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Solution Set of the Yang-Baxter-like Matrix Equation for an Idempotent Matrix

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Abstract: Given a complex idempotent matrix A, we derive simple, sufficient and necessary conditions for a matrix X being a nontrivial solution of the Yang-Baxter-like matrix equation $AXA = XAX$, discriminating commuting solutions from non-commuting ones. On this basis, we construct all the commuting solutions of the nonlinear matrix equation.

Keywords: nonlinear matrix equation; Yang-Baxter-like matrix equation; idempotent matrices

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1. Introduction

Nonlinear matrix equations arise in many scientific and engineering fields. Seeking their solutions is a difficult task [1]. In this paper, we are interested in solving the nonlinear matrix equation of the form

$$AXA = XAX \tag{1}$$

where A is a given nonzero complex matrix. Matrix Equation (1) is called the Yang-Baxter-like matrix equation (see [2–9]), termed YBME for short, since it originates from physics. Matrix Equation (1) has the same form as the well-known Yang–Baxter equation, which first arose in two independent papers by Yang [10] and Baxter [11]. Recently, Kumar et al. [12] investigated new solution sets for the Yang-Baxter-like matrix equation by using a class of generalized outer inverses of a matrix. Jiang et al. [13] proposed a zeroing neural network dynamical system approach for solving the time-varying Yang–Baxter matrix equation.

It is evident that YBME (1) is a symmetric matrix equation concerning a known matrix A and an unknown matrix X. Although YBME (1) looks simple in format, it is not easy to solve for a general matrix A since it is equivalent to solving a quadratic system of n^2 equations in n^2 variables. So far, only for a few matrices with a special structure (f.g. [2–6]) can all commutable solutions of Equation (1) be obtained, i.e., solutions satisfying $AX = XA$.

In this paper, our aim is to find all solutions of YBME (1) with an idempotent matrix A, that has already been provided in [5]. However, the idea of our approach is novel. We do not need to use and compute the Jordan canonical form of A as in some previous works, such as [2–9]. Our analysis and method to construct the solution set are based on simple sufficient and necessary conditions that we derive for a matrix to be a nontrivial solution of YBME (1). In particular, we construct all commuting solutions.

The rest of the paper is organized as follows. In Section 2, we present some properties of the idempotent matrices. In Section 3, we first derive simple sufficient and necessary conditions for a matrix being a nontrivial (commuting, non-commuting) solution of YBME (1), and then analyze and construct the set that contains all the (commuting, non-commuting) solutions of YBME (1). Finally, in Section 4, we provide an example.

2. Preliminaries

In this section, we first provide some properties of the idempotent matrices that are needed for our analysis later.

Let A be an $n \times n$ idempotent matrix with rank r, satisfying $A^2 = A$ and $rank(A) = r$. We assume that $0 < r < n$ to avoid the two trivial cases $A = \mathbf{0}$ or $A = I$. In fact, when $A = \mathbf{0}$, YBME (1) is zero. When $A = I$, YBME (1) reduces to $X^2 = X$, whose solution is any idempotent matrix. That is to say, if we denote the set of all the solutions of the matrix for equation $X^2 = X$ as

$$S_{X^2=X} = \{X \in C^{n\times n} | X^2 = X\}, \tag{2}$$

we know that $S_{X^2=X}$ is the set of all the idempotent matrices of order n.

It is widely known that any square matrix A with rank r has the spectral decomposition in the form

$$A = UV^T = u_1 v_1^T + \cdots + u_r v_r^T, \tag{3}$$

where $U = [u_1, \cdots, u_r]$ and $V = [v_1, \cdots, v_r]$ are of full rank. For notational convenience, in the following, we will denote

$$\tilde{U} = [u_{r+1}, \ldots, u_n], \quad \tilde{V} = [v_{r+1}, \ldots, v_n], \tag{4}$$

where $u_{r+1}, \ldots, u_n$ and $v_{r+1}, \ldots, v_n$ form the basis of the null space of U^H and V^H, respectively. Therefore, we have $U^H\tilde{U} = \mathbf{0}$ and $V^H\tilde{V} = \mathbf{0}$.

