symmetry

MDPI

Article

Terracini Loci of Segre Varieties

Edoardo Ballico †

Department of Mathematics, University of Trento, 38123 Povo, TN, Italy; edoardo.ballico@unitn.it
† The author is a member of GNSAGA of INdAM (Italy).

Abstract: Fix a format $(n_1+1)\times\cdots\times(n_k+1)$, $k>1$, for real or complex tensors and the associated multiprojective space Y. Let V be the vector space of all tensors of the prescribed format. Let $S(Y,x)$ denote the set of all subsets of Y with cardinality x. Elements of $S(Y,x)$ are associated to rank 1 decompositions of tensors $T\in V$. We study the dimension $\delta(2S,Y)$ of the kernel at S of the differential of the associated algebraic map $S(Y,x)\to\mathbb{P}V$. The set $\mathbb{T}_1(Y,x)$ of all $S\in S(Y,x)$ such that $\delta(2S,Y)>0$ is the largest and less interesting x-Terracini locus for tensors $T\in V$. Moreover, we consider the one (minimally Terracini) such that $\delta(2A,Y)=0$ for all $A\nsubseteq S$. We define and study two different types of subsets of $\mathbb{T}_1(Y,x)$ (primitive Terracini and solution sets). A previous work (Ballico, Bernardi, and Santarsiero) provided a complete classification for the cases $x=2,3$. We consider the case $x=4$ and several extremal cases for arbitrary x.

Keywords: Terracini locus; secant variety; Segre variety; multiprojective space

MSC: 15A69; 14N05; 14N07

check for updates

Citation: Ballico, E. Terracini Loci of Segre Varieties. *Symmetry* **2022**, *14*, 2440. https://doi.org/10.3390/sym14112440

Academic Editors: Muhammed Mursaleen and Mikail Et

Received: 30 October 2022
Accepted: 14 November 2022
Published: 17 November 2022

1. Introduction

Fix a format $(n_1+1)\times\cdots\times(n_k+1)$, $k>1$, for real or complex tensors and the associated multiprojective space Y. Let V be the vector space of all tensors of the prescribed format. Let $S(Y,x)$ denote the set of all finite subsets of Y with cardinality x. Elements of $S(Y,x)$ are associated to rank 1 decompositions of tensors of that format with x non-zero terms and the associated has a differential $S(Y,x)\to\mathbb{P}V$, and we call $\delta(2S,Y)$ the kernel of the differential of this algebraic map.

Let $Y=\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$ be a multiprojective space and $\nu: Y\to\mathbb{P}^r$, $r=1+\prod_{i=1}^k(n_i+1)$, its Segre embedding, i.e., the embedding of Y induced by the complete linear system $|\mathcal{O}_Y(1,\dots,1)|$. An element $q\in\mathbb{P}^r$ is an equivalence class of non-zero tensors of format $(n_1+1)\times\cdots\times(n_k+1)$, up to a non-zero scalar multiple. For any $p\in Y$ let $2p$ or $(2p,Y)$ denote the closed subscheme of Y with $(\mathcal{I}_p)^2$ as its ideal sheaf. For any finite set $S\subset Y$ set $2S:=\cup_{p\in S}2p$. Note that $\deg(2p)=1+\dim Y$. As in [1] for any positive integer x let $\mathbb{T}_1(Y,x)$ denote the set of all $S\in S(Y,x)$ such that $h^0(\mathcal{I}_{2S}(1,\dots,1))>0$ and $h^1(\mathcal{I}_{2S}(1,\dots,1))>0$. Let $\mathbb{T}(Y,x)$ denote the set of all $S\in\mathbb{T}_1(Y,x)$ such that Y is the minimal multiprojective space containing S.

The paper publised by [1] considered the set $\mathbb{T}(Y,3)$. Herein, we mostly study $\mathbb{T}(Y,4)$ but also provide some general results, and study 3 remarkable subsets of $\mathbb{T}(Y,x)$. The following results describe all multiprojective spaces Y such that $\mathbb{T}(Y,4)\neq\emptyset$.

Theorem 1. *Set $Y:=\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$ with $k\geq 1$ and $n_1\geq\cdots\geq n_k>0$. We have $\mathbb{T}(Y,4)\neq\emptyset$ if and only if $k\geq 3$, $n_1\leq 3$ and $n_3\leq 2$.*

For an arbitrary integer $x>4$, we prove the following existence theorem.

Theorem 2. *Set $Y:=\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$ with $k\geq 3$ and $n_1\geq\cdots\geq n_k>0$. Fix an integer $x\geq 5$ and assume $n_1\leq x-1$ and one of the following set of conditions:*

(i) $n_2 \leq x - 2$.
(ii) $k \geq 4$ and $n_3 \leq x - 2$.

Then, $\mathbb{T}(Y, x) \neq \emptyset$.

Consider the following highly useful definition ([1], Definition 2.2).

Definition 1. *Let Y be a multiprojective space and $S \subset Y$ a finite set. The set S is said to be minimally Terracini if $\delta(2S, Y) > 0$ and $\delta(2A, Y) = 0$ for all $A \subsetneq S$.*

For each positive integer x, let $\mathbb{T}(Y, x)'$ be the set of all $S \in \mathbb{T}(Y, x)$ which are minimal Terracini.

In Section 6, we prove the following results.

Theorem 3. *Fix integers $k \geq 4$, $x \geq 4$ and $n_1 \geq \cdots \geq n_k > 0$, $1 \leq i \leq k$, such that $n_1 \leq x - 1$ and $n_1 + \cdots + n_k = 2x - 2$. Set $Y := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Then, $\mathbb{T}(Y, x)' \neq \emptyset$ and $\dim \mathbb{T}(Y, x)' \geq x - 4 + \sum_{i=1}^{k}(n_i^2 + 2n_i)$.*

Theorem 4. *Fix integers $x \geq 3$, $k \geq 3$ and $n_1 \geq \cdots \geq n_k > 0$ such that $n_1 = n_2 = x - 1$. Set $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Then, $\mathbb{T}(Y, x)' = \emptyset$.*

In Section 7, we prove the following result.

Theorem 5. *Let Y be a multiprojective space with at least three factors and $\dim Y \geq 7$. Then, $\mathbb{T}(Y, 4)' = \emptyset$.*

Theorem 5 together with the results of Section 6 gives the following list of all multiprojective spaces Y such that $T(Y, 4)' \neq \emptyset$.

Theorem 6. *Let $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with $n_1 \geq \cdots \geq n_k > 0$ for all i. We have $\mathbb{T}(Y, 4)' \neq \emptyset$ if and only if $k \geq 3$, $n_1 \leq 3$ and either $\dim Y = 6$ or $Y \in \{(\mathbb{P}^1)^4, (\mathbb{P}^1)^5, \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1\}$.*

We introduce the following definition.

Definition 2. *Take $S \in \mathbb{T}(Y, x)$. We say that S is primitive if $S' \notin \mathbb{T}(Y, \#S')$ for any $S' \subsetneq S$. Let $\tilde{\mathbb{T}}(Y, x)$ denote the set of all primitive $S \in \mathbb{T}(Y, x)$. For any $S \in \mathbb{T}(Y, x) \setminus \tilde{\mathbb{T}}(Y, x)$ any set $A \subseteq S$ such that $A \in \tilde{\mathbb{T}}(Y, \#A)$ is called a primitive reduction of S.*

Clearly, $\mathbb{T}(Y, x) \supseteq \tilde{\mathbb{T}}(Y, x) \supseteq \mathbb{T}(Y, x)'$. By [1] (Proposition 1.8) $\mathbb{T}(Y, 2) = \emptyset$. By [1] (Theorem 4.12) $\mathbb{T}(Y, 3)' = \emptyset$ if $Y \neq (\mathbb{P}^1)^4$. Remark 16 gives $\mathbb{T}((\mathbb{P}^1)^4, 3)' \neq \emptyset$ and that $S \in \mathbb{T}((\mathbb{P}^1)^4, 3)'$ if and only if $\#\pi_i(S) = 3$ for all $i = 1, 2, 3, 4$, where $\pi_i : (\mathbb{P}^1)^4 \to \mathbb{P}^1$ is the i-th projection.

For any set E in a projective space, $\mathbb{P}^m$, let $\langle E \rangle$ denote the linear span of E in $\mathbb{P}^m$.

For any $q \in \langle \nu(Y) \rangle$, i.e., for any equivalence class of non-zero tensors, the rank $\mathrm{rank}(q)$ of q is the minimal cardinality of a set $S \subset Y$ such that $q \in \langle \nu(S) \rangle$. Let $\mathcal{S}(Y, q)$ denote the set of all $S \in S(Y, \mathrm{rank}(q))$ such that $q \in \langle \nu(S) \rangle$. The set $\mathcal{S}(Y, \mathrm{rank}(q))$ is often called the *solution set* of q. Concision ([2], Proposition 3.1.3.1) says that if $S \in \mathcal{S}(Y, q)$ for some q, then Y is the minimal multiprojective subspace containing S.

Let $\mathbb{S}(Y, x)$ denote the set of all $S \in \mathbb{T}(Y, x)$ such that $S \in \mathcal{S}(Y, q)$ for some q with rank x. An element $q \in \langle \nu(Y) \rangle$ is said to be *concise* if there is no multiprojective space $Y' \subsetneq Y$ such that $q \in \langle \nu(Y') \rangle$. If q is concise, then each $S \in \mathcal{S}(Y, \mathrm{rank}(q))$ has the property that Y is the minimal multiprojective space containing S ([2], Proposition 3.1.3.1). If $S \in \mathcal{S}(Y, q)$ for some q and $\delta(2S, Y) = 0$, then Terracini lemma gives that S is an isolated point of the constructible algebraic set $\mathcal{S}(Y, q)$. This observation provided the main geometric reason to study the Terracini loci.

Using the tangential variety of the Segre variety, we prove the following result.

Theorem 7. *Take $Y = (\mathbb{P}^1)^k$ with $k \ge 5$. Then $\mathbb{S}(Y,k) \cap \mathbb{T}(Y,k) \neq \emptyset$ and $\mathcal{S}(Y,k) \cap \tilde{\mathbb{T}}(Y,k)$ contains an element of the solution set of any concise $q \in \tau(\nu(Y))$.*

We also prove some more precise results for $(\mathbb{P}^1)^k$ with low k. In the section "Conclusions and open questions", we raise and discuss 3 open questions.

We work over an algebraically closed field with characteristic zero $\mathbb{K}$. The reader may assume $\mathbb{K} = \mathbb{C}$. However, the non-existence results are clearly then true for all fields contained in $\mathbb{K}$, i.e., for all fields containing $\mathbb{Q}$. When we mentioned a "general $S \in S(Y,x)$" it is sufficient to take S in a Zariski dense subset of $S(Y,x)$ and in particular, we may take general real rank 1 decompositions of real tensors. For the existence results which use rational normal curves, again we may find solution over $\mathbb{R}$ or over $\mathbb{Q}$.

2. Preliminaries

For any variety W and any positive integer x let $S(W,x)$ denote the sets of all subsets of W with cardinality x. Let $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, $k > 0$, $n_i > 0$ for all i. Let $\nu : Y \to \mathbb{P}^r$, $r = -1 + \prod_{i=1}^k (n_i + 1)$, denote the Segre embedding of i, i.e., the embedding of Y induced by the complete linear system $|\mathcal{O}_Y(1,\dots,1)|$. Let $\pi_i : Y \to \mathbb{P}^{n_i}$ denote the projection of Y onto its i-th factor. For any $S \in S(Y,x)$ the multiprojective space $\prod_{i=1}^k \langle \pi_i(S) \rangle$ is the minimal multiprojective subspace containing S. If $k \ge 2$, let Y_i be the product of all factors of Y, except the i-th one, and let $\eta_i : Y \to Y_i$ denote the projection (η_i is the map that forgets the i-th component of the Y elements).

For any $E \subsetneq \{1,\dots,k\}$, let Y_E be the product of all factors of Y associated to the integer $\{1,\dots,k\} \setminus E$ and $\eta_E : Y \to Y_E$ the projection. If $E = \{1,2\}$, we may write $\eta_{1,2}$ instead of $\eta_{\{1,2\}}$.

For any $(a_1,\dots,a_k) \in \mathbb{Z}^k$ set $\mathcal{O}_Y(a_1,\dots,a_k) := \otimes_{i=1}^k \pi_i^*(\mathcal{O}_{\mathbb{P}^{n_i}}(a_i))$. For any $i \in \{1,\dots,k\}$, let ε_i (resp. $\hat{\varepsilon}_i$) be the element $(a_1,\dots,a_k) \in \mathbb{N}^k$ such that $a_i = 1$ and $a_j = 0$ for all $j \neq i$ (resp. $a_i = 0$ and $a_j = 0$ for all $j \neq i$). We will often use the line bundles $\mathcal{O}_Y(\varepsilon_i)$ and $\mathcal{O}_Y(\hat{\varepsilon}_i)$. For any zero-dimensional scheme $Z \subset Y$ set $\delta(Z,Y) := h^1(\mathcal{I}_Z(1,\dots,1))$. We often write $\delta(Z)$ instead of $\delta(Z,Y)$. For any $p \in Y$, let $2p$ or $(2p,Y)$ denote the closed subscheme of Y with $(\mathcal{I}_p)^2$ as its ideal sheaf. Note that if W is a hypersurface of Y and $p \in \mathrm{Sing}(W)$, then $2p \subset W$. Fix Y and the positive integer x. Terracini lemma and the semicontinuity theorem for cohomology say that $\delta(2S,x) > 0$ and $h^0(\mathcal{I}_{2S}(1,\cdots,1)) > 0$ for all $S(Y,x)$ if and only if the x-secant variety $\sigma_x(\nu(Y))$ of the Segre variety $\nu(Y)$ is defective, i.e., $\sigma_x(\nu(Y)) \subsetneq \langle \nu(Y) \rangle$ and $\dim \sigma_x(\nu(Y)) \le x(\dim Y + 1) - 2$.

Remark 1. *Let $S \subset Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a general subset of Y with cardinality s. The s-secant variety $\sigma_s(\nu(Y))$ is said to be defective if $\sigma_s(Y) \subsetneq \langle \nu(Y) \rangle$ and $\dim \sigma_x(\nu(Y)) \le x(\dim Y + 1) - 2$. We recall that $\sigma_s(\nu(Y))$ is not defective if and only if either $\delta(2S,Y) = 0$ or $h^0(\mathcal{I}_{2S}(1,\dots,1)) = 0$ (or both if $h^0(\mathcal{O}_Y(1,\cdots,1)) = s(1+\dim Y)$). We assume $k \ge 3$ and we use the convention $n_1 \ge \cdots \ge n_k > 0$.*

(a) $\sigma_3(\nu(Y))$ is defective if and only if either $Y = (\mathbb{P}^1)^4$ or $k = 3$, $n_1 \ge 3$ and $n_2 = n_3 = 1$ ([3], Theorem 4.5).

(b) $\sigma_4(\nu(Y))$ is defective if and only if either $Y = (\mathbb{P}^2)^3$ or $k = 3$, $n_2 = 2$, $n_3 = 1$ and $n_1 \ge 4$ ([3], Theorem 4.6).

Remark 2. *By the semicontinuity theorem for cohomology, $\sigma_x(\nu(Y))$ is defective if and only if $\mathbb{T}_1(Y,x) = S(Y,x)$. Fix a general $S \in S(Y,x)$. The multiprojective space Y is the minimal multiprojective space containing S, i.e., $S \in \mathbb{T}(Y,x)$, if and only if each factor of Y has dimension $\le x-1$.*

For any zero-dimensional scheme $Z \subset Y$ and every effective divisor $M \subset Y$, let $\mathrm{Res}_M(Z)$ denote the closed subscheme of Y with $\mathcal{I}_Z : \mathcal{I}_M$ as its ideal sheaf. We have

$\mathrm{Res}_M(Z) \subseteq Z$, $\deg(Z) = \deg(Z \cap M) + \deg(\mathrm{Res}_M(Z))$ and for every line bundle $\mathcal{L}$ on Y we have the following exact sequence, which we call the residual sequence of M:

$$0 \to \mathcal{I}_{\mathrm{Res}_M(Z)} \otimes \mathcal{L}(-M) \to \to \mathcal{I}_{Z\cap M,M} \otimes \mathcal{L}_{|M} \to 0. \quad (1)$$

We have $\mathrm{Res}_M(2p) = p$ if p is a smooth point of M, $\mathrm{Res}_M(2p) = \emptyset$ if $p \in \mathrm{Sing}(M)$ and $\mathrm{Res}_M(2p) = 2p$ if $p \notin M$. If $Z = Z' \cup Z''$ with $Z' \cap Z'' = \emptyset$, then $\mathrm{Res}_M(Z) = \mathrm{Res}_M(Z') \cup \mathrm{Res}_M(Z'')$.

Remark 3. *Fix any multiprojective space $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, $k > 0$, $n_i > 0$ for all i and let $w \subset Y$ be any connected degree 2 zero-dimensional scheme. Fix any $q \in \langle \nu(w) \rangle$ such that $q \neq \nu(w_{\mathrm{red}})$. Set $m := \mathrm{rank}(q)$. We have $1 \le m \le k$ and the minimal multiprojective space Y' containing w is isomorphic to $(\mathbb{P}^1)^m$. If $m > 1$ (and hence $k > 1$), then $\eta_{i|w}$ is an embedding for all $i = 1, \dots, k$. Now assume $m = 1$ and $k > 1$. Let i be the only element of $\{1, \dots, k\}$ such that $\pi_1(Y')$ is isomorphic to $\mathbb{P}^1$ or, equivalently, such that $\#\pi_i(Y') \neq 1$. The map $\eta_{j|w}$ is an embedding if and only if $j \neq i$.*

Lemma 1. *Take any Y, any q and any $S \in \mathcal{S}(Y, q)$. Then all maps $\eta_{i|S}$, $i = 1, \dots, k$, are injective.*

Proof. Assume the existence of $i \in \{1, \dots, k\}$ and $a, b \in S$ such that $a \neq b$ and $\eta_i(a) = \eta_i(b)$, i.e., $\pi_j(a) = \pi_j(b)$ for all $j \in \{1, \dots, k\} \setminus \{i\}$. Set $S' := S \setminus \{a, b\}$. Since $a \neq b$, $\pi_i(a) \neq \pi_i(b)$. Let $L \subset \mathbb{P}^{n_i}$ be the line spanned by $\pi_i(a)$ and $\pi_i(b)$. Let $Y' \subset Y$ be the dimension 1 multiprojective subspace of Y with L as its i-th factor and $\pi_j(a)$ as its j-th factor for all $j \neq i$. Note that $\nu(Y')$ is a line containing $\{\nu(a), \nu(b)\}$. Therefore, there is $e \in L$ such that $q \in \langle \nu(S') \cup \{\nu(e)\} \rangle$. Thus, $\mathrm{rank}(q) < \#S$, is a contradiction. □

Remark 4. *Take any Y with $k \ge 3$ factors, any integer $x > 2$ and any $S \in \mathbb{T}(Y, x)'$. Fix any $A \subset S$ such that $\#A = 2$ and let Y' be the minimal multiprojective subspace containing A. We have $Y' \cong (\mathbb{P}^1)^m$ for some $m \le k$. The integer m is the number of integers $i \in \{1, \dots, k\}$ such that $\#\pi_i(A) > 1$. We have $m \ge 3$, because $\delta(2A, Y) \ge \delta(2A, Y')$ and $\delta(2E, (\mathbb{P}^1)^m) = 2$ for any $E \subset (\mathbb{P}^1)^m$ with $1 \le m \le 2$ and $(\mathbb{P}^1)^m$ the minimal multiprojective space containing E.*

Lemma 2. *Take any Y with $k \ge 3$ factors, $x > 2$, $S \in \mathbb{T}(Y, x)'$ and any $1 \le i < j \le k$. Then $\eta_{i,j|S}$ is injective.*

Proof. Assume that $\eta_{i,j|S}$ is not injective. Take $A \subset S$ such that $\#A = 2$ and $\#\eta_{i,j}(A) = 1$, i.e., $\pi_h(A) = 1$ for all $h \in \{1, \dots, k\} \setminus \{i, j\}$. Thus, the minimal multiprojective space Y' containing A is isomorphic to $\mathbb{P}^1$ or $\mathbb{P}^1 \times \mathbb{P}^1$. By [1] (Lemma 2.3) $\delta(2A, Y) \ge \delta(2A, Y') = 2$, contradicting the assumption $S \in \mathbb{T}(Y, x)'$. □

Remark 5. *Let Y be a multiprojectve space, and $Z \subset Y$ a zero-dimensional scheme. If $\deg(Z) \le 2$, then $\nu(Z)$ is linearly independent. Now assume $\deg(Z) = 3$. Since $\nu(Y)$ is scheme-theoretically cut out by quadrics, $\nu(Z)$ is linearly dependent, i.e., $\langle \nu(Z) \rangle$ is a line, if and only if $\langle Z \rangle \subseteq Y$, i.e., if and only if $\langle Z \rangle$ is a line contained in a ruling of Y.*

Proposition 1. *Take an integer $e \in \{1, 2, 3\}$, a set $E \subset Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ such that $\#E = e$ and a connected degree 2 scheme $v \subset Y \setminus E$. Set $Z := E \cup v$. Assume $h^1(\mathcal{I}_Z(1, \dots, 1)) > 0$. Let W be the minimal subscheme of Z such that $h^1(\mathcal{I}_W(1, \dots, 1)) > 0$. Assume that Y is the minimal multiprojective space containing W.*

(i) *If $e = 1$, then $k = 1$, $n_1 = 1$ and $Z = W$.*
(ii) *Assume $e = 2$ and $k > 1$. Then, $k = 2$, $n_1 = n_2 = 1$ and $W = Z$. Moreover, there is $C \in |\mathcal{O}_Y(1, 1)|$ containing W and the converse holds.*
(iii) *Assume $e = 3$ and $k > 2$. Then, $W = Z$, $k = 3$ and $n_1 = n_2 = n_3 = 1$.*

Proof. Note that $\deg(Z) = e + 2$. Part (a) is true by Remark 5. From now on we assume $k > 1$. We have $\deg(W) \le e + 2$ and $\deg(W) = e + 2$ if and only if $W = Z$. We just proved that $\deg(W) \ge 4$. If $W = W_{\mathrm{red}}$, then we use [4] (Proposition 5.2).

Write $W = v \cup W'$ with $W' \cap vs. = \emptyset$. Since $h^1(\mathcal{I}_W(1,\dots,1)) > 0$, there is $q \in \langle \nu(W') \rangle \cap \langle \nu(v) \rangle$. The minimality of W gives $q \notin \langle \nu(W_1) \rangle$ if either $W_1 \subsetneq W'$ or $W_1 \subsetneq W'$. Note that q is in the tangential variety of $\nu(Y)$. If $q \ne \nu(Y)$, then it has rank $\le \deg(W') \le 3$ and rank 3 only if $e = 3$ and $Z = W$. Thus, $Y \cong (\mathbb{P}^1)^k$ with $k \le 3$ and $k = 3$ only if $e = 3$ and $W = Z$. □

Lemma 3. *Take two-degree 2 connected zero-dimensional schemes $u, v \subset Y$ such that $u \cap v = \emptyset$, Y is the minimal mutiprojective space containing $Z := u \cup v$, $h^1(\mathcal{I}_Z(1,\dots,1)) > 0$ and $h^1(\mathcal{I}_{Z'}(1,\dots,1)) = 0$ for all $Z' \subsetneq Z$. Then, $k \le 2$ and $Y = \mathbb{P}^1 \times \mathbb{P}^1$ if $k = 2$.*

Proof. Assume $k \ge 3$. By assumption $\langle \nu(u) \rangle \cap \langle \nu(v) \rangle$ is a single point, q. Take $C \in |\mathcal{O}_Y(1,1)|$ such that $\deg(Z \cap C) \ge 3$. By [5] (Lemma 5.1) we have $Z \subset C$. Let i be any integer $i \in \{1,\dots,k\}$ such that there is $H_1 \in |\mathcal{O}_Y(\varepsilon_i)|$ such that $e_1 := \deg(Z \cap H_1)$ is maximal. Set $Z_1 := \mathrm{Res}_{H_1}(Z)$. Note that $\deg(Z_1) = z - e_1$. Set $E_1 := H_1 \cap Z$. Note that $\deg(E_1) = e_1$. Let e_2 be the maximal integer such that there is $j \in \{2,\dots,k\}$ and $H_2 \in |\mathcal{O}_Y(\varepsilon_j)|$ such that $e_2 := H_j \cap Z_1$ is maximal. With no loss of generality (we do not impose that the integer n_i is non-increasing) we may assume $j = 2$. We then continue in the same way, defining the integers $e_3, \dots$, the divisors $H_3, \dots$ and the zero-dimensional schemes $E_3, \dots$ and $Z_3, \dots$ such that $E_i := H_i \cap Z_i$, $e_i = \#E_i$, $Z_{i+1} = \mathrm{Res}_{H_1}(Z_i)$ and at each step the integer i is maximal. Note that $e_1 \ge e_2 \ge \cdots \ge e_i \ge e_{i+1}$ and that $e_i = 0$ if and only if $Z \subset H_1 \cup \cdots \cup H_{i-1}$. Since $k \ge \deg(Z) - 1$ there is a maximal integer $c \le k$ such that $e_c \le 1$ (it exists, because $k \ge \deg(Z) - 1$. Since $\mathcal{O}_Y$ is globally generated, [5] (Lemma 5.1) gives $e_c = 0$ and $e_{c-1} \ge 2$. We get $e_1 = e_2 = 2$ and $Z \subset H_1 \cup H_2$. By [5] (Lemma 5.1) we have $h^1(\mathcal{I}_{Z_1}(\hat{\varepsilon}_1)) > 0$. Since the Segre embedding of Y_1 is an embedding, we get $\deg(\eta_1(Z_1)) = 1$. Set $\{a\} := u_{\mathrm{red}}$ and $\{b\} = v_{\mathrm{red}}$. First assume that Z_1 is connected, say $Z_1 = v$. The set $\nu(\eta_1^{-1}(\eta_1(a)))$ is contained in a line contained in $\nu(Y)$, and hence $q \in \nu(Y)$. Since $h^1(\mathcal{I}_{Z'}(1,\dots,1)) = 0$ for all $Z' \subsetneq Z$, $q \ne \nu(a)$. Since $\nu(Y)$ is cut out by quadrics and the intersection of the line $\langle \nu(u) \rangle$ with $\nu(Y)$ contains the degree 3 scheme $\nu(u) \cup \{q\}$, we get $\langle \nu(u) \rangle \subset Y$, and hence $Y = \mathbb{P}^1 \times \mathbb{P}^1$. Now assume $Z_1 = \{a, b\}$. We get $\pi_i(a) = \pi_i(b)$ for all $i > 1$. We also get $\{a, b\} = E_1$, and hence if $n_1 = 1$ we obtain $\pi_1(a) = \pi_1(b)$. Hence, $a = b$, a contradiction, if $n_1 = 1$. Assume $\pi_1(a) \ne \pi_1(b)$, and hence $\pi_1(a)$ and $\pi_1(b)$ are linearly independent. Take $M \in |\mathcal{O}_Y(\varepsilon_3)|$ containing a. Since $\pi_3(a) = \pi_3(b)$, $b \in M$ and hence $\mathrm{Res}_M(Z) \subseteq \{a, b\}$. As above, we get $\pi_i(a) = \pi_i(b)$ for all $i \ne 3$. Thus, $\pi_1(a) = \pi_1(b)$, is a contradiction. □

We recall the following lemma which we learned from K. Chandler ([6,7]).

