1\big)\big(ec_{H_1}(u)+1\big)^\alpha - \big(d_{H_1}(u)+d_{H_2}(v)+1\big)\big(ec_{H_1}(u)+1\big)^\alpha \\ &\quad + \big(d_{H_2}(v)+1\big)\big(ec_{H_1}(u)+1\big)^\alpha - \big(ec_{H_1}(u)+1\big)^\alpha \\ &= \big(d_{H_1}(u)+d_{H_2}(v)+1\big)\big((ec_{H_1}(u)+1)^\alpha - ec_{H_1}(u)^\alpha\big) < 0,\end{aligned}$$

Now, take $ec_{H_1}(u) < ec_{H_2}(v)$, then we have $ec_{R^*}(u) = ec_{H_2}(v)+1, ec_{R^{**}}(u) = ec_{H_2}(v)$, $ec_{R^*}(v) = ec_{H_2}(v), ec_{R^{**}}(v) = ec_{H_2}(v)+1$. This implies that

$$\begin{aligned}\xi^{ec}_\alpha(R^*) - \xi^{ec}_\alpha(R^{**}) &\leq \big(d_{H_1}(u)+1\big)\big(ec_{H_2}(v)+1\big)^\alpha - \big(d_{H_1}(u)+d_{H_2}(v)+1\big)\big(ec_{H_2}(v)\big)^\alpha \\ &\quad + \big(d_{H_2}(v)+1\big)\big(ec_{H_2}(v)\big)^\alpha - \big(ec_{H_2}(v)+1\big)^\alpha \\ &= \big(d_{H_1}(u)\big)\big((ec_{H_2}(v)+1)^\alpha - ec_{H_2}(v)^\alpha\big) < 0,\end{aligned}$$

□

Theorem 8. *Let R be a graph of order n with $t \geq 1$ cut edges, then for $\alpha < 0$, we have*

$$\xi^{ec}_\alpha(R) \leq 2^\alpha\Big((r-t-1)^2 + t\Big) + r - 1$$

and the equality holds if and only if $R = K^t_r$.

Proof. Let R be the graph with the maximum general eccentric connectivity index, then by Lemma 2, all the cut edges are pendant edges in R. Now, the problem is to find the maximum GECI with given pendant edges, which is discussed in Theorem 7, hence the result. □

Let $C_{r,t}$ be a cactus by adding t independent edges among pendant vertices of S_r.

Theorem 9. *Let R be a cactus of order greater than four having t cycles. Then, for $\alpha < 0$, we have*

$$\xi^{ec}_\alpha(R) \leq n(2^\alpha + 1) + 2^{\alpha+1}t - (1 + 2^\alpha)$$

and the equality holds if and only if $R = C_{r,t}$.

Proof. Let V_1 and V_2 be the set of vertices of eccentricity one and greater than one, respectively. Clearly, $|V_1| \leq 1$. Otherwise assume that u and v are vertices of eccentricity one and these vertices must have degree $r-1$. Then, there exist cycles having common edges in RL,, which implies that R is not a cactus. Now, we have the following two cases:

Case 1. When $|V_1| = 1$.

Let u be the vertex of eccentricity one, this implies that each vertex of R is adjacent to u.

Hence, the cactus R is obtained by adding t independent edges among pendant vertices of S_r, in other words, $R = C_{r,t}$.

Case 2. When $|V_1| = 0$.

Here, every vertex of G has eccentricity greater than one and there are exactly $r+t-1$ edges in R. Then, $\xi^{ec}_{\alpha}(R) \leq 2^{\alpha}\sum_{u\in V(R)} d(u) = 2^{\alpha+1}(r+t-1)$. This implies that $\xi^{ec}_{\alpha}(R) - \xi^{ec}_{\alpha}(C_{r,t}) \leq 2^{\alpha+1}(n+t-1) - n(2^{\alpha}+1) - 2^{\alpha+1}t + (1+2^{\alpha}) = (r-1)(2^{\alpha+1} - 2^{\alpha} - 1) < 0$, since $\alpha < 0$ which is required proof. □

The following corollaries are direct consequences of the above result.

Corollary 4. *Let R be a tree of order n, then for $\alpha < 0$, we have*

$$\xi^{ec}_{\alpha}(R) \leq (r-1)(2^{\alpha}+1)$$

and the equality holds if and only $R = S_r$.

Corollary 5. *Let R be a unicyclic graph of order r, then for $\alpha < 0$, we have*

$$\xi^{ec}_{\alpha}(R) \leq r(2^{\alpha}+1) + 2^{\alpha+1} - 2^{\alpha} - 1$$

and the equality holds if and only $R = C_{r,1}$.

3. Conclusions

The general eccentric connectivity index is a newly introduced topological index. The study of this newly introduced topological index is a useful and interesting task. We put some lower and upper bounds on this topological index by using some graph parameters. It will be interesting to find the extremal graphs for other classes of graphs by using some graph parameters such as the number of vertices with the given parameter, the number of vertices with the given maximum degree, etc.

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