Lemma 1. *Let $A = UV^H$, where U and V are two $n \times r$ ($r < n$) complex matrices of full column rank; then, A is a singular idempotent matrix if and only if $V^H U = I$.*

Proof. Since $A^2 = U(V^H U)V^H$, $A = UV^H$. Thus, A is an idempotent matrix if and only if $V^H U = I$. □

According to Lemma 1, we easily verify that

$$S_{X^2=X} = \{I\} \cup \{\mathbf{0}\} \cup S_{id}, \tag{5}$$

where

$$S_{id} = \{X = UV^H | U, V \in C^{n\times r},\ rank(U) = rank(V) = r,\ V^H U = I,\ 1 \le r \le n-1.\}. \tag{6}$$

Now, we begin to find solutions of YBME (1) with an idempotent matrix.

Lemma 2. *Let $A = UV^H$ be an $n \times n$ idempotent matrix with rank r, and let E be a $r \times r$ matrix, then $X = UEV^H$ is a (commuting) solution of YBME (1) if and only if E is an ($r \times r$) idempotent matrix.*

Proof. Let $X = UEV^H$, notice that $V^H U = I$. Thus we have

$$XAX = UEV^H UV^H UEV^H = UE^2V^H, \quad AXA = UV^H UEV^H UV^H = UEV^H.$$

So, X is a solution of YBME (1) if and only if $E^2 = E$, i.e., E is an idempotent matrix. Further, since $XA = UEV^H UV^H = UEV^H$, $AX = UV^H UEV^H = UEV^H$, X commutes with A. □

Lemma 3. *Let $A = UV^H$ be an $n \times n$ idempotent matrix with rank r, then for any $(n-r) \times (n-r)$ matrix F, $X = \tilde{V}F\tilde{U}^H$ is a commuting solution of YBME (1).*

Proof. In fact, let $X = \tilde{V}F\tilde{U}^H$, notice that $U^H\tilde{U} = \mathbf{0}$ and $V^H\tilde{V} = \mathbf{0}$, so we have

$$XA = \tilde{V}F\tilde{U}^H UV^H = \mathbf{0}, \quad AX = UV^H\tilde{V}F\tilde{U}^H = \mathbf{0}.$$

So, X is a commuting solution of YBME (1). $\square$

3. All the Solutions

In this section, we find all the solutions X of YBME (1) with an idempotent matrix based on simple sufficient and the necessary conditions for a matrix being a nontrivial (commuting, non-commuting) solution. We first provide a lemma.

Lemma 4. *If $A = UV^H$ where U and V are two $n \times r$ complex matrices of full rank such that $V^T U = I$, then the two matrices*

$$P = [U, \tilde{V}], \quad Q = [V, \tilde{U}] \tag{7}$$

are nonsingular.

Proof. To show the matrix $P = [U, \tilde{V}]$ is nonsingular, we only need to show that the linear system $Pz = Ux + \tilde{V}y = 0$ only has a zero solution. In fact, since $V^H U = I$ and $V^H \tilde{V} = \mathbf{0}$, we have $V^H Pz = V^H Ux + V^H \tilde{V}y = V^H Ux = x = 0$. The linear system $Pz = 0$ becomes $\tilde{V}y = 0$. Further, since $\tilde{V}$ is of full column rank, we derive from $\tilde{V}y = 0$ that $y = 0$. So, $Pz = 0$ has only a zero solution; thus, P is nonsingular. Similarly, we can show that Q is nonsingular. $\square$

Since P and Q, defined in (7), are nonsingular, we can easily show that for an arbitrary $n \times n$ matrix X, there is an $n \times n$ matrix Z such that

$$X = PZQ^H = [U, \tilde{V}] \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} \begin{bmatrix} V^H \\ \tilde{U}^H \end{bmatrix}, \tag{8}$$

where Z_1, Z_2, Z_3 and Z_4 are $r \times r$, $r \times (n-r)$, $(n-r) \times r$ and $(n-r) \times (n-r)$ matrices, respectively. Now, we present our main results.