Lemma 4. *Let W be an integral projective variety, $\mathcal{L}$ a line bundle on W with $h^1(\mathcal{L}) = 0$ and $S \subset W_{\mathrm{reg}}$ a finite set. Then:*

(i) *$h^1(\mathcal{I}_{(2S,W)} \otimes \mathcal{L}) > 0$ if and only if for each $a \in S$ there is a degree 2 scheme $v(a) \subset W$ such that $v(a)_{\mathrm{red}} = 2$ and $h^1(\mathcal{I}_Z \otimes \mathcal{L}) > 0$, where $Z := \cup_{a \in S} v(a)$.*

(ii) *Assume $h^1(\mathcal{I}_{S,W} \otimes \mathcal{L}) = 0$. Take a minimal $Z' \subseteq Z$ containing S and such that $h^1(\mathcal{I}_{Z'} \otimes \mathcal{L}) > 0$. Then, $h^1(\mathcal{I}_{Z'} \otimes \mathcal{L}) = 1$.*

Lemma 5. *Fix $S \in \mathbb{T}(Y, x)'$ and take Z as in Lemma 4, i.e., assume $Z_{\mathrm{red}} \supseteq S$, that each connected component of Z has degree ≤ 2, $h^1(\mathcal{I}_Z(1,\dots,1)) = 1$ and $h^1(\mathcal{I}_{Z'}(1,\dots,1)) = 0$ for all $Z' \subsetneq Z$. Then $Z_{\mathrm{red}} = S$.*

Proof. Assume $S' := Z_{\mathrm{red}} \ne S$. The "if" part of Lemma 4 gives $\delta(2S', Y) > 0$. Thus, $S \nsubseteq \mathbb{T}(Y, x)'$, is a contradiction. □

Remark 6. *Take Z as in Lemma 4 for $x = 4$ and assume $S \in \mathbb{T}(Y,4)'$. Take a closed subscheme $W \subsetneq Z$ such that $3 \le \deg(W) \le 4$ and $\#W_{\mathrm{red}} = 3$. Let Y' be the minimal multiprojective space containing W_{red}. Assume the existence of at least $k-3$ indices such that $\#\pi_i(W_{\mathrm{red}}) = 1$, i.e., $Y' \cong \mathbb{P}^{m_1} \times \mathbb{P}^{m_2} \times \mathbb{P}^{m_3}$ with $0 \le m_i \le 2$ for all i. By [1] (Theorem 4.12) and a dimensional count, we get $\#W_{\mathrm{red}} \neq 3$.*

Remark 7. *Take $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, $k \ge 2$. As in [8] (Examples 2 and 3), let $\mathcal{C}(Y)$ denote the set of all curves $f(\mathbb{P}^1)$, where $f : \mathbb{P}^1 \to Y$ is a morphism with $\pi_i \circ f$ an isomorphism if $n_i = 1$, while $\pi_i \circ f$ is an embedding with as its image a rational normal curve if $n_i \ge 2$. Each $C \in \mathcal{C}(Y)$ is called a rational normal curve of Y. The set $\mathcal{C}(Y)$ is an integral quasi-projective variety and* $\dim \mathcal{C}(Y) = -3 + \sum_{i=1}^{k}(n_i^2 + 2n_i)$.

3. The Tangential Variety

Among the Terracini loci we obtain an interesting family from the tangential variety $\tau(\nu(Y))$ of the Segre variety. Since $\nu(Y)$ is smooth, $\tau(\nu(Y))$ is the union of all lines $L \subset \langle \nu(Y) \rangle$ such that $L \cap \nu(Y)$ contains a degree 2 connected zero-dimensional scheme.

From now on in this section, we only consider concise $q \in \tau(\nu(Y))$, i.e., we take $Y = (\mathbb{P}^1)^k$, $k \ge 2$, and take $q \in \tau(\nu(Y))$ such that $\mathrm{rank}(q) = k$.

Lemma 6. *Take $Y = (\mathbb{P}^1)^k$, $k \ge 3$. Take $q \in \tau(\nu(Y))$ such that* $\mathrm{rank}(q) = k$. *Then there is a unique connected degree* 2 *zero-dimensional scheme v such that $q \in \langle \nu(v) \rangle$.*

Proof. The existence part is true because $\nu(Y)$ is smooth. Assume the existence of another such a scheme w and set $Z := v \cup w$. Thus, $3 \le \deg(Z) \le 4$. The case $\deg(Z) = 4$, i.e., $u \cap v = \emptyset$ is excluded by Lemma 3. The case $\deg(Z) = 3$, i.e., $u_{\mathrm{red}} = v_{\mathrm{red}}$ is excluded, because in this case Z is not Gorenstein ([9], Lemma 2.3). □

Lemma 7. *Take Y with $k \ge 3$ factors. Let $Z \subset Y$ be the union of two degree* 2 *connected zero-dimensional scheme, u and v, and a point, c. Let Y' be the minimal multiprojective space containing Z. Assume $h^1(\mathcal{I}_Z(1,\dots,1)) > 0$ and take a minimal subscheme $W \subseteq Z$ such that $h^1(\mathcal{I}_W(1,\dots,1)) > 0$. Then., $W = Z$ and $Y' \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. If $W \neq Z$, we obtain a contradiction by Lemma 3 and Proposition 1. Thus, we may assume $W = Z$ and that either $k \ge 4$ or $n_i \ge 2$ for at least one integer i. We do not assume that the dimensions of the Y factors are non-increasing and hence we may permute the factors of Y to simplify the notation. Let e_1 be the maximal integer $\deg(Z \cap H_1)$ for some $i \in \{1,\dots,k\}$ and some $H_1 \in |\mathcal{O}_Y(\varepsilon_i)|$. Note that $e_1 \ge \max\{n_1,\dots,n_k\}$. Permuting the factors of Y, we may assume $i = 1$. Set $Z_1 := \mathrm{Res}_{H_1}(Z)$. Let e_2 be the maximal integer $\deg(Z_1 \cap H_2)$ for some $i \in \{2,\dots,k\}$ and some $H_2 \in |\mathcal{O}_Y(\varepsilon_i)|$. With no loss of generality, we may assume $i = 2$. Set $Z_2 := \mathrm{Res}_{H_2}(Z_1)$. We define in the same way e_3, e_4, Z_3, Z_4. Since either $k \ge 4$ or $n_i \ge 2$ for at least one integer i, $e_1 + \cdots + e_4 \ge 4$, and hence $\deg(Z_4) \le 1$. Thus, $h^1(\mathcal{I}_{Z_4}) = 0$. By [5] (Lemma 5.1) we have $Z \subset H_1 \cup \cdots \cup H_4$. We also get that the last integer i with $e_i > 0$ satisfies $e_i \ge 2$. Thus, $e_1 = 3$ and $e_2 = 2$. Since $h^1(\mathcal{I}_{Z_1}(\hat{\varepsilon}_1)) > 0$, $\deg(\pi_i(Z_1)) = 1$ for al $i > 1$. Set $W_1 := \mathrm{Res}_{H_2}(Z)$. Since $h^1(\mathcal{I}_{W_1}(\hat{\varepsilon}_2)) > 0$, Remark 4 gives that there is either $G \subseteq W_1$ with $\deg(G) = 2$ and $\deg(\eta_2(G)) = 1$ or $\deg(W_1) = 3$ and there is $i \in \{1,\dots,k\} \setminus \{2\}$ with $\deg(\pi_j(W_1)) = 1$ for all $j \in \{1,\dots,k\} \setminus \{i,2\}$ and $\dim\langle \pi_1(W_1) \rangle = 1$. Since $\deg(\eta_2(Z_1)) = 1$, $\langle \nu(Z_1) \rangle$ is contained in the second ruling of $\nu(Y)$. Thus, the plane $\langle \nu(Z \cap H_1) \rangle$ intersects another point $\alpha = \nu(\beta)$ of $\nu(Y)$. Proposition 1 implies that the minimal multiprojective space Y'' containing $Z \cap H_1$ is contained in $\mathbb{P}^1 \times \mathbb{P}^1$ and that $Z \cap H_1 \cup \beta$ is contained in a curve of bidegree $(1,1)$ of $\mathbb{P}^1 \times \mathbb{P}^1$. Thus, $n_1 = 1$ and, since $k \ge 3$, there are $a_i \in \mathbb{P}^{n_i}$, $1 \le i \le k$, $a_1 \in \pi_1(Y')$, $a_2 \in \pi_2(Y')$ such that $Y' = \mathbb{P}^1 \times \mathbb{P}^1 \times \{a_3\} \times \cdots \times \{a_k\}$ and $\beta = (a_1,\dots,a_k)$. The line $\langle \nu(Z_1) \rangle$ contains α. Hence, $\pi_i(Z_1) = a_i$, except for at most one i. Since $k \ge 3$, we get $Y' \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. □

We recall the following result ([8], Proposition 7).

Lemma 8. *Fix a concise $q \in \tau(\nu(Y)) \setminus \nu(Y)$ and set $k := \mathrm{rank}(q)$. Then, $Y = (\mathbb{P}^1)^k$ and $\dim \mathcal{S}(Y, q) \geq 2k - 2$.*

Proposition 2. *Take $Y = (\mathbb{P}^1)^4$. Then $\mathbb{S}(Y, 4) \cap \mathbb{T}(Y, 4)'$ contains a 9-dimensional family associated to rank 4 points $q \in \tau(\nu(Y))$ and each $S \in \mathcal{S}(Y, q)$ satisfies $\delta(2S) \geq 6$, $h^0(\mathcal{I}_{2S}(1, 1, 1, 1)) \geq 2$.*

Proof. Since $h^0(\mathcal{O}_Y(1, 1, 1, 1)) = 16$ and $4(1 + \dim Y) = 20$, the proposition follows from Lemma 8, Terracini lemma and the fact that Y is the minimal multiprojective space containing a set evincing the rank of a concise $q \in \langle \nu(Y) \rangle$. □

Proof of Theorem 7. Fix any $q \in \tau(\nu(Y))$ with is concise, i.e., $\mathrm{rank}(q) = k$, and let $v \subset Y$ be the only degree 2 connected zero-dimensional scheme such that $q \in \langle \nu(v) \rangle$ (Lemma 6). Set $\{o\} := v_{\mathrm{red}}$, say $o = (o_1, \dots, o_k)$. Take $\Sigma = \Sigma_1 \cup \dots \cup \Sigma_k$ as in Remark 7. Take a general hyperplane M of $\langle \nu(Y) \rangle$ passing through q and let $H \in |\mathcal{O}_Y(1, \dots, 1)|$ be the element corresponding to M. Since $\nu(o) \neq q$ and M is general, $o \notin H$. Thus, for $i = 1, \dots, k$ there is a unique $a(i) \in \Sigma_i \cap H$, and $a(i) \neq o$. Set $S := \{a(1), \dots, a(k)\}$. Note that $\langle \Sigma \rangle = T_{\nu(o)}\nu(Y)$, and that $\langle \Sigma \rangle = \{\nu(o) \cup \nu(S)\}$. Since q is contained in the hyperplane $M \cap \langle \Sigma \rangle$, and M is associated to H, $q \in \langle S \rangle$. Since $\mathrm{rank}(q) = k$, $S \in \mathcal{S}(Y, q)$. Varying M among the hyperplanes of $\langle \nu(Y) \rangle$ containing q, we get that S is not an isolated point of $\mathcal{S}(Y, q)$. Thus, $\delta(2S) > 0$. Since $\dim Y = k \geq 5$, we have $k(k+1) \leq 2^k$, and hence $h^0(\mathcal{I}_{2S}(1, \dots, 1)) > 0$. Thus, $S \in \mathbb{T}(Y, k)$. To check that $S \in \tilde{\mathbb{T}}(Y, k)$ it is sufficient to observe that for any $a(i) \in S$ the $(k-1)$-dimensional multiprojective space $\pi_i^{-1}(o_i)$ contains the set $S \setminus \{a(i)\}$. □

4. The Usual Terracini Sets and the Solution Sets

Remark 8. *We have $\mathbb{S}(Y, 2) = \emptyset$ for any Y, because $\mathbb{T}(Y, 2) = \emptyset$ ([1], Proposition 1.8).*

Remark 9. *Obviously $\mathbb{T}(\mathbb{P}^n, x) = \emptyset$ for all $x > 0$.*

Lemma 9. *Take $Y = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ with $n_1 \geq n_2 > 0$. Then, $\mathbb{T}(Y, n_1 + 1) = \emptyset$*

Proof. First assume $n_1 = n_2$. Since Y is the minimal multiprojective space containing Y, $\mathbb{T}(Y, n_1 + 1) = \emptyset$ by [1] (Lemma 2.4).

Now assume $n_1 > n_2$. We use induction on the non-negative integer $n_1 - n_2$. Assume the existence of $S \in \mathbb{T}(Y, n_1 + 1)$. To obtain a contradiction, it is sufficient to prove that $h^0(\mathcal{I}_{2S}(1, 1)) = 0$. Since Y is the minimal multiprojective space containing S, $\langle \pi_1(S) \rangle = \mathbb{P}^{n_1}$, i.e., $\pi_{1|S}$ is injective and $\pi_1(S)$ is linearly independent. Since $\#S > n_2 + 1$, there is $S' \subset S$ such that $\#S' = n_1$ and $\langle \pi_2(S') \rangle = \mathbb{P}^{n_2}$. Set $\{p\} := S \setminus S'$. Let H be the only element of $|\mathcal{O}_Y(\varepsilon_1)|$ containing S'. Since $\langle \pi_2(S') \rangle = \mathbb{P}^{n_2}$, H is the minimal multiprojective space containing S'. Hence, the inductive assumption gives $h^0(H, \mathcal{I}_{2(S \cap H, H}(1, 1)) = 0$. We have $\mathrm{Res}_H(2S) = 2p \cup S'$. Since $h^0(\mathcal{I}_{2p}(0, 1)) = 0$, the residual exact sequence of H gives $h^0(\mathcal{I}_{2S}(1, 1)) = 0$. □

Theorem 8. *If $Y = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$, then $\mathbb{T}(Y, x) = \emptyset$ for all x.*

Proof. We may assume $n_1 \geq n_2 > 0$. Assume the existence of $S \in \mathbb{T}(Y, x)$. The definition of $\mathbb{T}(Y, x)$, gives $x \geq n_1 + 1$ and the existence of $A \subseteq S$ such that $\#A = n_1 + 1$ and $\langle \pi_1(A) \rangle = \mathbb{P}^{n_1}$, i.e., $\pi_{1|A}$ is injective and $\pi_1(A)$ is linearly independent. To obtain a contradiction, it is sufficient to find $S' \subseteq S$ such that $h^0(\mathcal{I}_{2S'}(1, 1)) = 0$. Let Y' be the minimal multiprojective space containing A. Since $\langle \pi_1(A) \rangle = \mathbb{P}^{n_1}$, $Y' \cong \mathbb{P}^{n_1} \times \mathbb{P}^s$ for some integer $s \in \{0, \dots, n_2\}$. If $s = n_2$, then we may take $S' = A$ by Lemma 9. Assume $s < n_2$. We use induction on $n_2 - s$ allowing the case $s = 0$. Thus, we reduce to prove the existence of S' in the case $s = n_2 - 1$ for some $n_2 \geq 1$. In this case $Y' \in |\mathcal{O}_Y(0, 1)|$. Since Y is the minimal multiprojective space containing S there is $o \in S \setminus A$ such that $o \notin H$. We claim that we may take $S' = A \cup \{o\}$. Consider the residual exact sequence

$$0 \to \mathcal{I}_{2o\cup A}(1,0) \to \mathcal{I}_{2S'}(1,1) \to \mathcal{I}_{(2A,H),H}(1,1) \to 0 \tag{2}$$

of H. Lemma 9 gives $h^0(H, \mathcal{I}_{(2A,H),H}(1,1)) = 0$. Clearly $h^0(\mathcal{I}_{2o}(1,0)) = 0$ (Remark 9). □

We recall the following result ([10], Proposition 2.3).

Lemma 10. *Take $Y := \mathbb{P}^m \times \mathbb{P}^m \times \mathbb{P}^m$, $m \geq 3$. Then, each secant variety of $\nu(Y)$ has the expected dimension.*

Proposition 3. *Take $k \geq 3$ and $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with $n_1 = n_2 = n_3 = m \geq 3$. Then, $\mathbb{T}(Y, m+1) = \emptyset$.*

Proof. Let $\pi_{1,2,3} : Y \to Y' := (\mathbb{P}^m)^3$ denote the projection of Y onto its first three factors. Assume the existence of $S \in \mathbb{T}(Y, m+1)$. In particular, Y is the minimal multiprojective space containing Y and hence $\#\pi_{4,\dots,4}(S) = m+1$ and Y' is the minimal multiprojective space containing $S' := \pi_{1,2,3}(S)$. Thus, S' is in the open orbit for the action of $(\mathrm{Aut}(\mathbb{P}^m))^3$ of $S(Y', m+1)$. Lemma 10 gives $\dim \sigma_{m+1}(\nu(Y')) = (m+1)(3m+1) - 1 < (m+1)^3$. Hence, $\delta(2S', Y') = 0$. If $k = 3$, then $Y = Y'$. If $k \geq 4$ we see Y' as a multiprojective subspace of Y fixing $p_i \in \mathbb{P}^{n_i}$, $4 \leq i \leq k$, and applying $k-3$ times [1] (Proposition 2.7), we get $\delta(2S, Y) = 0$. □

Lemma 11. *Fix a finite set $S \subset Y$ and $a \in Y \setminus S$. Assume the existence of $i \in \{1, \dots, k\}$ such that $\pi_i(a) \in \pi_i(S)$. Then, $\delta(2S, Y) < \delta(2(S \cup \{a\}), Y)$.*

Proof. The thesis of the lemma is equivalent to proving the following statement: $T_{\nu(a)}\nu(Y) \cap \langle \cup_{b\in S} T_{\nu(b)}\nu(Y)\rangle \neq \emptyset$. By assumption, there are $i \in \{1, \dots, k\}$ and $b \in S$ such that $\pi_i(a) = \pi_i(b)$. Thus, $T_{\nu(a)}\nu(Y) \cap T_{\nu(b)}\nu(Y)$ contains a point of $\nu(Y)$. □

Lemma 12. *Fix integers $x > m > 0$ and $E \subset \mathbb{P}^m$ such that $\#E = x$ and $\langle E \rangle = \mathbb{P}^m$. Set $Y := \mathbb{P}^m \times (\mathbb{P}^1)^{k-1}$ for some $k \geq 2$. Fix $o_2, \dots, o_k \in \mathbb{P}^1$ and let $A \subset Y$ be the set of all $(a, o_2, \dots, a_k)$, $a \in E$. Fix $u \in Y \setminus A$ such that $\pi_1(u) \in E$. Then, $\delta(2(A \cup \{u\}), Y) > \delta(2A, Y) \geq (x-1)(m+1)$.*

Proof. The first inequality is true by Lemma 11. We have $\delta(2A, Y) \geq \delta(2E, \mathbb{P}^m)$ ([1], Lemma 2.3). Clearly, $\delta(2E, \mathbb{P}^m) = (x-1)(m+1)$. □

Remark 10. *Take $k = 4$, $m = 1$ and $x = 3$ in the set-up of Lemma 15. Thus, $Y = (\mathbb{P}^1)^4$. We get elements of $\mathbb{T}(Y, 4)$, because $h^0(\mathcal{O}_Y(1,1,1,1)) = 16$, $4(\dim Y + 1) = 20$ and $1 + (x-1)(m+1) = 5$.*

Lemma 13. *Take $Y = (\mathbb{P}^1)^3$ and any $S \in S(Y, 3)$. Let Y' be the minimal multiprojective space containing S.*

1. *If $Y' = Y$, then $h^0(\mathcal{I}_{2S}(1,1,1)) \leq 1$; $h^0(\mathcal{I}_Y(1,1,1)) > 0$, if and only if S is as in [1] (Proposition 3.2 (iv)). If S is as in [1] (Proposition 3.2 (iv)) with $\{i, j\} = \{1, 2\}$, then the only element, W, of $|\mathcal{I}_{2S}(1,1,1)|$ is of the form $W = W_1 \cup W_2 \cup W_3$ with $W_i \in |\mathcal{O}_Y(\varepsilon_i)|$. If S is as in [1] (Proposition 3.2 (iv)) with $W_i \in |\mathcal{O}_Y(\varepsilon_i)|$. Moreover, $\dim \mathrm{Sing}(W) = 1$.*
2. *If $Y' \cong \mathbb{P}^1 \times \mathbb{P}^1$, then $h^0(\mathcal{I}_{2S}(1,1,1)) = 1$.*
3. *If $Y' \cong \mathbb{P}^1$, then $1 \leq h^0(\mathcal{I}_{2S}(1,1,1)) \leq 2$.*

Proof. The case $Y' = Y$ is proved in the proof of [1] (Lemma 4.2) with the description of all cases with $h^0(\mathcal{I}_{2S}(1,1,1)) = 1$. It is easy to see that a reducible surface $W = W_1 \cup W_2 \cup W_3$

is singular at all points of S. Thus, W is the only element of $|\mathcal{I}_{2S}(1,1,1)|$. Note that $\mathrm{Sing}(Y)$ is the union of 3 curves.

Assume $Y' \cong \mathbb{P}^1 \times \mathbb{P}^1$. Obviously, $h^0(Y', \mathcal{I}_{(2S,Y')}(1,1,1)) = 0$. Thus, $\delta(2S, Y') = 5$. With no loss of generality, we may assume $\#\pi_3(S) = 1$, i.e., $Y' \in |\mathcal{O}_Y(\varepsilon_3)$. Consider the residual exact sequence of Y':

$$0 \to \mathcal{I}_S(1,1,0) \to \mathcal{I}_{2S}(1,1,1) \to \mathcal{I}_{(2S,Y')}(1,1,1) \to 0. \tag{3}$$

We have $h^0(\mathcal{I}_S(1,1,0) = 1$, because Y' is the minimal multiprojective space containing S. Therefore, $h^1(\mathcal{I}_S(1,1,0)) = 0$. Thus, $\delta(2S, Y) = \delta(2S, Y') = 5$ and $h^0(\mathcal{I}_{2S}(1,1,1)) = 1$, concluding the proof of this case.

Assume $Y' \cong \mathbb{P}^1$. Clearly, $\delta(2S, Y') = 4$. With no loss of generality, we may assume $\#\pi_i(S) = 1$ for $i = 2,3$, i.e., the existence of $o_2, o_3 \in \mathbb{P}^1$ such that $Y' = \mathbb{P}^1 \times \{o_2\} \times \{o_3\}$. Set $Y'' := \mathbb{P}^1 \times \mathbb{P}^1 \times \{o_3\}$. Thus, $Y' \in |\mathcal{O}_{Y''}(0,1)|$. Consider the residual exact sequence of Y' in Y'':

$$0 \to \mathcal{I}_{S,Y''}(1,0) \to \mathcal{I}_{2S,Y''}(1,1) \to \mathcal{I}_{(2S,Y')}(1,1,1) \to 0. \tag{4}$$

Since $h^0(Y'', \mathcal{I}_{(S,Y'')}(1,0)) = h^0(Y', \mathcal{I}_{2S,Y'}(1,1)) = 0$, (4) gives $h^0(Y'', \mathcal{I}_{2S,Y''}(1,1)) = 0$, and hence $\delta(2S, Y'') = 5$. Then, the exact sequence (3) with Y'' instead of Y' gives $1 \le h^0(\mathcal{I}_{2S}(1,1,1)) \le 2$. $\square$

Remark 11. *Take any multiprojective space Y and any positive integer x. Assume the existence of $S \in \mathbb{T}(Y,x)$ and $W \in |\mathcal{I}_{2S}(1,\dots,1)|$ such that* $\mathrm{Sing}(W) \supsetneq S$ *and take any $p \in \mathrm{Sing}(W) \setminus S$. Since $\delta(2(S \cup \{p\}), Y) \ge \delta(2S, Y) > 0$, Y is the minimal multiprojective space containing $S \cup \{p\}$ and $W \in |\mathcal{I}_{2(S\cup\{p\})}(1,\dots,1)|$, $S \cup \{p\} \in \mathbb{T}(Y, x+1)$. Hence, if* $\dim \mathrm{Sing}(W) > 0$, *then $\mathbb{T}(Y,y) \neq \emptyset$ for all $y > x$.*

Remark 12. *We claim that $(\mathbb{P}^2)^3$ is the only multiprojective space such that $\mathbb{T}_1(Y,4) = S(Y,4)$. If $k \ge 3$ it is sufficient to use part (b) of Remark 1. If $k \le 2$ use Remark 9 and Theorem 8.*

Proposition 4. *Fix any multiprojective space Y. Set $n := \dim Y$,*

$$w := \lceil (1 + h^0(\mathcal{O}_Y(1,\dots,1)))/(n+1) \rceil, \ \ z := \max\{n+1, w\}.$$

Then, $\tilde{\mathbb{T}}(Y,x) = \emptyset$ for all $x > z$.

Proof. Fix $A \subset Y$ such that $\#A \ge z$. Since $\dim \sigma_x(\nu(Y)) \le (x+1)(n+1) - 1$, the semicontinuity theorem for cohomology gives $h^1(\mathcal{I}_{2A}(1,\dots,1)) > 0$. Take any $x > z$ and any $S \in \mathbb{T}(Y,x)$. We saw that every $A \subset S$ with $\#A = z$ has $h^1(\mathcal{I}_{2A}(1,\dots,1)) > 0$. Since $A \subset S$, $h^0(\mathcal{I}_{2A}(1,\dots,1)) > 0$. Thus, to prove that $S \notin \tilde{\mathbb{T}}(Y,x)$ it is sufficient to find A with the additional condition that Y is the minimal multiprojective space containing A. We claim the existence of $E \subset S$ such that $\#E \le n+1$ and Y is the minimal multiprojective space containing E. Take any $a_1 \in S$. The set $Y(1) := \{a_1\}$ is the minimal multiprojective space containing a_1. Since Y is the minimal multiprojective space containing S, there is $a(2) \in S$ such that the minimal multiprojective space $Y(2)$ containing $\{a_1, a_2\}$ strictly contains $Y(1)$, and hence, $\dim Y(2) > \dim Y(1)$. Furthermore, so on to get E after at most $n-1$ steps. $\square$

Almost always $w \ge n+1$. For instance, if $n_i = 1$ for all i (and hence $n = k$) we have $w \ge n+1$ if and only if $k \ge 5$.

Proposition 5. *Fix integers $x \ge 3$ and $k \ge 3$. Fix $n_1 \ge \cdots \ge n_k > 0$ such that $n_1 \le x-1$, $n_2 \le x-1$ and $n_3 \le x-2$. Set $Y := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Assume $\sigma_{x-1}(\nu(Y)) \neq \langle \nu(Y) \rangle$. Fix lines $L \subset \mathbb{P}^{n_1}$, $R \subset \mathbb{P}^{n_2}$ and points $o_i \in \mathbb{P}^{n_i}$, $3 \le i \le k$. Let $Y' \subset Y$ the multiprojective space with L as its first factor R as its second factors and $\{o_i\}$ as its i-th factor $3 \le i \le k$. Fix a general $(a,b) \in Y' \times Y'$ and a general $S' \subset Y$ with $\#S' = x-2$. Set $S := S' \cup \{a,b\}$. Let q be a general element of $\langle \nu(S) \rangle$. Then,* $\mathrm{rank}(q) = x$ *and Y is the minimal multiprojective space containing S.*

Proof. Y' is the minimal multiprojective space containing $\{a,b\}$. Since $n_1 \ge \cdots \ge n_k > 0$, $n_1 \le x-1$, $n_2 \le x-1$, $n_3 \le x-2$, and S' is general, Y is the minimal multiprojective space containing S. Assume $\mathrm{rank}(q) \le x-1$. Thus, $q \in \sigma_{x-1}(\nu(Y))$. Since $\mathrm{Aut}(\mathbb{P}^{n_h})$, $h=1,2$, acts transitively on the Grassmannian of the lines of $\mathbb{P}^{n_h}$, $\mathrm{Aut}(\mathbb{P}^{n_i})$, $i=3,\dots,k$, a is general in Y' and S' is general in Y, $S' \cup \{a\}$ is a general subset of Y with cardinality $x-1$. Hence, varying S' and a the union of the sets $\langle \nu(S' \cup \{a\})\rangle$ covers a non-empty open subset of $\sigma_{x-1}(\nu(Y))$. Since for a fixed $S' \cup \{a\}$ the point b is a general point of Y', the closure of the union of all $\langle \nu(S)\rangle$ is the join, J, of $\nu(Y')$ and $\sigma_{x-1}(\nu(Y))$. Since $q \in \sigma_{x-1}(\nu(Y))$, we get that $\sigma_{x-1}(\nu(Y))$ is a cone with vertex containing $\nu(Y')$. Since Y is the image of Y' by the action of the group $\prod_{h=1}^{k} \mathrm{Aut}(\mathbb{P}^{n_h})$, we get that $\sigma_{x-1}(\nu(Y))$ is a cone with vertex containing $\nu(Y)$. Thus, $\sigma_{x-1}(\nu(Y)) = \langle \nu(Y)\rangle$, a contradiction. □

Remark 13. *Note that* $\mathbb{T}(Y, n_1+1) = \tilde{\mathbb{T}}(Y, n_1+1)$.

Lemma 14. *Take* $Y = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3}$. *We have* $\mathbb{T}(Y,4) \neq \emptyset$ *if* $(n_1,n_2,n_3) \in \{(3,1,1),(2,1,1),(2,2,1)\}$.

Proof. Fix a line $L \subset \mathbb{P}^{n_1}$, $a_1, b_1 \in L$ such that $a_1 \neq b_1$ and $o_i \in \mathbb{P}^{n_i}$, $i=2,3$. Set $Y' := L \times \{o_2\} \times \{o_3\}$, $a = (a_1,o_2,o_3)$, and $b = (b_1,o_2,o_3)$. Since $\delta(2\{a,b\},Y') = 2$, $\delta(2\{a,b\},Y) \ge 2$.

Take H_i, $i=2,3$, such that $o_i \in \pi_i(H_i)$. Take a general $H_1 \in |\mathcal{O}_Y(\varepsilon_1)|$ and set $W := H_1 \cup H_2 \cup H_3$. Note that $\{a,b\} \subset H_2 \cap H_3$ and hence $\{a,b\} \subset \mathrm{Sing}(H_2 \cup H_3) \subset \mathrm{Sing}(W)$. By Remark 11, it is sufficient to find $c,d \in \mathrm{Sing}(W) \setminus \{a,b\}$ such that $c \neq d$ and Y is the minimal multiprojective space containing $S := \{a,b,c,d\}$. Since M is general, $\mathbb{P}^{n_3} = \langle L \cup \pi_1(M)\rangle$.

Assume $(n_1,n_2,n_3) \in \{(3,1,1),(2,1,1),(2,2,1)\}$. Take a general $c \in M \cap H_2$ and a general $d \in M \cap H_3$. Since $\pi_1(H_2) = \pi_1(H_3) = \mathbb{P}^{n_1}$, $\mathbb{P}^{n_3} = \langle L \cup \pi_1(M)\rangle$ and c, d are general, $\langle \pi_1(S)\rangle = \mathbb{P}^{n_1}$. Since c is general and $\pi_3(H_2) = \mathbb{P}^1$, $\pi_3(S)$ spans $\mathbb{P}^1$. Since c and d are general, $\pi_2(H_3) = \mathbb{P}^{n_2}$ and $\langle \pi_2(S)\rangle = \mathbb{P}^{n_2}$. □

Lemma 15. *Assume* $k \ge 3$, $n_1 \in \{2,3\}$ *and* $n_i \le 2$ *for all* $i = 2,\dots,k$. *Then,* $\mathbb{T}(Y,4) \neq \emptyset$. *If* $n_1 = 3$, *then* $\tilde{\mathbb{T}}(Y,4) \neq \emptyset$.

Proof. With no loss of generality, we may assume $n_1 \ge \cdots \ge n_k > 0$. Since $\mathbb{T}(Y,4) = \tilde{\mathbb{T}}(Y,4)$ if $n_1 = 3$ (Remark 13) it is sufficient to prove that $\mathbb{T}(Y,4) \neq \emptyset$. Fix a line $L \subset \mathbb{P}^{n_1}$, $a_1, b_1 \in L$ such that $a_1 \neq b_1$ and $o_i \in \mathbb{P}^{n_i}$, $2 \le i \le k$. Set $a := (a_1, o_2, o_k)$, $b := (b_1, o_2, \dots, o_k)$ and $Y' := L \times \{o_2\} \times \cdots \times \{o_k\}$. Since $\delta(2(\{a,b\}),Y') = 2$, $\delta(2(\{a,b\},Y)) \ge 2$. Fix a general $(c,d) \in Y \times Y$ and set $S := \{a,b,c,d\}$. Note that Y is the minimal multiprojective space containing S. We have $\delta(2S,Y) \ge \delta(2(\{a,b\}),Y) \ge 2$. Thus, to prove that $S \in \mathbb{T}(Y,4)$ it is sufficient to prove that $h^0(\mathcal{I}_{2S}(1,\dots,1)) > 0$. Since $\delta(2S,Y) \ge 2$, it is sufficient to prove that

$$4(n_1 + \cdots + n_k + 1) \le 1 + \prod_{i=1}^{k}(n_i+1). \tag{5}$$

Since $k \ge 3$, the difference $\psi(n_1,\dots,n_k)$ between the right-hand side and the left-hand side of (5) is a non-decreasing function of each n_i. If $n_1 = 1$ (and hence $n_i = 1$ for all i, then (5) is satisfied if and only if $k \ge 5$. Theorems 10 and 11 in the next section give $\mathbb{T}((\mathbb{P}^1)^k,4) \neq \emptyset$ for $k = 3,4$. For $k \ge 3$ we have $\psi(n_1,\dots,n_k) < \psi(n_1,\dots,n_k,1)$. We have $\psi(3,3,1) = 1$, $\psi(3,2,2) = 5$, $\psi(3,2,1,1) = 17$, $\psi(2,2,1,1) = 8$, $\psi(3,1,1,1) = 1$. Thus, it is sufficient to check all $(n_1,\dots,n_k)$ in the following list $(2,1,1)$, $(2,2,1)$, $(3,1,1)$, $(2,1,1,1)$. This is done in Lemma 14. □

Lemma 16. *Assume* $k \ge 3$, $n_1 = n_2 = 3$ *and* $n_3 \le 2$. *Then,* $\mathbb{T}(Y,4) \neq \emptyset$.

Proof. Fix lines $L, R \subset \mathbb{P}^3$ and $o_i \in \mathbb{P}^{n_i}$, $3 \le i \le k$. Set $Y' := L \times R \times \{o_3\} \times \cdots \times \{o_k\}$. Fix a general $(a,b) \in Y' \times Y'$. Since $\delta(2\{a,b\}, Y') = 2$, we have $\delta(2\{a,b\}, Y) \ge 2$. Fix a general $(c,d) \in Y \times Y$ and set $S := \{a,b,c,d\}$. Note that Y is the minimal multiprojective space containing S and that $\delta(S,Y) \ge \delta(2\{a,b\}, Y) \ge 2$. Thus, to prove that $S \in \mathbb{T}(Y,4)$ it is sufficient to prove that the inequality (5) is satisfied. As in the proof of Lemma 15, it is sufficient to observe that it is satisfied if $k = 3$ and $n_3 = 1$. □

Proposition 6. *Take $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with $k > 0$ and $n_1 \ge \cdots \ge n_k > 0$.*

(i) We have $\mathbb{S}(Y,3) \neq \emptyset$ if and only if $k \ge 3$ and $n_1 \le 2$.

(ii) If $Y \neq (\mathbb{P}^1)^4$, all $S \in \mathbb{S}(Y,3)$ are as in [1] (Proposition 3.2).

(iii) Assume $Y = (\mathbb{P}^1)^4$; $S \in \mathbb{S}(Y,3)$ if and only if either $S \in \mathcal{S}(Y,q)$ for some q such that $\mathrm{rank}(q) = 3$ *or it is as in [1] (Proposition 3.2).*

Proof. Since $\mathbb{T}(Y,3) \neq \emptyset$, $k \ge 3$ and $n_1 \le 2$ ([1], Theorem 4.12) and all $S \in \mathbb{T}(Y,3)$ are as described in [1] (Theorem 4.12).

(a) If $Y \neq (\mathbb{P}^1)^4$, $S \in \mathbb{T}(Y,3)$ if and only if either $S \in \mathcal{S}(Y,q)$ for some q such that $\mathrm{rank}(q) = 3$ or it is as in [1] (Propositions 3.1 and 3.2, Theorem 4.12). The case [1] (Proposition 3.1) is excluded by Lemma 1, because in this case $\eta_{1|S}$ is not injective. Proposition 5 proves that a general S as in [1] (Proposition 3.2) is an element of $\mathbb{T}(Y,3)$. In (iii), we claim a stronger statement. Fix S as in [1] (Proposition 3.2) and a general $q \in \langle \nu(S) \rangle$. We need to prove that $\mathrm{rank}(q) = 3$. Assume $\mathrm{rank}(q) \le 2$ and take $A \in \mathcal{S}(Y,q)$. Set $U := S \cup A$. We have $\#U \le 5$ and $h^1(\mathcal{I}_U(1,\dots,1)) > 0$. Note that $h^1(\mathcal{I}_S(1,\dots,1)) = 0$. Let V be the minimal subset of U containing S and with $h^1(\mathcal{I}_V(1,\dots,1)) > 0$. Since V contains S, Y is the minimal multiprojective space containing V. Since $k \ge 3$, [4] (Theorem 1.1 and Proposition 5.2) gives $\#V = 5$ (hence $V = U$, $\mathrm{rank}(q) = 2$ and $A \cap S = \emptyset$) and $Y = (\mathbb{P}^1)^3$. In this case, all possible sets V are described in [4] (Lemma 5.8) and $\pi_{i|V}$ is injective for all i. However, $\pi_{i|S}$ is not injective for one i by the definition of the example described in [4] (Proposition 3.1), a contradiction.

(b) Now assume $Y = (\mathbb{P}^1)^4$. $S \in \mathbb{T}(Y,3)$ if and only if either $S \in \mathcal{S}(Y,q)$ for some q such that $\mathrm{rank}(q) = 3$ or it is described in part (a) ([1], Theorem 4.12).

□

Theorem 9. *Take $Y = (\mathbb{P}^2)^3$. Then, $\mathbb{T}_1(Y,4) = S(Y,4)$, $\mathbb{T}(Y,4)' \neq \emptyset$ and $\mathbb{S}(Y,4) \neq \emptyset$. Moreover, $S \in \mathbb{T}(Y,4)'$ if and only if the following conditions are satisfied:*

(i) $\pi_{i|S}$ is injective for all $i = 1,2,3$;

(ii) for each $A \subset S$ such that $\#A = 3$, we have $\langle \pi_i(A) \rangle = \mathbb{P}^2$ for at least two $i \in \{1,2,3\}$.

Proof. Take a general $U \subset Y$ such that $\#U = 4$. Since $\sigma_4(Y)$ is defective (Remark 1), $U \in \mathbb{T}(Y,4)$. The semicontinuity theorem for cohomology gives $\mathbb{T}_1(Y,4) = S(Y,4)$. The solution set of any $q \in \langle \nu(Y) \rangle$ with rank 4 is an element of $\mathbb{S}(Y,4)$. Since $\sigma_3(Y)$ is not defective and $3(1 + \dim Y) < h^0(\mathcal{O}_Y(1,1,1))$, $\delta(2A, Y) = 0$ for all $A \subsetneq U$, and hence, $U \in \mathbb{T}(Y,4)'$. Fix $S \in S(Y,3)$. By Remark 15 the injectivity of all $\pi_{i|S}$ is a necessary condition to have $S \in \mathbb{T}(Y,4)'$. Condition (ii) is also necessary by Terracini Lemma and the inequalities $h^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}(1,1,1)) = 8 < 3(1 + \dim \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ and $h^0(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1}(1,1,1)) = 12 < 3(1 + \dim \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1)$. Now assume (i) and (ii) for the set S. By (i) $\delta(2A, Y) = 0$ for all $A \subset S$ such that $\#A = 2$. Now take $A \subset S$ such that $\#A = 3$. First assume $\langle \pi_i(A) \rangle = \mathbb{P}^2$. In this case A is the open orbit of $S(Y,3)$ for the action of $\mathrm{Aut}(\mathbb{P}^2) \times \mathrm{Aut}(\mathbb{P}^1) \times \mathrm{Aut}(\mathbb{P}^1)$. Since $\sigma_3(Y)$ is not defective, we get $\delta(2A, Y) = 0$. Now assume $\dim \langle \pi_i(A) \rangle = 1$ for exactly one i, say for $i = 3$. Thus, the minimal multiprojective space Y' containing A is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$. Since $\#\pi_3(A) = 3$ and $\langle \pi_i(A) \rangle = \mathbb{P}^2$ for $i = 1,2$, A is in the open orbit for the action on $S(Y',3)$ of the connected component $\mathrm{Aut}(Y')$ and $\dim\sigma_3(Y') = 17$ (Remark 1), we get $\delta(2A, Y') = 0$. We have $Y' \in |\mathcal{O}_Y(\varepsilon_3)|$. Consider the residual exact sequence of Y':

$$0 \to \mathcal{I}_A(1,1,0) \to \mathcal{I}_{2A}(1,1,1) \to \mathcal{I}_{(2A,Y'),Y'}(1,1,1) \to 0. \quad (6)$$

Since $\langle \pi_1(A) \rangle = \mathbb{P}^2$, we have $h^1(\mathcal{I}_A(1,1,0)) = 0$. Since $\delta(2A, Y') = 0$, (6) gives $\delta(2A, Y) = 0$. Thus, S is minimally Terracini. □

5. Proofs of Theorems 1 and 2

Lemma 17. *Fix integers $k \geq 4$, $x \geq 4$ and $n_1 \geq \cdots \geq n_k > 0$, $1 \leq i \leq k$, such that $n_1 \leq x - 1$ and $n_1 + \cdots + n_k = 2x - 2$. Then,*

$$\prod_{i=1}^{k}(n_i + 1) \geq x(2x - 1). \quad (7)$$

Proof. We fix the integer $x \geq 4$.

Observation 1: Fix an integer $a \geq 3$. The real function $h(t) := t(a - t)$ has a unique maximum in the interval $[1, a - 1]$ and the integers $\lfloor a/2 \rfloor$ and $\lceil a/2 \rceil$ are the only one with maximum value for the integers $1 \leq x \leq a - 1$.

First assume $k = 4$. Applying several times Observation 1, we see that the right hand side of (7) has a minimum with $n_1 = x - 1$, $n_2 = x - 3$ and $n_3 = n_4 = 1$. For these integers, (7) is satisfied.

Now assume $k \geq 5$. Since $n_1 \geq \cdots \geq n_k > 0$ and $n_1 + \cdots + n_k = 2x - 2$, $n_{k-1} + n_k \leq x - 1$. We apply Observation 1 to the integer $a = n_1 + n_k$ and the inductive assumption for the integers $n_1, \ldots, n_{k-2}, n_{k-1} + n_k$ (after permuting them to get a non-increasing sequence). □

Proof of Theorem 1. Assume $\mathbb{T}(Y, 4) \neq \emptyset$. For any $S \in \mathbb{T}(Y, 4)$, Y is the minimal multiprojective space containing S, and hence, $n_1 \leq 3$. Obviously $k > 1$ (Remark 9). Theorem 8 excludes the case $k = 2$. Proposition 3 gives $n_3 \leq 2$.

If $k \geq 3$ and $n_i = 1$ for all i, then $\mathbb{T}(Y, 4) \neq \emptyset$ by Theorem 10 (the case $k = 3$) and the case $m = 1$ of Theorem 11. If $k \geq 3$, $2 \leq n_1 \leq 3$ and $n_2 \leq 2$, then $\mathbb{T}(Y, 4) \neq \emptyset$ by Lemma 16. If $k \geq 3$, $n_1 = n_2 = 3$ and $n_3 \leq 2$, then $\mathbb{T}(Y, 4) \neq \emptyset$ by Lemma 16. □

Theorem 10. *Take $Y = (\mathbb{P}^1)^3$. Then, $\mathbb{T}(Y, x) \neq \emptyset$ and $\tilde{\mathbb{T}}(Y, x) = \emptyset$ for all $x \geq 4$. Moreover, for all $x \geq 4$ each set $A \in \mathbb{T}(Y, 3)$ as in [1] (Proposition 3.2) is a primitive reduction of some $S \in \mathbb{T}(Y, x)$.*

Proof. We have $\mathbb{T}(Y, x) \neq \emptyset$ for all $x \geq 4$ by Remark 11 and part (1) of Lemma 13. Thus, the "Moreover" part is proved.

Take $S \in \mathbb{T}(Y, x)$, $x \geq 4$. For each $S' \subset S$ such that $\#S' = 3$ we have $\delta(S', Y) > 0$. Thus, to prove that $\tilde{\mathbb{T}}(Y, x) = \emptyset$ it is sufficient to find S' such that Y is the minimal multiprojective space containing Y.

Claim 1. There is $u, v \in S$ such that $u \neq v$ and the minimal multiprojective space containing $\{u, v\}$ is not isomorphic to $\mathbb{P}^1$.

Proof of Claim 1. Assume that Claim 1 is not true, i.e., assume that for all $a, b \in S$ such that $a \neq b$, there is $A(a, b) \subset \{1, 2, 3\}$ such that $\#A(a, b) = 2$ and $\pi_i(a) = \pi_i(b)$ for all $i \in A(a, b)$. For any $u \in S$ set $u_i := \pi_i(u)$. By assumption $\#\pi_i(S) \geq 2$ for all $i = 1, 2, 3$. Start with any $a = (a_1, a_2, a_3) \in S$. There is $b \in S$ such that $b_1 \neq a_1$. Assume $b = (b_1, a_2, a_3)$. There is $c \in S$ such that $c_2 \neq a_2$. If $c_1 = b_1$ take $u = a$ and $v = c$. □

Fix $u, v \in S$ as in Claim 1 and let W be the minimal multiprojective space containing $\{u, v\}$. If $W = Y$, then any $w \in S \setminus \{u, v\}$ shows that S is not primitive. If $W \cong \mathbb{P}^1 \times \mathbb{P}^1$, then any $w \in S$ such that $w \notin W$ shows that S is not primitive. □

Theorem 11. *Take $Y = \mathbb{P}^m \times (\mathbb{P}^1)^{k-1}$ with $m \in \{1,2\}$ and $k \geq 4$. For any integer $x \geq 4$, there is $S \in \mathbb{T}(Y,x)$ with as a primitive reduction an element $A \in \mathbb{T}(Y,3)$ described in [1] (Proposition 3.1).*

Proof. Take any $A \in \mathbb{T}(Y,3)$ described by [1] (Proposition 3.1). By Remark 11 it is sufficient to find $W \in |\mathcal{I}_{2S}(1,\dots,1)|$ such that $\dim \mathrm{Sing}(W) > 0$. As in [1] (Proposition 3.2) take $A = \{a,b,c\}$ with $a = (a_1, u_2 \dots, u_k)$, $b = (b_1, u_2, \dots, u_k)$, $c = (c_1, \dots, c_k)$, $c_i \neq u_i$ for all $i > 1$, $\#\{a_1, b_1, c_1\} = 3$ and a_1, b_1, c_1 spanning $\mathbb{P}^m$. Set H_2 and H_3) be the only element of $|\mathcal{O}_Y(\varepsilon_i)|$, $i = 2,3$, containing a. Note that $b \in H_2 \cap H_3$, and hence, $\{a,b\} \in \mathrm{Sing}(H_2 \cup H_3)$. Let H_4 be the only element of $|\mathcal{O}_Y(\varepsilon_4)|$ containing c. Let H_1 be an element of $|\mathcal{O}_Y(\varepsilon_1)|$ containing c. Note that $A \subset \mathrm{Sing}(H_1 \cup H_2 \cup H_3 \cup H_4)$ and that $\mathrm{Sing}(H_1 \cup H_2 \cup H_3 \cup H_4)$ has codimension 2 in Y. If $k > 4$, use the union of $H_1 \cup H_2 \cup H_3 \cup H_4$ and an arbitrary element of $|\mathcal{O}_Y(0,0,0,0,1\dots,1)|$. □

Theorem 12. *Fix integers $x \geq 4$, $k \geq 3$, $n_1 \in \{1,2\}$ and $n_2 \in \{1,2\}$. Set $Y := \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times (\mathbb{P}^1)^{k-2}$. Fix any $A \in \mathbb{T}(Y,3)$ as in [1] (Proposition 3.2). Then, there is $S \in \mathbb{T}(Y,x)$ such that A is a primitive reduction of S.*

Proof. The set A is primitive, because $\mathbb{T}(Y,y) = \emptyset$ for $y < 3$ (the case $y = 1$ is trivial and [1] (Proposition 1.8) gives the case $y = 2$. By Remark 11 it is sufficient to find $W \in |\mathcal{O}_Y(1,\dots,1)|$ such that $A \subset \mathrm{Sing}(W)$ and $\dim \mathrm{Sing}(W) > 0$. Write $A = \{u,v,o\}$ with u,v,o as in [1] (Proposition 3.2).

(a) Assume $k \geq 4$. For $i = 3,4$ let H_i be the only element of $|\mathcal{O}_Y(\varepsilon_i)|$ containing u. Note that $v \in H_2 \cap H_3$ and hence $\{u,v\} \subset \mathrm{Sing}(H_2 \cup H_3)$. Take $H_i \in |\mathcal{O}_Y(\varepsilon_i)|$, $i = 1,2$, containing o. Thus, $A \subset \mathrm{Sing}(H_1 \cup H_2 \cup H_3 \cup H_4)$. The set $\mathrm{Sing}(H_1 \cup H_2 \cup H_3 \cup H_4)$ has codimension 2 in Y. If $k > 4$, use the union of $H_1 \cup H_2 \cup H_3 \cup H_4$ and an arbitrary element of $|\mathcal{O}_Y(0,0,0,0,1\dots,1)|$.

(b) Assume $k = 3$. Since the case $n_1 = n_2 = 1$ is true by Theorem 10, we may assume $n_1 + n_2 \geq 3$, say $n_1 = 2$. Let H_3 be the only element of $|\mathcal{O}_Y(\varepsilon_1)|$ containing u. Note that $v \in H_3$.

(b1) Assume $n_1 = n_2 = 2$. Take $H_1 \in |\mathcal{O}_Y(\varepsilon_1)|$ containing $\{o,u\}$ and $H_2 \in |\mathcal{O}_Y(\varepsilon_2)|$ containing $\{o,v\}$. Use $H_1 \cup H_2 \cup H_3$.

(b2) Assume $n_1 = 2$ and $n_2 = 1$. Since o is as in [1] (Proposition 3.2 (v)), there $\pi_2(o) \in \{\pi_2(u), \pi_2(v)\}$, say $\pi_2(o) = \pi_2(v)$. Take $H_1 \in |\mathcal{O}_Y(\varepsilon_1)|$ containing $\{o,u\}$ and $H_2 \in |\mathcal{O}_Y(\varepsilon_2)|$ containing o and hence containing v. Use $H_1 \cup H_2 \cup H_3$.

□

Proof of Theorem 2. Assume $n_2 \leq x-2$. Fix a line $L \subset \mathbb{P}^{n_1}$ and points $o_i \in \mathbb{P}^{n_i}$, $2 \leq i \leq k$. Let $Y' \subset Y$ the multiprojective space with L as its first factor and $\{o_i\}$ as its i-th factor $2 \leq i \leq k$. Fix a general $(a,b) \in Y' \times Y'$. Since $h^1(Y', \mathcal{I}_{2\{a,b\},Y'}) = 2$, $\delta(2\{a,b\},Y) \geq 2$.

Claim 1. We have $2(n_1 + \cdots + n_k + 1) \leq 1 + \prod_{i=1}^k (n_i + 1)$.

Proof of Claim 1. Let $\psi(n_1,\dots,n_k)$ be the difference between the right hand side and the left hand side of the inequality in Claim 1. Since $k \geq 3$, $\psi(n_1,\dots,n_k)$ is an increasing function in $[1,+\infty)^k$. Thus, it is sufficient to check that $\varphi(k) := \psi(1,\dots,1) \geq 0$. Since the function φ is an increasing function of k, it is sufficient to observe that $\varphi(3) = 1$. □

Claim 1 and the inequality $\delta(2\{a,b\},Y) \geq 2$ give $h^0(\mathcal{I}_{2\{a,b\},Y}(1,\dots,1)) > 0$. By Remark 2 it is sufficient to find $W \in |\mathcal{I}_{2\{a,b\}}(1,\dots,1)|$ such that $\mathrm{Sing}(W)$ contains a set S' such that $\#S' = x-2$, $S' \cap \{a,b\} = \emptyset$ and Y is the minimal multiprojective space containing $S := S' \cup \{a,b\}$. Take a general $H_i \in |\mathcal{I}_a(\varepsilon_i)|$, $i = 2,3$. Since $\{a,b\} \subset H_2 \cap H_3$, $\{a,b\} \subset \mathrm{Sing}(H_2 \cup H_3)$. Fix general $H_1 \in |\mathcal{O}_Y(\varepsilon_1)$ and set $W := H_1 \cup H_2 \cup H_3$. Since H_1 is general, $\langle L \cup \pi_1(H_1)\rangle = \mathbb{P}^{n_1}$. Fix a general $S'' \subset H_1 \cap H_2$ such that $\#S'' = x-3$ and a general $c \in H_1 \cap H_3$. Set $S' := S'' \cup \{c\}$. Obviously, $S' \cap \{a,b\} = \emptyset$ and $S := S' \cup \{a,b\} \subset \mathrm{Sing}(W)$. Note that $L = \langle \{\pi_1(a), \pi_1(b)\rangle$ and that $\pi_1(H_1 \cap H_2) = \pi_1(H_1 \cap H_3) = \pi_1(H_1)$. Hence $\langle \pi_1(S)\rangle = \mathbb{P}^{n_1}$. Since $R = \langle \{p_2(a), \pi_2(b)\}\rangle$, $\langle R \cup \pi_2(H_2)\rangle \mathbb{P}^{n_2}$ and S' is general,

$\langle \pi_2(S)\rangle = \mathbb{P}^{n_2}$. Obviously, $\langle \pi_i(o_i) \cup \pi_i(S')\rangle = \mathbb{P}^{n_i}$ for all $i > 2$. Thus, Y is the minimal multiprojective space containing S.