Theorem 1. *Let $A = UV^H$ be an $n \times n$ idempotent matrix with rank r, and $X = PZQ^H$, defined in (8), an $n \times n$ complex matrix. Then, X is a solution of YBME (1) if and only if Z_1, Z_2, and Z_3 satisfy the following equations simultaneously:*

$$Z_1^2 = Z_1, \quad Z_1 Z_2 = \mathbf{0}, \quad Z_3 Z_1 = \mathbf{0}, \quad Z_3 Z_2 = \mathbf{0}. \tag{9}$$

Proof. Notice that $V^H U = I$, $\tilde{U}^H U = \mathbf{0}$ and $\tilde{V}^H V = \mathbf{0}$, in a simple calculation lead to

$$XAX = XUV^H X = P \begin{bmatrix} Z_1 \\ Z_3 \end{bmatrix} [Z_1, Z_2] Q^H = P \begin{bmatrix} Z_1^2 & Z_1 Z_2 \\ Z_3 Z_1 & Z_3 Z_2 \end{bmatrix} Q^H$$

and $AXA = UV^H XUV^H = UZ_1 V^H$. Thus,

$$XAX - AXA = XUV^H X - UV^H XUV^H = P \begin{bmatrix} Z_1^2 - Z_1 & Z_1 Z_2 \\ Z_3 Z_1 & Z_3 Z_2 \end{bmatrix} Q^H.$$

Notice that P and Q are nonsingular; therefore, X is a solution of YBME (1) if and only if Z_1, Z_2 and Z_3 satisfy (9). $\square$

In light of the previous result, note that the matrix Z_4 is completely arbitrary. In terms of Theorem 1, we can obtain all the solutions of YBME (1) by solving (9) for the matrices Z_i ($i = 1, 2, 3$). Denote the solution set of (9) as

$$S_Z(Z_1, Z_2, Z_3) = \{(Z_1, Z_2, Z_3) | Z_1, Z_2, Z_3 \ \ satisfy \ \ (9)\},$$

or equivalently,

$$S_Z(Z_1, Z_2, Z_3) = \{(Z_1, Z_2, Z_3) | Z_1 \in S_{Z_1^2 = Z_1}, \ Z_1 Z_2 = \mathbf{0}, \ Z_3 Z_1 = \mathbf{0}, \ Z_3 Z_2 = \mathbf{0}\}.$$

In the following, we derive the set $S_Z(Z_1, Z_2, Z_3)$ using the set $S_{Z_1^2 = Z_1} = \{I\} \cup \{\mathbf{0}\} \cup S_{id}$.
(i) If $Z_1 = I$, then all the solutions of (9) are $(I, \mathbf{0}, \mathbf{0})$.
(ii) If $Z_1 = \mathbf{0}$, then all the solutions of (9) are $\{(\mathbf{0}, Z_2, Z_3)) | Z_3 Z_2 = \mathbf{0}\}$.
(iii) If $Z_1 \in S_{id}$, then we can assume that $Z_1 = EF^H$k where E, and F are two $r \times s$ matrices of full column rank such that $F^H E = I$, $s = 1, 2, \ldots, r-1$.

Let $Z_2 = \tilde{F} Y_2$ and $Z_3 = Y_3 \tilde{E}^H$, where $\tilde{E}, \tilde{F}$ are two $r \times (r-s)$ matrices whose columns form the basis of the null space of E^H and F^H, respectively. Then, we have $Z_1 Z_2 = \mathbf{0}$, $Z_3 Z_1 = \mathbf{0}$ for the arbitrary $(r-s) \times (n-r)$ matrix Y_2 and arbitrary $(n-r) \times (r-s)$ matrix Y_3. Thus, when $Z_1 \in S_{id}$, we obtain the set of all the solutions of (9) is

$$\{(Z_1 = EF^H, Z_2 = \tilde{F} Y_2, Z_3 = Y_3 \tilde{E}^H) | Y_3 \tilde{E}^H \tilde{F} Y_2 = \mathbf{0}\}.$$

So, we find that the set $S_Z(Z_1, Z_2, Z_3)$ is

$$\{(I, \mathbf{0}, \mathbf{0})\} \cup \{(\mathbf{0}, Z_2, Z_3)) | Z_3 Z_2 = \mathbf{0}\} \cup \{(Z_1 = EF^H, Z_2 = \tilde{F} Y_2, Z_3 = Y_3 \tilde{E}^H) | Y_3 \tilde{E}^H \tilde{F} Y_2 = \mathbf{0}\}. \quad (10)$$

Corollary 1. *Let $A = UV^H$ be an idempotent matrix with rank r; then, all the solutions of YBME (1) can be expressed as $X = PZQ^H$ in (8), where $(Z_1, Z_2, Z_3) \in S_Z(Z_1, Z_2, Z_3)$, and Z_4 are arbitrary.*

For the commuting solutions of YBME (1), we have an optimal result.