Now assume $k \ge 4$ and $n_3 \le x-2$.

By Claim 1 and the inequality $\delta(2\{a,b\},Y) \ge 2$ we have $h^0(\mathcal{I}_{2\{a,b\},Y}(1,\dots,1)) > 0$. By Remark 2 it is sufficient to find $W \in |\mathcal{I}_{2\{a,b\}}(1,\dots,1)|$ such that $\mathrm{Sing}(W)$ contains a set S' such that $\#S' = x-2$, $S' \cap \{a,b\} = \emptyset$ and Y is the minimal multiprojective space containing $S := S' \cup \{a,b\}$. Take a general $H_i \in |\mathcal{I}_a(\varepsilon_i)|$, $i = 3,4$. Since $\{a,b\} \subset H_3 \cap H_4$, $\{a,b\} \subset \mathrm{Sing}(H_3 \cup H_4)$. Fix general $H_i \in |\mathcal{O}_Y(\varepsilon_i)$, $i = 1,2$, and set $W := H_1 \cup H_2 \cup H_3 \cup H_4$. Since H_1 and H_2 are general, $\langle L \cup \pi_1(H_1)\rangle = \mathbb{P}^{n_1}$ and $\langle \pi_2(H_2)\rangle = \mathbb{P}^{n_2}$. Fix a general $S' \subset H_1 \cap H_2$ such that $\#S' = x-2$. Obviously $S' \cap \{a,b\} = \emptyset$ and $S := S' \cup \{a,b\} \subset \mathrm{Sing}(W)$. Note that $L = \langle \{\pi_1(a), \pi_1(b)\rangle$ and that $\pi_1(H_1 \cap H_2) = \pi_1(H_1)$. We conclude as in the proof of (i). □

6. Minimally Terracini

Remark 14. *Take $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Fix any $S \subset Y$ such that $\#S = 4$ and Y is the minimal multiprojective space containing S, i.e., $\langle \pi_i(S)\rangle = \mathbb{P}^{n_i}$ for all i. If $n_i = 1$, then $\#\pi_i(S) > 1$. If $n_i = 2$, then $\#\pi_i(S) > 2$ and $\pi_i(S)$ is not contained in a line. If $n_i = 3$, then $\pi_{i|S}$ is injective and $\pi_i(S)$ is linearly independent. If $E \in \mathbb{T}(Y,x)'$ and $n_i = x+1$, then $\pi_{i|E}$ is injective and $\pi_i(E)$ is linearly independent.*

Proof of Theorem 3. Since $h^0(\mathcal{O}_Y(1,\dots,1)) \ge x(1+\dim Y)$ (Lemma 17), we have $h^0(\mathcal{I}_{2S}(1,\dots,1)) > 0$ if $\#S = x$ and $\delta(2S,Y) > 0$. Fix $C \in \mathcal{C}(Y)$ (Remark 7) and a general $S \subset C$ such that $\#S = x$. For any $o \in S$, let $W(o)$ be the degree 2 zero-dimensional subscheme of the smooth curve C with o has its reduction. Set $W := \cup_{o\in S} W(o)$. Note that $\deg(W) = 2x$, $W \subset C$. Since $x \ge n_i+1$ for all i and either $\pi_i(C) = \mathbb{P}^1$ (case $n_i = 1$) or $\pi_i(C)$ is a rational normal curve of $\mathbb{P}^{n_i}$ if $n_i > 1$, Y is the minimal multiprojective space containing S. Since $\nu(C)$ is a degree $\dim Y = 2x-2$ rational normal curve in its linear span and $\deg(W) = 2x$, $h^1(C, \mathcal{I}_{Z,C}(1,\dots,1)) > 0$. Thus, $\delta(2S,Y) > 0$, and hence, $S \in \mathbb{T}(Y,x)$. Assume $S \notin \mathbb{T}(Y,x)'$ and take a minimal $S' \subsetneq S$ such that $\delta(2S',Y) > 0$. Set $y := \#S'$. We have $2 \le y \le x-1$. By Lemma 4 and the minimality of y there is a zero-dimensional scheme $Z = \cup_{o\in S'} Z(o) \subset Y$ with $Z(o)_{\mathrm{red}} = \{o\}$, $\deg(Z(o)) \le 2$ for all $o \in S'$, $h^1(\mathcal{I}_Z(1,\dots,1)) > 0$ and $h^1(\mathcal{I}_{Z'}(1,\dots,1)) = 0$ for all $Z' \subsetneq Z$.

Observation 1: Each $\pi_{i|S'}$ is injective and each $\pi_i(S')$ is in linear independent position in $\mathbb{P}^{n_i}$, i.e., each subset of $\pi_i(S')$ with cardinality $\le n_i+1$ is linearly independent.

Observation 1 gives $h^1(\mathcal{I}_{S'}(1,\dots,1)) = 0$. Thus, $Z \ne S'$, i.e., there is $o \in S'$ such that $\deg(Z(o)) = 2$.

Take $H_1 \in |\mathcal{I}_o(\varepsilon_1)|$ containing $\min\{n_i, y\}$ points of S'. Since $\pi_{1|S'}$ is injective and each $\pi_1(S')$ is in linear independent position in $\mathbb{P}^{n_1}$, $\#(H_1 \cap S') = \min\{y, n_1\}$. If $y > n_1$, we take in $H_1 \cap S'$ as much points $x \in S'$ with $\deg(Z(x)) = 2$ as possible. Set $Z_1 := \mathrm{Res}_{H_1}(Z)$ and $S_1 := (Z_1)_{\mathrm{red}}$.

(a) Assume $Z(o) \nsubseteq H_1$. Note that $\{o\}$ is a connected component of Z_1. We take $H_2 \in |\mathcal{O}_Y(\varepsilon_2)|$ such that $o \notin H_2$ and H_2 contains $\min\{-1+\#S_1, n_2\}$ points of S_1, taking first the ones which are not connected components of Z_1. Set $Z_2 := \mathrm{Res}_{H_2}(Z_1)$. Note that o is a connected component of Z_2. We continue in this way, until we get Z_c, S_c and $H_c \in |\mathcal{O}_Y(\varepsilon_c)|$ with $\#S_c \le n_c$ and $o \notin H_c$ (we find $c \le k$, because $n_1 + \cdots + n_k \ge x > y$). Set $Z_{c+1} := \mathrm{Res}_{H_c}(Z_c)$. First assume $Z_c \setminus \{o\} \subset H_c$. In this case $Z_{c+1} = \{o\}$ and since $h^1(\mathcal{I}_o) = 0$ we obtain a contradiction. Now assume $Z_c \setminus \{o\} \nsubseteq H_c$. In this case, Z_{c+1} is a reduced set containing o and with cardinality at most n_c. Set $u := (u_1,\dots,u_n) \in \mathbb{N}^k$ with $u_1 = 0$ if $i \le c$ and $u_i = 1$ if $c+1 \le i \le k$. By Observation 1, to prove that $h^1(\mathcal{I}_{Z_{c+1}}(u)) = 0$ (and hence to conclude the proof of this case) it is sufficient to prove that $\#Z_{c+1} \le n_{c+1} + \cdots + n_k + 1$. We started with Z such that $\deg(Z) \le 2y \le 2x-2$. We have $\#Z_{c+1} \le \deg(Z) - n_1 - \cdots - n_{c-1} - \deg(H_c \cap Z_c)$ and $\#Z_{c+1} \le 1 + \deg(H_c \cap Z_c)$. Since $n_1 + \cdots + n_k \ge 2x-2$, we conclude.

(b) Assume $Z(o) \subset H_1$ and $Z \nsubseteq H_1$. Since we required that H_1 contains as much points $x \in S'$ with $\deg(Z(x)) = 2$, Z_1 has at least one connected component, o', of degree 1. We continue as in Step (a), using o' instead of o.

(c) Assume $Z \subset H_1$. Hence $S' \subset H_1$. Thus, $y \le n_1$. First assume $\deg(Z) = 2y$ and $\deg(\eta_1(Z(x))) = 1$ for all $x \in S'$.

(c1) Assume the existence of $x \in S'$ such that either $\deg(Z(x)) = 1$ or $\deg(\eta_1(Z(x))) = 2$. The latter condition is equivalent to the existence of $i > 1$ such that $\deg(\pi_i(Z(x))) = 2$. Instead of H_1, we take $M_1 \in |\mathcal{I}_{S'\setminus\{x\}}(\varepsilon_1)|$ such that $x \notin M_1$. The scheme $E := \mathrm{Res}_{M_1}(Z)$ is the union of $Z(x)$ and a subset S'' of $S' \setminus \{x\}$. Thus, $\deg(E) \le n_1 + 1$. Lemma 2 and the assumption on $Z(x)$ give that $\eta_{1|E}$ is an embedding. Since $n_2 + \cdots + n_k \ge x - 1 \ge y$ and $\deg(\pi_i(Z(x))) = 2$ for some $i > 1$, Observation 1 and step (a) applied to $\eta_1(E) \subset Y_1$ prove this case.

(c2) Assume $\deg(Z) = 2y$ and $\deg(\eta_1(Z(x)) = 1$ for all $x \in S'$. Thus, $\deg(\pi_1(Z(x))) = 2$ for all $x \in S'$. We order the points $o_1, \dots, o_y$ of S' and use $M_i \in |\mathcal{O}_Y(\varepsilon_i)|, 2 \le i \le k$, first with M_k, but never taking a divisor M_i containing o_1. Set $Z^k := \mathrm{Res}_{M_k}(Z)$, $Z^{k-1} := \mathrm{Res}_{M_{k-1}}(Z^k)$, and so on. Note that all the connected components of all schemes Z^i have degree 2 and that either $\deg(Z^i) = 2y - 2n_k - \cdots - 2n_{i+1}$ or $Z^i = Z(x)$. Then, we use that $h^1(\mathcal{I}_{Z(x)}(\varepsilon_1)) = 0$, because $\deg(\pi_1(Z(x))) = 2$.

We have $\dim \mathcal{C}(Y) = -3 + \sum_{i=1}^{k}(n_i^2 + 2n_i)$ (Remark 7) and each $C \in \mathcal{C}(Y)$ has ∞^{x-1} subsets with cardinality $x - 1$. Take $C, C' \in \mathcal{C}$ such that $C \neq C'$. Since $C \cap C'$ is a finite set, 2 different rational normal curves may only have finitely many common elements of $\mathbb{T}(Y, x)'$. Thus, $\dim \mathbb{T}(Y, x) \ge x - 4 + \sum_{i=1}^{k}(n_i^2 + 2n_i)$. $\square$

Remark 15. *Take any Y with three factors and take $A \subset Y$ such that $\#A = 2$ and $\delta(2A, Y) = 0$. Then, [1] (Propositions 3.1 and 3.2) show that $\pi_{i|A}$ is injective for all $i = 1, 2, 3$. Hence, for every every $S \in \mathbb{T}(Y, x)'$, $x \ge 4$, all $\pi_{i|S}$, $1 \le i \le 3$, are injective.*

Remark 16. *Take $Y = (\mathbb{P}^1)^4$ and any $S \subset Y$ such that $\#S = 3$. We have $S \in \mathbb{T}(Y, 3)$, and in particular, $h^1(\mathcal{I}_{2S}(1, 1, 1, 1)) > 0$ and $h^0(\mathcal{I}_{2S}(1, 1, 1, 1)) > 0$. Thus, S is minimally Terracini if and only if each $A \subset S$ such that $\#A = 2$ satisfies $h^1(\mathcal{I}_{2A}(1, \dots, 1)) = 0$. By [1] (Propositions 3.1 and 3.2) this is the case if and only if for each $A \subset S$ such that $\#A = 2$ we have $\#\pi_i(A) = 2$ for at least 3 indices $i \in \{1, 2, 3, 4\}$. Thus, $S \in \mathbb{T}(Y, 3)'$ if and only if $\pi_{i|S}$ is injective for all $i = 1, 2, 3, 4$.*

Proposition 7. *Take as Y one of the following multiprojective spaces: $\mathbb{P}^3 \times (\mathbb{P}^1)^3$, $\mathbb{P}^2 \times (\mathbb{P}^1)^4$, $(\mathbb{P}^1)^6$. Then, $\mathbb{T}(Y, 4)' \neq \emptyset$. In the first (resp. second, resp. third) case we have $\dim \mathbb{T}(Y, 4)' \ge 25$ (resp. 21, resp. 19).*

Proof. In all cases, we have $\dim Y = 6$ and $h^0(\mathcal{O}_Y(1, \dots, 1)) \ge 4(1 + \dim Y)$. Thus, $S \in \mathbb{T}(Y, 4)$ if and only if $\delta(2S, Y) > 0$. Let $C \subset Y$ be a rational normal curve (Remark 7). Fix a general $S \in S(C, 4)$. Since $h^0(\mathcal{O}_C(1, \dots, 1)) = \dim Y + 1 = 7$ and $\deg((2S, Y) \cap C) = \deg((2S, C)) = 8$, we have $h^1(\mathcal{I}_{(2S,C)}(1, \dots, 1)) > 0$. Since $(2S, C)$ is a subscheme of the zero-dimensional scheme $(2S, Y)$, $h^1(\mathcal{I}_{2S}(1, \dots, 1)) > 0$. Thus, $S \in \mathbb{T}(Y, 4)$. Fix $A \subset S$ such that $a := \#A \in \{2, 3\}$. Fix $i \in \{1, \dots, k\}$. If $n_i = 1$, then $\pi_i(C) = \mathbb{P}^1$. The generality of S gives that $\pi_i(A)$ are x general points of $\mathbb{P}^1$. Recall that $\mathrm{Aut}(\mathbb{P}^1)$ is 3-transitive. If $n_i \ge 2$, then $\pi_i(C)$ is a rational normal curve of $\mathbb{P}^{n_i}$, and hence, the generality of $S \subset C$ gives that $\pi_i(A)$ is in the open orbit for the action of $\mathrm{Aut}(\mathbb{P}^{n_i})$. Thus, A is in the open orbit for the action on $S(Y, x)$ of the connected component of the identity of $\mathrm{Aut}(Y)$. Since $\sigma_2(Y)$ and $\sigma_3(A)$ are not defective (Remark 1), $S \in \mathbb{T}(Y, 4)'$.

Since in the first (resp. second, resp. third) case we have $\dim \mathcal{C} = 21$ (resp. 17, resp. 15), we get the last assertion of the proposition. $\square$

We do not claim that all $S \in \mathbb{T}(Y, 4)'$ are the ones described in the proof of Proposition 7. The following example for $Y = (\mathbb{P}^1)^6$ is in the limit of the family constructed to prove Proposition 7.

Example 1. *Take $(\mathbb{P}^1)^6$. Fix a partition $E \sqcup F$ of $\{1,2,3,4,5,6\}$ such that $\#E = \#F = 3$. Take $a_E := (a_1,\dots,a_6)$ with $a_i = 1$ if $i \in E$ and $a_i = 0$ if $i \in F$. Let a_F be the multidegree $(1,\dots,1) - a_E$. Let C_1 be an integral curve of multidegree a_E (all of them are in the same orbit for the action of $(\mathrm{Aut}(\mathbb{P}^1))^6$ and the stabilizer for this action acts transitively on C_1). Using π_i for some $i \in E$, we see that $C_1 \cong \mathbb{P}^1$. Let $C_2 \subset Y$ be an integral curve of multidegree a_F such that $C_1 \cap C_2 \neq \emptyset$. It is easy to see that $\#(C_1 \cap C_2) = 1$ and that $C_1 \cup C_2$ is a nodal curve of arithmetic genus 0. Fix a general $(E_1,E_2) \subset C_1 \times C_2$ such that $\#E_1 = \#E_2 = 2$. Note that C_1 and C_2 are isomorphic to rational normal curves of $(\mathbb{P}^1)^3$. Since 2 general points of $(\mathbb{P}^1)^3$ are contained in a rational normal curve of $(\mathbb{P}^1)^3$ and $\sigma_2((\mathbb{P}^1)^3) = \mathbb{P}^7$ ([10], Example 2.1), $E_i \notin \mathbb{T}(Y,2)$. Fix $A \subset S$ such that $x := \#A \le 3$, $A \cap E_1 \neq \emptyset$ and $E_2 \neq \emptyset$. Y is the minimal multiprojective space containing A. Since $\#\pi_j(E_i) = 2$ for 3 indices j, [1] (Theorem 4.12) gives $A \notin \mathbb{T}(Y,x)$. Thus, $S \in \mathbb{T}(Y,4)'$.*

Proposition 8. *Take $Y = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then:*

1. *$\mathbb{T}(Y,4)' \neq \emptyset$;*
2. *for a general $A \in S(Y,3)$, there are ∞^5 $S \in \mathbb{T}(Y,4)'$ containing A;*
3. *$\dim \mathbb{T}(Y,4)' = 23$.*

Proof. Fix any smooth $C \in |\mathcal{O}_Y(0,0,1,1)|$ and a general $S \subset C$ such that $\#S = 4$. Obviously, $h^0(\mathcal{I}_{2S}(1,1,1,1)) \ge 32 - 4 \times 8 > 0$. Note that $C \cong \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$ and that $\nu_{|C}$ is the embedding of C by the complete linear system $|\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^1}(1,1,2)|$. We have $h^0(\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^1}(1,1,2)) = 27$ and $4(1+\dim C) = 24$. Since the fourth secant variety of $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$ embedded by $|\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^1}(1,1,2)|$ is defective ([11], Theorem 4.13), $\delta(2S,\nu(C)) > 0$. Since the scheme $2S \cap C$ does not impose independent conditions to $|\mathcal{O}_Y(1,1,1,1)|$, $\delta(2S,Y) > 0$. Thus, $S \in \mathbb{T}(Y,4)$. Since S is general in C, $\#\pi_i(S) = 4$ for all $i = 1,2,3,4$ and no 3 of the points of $\pi_i(S)$, $i = 1,2$, are collinear. Thus, every subset of S with cardinality $x \le 3$ is the open orbit for the action of the connected component of the identity of $\mathrm{Aut}(Y)$ on $S(Y,x)$. Since the second and third secant varieties of Y are not defective (Remark 1), $S \in \mathbb{T}(Y,4)'$.

Fix a general $A \in S(Y,3)$. Since $h^0(\mathcal{O}_Y(0,0,1,1)) = 4$ and A is general, there is a unique $C \in |\mathcal{I}_A(0,0,1,1)|$ and C is smooth. We proved that $A \cup \{p\} \in \mathbb{T}(Y,4)'$. Thus, $\dim \mathbb{T}(Y,4)' \ge 23$. Since $\dim(Y) = 6$ and $\dim \sigma_4(Y) = 27$, the set of all $S \in \mathbb{T}(Y,4)$ has dimension ≤ 23. We get parts (ii) and (iii) with equality, not just the inequality ∞^x with $x \ge 5$. □

Lemma 18. *Take either $Y = \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1$ or $Y = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then, $\mathbb{T}(Y,4)' = \emptyset$.*

Proof. Assume the existence of $S \in \mathbb{T}(Y,4)'$. By Remark 15 each $\pi_{i|S}$ is injective. Fix $A \subset S$ such that $\#A = 3$ and let Y' be the minimal multiprojective space containing A. Since $\delta(2A,Y) \ge \delta(2A,Y')$ ([1], Lemma 2.3), to a contradiction it is sufficient to prove that $\delta(2A,Y') > 0$.

(a) Assume $Y = \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1$. Since Y is the minimal multiprojective space containing S, $\langle \pi_1(S) \rangle = \mathbb{P}^3$. Thus, $Y' \cong \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$. Since $h^0(\mathcal{O}_{Y'}(1,1,1)) = 12 < 3(1+\dim Y')$, $\delta(A,Y') > 0$.

(b) Assume $Y = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$. If $Y' \cong (\mathbb{P}^1)^3$, then $\delta(2A,Y') > 0$, because $h^0(\mathcal{O}_{Y'}(1,1,1)) = 8 < 3(1+\dim Y')$. Now assume $Y' = Y$. Since $h^0(\mathcal{O}_Y(1,1,1)) = 12 < 3(1+\dim Y)$, $\delta(2A,Y) > 0$.

□

Proposition 9. *Take either $Y = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$ or $Y = \mathbb{P}^2 \times (\mathbb{P}^1)^3$ or $Y = (\mathbb{P}^1)^5$. Then, $\mathbb{T}(Y,4)' \neq \emptyset$.*

Proof. Write $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with $n_1 \ge \cdots \ge n_k > 0$. Let $f : \mathbb{P}^1 \to Y$ be the embedding induced by $f = (f_1,\dots,f_k)$, $f_i : \mathbb{P}^1 \to \mathbb{P}^{n_i}$ with f_i an isomorphism if $n_i = 1$, while f_i is an embedding with $f_i(\mathbb{P}^1) \subset \mathbb{P}^{n_i}$ a degree n_i rational normal curve. Set $C := f(\mathbb{P}^1)$. Note that $\nu(C)$ is a degree 5 rational normal curve in its linear span. Let $W \subset C$ be a connected degree 3 zero-dimensional scheme. Fix a general $q \in \langle \nu(W) \rangle$. A theorem of Sylvester gives the existence of a one-dimensional family $\mathcal{U}$ of set $S \subset C$ such that $\#S = 4$ and each S evinces the $\nu(C)$-rank of q. Since $\dim \mathcal{U} > 0$ and each $\nu(S)$, $S \in \mathcal{U}$ irredundantly span q, Terracini lemma gives $\delta(2S, Y) > 0$. Fix $A \subset Y$ such that $\#A \le 3$ and let Y' be the minimal multiprojective space containing A. First assume $\#A = 2$. Since each f_i is injective, $Y' \cong (\mathbb{P}^1)^k$ and A is in the open orbit for the action on $S(Y', 2)$ of $(\mathrm{Aut}(\mathbb{P}^1)^k$. Since $\dim \sigma_2(Y') = 2k + 1$, we get $\delta(2A, Y') = 0$. If $n_1 = 1$ we get $\delta(2A, Y') = 0$. Now assume $n_1 = n_2 = 2$. Since $\#\pi_i(A) = 2$, $h^1(\mathcal{I}_A(\varepsilon_i)) = 0$ for all i. Take $H_1 \in |\mathcal{O}_Y(\varepsilon_1)|$ and $H_2 \in |\mathcal{O}_Y(\varepsilon_2)|$ such that $Y' = H_1 \cap H_2$. Taking the residual exact sequence of Y' in H_1 and using that $h^1(\mathcal{I}_A(\varepsilon_3)) = 0$, we get $\delta(2A, H_1) = 0$. Then, using the residual exact sequence of H_1 in Y we get $\delta(2A, Y) = 0$.

Now assume $\#A = 3$. Since each f_i is injective, $n_i \le 2$ for all i and $f_i(C)$ is a rational normal curve if $n_i = 2$, then $Y' = Y$ and A is in the open orbit of $S(Y, 3)$ for the action of the connected component of the identity of $\mathrm{Aut}(Y)$. Since $\dim \sigma_3(Y) = 17$ (Remark 1), we get $\delta(2A, Y) = 0$. Thus, S is minimally Terracini. □

Lemma 19. *Take $Y = \mathbb{P}^m \times \mathbb{P}^m \times \mathbb{P}^s$ with $s > 0$. Fix $S \subset Y$ such that $\#S = m + 1$, Y is the minimal multiprojective space containing S and $\#\pi_3(S) = m + 1$. Then $h^1(\mathcal{I}_{2S}(1, 1, 1)) = 0$.*

Proof. Taking linear projections in the 3-rd coordinate, if necessary we reduce to the case $s = 1$. In this case, Y is the minimal multiprojective space containing S and $(m + 1)(\dim Y + 1) = h^0(\mathcal{O}_Y(1, 1, 1))$. Thus, if the lemma fails, then $S \in \mathbb{T}(Y, m + 1)$. The case $m = 1$ follows from [1] (Proposition 1.8). Assume $m > 1$. Fix a general $q \in \langle \nu(S) \rangle$. By Terracini's lemma, it is sufficient to prove that $\mathcal{S}(q) = \{S\}$. This is a simple consequence of [8] (Theorem 3). □

Proof of Theorem 4. Assume the existence of $S \in \mathbb{T}(Y, x)'$. Since $S \in \mathbb{T}(Y, x)$, $h^0(\mathcal{I}_{2S}(1, \dots, 1)) > 0$ and $\delta(2S, Y) > 0$. Since $S \in \mathbb{T}(Y, x)'$, Y is the minimal multiprojective space containing S, $\pi_{i|S}$ is injective and $\pi_i(S)$ is linearly independent for $i = 1, 2$. Assume for the moment $k = 3$. Since $\delta(2A, Y) = 0$ for all $A \subset S$ such that $\#A = 2$, Remark 15 gives that $\pi_{3|S}$ is injective. Lemma 19 gives $h^1(\mathcal{I}_{2S}(1, \dots, 1)) = 0$, a contradiction. Now assume $k \ge 4$. Let $\pi_{1,2,3} : Y \to \mathbb{P}^m \times \mathbb{P}^m \times \mathbb{P}^s$ denote the projection onto the first three factors of Y. Since $\pi_{1|S}$ is injective, $\#\pi_{1,2,3}(S) = m + 1$. The case $k = 3$ of Lemma 19 shows that $\{\pi_{1,2,3}(S)\} = \mathcal{S}(\mathbb{P}^m \times \mathbb{P}^m \times \mathbb{P}^s, q')$ for a general $q' \in \langle \nu(\pi_{1,2,3}(S)) \rangle$. Since $\#\pi_{1,2,3}(S) = \#S$, we get $\{S\} = \mathcal{S}(Y, q)$ for a general $q \in \langle \nu(S) \rangle$. Thus, Terracini Lemma gives $h^1(\mathcal{I}_{2S}(1, \dots, 1)) = 0$. □

7. Proof of Theorems 5 and 6

We divide the long proof of Theorem 5 into five different propositions, and then join them together. In Section 6 we proved Theorem 4, which covers some cases of Theorem 5. Since the proofs of Propositions 10–14 have the same beginning, we write here the starting sentences of all 5 proofs and avoid duplications.

Notation 1. *Assume the existence of $S \in \mathbb{T}(Y, 4)'$. By Lemmas 4 and 5, there is a zero-dimensional scheme $Z \subset Y$ such that $Z_{\mathrm{red}} = S$, each connected component of Z has degree ≤ 2, $h^1(\mathcal{I}_Z(1, \dots, 1)) > 0$ and $h^1(\mathcal{I}_{Z'}(1, \dots, 1)) = 0$ for all $Z' \subsetneq Z$. Set $z := \deg(Z) \le 8$. For each $p \in S$, let $Z(p)$ denote the connected component of Z containing p.*

Proposition 10. *Take $Y = (\mathbb{P}^1)^k$, $k \ge 7$. Then $\mathbb{T}(Y, 4)' = \emptyset$.*

Proof. For any $a \in S$, let $e(a)$ be the dimension of the minimal multiprojective space containing $Z(a)$ with the convention $e(a) = 0$ if $Z(a) = \{a\}$. We take a partition $S = S' \sqcup S''$ of S with $\#S' = \#S'' = 2$ and set $Z' := Z \cap (\cup_{a \in S'} Z(a))$ and $Z'' := Z \cap (\cup_{a \in S'} Z(a))$. Note that $Z' \cap Z'' = \emptyset$, $Z' \ne \emptyset$ and $Z'' \ne \emptyset$. Since $S \in \mathbb{T}(Y, 4)'$, $h^1(\mathcal{I}_{Z'}(1, \dots, 1)) =$

$h^1(\mathcal{I}_{Z''}(1,\dots,1)) = 0$. Since $h^1(\mathcal{I}_Z(1,\dots,1)) > 0$, $\langle \nu(Z')\rangle \cap \langle \nu(Z'')\rangle \neq \emptyset$. Fix a general $q \in \langle \nu(Z')\rangle \cap \langle \nu(Z'')\rangle$. There are minimal $V' \subseteq Z'$ and $V'' \subseteq Z''$ such that $q \in \langle \nu(V')\rangle \cap \langle \nu(V'')\rangle$. The minimality property of Z gives $V' = Z'$ and $V'' = Z''$; however, we typically do not utilize it. Instead, we use $U' \cup U''$ in place of Z in the construction we provided.

Write $S = \{p(1), p(2), p(3), p(4)\}$. Fix a divisor $C \in |\mathcal{O}_Y(\varepsilon_1 + \varepsilon_2)|$ containing $\{p(1), p(2), p(3)\}$ and set $U := \mathrm{Res}_C(Z)$. We have $h^1(\mathcal{I}_U(1,\dots,1)(-\varepsilon_1 - \varepsilon_2)) > 0$ ([5], Lemma 5.1). Note that $U \subsetneq \{p(1), p(2), p(3)\} \cup Z(p(4))$. By [5] (Lemma 5.1), either $U = \emptyset$ or $h^1(\mathcal{I}_U(1,\dots,1)) > 0$. In steps (a), (b) and (c), we assume $h^1(\mathcal{I}_U(1,\dots,1)) > 0$, while step (d) handles the case $U = \emptyset$.

(a) Assume for the moment that $\eta_{1,2|U}$ is an embedding and that $U \supseteq S$. We get $h^1(Y_{1,2}, \mathcal{I}_{\eta_{1,2}(U)}(1,\dots,1)) > 0$. Proposition 1 gives that the minimal multiprojective space containing $\eta_{1,2}(U)$ contains at most three factors, and hence, the minimal multiprojective space containing S has at most five factors, a contradiction.

(b) Assume that $\eta_{1,2|U}$ is not an embedding. This assumption occurs for exactly two reasons: either $U \supseteq Z(p(4))$, $\deg(Z(p(4))) = 2$ and $\deg(\eta_{1,2}(Z(p(4)))) = 1$ or there are i,j such that $1 \le i < j \le 4$ and $\eta_{1,2}(p(i)) = \eta_{1,2}(p(j))$. The latter possibility is excluded by Lemma 2. If $\deg(Z(p4)) = 2$ and $\deg(\eta_{1,2}(Z(p(4)))) = 1$, then $e(p(4)) \le 2$ and $\deg(\pi_i(Z(p(4)))) = 1$ for all $i > 2$. We may avoid this case by instead taking the first two factors, the factor associated to two of the integers in $\{1,\dots,k\}$, say i_1 and i_2, such that $v(p(4))$ depends on at least one factor of $\{1,\dots,k\} \setminus \{i_1, i_2\}$ (Lemma 2).

(c) Assume $S \nsubseteq U$. Note that either $U_{\mathrm{red}} = U$ or $\deg(U) = \deg(U_{\mathrm{red}}) + 1$. We have $U_{\mathrm{red}} \neq U$ if and only if $p(4) \notin C$. Since $\#U_{\mathrm{red}} \le 3$ and $h^1(\mathcal{I}_U(1,\dots,1)(-\varepsilon_1 - \varepsilon_2)) > 0$, we get that $\eta_1(U)$ depends on at most three factors of Y_1 (Remark 5 and Proposition 1), and hence, U depends on four factors at most. Thus, $\delta(2U, Y) > 0$ (Remark 1) and hence S is not minimally Terracini.

(d) Assume $U = \emptyset$, i.e., $Z \subset C$. Set $C_{1,2} := C$. Fix integer $1 \le i < j \le k$ and take $C_{i,j} \in |\mathcal{I}_{p(1),p(2),p(3)}(\varepsilon_i + \varepsilon_j)|$. By steps (a), (b) and (c) we get (by exclusion) $Z \subset C_{i,j}$.

(e) Up to now, we only used (roughly speaking) that $k \ge 6$, and we know (Proposition 7 and Example 1) that the statement of the theorem is not true if $k = 6$. From now on, we use that $k \ge 7$. More precisely, we use that $z := \deg(Z) \le k+1$. In steps (a)–(d), we did not use any ordering of the set $\{1,\dots,k\}$, the only possible difference being whether $C_{i,j}$ is reducible or not. In the following steps, we freely permute the factors of Y. Let i be any integer $i \in \{1,\dots,k\}$ such that there is $H_1 \in |\mathcal{O}_Y(\varepsilon_i)|$ such that $e_1 := \deg(Z \cap H_1)$ is maximal. Set $Z_1 := \mathrm{Res}_{H_1}(Z)$. Note that $\deg(Z_1) = z - e_1$. Set $E_1 := H_1 \cap Z$. Note that $\deg(E_1) = e_1$. Let e_2 be the maximal integer such that there is $j \in \{2,\dots,k\}$ and $H_2 \in |\mathcal{O}_Y(\varepsilon_j)|$ such that $e_2 := H_j \cap Z_1$ is maximal. With no loss of generality, we may assume $j = 2$. Then, we continue in the same way, defining integers $e_3, \dots$, the divisors $H_3, \dots$ and zero-dimensional schemes $E_3, \dots$ and $Z_3, \dots$ such that $E_i := H_i \cap Z_i$, $e_i = \#E_i$, $Z_{i+1} = \mathrm{Res}_{H_1}(Z_i)$ and at each step the integer i is maximal. Note that $e_1 \ge e_2 \ge \cdots \ge e_i \ge e_{i+1}$ and that $e_i = 0$ if and only if $Z \subset H_1 \cup \cdots \cup H_{i-1}$. Since $k \ge \deg(Z) - 1$ there is a maximal integer $c \le k$ such that $e_c \le 1$. Assume for the moment $e_c = 1$. We have $\deg(Z_c) = 1$, and hence, $h^1(\mathcal{I}_{Z_c}) = 0$, contradicting [5] (Lemma 5.1). Thus, $e_c = 0$. In the same way, we get $e_{c-1} \ge 2$. Since $e_1 \ge \cdots \ge e_c \ge 2$, we have the following possibilities (for $z = 8$, for $z < 8$, the first one does not arise, and the second, third, must be modified):

1. $c = 5$, $e_1 = e_2 = e_3 = e_4 = 2$;
2. $c = 4$, $e_1 = 4$, $e_3 = e_4 = 2$;
3. $c = 4$, $e_1 = 3$, $e_2 = 3$, $e_4 = 2$;
4. $c = 3$, $\lceil z/2 \rceil \le e_1 \le z - 2$, $e_2 = z - e_1$.

(e1) Assume $c = 5$, and thus, $e_1 = e_2 = e_3 = e_4 = 2$. By [1] (Lemma 5.1) we have $h^1(\mathcal{I}_{E_4}(0,0,0,1,1,1,1,1)) > 0$, and hence $\deg(\pi_i(E_4)) = 1$ for all $i \ge 4$. Fix $j \in \{1,2,3\}$. Using H_4 instead of H_j we get $\deg(\pi_i(E_j)) = 1$ for $i = j$ and for $i \ge 4$.

Then, we use $\{H\} = |\mathcal{I}_{E_4}(\varepsilon_7)|$ and we also get $\deg(\pi_4(E_j)) = 1$. Thus, $\deg(\pi_i(E_1)) = 1$ except at most for $i = 2,3$. Using $\{H'\} \in |\mathcal{I}_{E_2}(\varepsilon_7)|$ and $\{H''\} \in |\mathcal{I}_{E_3}(\varepsilon_7)|$, we get $\deg(\pi_2(E_1)) = 1$ and $\deg(\pi_3(E_1)) = 1$. Thus, $e_1 = 1$, a contradiction.

(e2) Assume $c < 5$, i.e., $e_1 + e_2 \geq 5$. Since each connected component of Z has degree at most 2, we get that $H_1 \cup H_2$ contains at least 3 points of S, and hence, $H_1 \cup H_2 = C_{1,2}$. Hence, we excluded case (2) and (3), $e_1 \geq 4$ and $e_2 = z - e_1$. By [5] (Lemma 5.1), we have $h^1(\mathcal{I}_{Z_1}(\hat{\varepsilon}_1)) > 0$. Therefore, either $\eta_{1|Z_1}$ is an embedding and $h^1(Y_1, \mathcal{I}_{\eta_1(Z_1)}(1,\dots,1)) > 0$ or there is a degree 2 scheme $w \subset Z_1$ such that $\deg(\eta_1(w)) = 1$. Lemma 2 gives that w is connected, i.e., $w = Z(p(i)$ for some i. Since $w \subseteq Z_1$, $p(i) \notin H_1$. Since $e_1 \geq \lceil z/2 \rceil$, H_1 contains at least two points of S. Take $j \in \{3,\dots,k\}$ and $M_j \in |\mathcal{I}_{p(i)}(\varepsilon_j)|$. Since $H_1 \cup M_j$ contains at least three points of S, steps (a)–(d) give $Z \subset H_1 \cup M_j$, and hence, $\deg(\pi_j(Z_1)) = 1$ for all $j > 2$. Since $\deg(\pi_2(Z_1)) = 1$, we also get $\#(Z_1)_{\mathrm{red}} = 1$ and hence $Z_1 = w$. Thus, $\#(S \cap H_1) = 3$, $S \cap H_1 = S \setminus \{p(i)\}$ and E_1 is the union of the connected components of Z with a point of $S \cap H_1$ as its reduction. For any $p \in (S \cap H_1)$ set $m(p) := \{2,\dots,k\}$ if $z(p) = \{p\}$, while if $\deg(Z(p)) = 2$ let $m(p)$ denote the set of all $j \in \{2,\dots,k\}$ such that $\eta_{j|Z(p)}$ is an embedding. Remark 3 gives $\#m(p) \geq k-2$ for all $p \in S \cap H_1$. Since $\#(S \cap H_1) = 3$ and $k \geq 5$, there is $j \in m(p)$ for all $p \in S \cap H_1$. Fix $j \in \cap_{p \in S \cap H_1} m(p)$ and take $M \in |\mathcal{I}_{p(i)}(\varepsilon_j)|$. Set $Z' := \mathrm{Res}_M(Z)$ and $Z'' := \eta_j(Z') \subset Y_j$. We have $h^1(\mathcal{I}_{Z'}(\hat{\varepsilon}_j)) > 0$ ([5], Lemma 5.1). Since $j \geq 2$, $w \subset M$ and hence $w \cap Z' = \emptyset$. By the definition of j each map $\eta_{j|Z(p)}$ is an embedding. Since $\delta(2A, Y) = 0$ for all $A \subset S \cap H_1$ such that $\#A = 2$, $\eta_{j|S \cap H_1}$ is injective. Thus, $\eta_{j|Z'}$ is an embedding and hence $h^1(Y_j, \mathcal{I}_{Z''}(1,\dots,1)) = h^1(\mathcal{I}_{Z'}(\hat{\varepsilon}_j)) > 0$. Let Y'' be the minimal multiprojective subspace of Y_j containing $\eta_j(S \cap H_1)$. By [1] (Theorem 4.12), we have $Y'' \cong (\mathbb{P}^1)^m$ for some $m \leq 4$. Thus, there is $h \in \{2,\dots,k\}$ and $D \in |\mathcal{O}_Y(\varepsilon_h)|$ such that $D \supseteq \eta_h^{-1}(Y'')|$. Since Y is the minimal multiprojective space containing S, $p(i) \notin D$. Thus, $\mathrm{Res}_D(Z) = w$. Since $\deg(\pi_i(w)) = 1$ for all $i > 1$, $\pi_{1|w}$ is an embedding. Therefore, $h^1(\mathcal{I}_w(\hat{\varepsilon}_h)) \leq h^1(\mathcal{I}_w(\varepsilon_1)) = 0$, contradicting [5] (Lemma 5.1).

□

Proposition 11. *Take $Y = \mathbb{P}^3 \times (\mathbb{P}^2)^m \times (\mathbb{P}^1)^s$ with $m \geq 2$ and $s \geq 0$. Then, $\mathbb{T}(Y,4)' = \emptyset$.*

Proof. We only use the case $Y = \mathbb{P}^3 \times (\mathbb{P}^2)^2$, because the proofs are extremely similar in all other cases, but far simpler.

Claim 1. $\#\pi_i(S) = 4$ for $i = 2,3$.

Proof of Claim 1. Assume for instance $\#\pi_3(S) \leq 3$, and take $a, b \in S$ such that $\pi_3(a) = \pi_3(b)$ and $a \neq b$. The minimal multiprojective space Y' containing is isomorphic to either $\mathbb{P}^1$ (case $\pi_2(a) = \pi_2(b)$) or to $\mathbb{P}^1 \times \mathbb{P}^1$ (case $\pi_2(a) \neq \pi_2(b)$). Since $(2\{a,b\}, Y) \geq (2\{a,b\}, Y') = 2$ ([1], Lemma 2.3), $S \notin \mathbb{T}(Y,4)'$, a contradiction. □

Claim 2. If $H \in |\mathcal{O}_Y(\varepsilon_2)|$, $M \in |\mathcal{O}_Y(\varepsilon_3)|$ and $S \subset H \cup M$, then $Z \subset H \cup M$.

Proof of Claim 2. Assume $Z \nsubseteq H \cup M$, i.e., assume $E := \mathrm{Res}_{H \cup M}(Z) \neq \emptyset$. Since $E \subseteq S$, $\pi_{1|S}$ is injective, $\pi_1(S)$ is linearly independent and $\#S = h^0(\mathcal{O}_Y(1,0,0), h^1(\mathcal{I}_E(1,0,0)) = 0$, contradicting [5] (Lemma 5.1). □

Claim 3. None of the three points of $\pi_i(S)$, $i \in \{2,3\}$ are collinear.

Proof of Claim 3. Suppose the existence of $A \subset S$ such that $\#A = 3$ and $L := \langle \pi_3(A) \rangle$ is a line. Set $\{p\} := S \setminus A$, $M := \pi_3^{-1}(L)$. Since Y is the minimal multiprojective space containing S, $M \cap S = A$. Take a general $H \in |\mathcal{I}_p(\varepsilon_2)|$. Since $S \subset H \cup M$, Claim 2 gives $Z \subset H \cup M$. Since $\#\pi_2(S) = 4$ (Claim 1) and H is general, $H \cap S = \{p\}$, and hence, $\cup_{o \in A} Z(o) \subset M$. Since $h^1(\mathcal{I}_{Z(p)}(1,1,0)) > 0$ ([5], Lemma 5.1), $\deg(Z(p)) = 2$ and $\deg(\eta_3(Z(p))) = 1$. Fix $o \in A$ and take $M' \in |\mathcal{O}_Y(\varepsilon_3)|$ containing $\{p, o\}$, and $H' \in |\mathcal{O}_Y(\varepsilon_2)|$ containing $A \setminus \{o\}$. Since $S \subset H' \cup M'$, Claim 1 gives $Z \subset H' \cup M'$. Claim 1 gives $Z(p) \cup Z(o) \subset M'$ and $Z' := \cup_{a \in A \setminus \{o\}} Z(a) \subset H'$. Since $\deg(Z(p)) = 2$ and $\deg(\eta_3(Z(p))) = 1$, $\deg(\pi_3(Z(p))) = 2$. Thus, the line $\langle \pi_3(Z(p)) \rangle$ contains $\pi_3(o)$. Taking another point $o' \in A$, we get $\langle \pi_3(Z(p)) \rangle = L$ and hence $S \subset M$, a contradiction. □

Claim 4. Fix $i \in \{2,3\}$ and $D \in |\mathcal{O}_Y(\varepsilon_i)|$ such that $\#(D \cap S) \geq 2$. Then, $\#(D \cap S) = 2$ and $\cup_{o \in D \cap S} Z(o) \subset H$.

Proof of Claim 4. Claims 1 and 3 give $\#(D \cap S) = 2$. The last assertion of Claim 4 was proved in the proof of Claim 3. □

Fix $a, b \in S$ such that $a \neq b$ and let M be the only element of $|\mathcal{O}_Y(0,0,1)|$ containing $\{a,b\}$ (Claim 1). Write $S = \{a,b,c,d\}$. We have $Z(a) \cup Z(b) \subset M$ and $\mathrm{Res}_M(Z) = Z(c) \cup Z(d)$ (Claim 3 and 4). Hence, $h^1(\mathcal{I}_{Z(c) \cup Z(d)}(1,1,0)) > 0$ ([5], Lemma 5.1). Take a general $D \in |\mathcal{I}_{\{c,d\}}(1,0,0)|$. We have $\mathrm{Res}_D(Z(c) \cup Z(d)) \subseteq \{c,d\}$. Claim 1 implies $h^1(\mathcal{I}_{\{c,d\}}(0,1,0)) = 0$. Thus, $Z(c) \cup Z(d) \subset D$ ([5], Lemma 5.1). Since D is general, we get $\pi_1(Z(c) \cup Z(d)) \subset \langle \{\pi_1(c), \pi_1(d)\} \rangle$. Taking different subsets of S with cardinality 2, we get $\pi_1(Z(c)) \subseteq \cap_{x \in S \setminus \{c\}} \langle \{c, x\} \rangle = \{c\}$, because $\#\pi_1(S) = 4$ and $\pi_1(S)$ is linearly independent. Therefore, $\deg(\pi_1(Z(y))) = 1$ for all $y \in S$. Take $y \in S$ such that $\deg(Z(y)) = 2$. Since $\deg(\pi_1(Z(y))) = 1$, there is $i \in \{2,3\}$ such that $\deg(\pi_i(Z(y))) = 2$, and hence, $h^1(\mathcal{I}_{Z(y)}(0,1,1)) = 0$. If $y \in S$ and $\deg(Z(y)) = 1$, then obviously $h^1(\mathcal{I}_{Z(y)}(0,1,1)) = 0$. Fix $A \subset S$ such that $\#A = 3$ and let D be the only element of $|\mathcal{O}_Y(1,0,0)|$ containing A because $\#\pi_1(S) = 4$ and $\pi_1(S)$ is linearly independent. Set $\{y\} := S \setminus A$. We saw that $\cup_{a \in A} Z(a) \subset D$, and hence, $\mathrm{Res}_D(Z) = Z(y)$. Since $h^1(\mathcal{I}_{Z(y)}(0,1,1)) = 0$, we conclude quoting [5] (Lemma 5.1). □

Proposition 12. *Take $Y = \mathbb{P}^3 \times (\mathbb{P}^1)^{k-1}$, $k \geq 5$. Then, $\mathbb{T}(Y,4)' = \emptyset$.*

Proof. Take $H_1 \in |\mathcal{O}_Y(\varepsilon_1)|$ containing 3 points of S. By Remark 14, H_1 is uniquely determined by $H_1 \cap S$ and $\#(H_1 \cap S) = 3$. Set $z_1 := \deg(Z \cap H_1)$, $Z_1 := \mathrm{Res}_{H_1}(Z)$ and $S_1 := \mathrm{Res}_{H_1}(S)$. Since $z_1 \geq 3$, $\deg(Z_1) = z - z_1 \leq 5$. Take $i \in \{2, \dots, k\}$ and $H_2 \in |\mathcal{O}_Y(\varepsilon_i)|$ such that $z_2 := \deg(Z_1 \cap H_2)$ is maximal, and set $Z_2 := \mathrm{Res}_{H_2}(Z_1)$. Permuting the last $k-1$ factors of Y, we may assume $i = 2$. Take $i \in \{3, \dots, k\}$ and $H_3 \in |\mathcal{O}_Y(\varepsilon_i)|$ such that $z_3 := \deg(Z_2 \cap H_3)$ is maximal, and set $Z_3 := \mathrm{Res}_{H_3}(Z_2)$. Permuting the last $k-2$ factors of Y, we may assume $i = 3$. Note that $z_2 \geq z_3$. We continue in the same way until we obtain an integer $c \geq 2$ such $z_c \leq 1$; since $k - 1 \geq z - z_1$, we find some $c \leq k$. Since $h^1(\mathcal{I}_W) = 0$ for any degree 1 zero-dimensional scheme, [5] (Lemma 5.1) gives $z_c = 0$, i.e., $Z \subset H_1 \cup \cdots \cup H_{c-1}$. Permuting the 1-dimensional factors of Y, we may assume $H_i \in |\mathcal{O}_Y(\varepsilon_i)|$ for all i. Since $z_{c-1} \geq 2$ and $z - z_1 \leq 5$, either $z - z_1 = 5$ and $z_2 = 3$ and $z_3 = 2$ or $z - z_1 = 4$ and $z_2 = z_3 = 2$ or $c = 2$ and $z_2 = z - z_1$. Since $h^1(\mathcal{I}_{Z_2}(0,0,1,\dots,1)) = 0$ and $\deg(Z_2) = z_3 = 2$, $\deg(\pi_i(Z_2)) = 1$ for all $i \geq 3$.

Claim 1. $z_1 > 3$.

Proof of Claim 1. Assume $z_1 = 3$, i.e., assume $Z \cap H_1 = S \cap H_1$. Thus, $h^1(\mathcal{I}_{Z \cap H_1}(\varepsilon_1)) = 0$ by Observation 1. Set $H := H_2 \cup \cdots \cup H_{c-1}$. Since $\mathrm{Res}_H(Z) \subseteq H_1 \cap Z$, [1] (Lemma 5.1) gives $Z \subset H$. Observation 1 gives $c > 2$, and hence, $c = 3$ and either $z = 8$, $z_2 = 3$ and $z_3 = 2$ or $z = 7$ and $z_2 = z_3 = 2$. Since $\mathrm{Res}_{H_1 \cup H_2}(Z) = Z_3$ has degree 2 and $h^1(\mathcal{I}_{Z_3}(0,0,1,\dots,1)) > 0$, $\deg(\pi_i(Z_3)) = 1$ for all $i \geq 3$. First assume $Z_3 = \{a,b\}$ with $a \neq b$, and call Y' the minimal multiprojective space containing $\{a,b\}$. Since $\deg(\pi_i(Z_3)) = 1$ for all $i \geq 3$, we get $\delta(2\{a,b\}, Y) \geq 2$ (Remark 4), a contradiction. Thus, Z_3 is connected. Since $Z_3 \subset \mathrm{Res}_{H_1}(Z)$, $Z_3 = Z(p)$, where $\{p\} := S \setminus S \cap H_1$. Since $Z \cap H_1 = S \setminus \{p\}$, we have $Z_2 = S \setminus \{p\}$. Applying [5] (Lemma 5.1), we get $h^1(\mathcal{I}_{S \setminus \{p\}}(0,1,0,1,\dots,1)) > 0$. For any $A \subset S \setminus \{p\}$ such that $\#A = 2$, there are at most $k-3$ integers i with $\#\pi_i(A) = 1$ by Lemma 2. Thus, there is $i, j \in \{4, \dots, k\}$ such that $i < j$, $M_i \in |\mathcal{O}_Y(\varepsilon_i)|$, $M_j \in |\mathcal{O}_Y(\varepsilon_i)|$ and $\#((S \setminus S \cap M)) = 2$. Since $\mathrm{Res}_{H_1 \cup H_3 \cup M_i \cup M_j}(Z)$ is a single point, $h^1(\mathcal{I}_{\mathrm{Res}_{H_1 \cup H_3 \cup M_i \cup M_j}(Z)}) = 0$, contradicting [5] (Lemma 5.1). □

Claim 1 excludes the case $c = z_2 = 3$, $z_3 = 2$. Note that Claim 1 is true for each $H_1 \in |\mathcal{O}_Y(\varepsilon_1)|$ containing 3 points of S.

Claim 2. $z = 8$.

Proof of Claim 2. Assume $z \leq 7$, and write $S = \{a,b,c,d\}$ with $Z(d) = \{d\}$. Let H_1 be the only element of $|\mathcal{O}_Y(\varepsilon_1)|$ containing $\{a,b,c\}$ (Observation 1). By Claim 1 $Z_1 := \mathrm{Res}_{H_1}(Z)$ is the union of d and at most 2 points of $\{a,b,c\}$, say $Z_1 = \{d\} \cup$

A with $A \subset \{a,b,c\}$ and $\#A \le 2$. Remark 4 gives that $\eta_{1|Z_1}$ is injective and hence $h^1(Y_1, \mathcal{I}_{\eta_1(Z_1)}(1,\dots,1)) = h^1(\mathcal{I}_{Z_1}(\hat{\varepsilon}_1)) > 0$. Proposition 1 gives that the minimal multiprojective space containing $\eta_1(Z_1)$ is isomorphic to $\mathbb{P}^1$ and hence the minimal multiprojective space containing the set Z_1 is isomorphic to $\mathbb{P}^{\#Z_1-1} \times \mathbb{P}^1$, contradicting Remark 4. □

Claim 3. $c = 2$.

Proof of Claim 3. Assume $c \neq 2$. By Claims 1 and 2 we get $c = 3$, $z_1 = 4$, $z_2 = 2$ and $z_3 = 2$. Fix $p \in S$ and set $B := S \setminus \{p\}$. Let H_1 be the only element of $|\mathcal{O}_Y(\varepsilon_1)|$ containing B. We have $Z_1 = Z(p) \cup A$ with $A \subset B$ and $\#A = 2$. There is $M_2 \in |\mathcal{O}_Y(\varepsilon_2)|$ containing $Z(p)$, and hence, $E := \mathrm{Res}_{H_1 \cup M_2}(Z) \subseteq A$. Since $h^1(\mathcal{I}_E(0,0,1\dots,1)) > 0$ ([5], Lemma 5.1), we first get $\#E = 2$ and then $\delta(2E, Y) > 0$ (Remark 4), contradicting the assumption $S \in \mathbb{T}(Y,4)'$. □

By Claim 3, $Z_1 \subset H_2$ for any choice of H_1 containing 3 points of S. Set $\{p\} := S \setminus S \cap H$. Since $z = 8$, Observation 1 gives $Z(p) \subseteq Z_1$ with $\deg(Z(p)) = 2$.

Claim 4. $z_2 = 4$.