Theorem 2. *Let $A = UV^H$, where U and V are two $n \times r$ complex matrices of full column rank such that $V^H U = I$. Then, an $n \times n$ complex matrix X is a commuting solution of YBME (1) if and only if*

$$X = UEV^H + \tilde{V} F \tilde{U}^H, \quad (11)$$

where E is any $r \times r$ idempotent matrix, and F is any $(n-r) \times (n-r)$ matrix.

Proof. Sufficiency. Using Lemmas 2 and 3, we easily verify that when E is an idempotent matrix; for arbitrary F, $X = UEV^H + \tilde{V} F \tilde{U}^H$ is a commuting solution of YBME (1).

Necessity. Let $X = PZQ^H$, where Z is partitioned as in (8). A simple calculation leads to

$$\mathbf{0} = XA - AX = U[Z_1 - Z_1]V^H + \tilde{V} Z_3 V^H - U Z_2 \tilde{U}^H = P \begin{bmatrix} \mathbf{0} & -Z_2 \\ Z_3 & \mathbf{0} \end{bmatrix} Q^H, \quad (12)$$

If X commutes with $A = UV^H$, then since P, Q are nonsingular, we obtain $Z_2 = \mathbf{0}$ and $Z_3 = \mathbf{0}$. Thus,

$$X = [U, \tilde{V}] \begin{bmatrix} Z_1 & \mathbf{0} \\ \mathbf{0} & Z_4 \end{bmatrix} \begin{bmatrix} V^H \\ \tilde{U}^H \end{bmatrix} = U Z_1 V^H + \tilde{V} Z_4 \tilde{U}^H \quad (13)$$

where Z_1 is an $r \times r$ matrix, Z_4 is an $(n-r) \times (n-r)$ matrix.

To prove necessity, we still need to show that if X in (13) is a commuting solution of YBME (1), Z_1 must be an idempotent matrix.

In fact, let $X = U Z_1 V^H + \tilde{V} Z_4 \tilde{U}^H$; we easily verify that $XAX = U Z_1^2 V^H$, $AXA = U Z_1 V^H$. Thus, it is clear that since X is assumed to be a solution of YBME (1), Z_1 needs to be an idempotent matrix. Let $E = Z_1$ and $F = Z_4$, whereby the necessity is proved. □

Remark 1. *Theorem 2 can be seen as a corollary of Theorem 1, and Lemmas 1 and 2 are corollaries of Theorem 2.*

4. An Illustration Example

We provide an example to illustrate the process of constructing the solution set of YBME (1).

Example 1. *(see [5]) Let*

$$A = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

We easily obtain a spectral decomposition of A: $A = UV^H$ *with* $U = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$ *and* $V^H = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$. *Since* $V^H U = I$, *we know that* A *is an idempotent matrix of rank 2.*

A simple calculation leads to $\tilde{U}^H = (1,\ 1,\ 1)$ *and* $\tilde{V}^H = (1,\ 1,\ 1)$.

Using Corollary 1, all the solutions of YBME (1) are

$$X = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} E & f \\ g^H & \alpha \end{pmatrix}\begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix}$$

where α *is an arbitrary number,* E *is a* 2×2 *matrix, and* $f,\ g$ *are 2-dimensional vectors.* (E, f, g^H) *is in the set*

$$\{(I, 0, 0)\} \cup \{(\mathbf{0}, f, g^H))|g^H f = 0\} \cup \{(E = ab^H, f = \beta\tilde{b}, g^H = \gamma\tilde{a}^H)|\beta\gamma\tilde{a}^H\tilde{b} = 0\}.$$

In the above set, a, b *are any two 2-dimensional vectors satisfying* $b^H a = 1$. $\tilde{a}, \tilde{b}$ *are two 2-dimensional vectors satisfying* $a^H\tilde{a} = 0,\ b^H\tilde{b} = 0$. β, γ *are numbers.*

According to Theorem 2, all the commuting solutions of YBME (1) are as follows:

$$X = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} E \begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\alpha(1,\ 1,\ 1)$$

where α *is an arbitrary complex number,* E *is any* 2×2 *idempotent matrix, that is,* $E \in S_{E^2=E} = \{I\} \cup \{\mathbf{0}\} \cup S_{id}$. *Here, more specifically,* $E \in S_{id}$ *means that* $E = ab^H$, *and* a, b *are arbitrary 2-dimensional vectors satisfying* $b^H a = 1$.

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