Proof of Claim 4. Recall that $z = 8$ and $z_1 \ge 4$. Assume $z_1 \ge 5$. We have $Z_1 = Z(p) \cup \{a\}$ with $a \in H_1 \cap S$. By Remark 4 there is $i > 2$ such that $\pi_i(p) \neq \pi_i(a)$. Take $M \in |\mathcal{O}_Y(\varepsilon_i)|$ such that $p \in M$. We have $\mathrm{Res}_{H_1 \cup M}(Z) \subseteq \{p,a\}$. Since $h^1(\mathcal{I}_{\mathrm{Res}_{H_1 \cup M}(Z)}(0,1,\dots,1)(-\varepsilon_i)) > 0$, we get $\mathrm{Res}_{H_1 \cup M}(Z) = \{p,a\}$ and $\pi_j(p) = \pi_j(a)$ for all $j \in \{2,\dots,k\} \setminus \{k\}$. Thus, $\delta(2\{p,a\}, Y) \ge 2$, a contradiction. □

The previous claims give the existence of $E \subset S \cap H_1$ such that $\#E = 2$ and $Z_1 = Z(p) \cup E \subset H_2$. Write $E = \{b,c\}$ and $\{a\} = H \cap H_1 \setminus E$. We have $\mathrm{Res}_{H_2}(Z) = Z(a) \cup \{b,c\}$. By Lemma 2 there is $j > 2$ such that $\pi_j(a) \neq \pi_j(b)$. Hence $W := \mathrm{Res}_{H_2 \cup M}(Z) \subset \{a,b,c\}$ and $W \neq \emptyset$. Observation 1 gives $h^1(\mathcal{I}_W(\varepsilon_1)) = 0$, and hence, $h^1(\mathcal{I}_W((1,\dots,1) - \varepsilon_2 - \varepsilon_j)) = 0$, contradicting [5] (Lemma 5.1). □

Proposition 13. *Take $Y = \mathbb{P}^3 \times (\mathbb{P}^2)^m \times (\mathbb{P}^1)^s$ with $m \ge 1$ and $s \ge 2$. Then $\mathbb{T}(Y,4)' = \emptyset$.*

Proof. To simplify the notation, we take $Y = \mathbb{P}^3 \times \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$, but the general case is very similar and all other cases are easier. Assume the existence of $S \in \mathbb{T}(Y,4)'$ and take $Z \subset Y$ such that $Z_{\mathrm{red}} = S$, for each $p \in S$ the connected component of Z with p as its reduction has degree ≤ 2, $h^1(\mathcal{I}_Z(1,\dots,1)) > 0$ and $h^1(\mathcal{I}_{Z'}(1,\dots,1)) = 0$ for every $Z' \subsetneq Z$ (Lemma 4). Set $z := \deg(Z)$.

Claim 1. Take any $C \in |\mathcal{O}_Y(1,1,0,0)|$ such that $S \subset C$ and $\deg(Z \cap C) \ge \min\{z, 5\}$. Then, $Z \subset C$.

Proof of Claim 1. Since the case $z \le 5$ is trivial, we may assume $z > 5$. Assume $Z \nsubseteq C$. The scheme $W := \mathrm{Res}_C(Z)$ is a subset of S with cardinality ≤ 3. Since $W \neq \emptyset$, $h^1(\mathcal{I}_W(0,0,1,1)) > 0$. Thus, either there is $A \subseteq E$ such that $\#A = 2$ and $\#\pi_3(A) = \#\pi_4(A) = 1$ (with $\delta(2A, Y) \ge 2$, a contradiction) or $\#E = 3$ and there is $i \in \{3,4\}$ such that $\#\pi_i(E) = 1$ (Proposition 1). In the latter case (with, say $\#\pi_4(E) = 1$), $\delta(2E, Y) > 0$, unless the minimal multiprojective space Y' containing E is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$, i.e., $\langle \pi_2(E) \rangle = \mathbb{P}^2$ and $\#\pi_3(E) > 1$. Set $\{p\} := S \setminus \{p\}$. Take $D \in |\mathcal{O}_Y(\varepsilon_2)|$ containing $Z(p)$ and let M be the only element of $|\mathcal{O}_Y(\varepsilon_4)|$ containing p. We have $\mathrm{Res}_{D \cup M}(Z) \subseteq E$. Since $h^1(\mathcal{I}_E(1,0,0,0)) = 0$, [5] (Lemma 5.1) gives $\mathrm{Res}_{D \cup M}(Z) = \emptyset$, i.e., $Z \subset D \cup M$. Set $W := \mathrm{Res}_M(Z)$. We have $W \subseteq Z(p) \cup E$. Since $\langle \pi_2(E) \rangle = \mathbb{P}^2$, there is $N \in |\mathcal{O}_Y(\varepsilon_1)|$ such that $p \in N$ and $E \nsubseteq N$. Since $\mathrm{Res}_{N \cup M}(Z) \neq \emptyset$, $\mathrm{Res}_{N \cup M}(Z) \subseteq S$ and $h^1(\mathcal{I}_S(1,0,0,0)) = 0$, [5] (Lemma 5.1) gives a contradiction. □

Fix $p \in S$ and set $B := S \setminus \{p\}$. Let H be the only element of $|\mathcal{O}_Y(\varepsilon_1)|$ containing B. Take $D \in |\mathcal{I}_{Z(p)}(\varepsilon_2)|$. Claim 1 gives $Z \subset H \cup D$. Note that $\mathrm{Res}_H(Z) = Z(p) \cup A$ with $A \subseteq B$. Since $\mathrm{Res}_H(Z) \subset D$ and Y is the minimal multiprojective space containing S, $A \neq E$, i.e., $\deg(Z \cap H) \ge 4$.

(a) Assume $\deg(Z \cap H) = 6$. Thus, $\mathrm{Res}_H(Z) = Z(p)$. Since $h^1(\mathcal{I}_{Z(p)}(0,1,1,1)) > 0$ ([5], Lemma 5.1), we get $z = 8$ and $\deg(\pi_i(Z(p))) = 1$ for all $i = 2,3,4$. Write $S \cap H = \{a,b,c\}$. Set $\{M_3\} := |\mathcal{I}_a(\varepsilon_3)|$ and $\{M_4\} := |\mathcal{I}_p(\varepsilon_4)|$. Note that $Z(p) \subset M_4$. Take $M_2 \in |\mathcal{O}_Y(\varepsilon_2)|$ containing $\{b,c\}$, except that if $b \in M_3 \cup M_4$ (resp. $c \in M_3 \cup M_4$)), we take M_2 not containing b (resp. c); this is possible unless $\pi_2(b) = \pi_2(c)$; if $\pi_2(b) = \pi_2(c)$ (and hence $\pi_2(a) \neq \pi_2(b)$), we reverse the role of a and b. Since $\mathrm{Res}_{M_2 \cup M_3 \cup M_4}(Z) \subset \{a, b, c\}$ and $h^1(\mathcal{I}_S(1,0,0,0)) = 0$, we get $Z \subset M_2 \cup M_3 \cup M_4$. Set $W := \mathrm{Res}_{M_2 \cup M_4}(Z)$. If $W = \emptyset$, we are in a case handles in the proof of Claim 1. Assume $W \neq \emptyset$. We get $h^1(\mathcal{I}_W(1,0,1,0)) > 0$. Hence $\pi_1(W)$ is linearly dependent. Note that $Z(p) \cap W = \emptyset$ and that $W \subseteq Z(a) \cup \{b, c\}$. By Observation 1, $\langle \pi_1(Z(a)) \rangle \cap \{\pi_1(b), \pi_1(c)\} \leq 1$, say $\pi_1(b) \notin \langle \pi_1(Z(a)) \rangle$.

(b) Assume $\deg(Z \cap H) = 4$. Write $B = \{a, b, c\}$ with $Z \cap H = Z(a) \cup \{b, c\}$ and $\deg(Z(a)) = 2$. By Remark 4 there is $i \in \{3, 4\}$, say $i = 4$, such that $\pi_4(a) \neq \pi_4(\{b, c\})$. Let N be the only element of $|\mathcal{O}_Y(\varepsilon_4)|$ containing a. We have $W := \mathrm{Res}_{D \cup N}(Z) \subset \{a, b, c\}$, and hence, $h^1(\mathcal{I}_W(1,0,0,0)) = 0$. Thus, $W = \emptyset$ ([5], Lemma 5.1), i.e., $Z \subset D \cup N$. We conclude as in the proof of Claim 1.

(c) Assume $\deg(Z \cap H) = 5$. Since we proved the other cases for every choice of $p \in S$, we may assume that $\deg(Z \cap H) = 5$ for every choice of $p \in S$. Write $Z \cap H = Z(a) \cup Z(b) \cup \{c\}$. We have $\mathrm{Res}_H(Z) = \{c\} \cup Z(p)$ and $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(0,1,1,1)) > 0$. By Lemma 2 there are at least two integers $i \in \{2, 3, 4\}$ such that $\pi_i(p) \neq \pi_i(c)$. Call i_1 and i_2 these integers with $i_1 < i_2$. Hence $i_2 \in \{3, 4\}$. With no loss of generality, we may assume $i_2 = 4$. Let M_4 denotes the only element of $|\mathcal{I}_p(\varepsilon_4)|$. We have $W := \mathrm{Res}_{H \cup M_4}(Z) = \{c, p'\}$ with $p' = p$ if $Z(p) \nsubseteq M_4$ and $p' = \emptyset$ if $Z(p) \subset M_4$. In both cases $W \neq \emptyset$. Using i_3 in both cases, we get $h^1(\mathcal{I}_W(0,1,1,0)) = 0$, contradicting [5] (Lemma 5.1).

□

Proposition 14. *Take $Y = (\mathbb{P}^2)^m \times (\mathbb{P}^1)^s$ with $m > 0$, $s \geq 0$ and $2m + s \geq 7$. Then, $\mathbb{T}(Y, 4)' = \emptyset$.*

Proof. The reader easily check (after the proof) that the proofs we give for $1 \leq m \leq 4$ and $s := \max\{0, 7 - 2m\}$ prove the general case in which s is larger. Moreover, the proof of the case $Y = (\mathbb{P}^2)^3 \times \mathbb{P}^1$ gives the case $Y = (\mathbb{P}^2)^4$. Thus, we only write the cases $1 \leq m \leq 3$ and $s = 7 - 3m$.

(a) Assume $Y = (\mathbb{P}^2)^3 \times \mathbb{P}^1$. Take $i \in \{1, 2, 3\}$ such that there is $H_1 \in |\mathcal{O}_Y(\varepsilon_i)|$ with $z_1 := \deg(Z \cap H_1)$ maximal. Since $\dim |\mathcal{O}_Y(\varepsilon_i)| = 2$, we have $z_1 \geq 2$. With no loss of generality, we may assume $i = 1$. Set $Z_1 := \mathrm{Res}_{H_1}(Z)$. Take $i \in \{2, 3\}$ such that there is $H_2 \in |\mathcal{O}_Y(\varepsilon_i)|$ with $z_2 := \deg(Z_1 \cap H_2)$ maximal. Since $\dim |\mathcal{O}_Y(\varepsilon_i)| = 2$, we have $z_2 \geq \min\{z - z_1, 2\}$. With no loss of generality, we may assume $i = 2$.
Set $Z_2 := \mathrm{Res}_{H_2}(Z_1)$. Take $H_3 \in |\mathcal{O}_Y(\varepsilon_3)|$ such that $z_3 := \deg(H_3 \cap Z_2)$ is maximal. Set $Z_3 := \mathrm{Res}_{H_3}(Z_2)$. Note that $z_1 \geq z_2 \geq z_3$. We have $z_3 \geq \min\{z - z_1 - z_2, 2\}$. Thus, $\deg(Z_3) = z - z_1 - z_2 - z_3 \leq 2$.

(a1) Assume $\deg(Z_3) \leq 1$. Since $h^1(\mathcal{I}_{Z_3}(0,0,0,1)) = 0$, [5] (Lemma 5.1) gives $Z_3 = \emptyset$, i.e., $Z \subset H_1 \cup H_2 \cup H_3$. In the same way, we get that either $z_3 = 0$, i.e., $Z \subset H_1 \cup H_2$, or $z_3 \geq 2$.

(a1.1) Assume $Z \subset H_1 \cup H_2$. Since $S \subseteq Z$, and Y is the minimal multiprojective space containing S, $z_2 > 0$. Since $h^1(\mathcal{I}_{Z_1}(0,1,1,1)) > 0$, $z_2 \geq 2$. Note that $z_2 \leq \lfloor z/2 \rfloor$.

(a1.2) Assume $z_3 \geq 2$. Since $z_1 \geq z_2 \geq z_3$ and $z \leq 8$, $z_3 = 2$. By [5] (Lemma 5.1), we have $h^1(\mathcal{I}_{Z_3}(0,0,1,1)) = 0$, i.e., $\deg(\pi_i(Z_3)) = 1$, for $i = 3, 4$. Since $z \leq 8$, either $z_2 = 2$ or $z = 8$ and $z_1 = z_2 = 3$.

(a1.2.1) Assume $z_2 = 2$. Note that $\deg(\mathrm{Res}_{H_1 \cup H_3}(Z)) \leq 2$. The minimality of H_2 gives $\deg(\mathrm{Res}_{H_1 \cup H_3}(Z)) = 2$. Using $H_1 \cup H_3$, we get $\deg(\pi_i(\mathrm{Res}_{H_1 \cup H_3}(Z))) = 1$ for $i = 2, 4$. Since $z_3 > 0$, there is $D \in |\mathcal{O}_Y(\varepsilon_2)|$ such that $\deg(D \cap Z_1) > 2$, contradicting the definition of z_2.

(a1.2.2) Assume $z = 8$ and $z_1 = z_2 = 3$. Remember that $\deg(\pi_i(Z_3)) = 1$, for $i = 3, 4$. Set $\{M_4\} := |\mathcal{I}_{Z_3}(\varepsilon_4)|$ and $W := \mathrm{Res}_{M_4}(Z)$. We have $w := \deg(W) \leq z - 2 = 6$. Take $i \in \{1, 2, 3\}$ such that there is $M_i \in |\mathcal{O}_Y(\varepsilon_i)|$ with $w_1 := \deg(W \cap M_i)$

maximal and set $W_1 := \mathrm{Res}_{M_i}(W)$. Take $j \in \{1,2,3\} \setminus \{i\}$ such that there is $M_j \in |\mathcal{O}_Y(\varepsilon_j)|$ with $w_2 := \deg(W_1 \cap M_j$ maximal and set $W_2 := \mathrm{Res}_{M_j}(W)$. Set $\{h\} := \{1,2,3\} \setminus \{i,j\}$. Take $M_h \in |\mathcal{O}_Y(\varepsilon_h)|$ with $w_3 := \deg(W_2 \cap M_i)$ maximal. We have $w_1 \ge w_2 \ge w_3 \ge 0$. Since $\dim |\mathcal{O}_{\mathbb{P}^2}(1)| = 2$, for any $i \in \{1,2\}$ if $w_i \le 1$, then $w_{i+1} = 0$. Thus, $w = w_1 + w_2 + w_3$. Assume $w_3 = 1$. Using $M_4 \cup M_i \cup M_j$ and [5] (Lemma 5.1), we get a contradiction. Thus, either $w_3 \ge 2$ or $w_3 = 0$.

(a1.2.2.1) Assume $w_3 \ge 2$. Thus, $w = 6$ and $w_1 = w_2 = w_2 = 2$. Using $M_4 \cup M_i \cup M_j$ and [5] (Lemma 5.1), we get $\deg(\pi_h(W_3)) = 0$. Since $w_1 = w_2 = w_3 = 2$, we may take a different ordering of $\{1,2,3\}$. Using $M_4 \cup M_j \cup M_h$, we get $\deg(\pi_i(W \cap M_i)) = 1$. If $W \nsubseteq M_i$, then there is $N \in |\mathcal{O}_Y(\varepsilon_i)|$ such that $W \cap N \supsetneq W \cap M_i$, contradicting the definition of w_1. Thus, $W \subset M_i$. Since Y is the minimal multiprojective space containing S, $S \nsubseteq M_i$. Thus, Z_3 is connected and W is the union of the 3 degree 2 connected components of Z with as its reduction the 3 points of $S \cap M_i$. Since $z_1 = z_2 = z_3 = 2$, we have $\deg(\pi_i(A)) = 2$ for all $i = 1,2,3$ and all $A \subset Z$ such that $\deg(A) = 2$. Thus, $\eta_{4|W}$ is an embedding, and hence, $h^1(Y_4, \mathcal{I}_{\eta_4(W)}(1,1,1)) = h^1(\mathcal{I}_W(1,1,1,0)) > 0$. Let Y' be the minimal multiprojective space containing $S \cap M_1$. If Y' is not isomorphic to $(\mathbb{P}^1)^4$, then there is $A \subset S \cap M_i$ such that $\delta(2A, Y') \ge 2$, and hence, $\delta(2A, Y) \ge 2$ ([5], Lemma 2.3). Assume $Y' \cong (\mathbb{P}^1)^4$. We would find $x \in \{1,2,3\}$ and $N \in |\mathcal{O}_Y(\varepsilon_x)|$ such that $\#(W_{\mathrm{red}} \cap N) = 3$, contradicting the assumption $w_1 = 2$.

(a1.2.2.2) Assume $w_3 = 0$, and hence, $W \subset M_i \cup M_j$. Since we are in the set up of (a1.2.2), we have $w_1 = w_2 = 3$, and we may take $i = 1$, $j = 2$, $M_i = H_1$ and $M_j = H_2$. We get $\deg(\pi_3(Z_3)) = 1$. By Remark 4 Z_3 is connected, say $Z_3 = Z(p)$ for some $p \in S$ and W is the union of the connected components of Z with $W_{\mathrm{red}} = S \setminus \{p\}$. As in step (a1.2.2.1), we get $W \nsubseteq M_i$. Thus, $Z \subset M_i \cup M_j \cup M_4$. Using [5] (Lemma 5.1), we get $w_2 \ge 2$. First assume $w_2 = 2$. Using $M_4 \cup M_i$, we get $\deg(\pi_x(W_1)) = 1$ for $x \in \{1,2,3\} \setminus \{i\}$ and hence $z_1 > 2$, a contradiction. Now assume $w_2 \ge 3$, and hence, $w_1 = w_2 = 3$ and $w = 6$.

(a2) Assume $\deg(Z_3) > 1$. Thus, $z = 8$, and $\deg(Z_3) = z_1 = z_2 = z_3 = 2$. Note that the role of the first three factors of Y are symmetric and that in this case if we take $D \in |\mathcal{O}_Y(\varepsilon_i)|$, $i = 1,2,3$ such that $\deg(D \cap Z) \ge 2$, then $\deg(D \cap Z) = 2$ and D is the only element of $|\mathcal{O}_Y(\varepsilon_i)|$ containing $D \cap Z$. Write $S = \{a,b,c,d\}$, and fix a point of S, say d. Set $\{M_1\} := |\mathcal{I}_{Z(a)}(\varepsilon_1)|$, $\{M_2\} := |\mathcal{I}_{Z(b)}(\varepsilon_1)|$, $\{M_3\} := |\mathcal{I}_{Z(c)}(\varepsilon_3)|$. We have $\mathrm{Res}_{M_1 \cup M_2 \cup M_3}(Z) = Z(d)$. By [5] (Lemma 5.1), $\deg(\pi_4(Z(d))) = 1$, and hence, there is $M_4 \in |\mathcal{O}_Y(\varepsilon_4)|$ containing $Z(p)$. Taking a instead of d, we get $\deg(\pi_4(Z(a))) = 1$. We have $\mathrm{Res}_{M_2 \cup M_3 \cup M_4}(Z) = Z(a)$. By [5] (Lemma 5.1), we have $h^1(\mathcal{I}_{Z(a)}(1,0,0,0)) > 0$, i.e., $\deg(\pi_1(Z(a))) = 1$. Take $\{N_1\} = |\mathcal{I}_{Z(b)}(\varepsilon_1)|$, $\{N_2\} = |\mathcal{I}_{Z(a)}(\varepsilon_2)|$. Using $N_1 \cup M_3 \cup M_4$, we get $\deg(\pi_2(Z(a))) = 1$. In a similar way, we get $\deg(\pi_3(Z(a))) = 0$. Since $\deg(Z(a)) = 2$, ν is not an embedding, a contradiction.

(b) Assume $Y = (\mathbb{P}^2)^2 \times (\mathbb{P}^1)^3$. Since $\dim |\mathcal{O}_Y(\varepsilon_1)| = \dim |\mathcal{O}_Y(\varepsilon_2)|$, there are $H_1 \in |\mathcal{O}_Y(\varepsilon_1)|$ and $H_2 \in |\mathcal{O}_Y(\varepsilon_2)|$ such that $S \subset H_1 \cup H_2$. Since $S \subset H_1 \cup H_2$ and each connected component of Z has degree ≤ 2, $W := \mathrm{Res}_{H_1 \cup H_2}(Z) \subseteq S$.

(b1) In this step, we prove that $W = \emptyset$. Assume $w := \#W > 0$. Since $W \neq \emptyset$, $h^1(\mathcal{I}_W(0,0,1,1,1)) > 0$. Fix $i \in \{3,4,5\}$ such that there is $H_3 \in |\mathcal{O}_Y(\varepsilon_i)|$ such that $w_1 := \deg(W \cap H_3)$ is maximal. Permuting the last three factors of Y, we may assume $i = 3$. Take $i \in \{4,5\}$ such that there is $H_4 \in |\mathcal{O}_Y(\varepsilon_i)|$ with $w_2 := \deg(\mathrm{Res}_{H_3}(W) \cap H_4)$ maximal. Permuting the last two factors of Y, we may assume $i = 4$. Take $H_5 \in |\mathcal{O}_Y(\varepsilon_5)|$ such that $w_3 := \deg(\mathrm{Res}_{H_3 \cup H_4}(W) \cap H_4)$ is maximal. Since $w \le 4$, $w - w_1 - w_2 - w_3 \le 1$. By [5] (Lemma 5.1), there is $c \in \{1,2,3\}$ such that $w_c \ge 2$ and $w_1 + \cdots + w_c = w$. Since $w \le 4$, $w_1 \ge w_2 \ge w_3$ and $w_c \ge 2$ either $c = 1$ or $c = 2$, $w_1 = w_2 = 2$ and $w = 4$.

(b.1) Assume $w_1 = w_2 = 2$ and $w = 4$, and hence, $W = S$ and $z = 8$. Since $H_4 \cap W = \mathrm{Res}_{H_3}(W) = \mathrm{Res}_{H_1 \cup H_2 \cup H_3}(Z)$, $h^1(\mathcal{I}_{\mathrm{Res}_{H_3}(W)}(0,0,0,1,1)) > 0$, i.e., $\pi_i(W \cap H_3)) =$

1 for $i = 4, 5$. By construction $\#\pi_3(W \cap H_3) = 1$. Thus, $W \cap H_3$ depends only on two factors of Y, contradicting Lemma 2.

(b1.2) Assume $c = 1$. Since Y is the minimal multiprojective space containing S, $2 \le w_1 \le 3$. First assume $w_1 = 2$. Since $h^1(\mathcal{I}_W(0,0,1,1,1)) > 0$, $W \cap H_3$ only depends on the first two factors of Y, contradicting Lemma 2. Now assume $w_1 = 3$. Since $h^1(\mathcal{I}_W(0,0,1,1,1)) > 0$, there is either $A \subset Y$ such that $\#A = 2$ and $\#\eta_{1,2}(A) = 1$ (excluded by Lemma 4) or $\eta_{1,2}(W)$ depends on only one factor of $Y_{1,2}$, say the last one. Thus, $\#\pi_i(W) = 1$ for $i = 3, 4$. Set $\{M_i\} := |\mathcal{I}_W(\varepsilon_i)|$, $i = 3, 4$. Note that $z = 7$. Set $\{p\} := S \setminus W$ and $\tilde{W} := \cup_{o \in W} Z(o)$. Since Sing$)M_3 \cup M_4) \supset W$, $\tilde{W} \subset M_3 \cup M_4$. Since Y is the minimal multiprojective space containing Y, $p \notin (M_3 \cup M_3)$. Thus, $\mathrm{Res}_{M_3 \cup M_4}(Z) = Z(p)$. Recall that $\deg(Z(p)) = 2$ and $\deg(\pi_i(Z(p))) = 1$ for all $i > 0$. Since $h^1(\mathcal{I}_{\mathrm{Res}_{M_3 \cup M_4}(Z)}(1,1,0,0,1)) > 0$ ([5], Lemma 5.1), we get $\deg(\pi_1(Z(p))) = 1$, contradicting the very ampleness of $\mathcal{O}_Y(1, \dots, 1)$.

(b2) By step (b1), $Z \subset H_1 \cup H_2$ for all $H_i \in |\mathcal{O}_Y(\varepsilon_i)|$, $i = 1, 2$, such that $S \subset H_1 \cup H_2$.

Claim 1. Assume $z = 8$ and $z_1 = 4$. For any $i = 1, 2$, and any $E \subset S$ such that $\#E = 3$, we have $\#\pi_i(S) = 4$, and $\pi_i(E)$ is linearly independent.

Proof of Claim 1. It is sufficient to prove the second statement of Claim 1. Since $\langle \pi_i(S) \rangle = \mathbb{P}^2$, any fiber of π_i contains two points of S at most. With no loss of generality, we prove the case $i = 1$. Assume that $\langle \pi_E \rangle$ is a line L and set $H_1 := \pi_1^{-1}(L)$. Write $S = \{a, b, c, d\}$ with $E = \{a, b, c\}$. Take a general $H_2 \in |\mathcal{I}_d(\varepsilon_2)|$. By step (b1), $Z \subset H_1 \cup H_2$. Since H_2 is general and each connected component of Z has degree ≤ 2, $H_2 \cap Z = \pi_2^{-1}(\pi_2(a)) \cap Z$. Since $z_1 = 4$ and $z = 8$, $\deg(H_1 \cap Z) = 4$, $\deg(H_2 \cap Z) = 4$ and $\deg(\mathrm{Res}_{H_i}(Z) = 4$. Since $\#((\pi_2^{-1}\pi_2(d) \cap S)) \le 2$, we get $\#((\pi_2^{-1}\pi_2(d) \cap S)) = 2$, say $(\pi_2^{-1}\pi_2(d)) \cap S = \{c, d\}$. Thus, $Z \cap H_2 = Z(c) \cup Z(d)$. Take $M \in |\mathcal{O}_Y(\varepsilon_2)$ containing d and b. We get $Z \cap M \supseteq Z(c) \cup Z(d) \cup \{b\}$, and hence, $z_1 > 4$, a contradiction. □

Claim 2. Assume $z = 8$ and $z_1 = 4$. Then, $\deg(\pi_i(Z(o))) = 1$ for all $i = 1, 2$, and all $o \in S$, and for each $U_i \in |\mathcal{O}_Y(\varepsilon_i)|$, $i = 1, 2$, such that $S \subset U_1 \cup U_2$, we have $\#(S \cap H_1) = \#(S \cap U_2) = 2$, $S \cap U_1 \cap U_2 = \emptyset$, and $Z \cap U_i = \cup_{o \in S \cap U_i} Z(o)$, $i = 1, 2$.

Proof of Claim 2. Claim 1 gives $\#\pi_i(S) = 4$, and that $\pi_i(S)$ is linearly independent. Thus, $\#(S \cap H_1) = \#(S \cap H_2) = 2$ and $S \cap H_1 \cap H_2 = \emptyset$. Since $Z \subset H_1 \cup H_2$, we get $Z \cap H_1 = Z(a) \cup Z(b)$ and $G = Z(c) \cup Z(d)$ with $S = \{a, b, c, d\}$. Set $\{M_2\} := |\mathcal{I}_{c,b}(\varepsilon_2)|$ and $\{M_1\} := |\mathcal{I}_{a,d}(\varepsilon_1)|$. Step (b1) and Claim 1 give $M_1 \cap Z = Z(a) \cup Z(d)$ and $M_2 \cap Z = Z(c) \cup Z(b)$. Hence $Z(a) \subset \pi_1^{-1}(\pi_1(a))$. Taking different partitions of S into two subsets of cardinality 2 we get $\deg(\pi_i(Z(o))) = 1$ for all $i = 0, 1$ and all $o \in S$. □

With no loss of generality, we may assume $z_1 := \deg(Z \cap H_1) \ge \deg(Z \cap H_2)$. Set $G := \mathrm{Res}_{H_1}(Z)$ and $g := \deg(G)$. Fix $i \in \{3, 4, 5\}$ such that there is $N_3 \in |\mathcal{O}_Y(\varepsilon_i)|$ with $e_1 := \deg(G \cap N_3)$ maximal. Permuting the last three factors of Y, we may assume $i = 3$. Take $i \in \{4, 5\}$ such that there is $N_4 \in |\mathcal{O}_Y(\varepsilon_i)|$ with $2_2 := \deg(\mathrm{Res}_{N_3}(G) \cap N_4)$ maximal. Permuting the last two factors of Y, we may assume $i = 4$. Take $N_5 \in |\mathcal{O}_Y(\varepsilon_5)|$ such that $e_3 := \deg(\mathrm{Res}_{N_3 \cup N_4}(W) \cap N_5)$ is maximal. Since $g \le 4$, $g - e_1 - e_2 - e_3 \le 1$. As in step (b1), we get that either $g = 4$, $e_1 = e_2 = 2$ and $e_3 = 0$ or $e_1 = g \in \{2, 3, 4\}$, and $e_2 = e_3 = 0$. The main difference with respect to step (b1) is that G is not a finite set, in general.

(b2.1) Assume $g = 4$, $e_1 = e_2 = 2$ and $e_3 = 0$. Thus, $z = 8$ and $\deg(Z \cap H_1) = 4$. Taking $H_1 \cup N_3$, we get $h^1(\mathcal{I}_{\mathrm{Res}_{N_3}(G)}(0,1,0,1,1)) > 0$. Since $\deg(\mathrm{Res}_{N_3}(G)) = 2$, Lemma 2 implies that $\mathrm{Res}_{N_3}(G)$ is connected, say $\mathrm{Res}_{N_3}(G) = Z(a)$ for some $a \in S$. Since $\mathrm{Res}_{N_4}(G) \subseteq G \cap N_3$, we get $\mathrm{Res}_{N_4}(G) = G \cap N_3$ and that $G \cap N_4 = Z(b)$ for some $b \in S \setminus \{a\}$. Since $G = Z(a) \cup Z(b)$, we obtain $Z \cap H_1 = Z(c) \cup Z(d)$ with $S = \{a, b, c, d\}$, $\deg(\pi_i(Z(a))) = 1$ for $i = 2, 4, 5$, and $\deg(\pi_i(Z(b))) = 1$ for $i = 2, 3, 5$. Taking $N_5 \in |\mathcal{I}_{Z(a)}(\varepsilon_5)|$ instead of N_3, we get $\deg(\pi_3(Z(a))) = 1$.

Using $N_5' \in |\mathcal{I}_{Z(b)}(\varepsilon_5)|$ instead of N_4, we get $\deg(\pi_4(Z(b))) = 1$. Recall that $\mathrm{Res}_{H_2}(Z)) = Z(c) \cup Z(d)$. Using $\mathrm{Res}_{H_2}(Z)$ instead G, we get $\deg(\pi_i(Z(c))) = \deg(\pi_i(Z(d)) = 1$ for $i = 1,3,4,5$. By Lemma 2 there is $i \in \{3,4,5\}$ such that $\pi_i(a) \neq \pi_i(b)$. Permuting the last three factors (we are allowed to do this at this point, since we run in a situation symmetric with respect to the last three factors), we may assume $i = 3$. Fix $M \in |\mathcal{O}_Y(\varepsilon_5)|$ containing $Z(c)$, $D \in |\mathcal{O}_Y(\varepsilon_4)|$ containing $Z(d)$, and $T \in |\mathcal{O}_Y(\varepsilon_3)|$ such that $T \cap \{a,b\} = \{b\}$. We have $\mathrm{Res}_{T \cup N_4 \cup D \cup M}(Z) = \{b\}$. Since $h^1(\mathcal{I}_b) = 0$, [5] (Lemma 5.1) gives a contradiction.

(b2.2) Assume $e_1 = g \in \{2,3,4\}$ and $e_2 = e_3 = 0$. We often use the inequality $h^1(\mathcal{I}_G(0,1,1,1,1)) > 0$.

(b2.2.1) Assume the non-existence of $A \subseteq G$ such that A is connected, $\deg(A) = 2$ and $\deg(\pi_i(A)) = 1$ for $i = 2,3,4,5$. Thus, $\#G_{\mathrm{red}} > 1$. By Lemma 2, $\eta_{1|G}$ is an embedding and hence $h^1(Y_1, \mathcal{I}_{\eta_1(G)}(1,1,1,1)) = h^1(\mathcal{I}_G(0,1,1,1,1)) > 0$. Since $\deg(G) \leq 4$, there are $j, h \in \{2,3,4,5\}$ such that $j \neq h$ and $\deg(\pi_j(G)) = \deg(\pi_h(G)) = 1$. Since $\langle \pi_2(S) \rangle = \mathbb{P}^2$, $j \neq 2$ and $h \neq 2$. If $g \leq 3$ there is a third index with the same property, contradicting Lemma 2. Now assume $g = 4$, and hence, $z = 8$ and $z_1 = 4$. Write $Z \cap H_1 = Z(a) \cup Z(b)$ and $G = Z(c) \cup Z(d)$ with $S = \{a,b,c,d\}$ and $\deg(\pi_i(Z(o))) = 1$ for all $i = 1,2$ and all $o \in S$ (Claims 1 and 2). Take a general $M_2 \in |\mathcal{I}_c(\varepsilon_2)|$. Since $\mathrm{Res}_{H_1 \cup M_2}(Z) = Z(d)$, we have $h^1(\mathcal{I}_{Z(c)}(0,0,1,1,1)) > 0$, and hence, $\deg(\pi_i(Z(d))) = 1$ for all $i > 2$. Thus, $\deg(\pi_i(Z(d))) = 1$ for all $1 \leq i \leq 5$, a contradiction.

(b2.2.2) Assume the existence of $A \subseteq G$ such that A is connected, $\deg(A) = 2$ and $\deg(\pi_i(A)) = 1$ for $i = 2,3,4,5$. We have $A = Z(p)$ for some $p \in S' := S \setminus S \cap H_1$.

(b2.2.2.1) Assume $g = 3$. Thus, $G = Z(p) \cup \{a\}$ for some $a \in S \setminus \{p\}$. By Lemma 4, there is $i \in \{2,3,4,5\}$ such that $\pi_i(a) \neq \pi_i(p)$. Take $M \in |\mathcal{I}_p(\varepsilon_i)|$. Since $\mathrm{Res}_{H_1 \cup M}(Z) = \{a\}$ and $h^1(\mathcal{I}_a) = 0$, we conclude quoting [5] (Lemma 5.1).

(b2.2.2.2) Assume $g = 4$, and hence, $z = 8$. Either $G = Z(p) \cup Z(a)$ or $G = Z(p) \cup \{a,b\}$. First assume $G = Z(p) \cup \{a,b\}$. By Lemma 4 there are $i \in \{3,4,5\}$ such that $\pi_i(p) \neq \pi_i(a)$ and $j \in \{2,3,4,5\} \setminus \{i\}$ such that $\pi_j(a) \neq \pi_j(b)$. Take $M \in |\mathcal{O}_Y(\varepsilon_i)|$ containing p and $D \in |\mathcal{O}_Y(\varepsilon_j)$ containing b. Note that $\mathrm{Res}_{H_1 \cup M \cup D}(Z) = \{a\}$. Since $h^1(\mathcal{I}_a) = 0$, we conclude by [5] (Lemma 5.1). Now assume $G = Z(p) \cup Z(a)$. Assume for the moment the existence of $i \in \{2,3,4,5\}$ such that $\deg(\pi_i(Z(a))) = 2$, and take $M_i \in |\mathcal{O}_Y(\varepsilon_i)|$ such that $a \in M_i$ and $Z(a) \nsubseteq M_i$. By Lemma 4 there is $j \in \{2,3,4,5\} \setminus \{i\}$ such that $\pi_j(p) \neq \pi_j(a)$. Take $M_j \in |\mathcal{O}_Y(\varepsilon_j)|$ such that $p \in M_j$ and $a \notin M_j$. Since $\mathrm{Res}_{H_1 \cup M_i \cup M_j}(Z) = \{a\}$, we conclude as above. Now assume $\deg(\pi_i(Z(a))) = 1$ for all $i > 1$. Note that $Z \cap H_1 = Z(b) \cup Z(c)$ and $Z \cap H_1 \cap H_2 = \emptyset$. Using H_2 instead of H_1, we get $\deg(\pi_i(Z(b))) = \deg(\pi_i(Z(c))) = 1$ for all $i = 1,3,4,5$. Note the $\deg(\pi_1(Z(p))) = \deg(\pi_1(Z(a))) = \deg(\pi_2(Z(b))) = \deg(\pi_2(Z(c))) = 2$. Take $U_1 \in |\mathcal{O}_Y(\varepsilon_1)|$ containing $\{p,b\}$ and $U_2 \in |\mathcal{O}_Y(\varepsilon_2)|$ containing $\{a,c\}$. Note that $Z \cap U_1 \supseteq Z(b) \cup \{p\}$ and $Z \cap U_2 \supseteq Z(a) \cup \{c\}$. By step (b1), $Z \subset U_1 \cup U_2$. Assume for the moment $p \notin U_2$ and $c \notin U_1$. We get $Z \cap U_1 = Z(p) \cup Z(a)$ and $Z \cap U_2 = Z(b) \cup Z(c)$. Thus, running the previous proof, we get $\deg(\pi_1(Z(b)) = 1$, contradicting the very ampleness of $\mathcal{O}_Y(1,1,1,1,1)$. Now assume for instance $p \in U_2$. Therefore, $U_2 \cap Z \supseteq Z(a) \cup \{p,c\}$. The maximality property of H_1 gives $U_2 \cap Z = Z(a) \cup \{p,c\}$ and $\mathrm{Res}_{U_2}(Z) = Z(b) \cup \{p,c\}$. We excluded all such cases.

(b2.2.2.3) Assume $g = 2$. We get $Z \cap H_1 = Z(a) \cup Z(b) \cup Z(c)$ with $S = \{a,b,c,p\}$. Since $S \nsubseteq H_1$, $p \notin H_1$, and hence, $\mathrm{Res}_{H_1}(Z) = Z(p)$. Set $Z' := Z(a) \cup Z(b) \cup Z(c)$. Recall that $h^1(\mathcal{I}_{Z(p)}(\hat{\varepsilon}_1)) > 0$, and hence, $\deg(\pi_i(Z(p))) = 1$ for all $i > 1$. Thus, $\deg(\pi_1(Z(p)) = 2$.

Claim 3. We have $\langle \pi_2(S')\rangle = \mathbb{P}^2$, i.e., $\#\pi_2(S') = 3$, and $\pi_2(S')$ is linearly independent.
Proof of Claim 3. Assume $L := \langle \pi_2(S')\rangle$ contained in a line. Since $\langle \pi_2(S)\rangle = \mathbb{P}^2$, L is a line. Set $M := \pi_2^{-1}(L) \in |\mathcal{O}_Y(\varepsilon_2)|$. Since $S \nsubseteq M$, $p \notin M$. Take a general line $R \subset \mathbb{P}^2$ containing $\pi_1(p)$. Set $D := \pi_1^{-1}(R)$. Since $S \subset M \cup D$, $Z \subset M \cup D$ (Claim 1). Since $p \notin M$, $Z(p) \subset D$. Since $\deg(\pi_1(Z(p))) = 2$ and R is general, $Z(p) \nsubseteq M$, a contradiction. □
Claim 4. Set $R := \langle \pi_1(Z(p))\rangle$. We have $\#(R \cap \pi_1(S')) = 1$.
Proof of Claim 4. Since $S \nsubseteq D := \pi_1^{-1}(R)$, $\#(R \cap \pi_1(S')) \le 2$. Assume $\#(R \cap \pi_1(S')) = 2$, say $\pi_1(b) \in R$ and $\pi_1(c) \in R$. Since $\langle \pi_1(S')\rangle$ is a line, $\pi_1(b) = \pi_1(c)$, and hence, $\langle \pi_1(S')\rangle = \langle \{\pi_1(a), \pi_1(b)\}\rangle$. Take a general line $L \subset \mathbb{P}^2$ containing $\pi_2(a)$, and set $M := \pi_2^{-1}(L)$. Since $S \subset D \cup M$, $Z \subset D \cup M$ (Claim 1). Since L is general, Claim 3 gives $\{b, c\} \cap M = \emptyset$. Since $a \notin D$, we get $Z(c) \cup Z(b) \cup Z(p) = Z \cap D$. Taking $\mathrm{Res}_D(Z)$, we get $\deg(\pi_i(a)))$ for all $i > 1$. Since $\deg(Z(a)) = 2$, we get $\deg(\pi_1(Z(a))) = 2$, and hence, $\langle \pi_1(Z(a)\rangle = \langle \pi_1(S')\rangle$. Using D instead of H_1 and M instead of H_2 in the proof of Claim 3, we get that $\langle \pi_2(\{b, c, p\})\rangle = \mathbb{P}^2$. Let M' be the only element of $|\mathcal{O}_Y(\varepsilon_2)|$ containing $\{b, c\}$. Take $D' \in |\mathcal{O}_Y(\varepsilon_1)|$ containing $\{a, p\}$. Claim 1 gives $Z \subset D' \cup M'$. Since $a \notin M'$, $Z(a) \subset D'$. Since $\langle \pi_1(Z(a))\rangle = \langle \pi_1(S')\rangle$, $p \notin \langle \pi_1(Z(a))\rangle$. Thus, $Z(a) \nsubseteq D'$, a contradiction. □
Now assume $\pi_1(S') \cap R = \emptyset$. Since $\mathbb{P}^2 = \langle \pi_2(S')\rangle$ there are $b', c' \in S'$ such that $b' \neq c'$ and $\pi_2(p) \notin \langle\{\pi_2(b'), \pi_2(c')\rangle$. With no loss of generality, we may assume $b' = b$ and $c' = c$. Take $\{D''\} := |\mathcal{I}_{p,a}(\varepsilon_1)|$ and $M'' := |\mathcal{I}_{c,b}(\varepsilon_2)|$ (Claim 3). Claim 1 gives $Z \subset D'' \cup M''$. Since $p \notin M''$, $Z(p) \subset D''$ contradicting the assumption $a \notin R$. □
We just proved that $\#(R \cap \pi_1(S')) = 1$, say $R \cap \pi_1(S') = \{\pi_1(c)\}$. Set $\{M_1\} := |\mathcal{I}_{\{b,c\}}(\varepsilon_2)|$ and note that $a \notin M_1$ (Claim 3). Set $\{D_1\} := |\mathcal{I}_{\{a,p\}}(\varepsilon_1)|$. Claim 1 gives $Z \subset D_1 \cup M_1$. Since $\pi_1(a) \notin R$, $p \in M_1$, i.e., $\pi_2(p) \in \langle\{\pi_2(b), \pi_2(c)\rangle$. Using a instead of b, we get $\pi_2(p) \in \langle\{\pi_2(a), \pi_2(c)\rangle$. Claim 3 gives $\langle\{\pi_2(b), \pi_2(c)\rangle \cap \langle\{\pi_2(a), \pi_2(c)\rangle = \{\pi_2(c)\}$. Therefore, $p_2(p) = \pi_2(c)$. Set $\{M_2\} := |\mathcal{I}_{c,b}(\varepsilon_2)|$. Claim 3 gives $a \notin M_2$. Take a general $D_2 \in |\mathcal{I}_a(\varepsilon_1)|$. Since $S \subset D_2 \cup M_2$, $Z \subset D_2 \cup M_2$ and $a \notin M_2$, $Z(a) \subset D_2$ and $Z(c) \subset M_2$. Since D_2 is general, $\deg(\pi_1(Z(a))) = 1$. Using $M_3 := |\mathcal{I}_{c,a}(\varepsilon_2)|$ instead of M_2, we get $\deg(\pi_1(Z(b))) = 1$ and $Z(c) \subset M_3$. Since $M_2 \cap M_3 = \pi_2^{-1}(c)$, we get $\deg(\pi_2(Z(c))) = 1$.
Fix a general $D_4 \in |\mathcal{I}_a(\varepsilon_1)|$ and a general $M_4 \in |\mathcal{I}_c(\varepsilon_2)|$. Since D_4 and M_4 are general, we just proved that $Z \cap (D_4 \cup M_4) = Z(a) \cup Z(c) \cup Z(p)$, and hence, $\deg(\pi_i(Z(b))) = 1$ for $i = 3, 4, 5$. Since $\deg(\pi_1(Z(b))) = 1$, $\deg(\pi_2(Z(b))) = 2$. Taking a general $D_5 \in |\mathcal{I}_b(\varepsilon_1)|$ and using $D_5 \cup M_4$, we get $\deg(\pi_i(Z(a))) = 1$ for $i = 3, 4, 5$. Since $\deg(\pi_1(Z(a))) = 1$, $\deg(\pi_2(Z(a))) = 2$. Thus, we proved that $h^1(\mathcal{I}_{Z(o)}(1, 1, 0, 0, 0)) = 0$ for all $o \in S$. Let e_1 be the maximal integer $e := \#(S \cap M)$ for some $i \in \{3, 4, 5\}$. Obviously $e \ge 1$. Since $S \nsubseteq M$, $e \le 3$. First assume $e = 3$. Thus, $\mathrm{Res}_M(Z) = Z(o)$ for some $o \in Z$. We conclude, because (since $i > 2$) $h^1(\mathcal{I}_{Z(o)}(\varepsilon_i)) \le h^1(\mathcal{I}_{Z(o)}(1, 1, 0, 0, 0)) = 0$. Now assume $e = 1$. The maximality of the integer e gives $\#\pi_i(S) = 4$ for all $i = 3, 4, 5$. Set $\{U_3\} := |\mathcal{I}_p(\varepsilon_3)|$, $\{U_4\} := |\mathcal{I}_a(\varepsilon_3)|$ and $\{U_5\} := |\mathcal{I}_b(\varepsilon_5)|$. Since $\mathrm{Res}_{U_3 \cup U_4 \cup U_5}(Z) = Z(c)$, it is sufficient to use that $h^1(\mathcal{I}_{Z(o)}(1, 1, 0, 0, 0)) = 0$. Now assume $e = 2$. With no loss of generality, we may assume $M \in |\mathcal{O}_Y(\varepsilon_3)|$. Set $S_1 := M \cap S$ and $S_2 := S \setminus S_1$. First assume the existence of $i \in \{4, 5\}$ such that $\#\pi_i(S_2) = 1$. Take $M' \in |\mathcal{O}_Y(\varepsilon_i)|$ containing exactly one point of S_2 and use that $\mathrm{Res}_{M \cup M'}(Z) = Z(o)$ for some $o \in S$. Now assume $\#\pi_i(S_2) = 1$ for $i = 4, 5$, and set $\{U_i\} := |\mathcal{I}_{S_2}(\varepsilon_i)|$, $i = 4, 5$. Using U_4 (resp. U_5), instead of M, and the maximality of the integer e, we get $\#(\pi_5(S_1)) = 2$ and $\#\pi_3(S_2) = 2$ (resp. $\#\pi_4(S_1) = 2$). Thus, $\#\pi_i(S) = 2$ for all $i = 3, 4, 5$ and $S_1 \sqcup S_2$ is the partition of S obtained as fibers of the maps $\pi_{i|S}$, $i = 3, 4, 5$. Since $\pi_2(c) = \pi_2(p)$, Lemma 2 gives that p and c are in different sets S_1 and S_2, say $p \in S_1$ and $c \in S_2$, and that $\pi_2(p) \notin \{\pi_2(a), \pi_2(b)\}$. Now the situation is

symmetric for a and b. Therefore, we may assume $S_1 = \{p,a\}$ and $S_2 = \{c,b\}$. Take $\{Q_3\} := |\mathcal{I}_p(\varepsilon_3)|$ and take a general $Q_2 \in |\mathcal{I}_b(\varepsilon_2)|$. Since $\pi_2(p) \neq \pi_2(b)$, $\deg(\pi_2(Z(b))) = 2$ and Q_2 is general, $\mathrm{Res}_{Q_2\cup Q_3}(Z) = Z(a)\cup\{b\}$. First assume $\pi_1(a) \neq \pi_1(b)$ and take a general $Q_1 \in |\mathcal{I}_a(\varepsilon_1)|$. Since $\deg(\pi_1(Z(a)) = 1$, we get $\mathrm{Res}_{Q_1\cup Q_2\cup Q_3}(Z) = \{b\}$, concluding because $h^1(\mathcal{I}_b(0,0,0,1,1)) = 0$. Now assume $\pi_1(a) = \pi_1(b)$ and set $\{U_1\} := |\mathcal{I}_{a,p}(\varepsilon_1)|$. Since. $\pi_1(a) \notin \langle \pi_1(Z(p))\rangle$, $\deg(\pi_1(Z(a))) = \deg(\pi_1(Z(b))) = 1$ and $\langle \pi_1(S')\rangle$ is a line, $\mathrm{Res}_{U_1}(Z) = Z(c)\cup\{p\}$. Take $\{U_3\} := |\mathcal{I}_c(\varepsilon_3)|$, and use that $\mathrm{Res}_{U_1\cup U_3}(Z) = \{p\}$.

(c) Assume $Y = \mathbb{P}^2\times(\mathbb{P}^1)^5$. Take $H_1 \in |\mathcal{O}_Y(\varepsilon_1)|$ such that $z_1 := \deg(Z\cap H_1)$ is maximal. Note that $z_1 \geq 2 = \dim|\mathcal{O}_Y(\varepsilon_1)|$. Set $W := \mathrm{Res}_{H_1}(Z)$ and $w := \deg(W) = z - z_1$. Fix $i \in \{2,3,4,5\}$ such that there is $H_2 \in |\mathcal{O}_Y(\varepsilon_i)|$ with $w_1 := \deg(W\cap H_i)$ maximal. Permuting the last five factors of Y we may assume $i = 2$. Set $W_2 := \mathrm{Res}_{H_2}(W)$. We continue defining the integers w_i and $H_i \in |\mathcal{O}_Y(\varepsilon_i)|$ (up to a permutation of the last $7-i$ factors of Y) with $w_1 \geq \cdots \geq w_5$. Let e be the last integer such that $w_e \geq 1$. Since $\dim Y = 7 \geq z-1$, e is well-defined. By [5] (Lemma 5.1), we have $w_e \geq 2$. Thus, either $z = 8$, $e = 3$ and $z_1 = w_1 = w_2 = w_3 = 2$ or $1 \leq e \leq 2$. We have $h^1(\mathcal{I}_W(\hat{\varepsilon}_1)) > 0$ ([5], Lemma 5.1). For any $o \in S$ set $\hat{o} := \{o\}$ if $\deg(Z(o)) = 2$ and $\hat{o} := \emptyset$ if $Z(o) = \{o\}$. For any $A \subset S$ such that $\#A \in \{2,3\}$ call $J(A)$ (resp. $I(A)$) the set of all $i \in \{3,4,5,6\}$ (resp. $i \in \{2,3,4,5,6\}$) such that $\#\pi_i(A) \geq 2$. Lemma 2 gives $\#J(A) \geq 3$, and $\#I(A) \geq 4$ for all A such that $\#A = 2$.

Observation 1: Fix $A \subset S$ such that $\#A = 3$. By [1] (Th. 4.12), $\#\pi_i(A) \geq 2$ for at least 5 integers $i \in \{1,2,3,4,5,6\}$.

Claim 5. There is $x \in S'$ such that $\langle \pi_1(Z(d))\cup\{\pi_1(x)\}\rangle = \mathbb{P}^2$ and x is unique if and only if $\#(\pi_1(S'\setminus\{x\})) = 1$.

Proof of Claim 5. We saw that $R := \langle \pi_1(Z(d))\rangle$ is a line. A point $x \in S'$ satisfies Claim 5 if and only if $\pi_1(x) \notin R$. Since Y is the minimal multiprojective space containing S and $\pi_1(d) \in S'$, there is at least one $x \in S'$ satisfying Claim 5. □

Since $\langle \pi_1(S)\rangle = \mathbb{P}^2$, $\langle \pi_1(S')\rangle$ is a line $L \neq R$. Since $\#(R\cap L) = 1$, x is unique if and only if $\pi_1(S'\setminus\{x\}) = L\cap R$. Let Σ be the set of all $x \in S'$ such that $\langle \pi_1(Z(d))\cup\{\pi_1(x)\}\rangle = \mathbb{P}^2$.

Observation 2: $z_1 = 2$ if and only if $\pi_{1|Z}$ is an embedding and $\langle \pi_1(E)\rangle = \mathbb{P}^2$ for every degree 3 subscheme of Z.

(c1) Assume $z = 8$, $e = 3$ and $z_1 = w_1 = w_2 = w_3 = 2$. Since $w_1 = w_2 = w_3 = 2$, we may permute the divisors H_2, H_3 and H_4, and still obtain residual schemes with the same degrees. Since $h^1(\mathcal{I}_{W_3}(0,0,0,1,1)) > 0$, we get $\deg(\pi_h(W\cap H_i)) = 1$ for $i = 2,3,4$, and $h = 5,6$ and for $h = i$. Hence there are $M_h \in |\mathcal{O}_Y(\varepsilon_h)|$, $h = 4,5,6$, such that $W \subset M_4\cup M_5\cup M_6$. Since $z_1 = 2$ and $\mathbb{P}^2 = \langle \pi_1(A)\rangle$ for all $A \subset Z$ such that $\deg(A) = 3$, we conclude, unless $Z \subset M_4\cup M_5\cup M_6$. Permuting the last three factors of Y, we may assume that $\deg(Z\cap M_i)$ has the maximum for $i = 4$ and that $\deg(\mathrm{Res}_{M_4}(Z)\cap M_5) \geq \deg(\mathrm{Res}_{M_4}(Z)\cap M_6)$. Since $z_1 = w_1 = 2$, $\deg(Z\cap M_4) \leq 4$. First assume $Z \subset M_4\cup M_5$ and hence $\deg(Z\cap M_i) = \deg(\mathrm{Res}_{M_i}(Z)) = 4$ for $i = 4,5$. We have $h^1(\mathcal{I}_{\mathrm{Res}_{M_i}(Z)}(\hat{\varepsilon}_i)) > 0$, $i = 4,5$. Since $S \nsubseteq M_i$, we get that either $\#((Z\cap M_i)) = 2$ for $i = 4,5$ or $\#((Z\cap M_i)) = 2$ for $i = 4,5$. First assume $\#((Z\cap M_i)) = 2$, say $Z\cap M_4 = Z(a)\cup Z(b)$ and $Z\cap M_5 = Z(c)\cup Z(d)$. Since $z_1 = w_1 = w_2 = 2$, and $\deg(\pi_5(Z\cap M_5)) = 1$, Remark 4 and Lemma 2 give the existence of at least one $i \in \{2,3,6\}$ such that $\deg(\pi_i(Z\cap M_5)) > 1$. Take $D_i \in |\mathcal{O}_Y(\varepsilon_i)|$ such that $Z\cap D_i \neq \emptyset$. Since $Z\cap D_i \neq Z\cap\mathrm{Res}_{M_4}(Z)$, we have $1 \leq \deg(\mathrm{Res}_{M_4\cup D_i}(Z)) \leq 3$ and hence $h^1(\mathcal{I}_{\mathrm{Res}_{M_4\cup D_i}(Z)}(\varepsilon_1)) = 0$. Now assume $\#((Z\cap M_i)) = 3$, say $Z\cap M_4 = Z(a)\cup\{b,c\}$ and $Z\cap M_5 = \{b,c\}\cup Z(d)$ with $\{b,c\} \in M_4\cap M_5$. There is $i \in \{2,3,6\}$ such that $\deg(\pi_i(Z\cap M_5)) > 1$. Take $U_i \in |\mathcal{O}_Y(\varepsilon_i)|$ and use $M_4\cup U_i$. Now assume $Z \nsubseteq M_4\cup M_5$. Since $\deg(Z) < 9$, we get $\mathrm{Res}_{M_4\cup M_5}(Z) \leq 2$, and hence, $h^1(\mathcal{I}_{\mathrm{Res}_{M_4\cup M_5}(Z)}(\varepsilon_1)) = 0$, concluding the proof.

(c2) Assume $e = 2$. Hence, $Z \subset H_1 \cup H_2 \cup H_3$. Since $w_2 \geq 2$, either $w_2 = 2$ or $z = 8$, $z_1 = 2$ and $w_1 = w_2 = 3$. We have $\mathrm{Res}_{H_2 \cup H_3}(Z) \subseteq Z \cap H_1$. Now assume $z_1 = 2$ and $Z \subset H_2 \cup H_3$. We conclude using $H_2 \cup H_3$ instead of $M_4 \cup M_5$ as in step (c1). Now assume $z_1 = 2$, and $\mathrm{Res}_{H_2 \cup H_3}(Z) \neq \emptyset$. Since $\mathrm{Res}_{H_2 \cup H_3}(Z) \subset H_1$, we have $\deg(\mathrm{Res}_{H_2 \cup H_3}(Z)) \leq 2$, and hence, we conclude by Observation 2.
Now assume $z_1 > 2$. Since $w_1 \geq w_2 \geq 2$ and $z \leq 8$, we get $w_2 = 2$, $w_1 + z_1 = z - 2$ and $(z_1, w_1) \in \{(4,2),(3,3),(3,2)\}$. Lemma 2 gives $\mathrm{Res}_{H_1 \cup H_2}(Z) = Z(d)$ for some $d \in S$ such that $\deg(Z(d)) = 2$ and $\deg(\pi_i(Z(d))) = 1$ for all $i > 2$. Hence, $\deg(\pi_i(Z(d))) = 2$ for at least one $i \in \{1,2\}$.

(c2.1) Assume $w_1 = z_1 = 3$, and hence, $z = 8$. By Remark 4, neither $Z \cap H_1$ nor $H_2 \cap \mathrm{Res}_{H_1}(Z)$ are reduced, and hence, $Z \cap H_1 = Z(a) \cup \{b\}$, $Z \cap H_2 = \mathrm{Res}_{H_1}(Z) \cap H_2 = Z(c) \cup \{b\}$ with $S = \{a,b,c,d\}$. Since $h^1(\mathcal{I}_{\mathrm{Res}_{H_1 \cup H_3}(Z)}(\hat{\varepsilon}_1 - \varepsilon_3)) > 0$, either $\deg(\pi_i(Z(c))) = 1$ for $i = 2,4,5,6$, or there are at least 3 indices $i \in \{2,4,5,6\}$ such that $\pi_i(Z(c)) = \pi_1(b)$ (Proposition 1). Since $h^1(\mathcal{I}_{\mathrm{Res}_{H_2 \cup H_3}(Z)}(\hat{\varepsilon}_2 - \varepsilon_3)) > 0$, either $\deg(\pi_i(Z(a))) = 1$ for $i = 1,4,5,6$ or there are at least 3 indices $i \in \{1,4,5,6\}$ such that $\pi_i(Z(a)) = \pi_1(b)$ as schemes (Proposition 1). First assume the existence of $i \in \{4,5,6\}$ such that $\deg(\pi_i(Z(a))) = 1$, and $\pi_i(a) \neq \pi_i(b)$. Set $\{U_i\} := |\mathcal{I}_a(\varepsilon_i)|$. Since $\mathrm{Res}_{H_2 \cup H_3 \cup U_i}(Z) = \{b\}$ and $h^1(\mathcal{I}_b(1,0,\dots)) = 0$, we conclude. Since $h^1(\mathcal{I}_a(1,0\dots,0)) = 0$, we also conclude if there is $j \in \{4,5,6\}$ such that
$\deg(\pi_j(Z(a))) = 2$ and $\pi_j(a) = \pi_j(b)$. Now assume that no such $i,j \in \{4,5,6\}$ exist. It implies $\deg(\pi_h(Z \cap H)) = 1$ for $h = 4,5,6$. Take $h \in \{4,5,6\}$ such that $\pi_h(d) \neq \pi_h(a)$ (Proposition 1). Set $\{U_h\} := |\mathcal{I}_a(\varepsilon_h)|$. We have $\mathrm{Res}_{H_2 \cup U_h}(Z) = Z(d)$. Since $\mathrm{Res}_{H_2 \cup U_h}(Z) = Z(d)$, we conclude if $\deg(\pi_1(Z(d))) = 2$. Now assume $\deg(\pi_1(Z(d))) = 1$, and hence, $\deg(\pi_2(Z(d))) = 2$. We use H_1 and $H_2 \cap Z$ instead of H_2 and $H_1 \cap Z$.

(c2.2) Assume $z_1 = 4$, and hence, $w_1 = 2$ and $z = 8$. Using $H_1 \cup H_3$, we get $Z \cap H_2 = Z(c)$ for some $c \in S \setminus \{d\}$ such that $\deg(\pi_i(Z(c))) = 1$ for all $i \in \{2,4,5,6\}$. Thus, $Z \cap H_1 = Z(a) \cup Z(b)$ with $S = \{a,b,c,d\}$. Using $H_2 \cup H_3$, we get $h^1(\mathcal{I}_{Z(a) \cup Z(b)}(1,0,0,1,1,1)) > 0$. First assume $\deg(\pi_1(Z(a))) = \deg(\pi_1(Z(b))) = 1$ and hence $\pi_1(b) \neq \pi_1(a)$. Taking $H_2 \cup M_i \cup D_j$ for some $3 < i < j$ we conclude, unless $Z(a) \subset M_i$, i.e., $\deg(\pi_i(Z(a))) = 1$, and $Z(b) \subset D_j$, i.e.,$\deg(\pi_j(Z(b))) = 1$. Thus, we may assume that $\deg(\pi_i(Z(a))) = \deg(\pi_i(Z(b))) = 1$ for all $i \in J(\{a,b\})$. First assume $\deg(\pi_1(Z(o))) = 2$ for at least one $o \in \{c,d\}$, say for $o = c$. We take $i,j \in J(\{a,b\})$ such that $i \neq j$ and set $U_i := |\mathcal{I}_a(\varepsilon_i)|$ and $\{U_j\} := |\mathcal{I}_b(\varepsilon_j)|$. We conclude using $H_3 \cup U_i \cup U_j$, unless $Z(c) \subset H_3 \cup U_i \cup U_j$. Since $w_1 = 2$, $c \notin H_3$. Thus, $Z(c) \subset H_3 \cup U_i \cup U_j$ if and only if either $c \in U_i \cap U_j$ or $Z(c) \subset U_i$ or $Z(c) \subset U_j$. To take i,j such that $c \notin U_i \cap U_j$, it is sufficient to use that $\#J(\{a,b\})) \geq 3$ and $\deg(\pi_i(Z(a))) = \deg(\pi_i(Z(b))) = 1$ for all $i \in J(\{a,b\})$. Now assume $\deg(\pi_1(Z(c))) = \deg(\pi_1(Z(d))) = 1$. We get $\deg(\pi_3(Z(c))) = 2$ and $\deg(\pi_2(Z(d))) = 2$. Take $i \in J(\{c,d\})$, say $i = 4$. Set $\{U_4\} := |\mathcal{I}_d(\varepsilon_4)|$. We conclude using $H_1 \cup U_4$, because $h^1(\mathcal{I}_{Z(c)}(0,0,0,0,1,1,1)) = 0$.

(c2.3) Assume $z_1 = 3$, and hence, $w_1 = 2$ and $z = 7$. We get that $\mathrm{Res}_{H_1}(Z) \cap H_2 = Z(c)$ and $Z \cap H_1 = Z(a) \cup \{b\}$ (up to the names of the elements of S'). Using $H_1 \cup H_3$ we get $\deg(\pi_i(Z(c))) = 1$ for $i = 2,4,5,6$. Hence $\deg(\pi_i(Z(c))) = 2$ for at least one $i \in \{1,3\}$. Using $H_2 \cup H_3$ we get that either $\deg(\pi_i(Z(a))) = 1$ for $i = 1,4,5,6$ or there are at least 3 indices $i \in \{1,4,5,6\}$ such that $\pi_i(Z(a)) = \pi_1(b)$ (Proposition 1). Since $z_1 < 4$, there is at most one $o \in S$ such that $\deg(\pi_1(Z(o))) = 1$.
Assume for the moment the existence of $i \in \{4,5,6\}$ such that $\pi_i(a) \neq \pi_i(b)$, say $i = 4$. First assume $\pi_1(a) \neq \pi_1(b)$. Take $\{T_4\} := |\mathcal{I}_a(\varepsilon_4)|$. We have $\{b\} \subseteq \mathrm{Res}_{T_4 \cup H_2 \cup H_3}(Z) \subseteq \{a,b\}$ and we use that $h^1(\mathcal{I}_{a,b}(\varepsilon_1)) = 0$ by the assumption $\pi_1(a) \neq \pi_1(b)$. Now assume $\pi_1(a) = \pi_1(b)$. Since $z_1 = 3$, $\deg(\pi_1(Z(x))) = 2$

for all $x \in S$. Thus, $\deg(\pi_1(Z(a))) = 2$, and hence $h^1(\mathcal{I}_{Z(a)}(\varepsilon_1)) = 0$. Set $\{D_4\} := |\mathcal{I}_b(\varepsilon_4)|$ and use that $\mathrm{Res}_{D_4\cup H_2\cup H_3}(Z) = Z(a)$. Now assume $\pi_i(a) = \pi_i(b)$ for all $i = 4,5,6$. Lemma 2 gives $\pi_i(a) \neq \pi_i(b)$ for all $i = 1,2,3$. Set $M_4 := |\mathcal{I}_a(\varepsilon_4)|$. Use the residual exact sequence with respect to $M_4 \cup E_2$ if $\deg(\pi_1(Z(d))) = 2$, and the residual exact sequence with respect to $M_4 \cup H_3$ if $\deg(\pi_1(Z(c))) = 2$.

(c3) Assume $e = 1$. We get $Z \subset H_1 \cup H_2$, and $w_1 = z - z_1$ with $w_1 \geq 2$ and $z_1 \geq 2$. Thus, $h^1(\mathcal{I}_{\mathrm{Res}_{H_2}(Z)}(\hat{\varepsilon}_2)) > 0$. If $z_1 = 2$ it is sufficient to use Observation 1. Thus, we only need to test the cases $3 \leq z_1 \leq 6$.

(c3.1) Assume $z_1 = 3$. Thus, (after changing the names of the elements of S) either $Z \cap H_1 = \{a,b,c\}$,and $\mathrm{Res}_{H_1}(Z) = Z(d) \cup \hat{a} \cup \hat{b} \cup \hat{c}$ or $Z \cap H_1 = Z(a) \cup \{b\}$, and $\mathrm{Res}_{H_1}(Z) = Z \cap H_2 = \hat{b} \cup Z(c) \cup Z(d)$ with $\deg(Z(a)) = 2$. First assume $Z \cap H_1 = \{a,b,c\}$ with, say, $\pi_1(c) \notin \{\pi_1(a), \pi_1(b)\}$. Since $S \nsubseteq H_2$, $\{a,b,c\} \nsubseteq H_2$. Take $j \in J(\{a,b\})$, set $\{M_j\} := |\mathcal{I}_a(\varepsilon_j)|$ and use $H_2 \cup M_j$. Now assume $Z \cap H_1 = Z(a) \cup \{b\}$. If $\pi_1(a) \neq \pi_1(b)$, use $H_2 \cup M_j$ with $\{M_j\} := |\mathcal{I}_a(\varepsilon_j)|$. If $\pi_1(a) = \pi_1(b)$, and hence, $\deg(\pi_1(Z(a))) = 2$ use $H_2 \cup D_j$ with $\{D_j\} := |\mathcal{I}_b(\varepsilon_j)|$.

(c3.2) Assume $z_1 = 4$. Since $S \nsubseteq H_1$, after changing the names of the elements of S, either $Z \cap H_1 = Z(a) \cup \{b,c\}$, and $\mathrm{Res}_{H_1}(Z) = \hat{b} \cup \hat{c} \cup Z(d)$ with $\deg(Z(a)) = 2$ or $Z \cap H_1 = Z(a) \cup Z(b)$ with $\deg(Z(a)) = \deg(Z(b)) = 2$, and $\mathrm{Res}_{H_1}(Z) = Z \cap H_2 = Z(c) \cup Z(d)$. There are at least 3 indices $j > 2$ such that $\pi_j(a) \neq \pi_j(b)$, say j_1, j_2, j_3. Set $\{M_h\} := |\mathcal{I}_a(\varepsilon_h)|$ and $\{D_h\} := |\mathcal{I}_b(\varepsilon_h)|$. If $Z \cap H_1 = Z(a) \cup \{b,c\}$, $\pi_1(b) \neq \pi_1(c)$, $\hat{b} = \hat{c} = \emptyset$), and $\{b,c\} \subset H_2$, it is sufficient to use $H_2 \cup M_{j_1}$. Now assume $Z \cap H_1 = Z(a) \cup \{b,c\}$ and $\pi_1(b) = \pi_1(c)$. Thus, $\pi_1(a) \neq \pi_1(b)$. It is sufficient to use H_2 (case $\hat{b} = \hat{c} = \emptyset$), $\{b,c\} \subset H_2$ and $\deg(\pi_1(Z(a)) = 2)$, $H_2 \cup M_{j_1}$ (case $\hat{b} = \hat{c} = \emptyset$) and $\{b,c\} \subset H_2$ and $\deg(\pi_1(Z(a)) = 1)$ and $H_2 \cup M_{j_1} \cup M_{j_2} \cup D_{j_3}$ (all other cases with $c \notin H_2 \cup M_{j_1} \cup M_{j_2} \cup D_{j_3}$). If $c \in H_2$ and $\deg(Z(c)) = 1$, we exchange the role of b and c.
Now assume $Z \cap H_1 = Z(a) \cup Z(b)$ and $H_2 \cap \{a,b\} = \emptyset$. Assume for the moment $\deg(\pi_1(Z(o))) = 2$ for at least one $o \in \{a,b\}$, say for $o = a$. We use $H_2 \cup D_{j_1} \cup D_{j_2}$. Now assume $\deg(\pi_1(Z(a))) = \deg(\pi_1(Z(b))) = 1$, and hence, $\pi_1(a) \neq \pi_1(b)$ (by the definition of z_1). We use $H_2 \cup M_{j_1} \cup D_{j_2}$. If $H_2 \cap \{a,b\} \neq \emptyset$ (and hence, $\#(H_2 \cap \{a,b\}) = 1$ because $S \nsubseteq H_2$), then we omit one or two of the divisors M_h, D_h.

(c3.3) Assume $z_1 = 5$, and hence, $w_1 = 3$. Since $S \nsubseteq H_1$, (after changing the names of the elements of S) we have $Z \cap H_1 = Z(a) \cup Z(b) \cup \{c\}$ and $\mathrm{Res}_{H_1}(Z) = \hat{c} \cup Z(d)$ with $\deg(Z(a)) = \deg(Z(b)) = 2$. Since $w_1 = 3$, $\hat{c} = \{c\}$ anddeg$(Z(d)) = 2$. Fix $i, j \in J(\{c,d\})$ such that $i \neq j$ and use $H_1 \cup M_i \cup M_j$ with $\{M_h\} := |\mathcal{I}_a(\varepsilon_h)|$.

(c3.4) Assume $z_1 = 6$, and hence, $w_1 = 2$ and $z = 8$. By Lemma 2, the scheme $\mathrm{Res}_{H_1}(Z)$ is a connected component $Z(d)$ of Z, and hence, $Z \cap H_1 = Z(a) \cup Z(b) \cup Z(c)$ with $S = \{a,b,c,d\}$. Set $S' := \{a,b,c\}$. Since $\deg(Z(d)) = 2$ and $\deg(\pi_i(Z(d))) = 1$ for all $i > 1$, $\deg(\pi_1(Z(d))) = 2$. Note that this case is symmetric with respect to the permutation of the last five factors of Y.

(c3.4.1) Assume $\pi_i(d) \notin \pi_i(S')$ for all $i = 2,3,4,5,6$. Fix $x \in \{a,b,c\}$ such that $\pi_1(x) \notin \langle \pi_1(Z(d)) \rangle$. Since $\pi_i(d) \notin \pi_i(S')$ for all $i = 2,3,4,5,6$, there are $M_i \in |\mathcal{O}_Y(\varepsilon_i)|$, $2 \leq i \leq 6$, such that $d \notin D := M_2 \cup M_3 \cup M_4 \cup M_5 \cup M_6$ (and hence, $\mathrm{Res}_D(Z) \supseteq Z(d)$) and $\mathrm{Res}_D(Z) \subseteq Z(d) \cup \{x\}$. Since $h^1(\mathcal{I}_{\{x\}\cup Z(d)}(\varepsilon_1)) = 0$, we conclude.

Claim 6. Let $G_1 \subseteq G$ be the minimal subscheme such that $h^1(\mathcal{I}_{G_1}(\hat{\varepsilon}_6)) > 0$. There is $o \in S'$ such that $Z(o) \subseteq G_1$ and $\deg(\eta_6(Z(o))) = 1$.

Proof of Claim 6. Assume the non-existence of any o. By Remark 4, the map $\eta_{6|G_{\mathrm{red}}}$ is injective. Thus, the map $\eta_{6|G_1}$ is an embedding and we have $h^1(Y_6, \mathcal{I}_{\eta_6(G_1)}(1,1,1,1,1)) = h^1(\mathcal{I}_{G_1}(\hat{\varepsilon}_6))) > 0$. Let Y' be the minimal multiprojective subspace of Y_6 containing $\eta_6((G_1)_{\mathrm{red}})$, and Y'' the minimal multiprojective space containing G_1. By [1] (Th. 4.14) either there are $u, v \in (G_1)_{\mathrm{red}}$ such that $u \neq v$ and $\pi_i(u) = \pi_i(v)$ for at least 3 integers $i \in \{1,2,3,4,5\}$ or $\#(G_1)_{\mathrm{red}} = 3$

and $Y' \cong (\mathbb{P}^1)^4$. Since $\deg(G) \le 5$ and $h^1(Y_6, \mathcal{I}_{\eta_6(G_1)}(1,1,1,1,1)) > 0$, Proposition 1 and Lemma 7 exclude the latter case. Assume the existence of u and v. Since $h^1(Y_6, \mathcal{I}_{\eta_6(G_1)}(1,1,1,1,1)) > 0$, the minimality of G_1 and the injectivity of $\eta_{6|G_{\mathrm{red}}}$ gives that G_1 contains $Z(u) \cup Z(v)$ and that the minimal multiprojective space containing $\eta_6(Z(u) \cup Z(v))$ is isomorphic either to $\mathbb{P}^1$ or to $\mathbb{P}^1 \times \mathbb{P}^1$. Thus, we get $\deg(\pi_i(Z(u) \cup Z(v))) = 1$ for at least 3 integers $i \in \{1, \dots, 5\}$ such that $\pi_i(u) = \pi_i(v)$. We may assume $\pi_2(Z(u)) = \pi_2(Z(v)) = \pi_2(u)$ and $\pi_3(Z(u)) = \pi_3(Z(v)) = \pi_3(u)$, but we need to distinguish the case $\pi_1(u) = \pi_1(v)$ and the case $\pi_4(u) = \pi_4(v)$. Write $S' = \{u, v, z\}$ with $\pi_6(z) = \pi_6(d)$. Lemma 2 gives the existence of at least 3 indices $i \in \{1,2,3,4,5\}$ such that $\pi_i(z) \ne \pi_i(d)$. Remark 4 gives the existence of at least 2 indices $i \in \{1,2,3,4,5\}$ such that $\#(\pi_1(S')) > 1$. Set $M_2 := |\mathcal{I}_u(\varepsilon_2)|$ and $W := \mathrm{Res}_{M_2}(Z)$. We have $W \ne \emptyset$ and $W \subseteq Z(z) \cup Z(d)$. Since $\deg(\eta_1(Z(d))) = 1$, either $Z(d) \subseteq W$ or $W \subseteq Z(z)$. If $W = Z(d)$, we use that $h^1(\mathcal{I}_{Z(d)}(\varepsilon_1)) = 0$. We also conclude if $W = \{z\}$ or if $W = Z(d) \cup \{z\}$ and $z \notin D_1 := |\mathcal{I}_{Z(d)}(\varepsilon_1)|$. By Lemma 2, there is $i > 2$ such that $\pi_i(z) \ne \pi_i(d)$. Set $\{D_i\} := |\mathcal{I}_z(\varepsilon_1)|$. Using $M_2 \cup D_i$, we conclude if $W = Z(d) \cup \{z\}$. Now assume $W = Z(z) \cup Z(d)$. Using $M_2 \cup D_i$, we conclude if either $z \notin D_1$ or if $\deg(\pi_i(Z(z))) = 2$. If $z \in D_1$ and $\deg(\pi_1(Z(z))) = 2$, we conclude using $M_2 \cup D_1$. Now assume $z \in D_1$ and $\deg(\pi_j(Z(z))) = 1$ for $j = 1, i$. Since $\pi_i(d) \ne \pi_i(z)$, $\mathrm{Res}_{M_2 \cup D_i}(Z) = Z(d)$, and hence, $h^1(\mathcal{I}_{\mathrm{Res}_{M_2 \cup D_i}(Z)}(\varepsilon_1)) = 0$. □

(c3.4.2) Assume $\pi_i(d) \notin \pi_i(S')$ for all $i = 2,3,4,5$. Note that $\pi_6(o) \ne \pi_6(d)$ and that $\deg(\pi_6(Z(o)) = 2$. Write $S' = \{u, v, o\}$. Set $\{U_6\} := |\mathcal{I}_o(\varepsilon_6)|$. We have $\mathrm{Res}_{U_6}(Z) = Z(d) \cup \{o\} \cup Z(u)' \cup Z(v)'$ with $\deg(Z(u)') \le 2$, $\deg(Z(v)') \le 2$, $Z(u)'$ (resp. $Z(v)'$) with u (resp. v) as its reduction, unless it is empty. By Claim 5 there is $x \in S'$ such that $h^1(\mathcal{I}_{Z(d) \cup \{x\}}(\varepsilon_1)) = 0$. Assume for the moment that we may take $x = o$. Set $\{U_i\} := |\mathcal{I}_u(\varepsilon_i)|$ for $i = 2,3$, and $\{U_i\} := |\mathcal{I}_v(\varepsilon_i)|$ for $i = 4,5$. Since $Z(d) \subseteq \mathrm{Res}_{U_2 \cup U_3 \cup U_4 \cup U_5 \cup U_6}(Z) \subseteq Z(d) \cup \{o\}$, we conclude in this case. We may use two different multidegrees among ε_i, $2 \le i \le 5$, for u and the remaining ones for v. We also conclude if $\deg(\pi_i(Z(w))) = 1$ for at least one $w \in S' \setminus \{z\}$, and at least one $i \in \{2,3,4,5\}$ (for instance if $\deg(\pi_2(Z(u))) = 1$ instead of U_3 we take the element $\{U_3'\} := |\mathcal{I}_z(\varepsilon_3)|$). Assume $\deg(\pi_i(Z(w))) = 1$ for all $w \in \{u, v\}$ and all $2 \le i \le 5$. Assume for instance $\pi_1(v) \notin \langle \pi_1(Z(d)) \rangle$. Set $\{Q_4\} := |\mathcal{I}_o(\varepsilon_2)|$ and use $U_2 \cup U_3 \cup Q_4 \cup U_5 \cup U_6$ to conclude this case.

(c3.4.2.1) By step c3.4.2, we may assume $\pi_i(d) \in \pi_i(S')$ for at least one $i \in \{2,3,4,5\}$, say for $i = 5$. Using $|\mathcal{I}_d(\varepsilon_5)|$ instead of M_6 in Claim 6 we get the existence of $o_1 \in S'$ such $\deg(\eta_5(Z(o_1))) = 1$. Since $\deg(\pi_5(Z(o_1))) = 2$, $o_1 \ne o$. Write $S' = \{o, o_1, o_2\}$.

(c3.4.2.2) Assume $\pi_i(d) \notin \pi_i(S')$ for $i = 2,3,4$. Set $\{D_2\} := |\mathcal{I}_o(\varepsilon_2)|$, $\{D_3\} := |\mathcal{I}_{o_1}(\varepsilon_3)|$ and $\{D_4\} := |\mathcal{I}_{o_2}(\varepsilon_4)|$. We have $Z(d) \subseteq \mathrm{Res}_{D_2 \cup D_3 \cup D_4}(Z) \subseteq Z(d) \cup \{o_2\}$. It would be sufficient to prove that $h^1(\mathcal{I}_{\mathrm{Res}_{D_2 \cup D_3 \cup D_4}(Z)}(1,0,0,0,1,1)) = 0$. This vanishing is true if either $\deg(\pi_4(Z(o_2))) = 1$ or $o_2 \in \Sigma$ or $\pi_i(o_2) \ne \pi_i(d)$ for at least one $i \in \{5,6\}$. Assume $\pi_5(o_2) = \pi_5(d)$, $\pi_6(o_2) = \pi_6(d)$, $o_2 \notin \Sigma$ and $\deg(\pi_4(Z(o_2))) = 2$. Permuting the set $\{2,3,4\}$, we may assume $\deg(\pi_2(Z(o_2))) = \deg(\pi_3(Z(o_2))) = 2$. By Lemma 2, there is a set $J \subset \{1,2,3,4,5,6\}$ such that $\#J \ge 3$ and $\pi_i(o) \ne \pi_i(o_1)$ for all $i \in J$. Note that $\{5,6\} \subset J$. Set $H := |\mathcal{I}_{Z(d)}(\varepsilon_1)|$. Note that $\mathrm{Res}_H(Z) \subseteq \{o_2\} \cup Z(o) \cup Z(o_1)$. First assume $\{o_2\} \subseteq \mathrm{Res}_H(Z)$. Set $N_i := |\mathcal{I}_o(\varepsilon_i)|$ and $Q_i := |\mathcal{I}_{o_1}(\varepsilon_i)|$. Since $\pi_i(o_2) = \pi_i(d) \notin \{o, o_1\}$ for $i = 5,6$, it is sufficient to use $H \cup N_5 \cup Q_6$. Now assume $\mathrm{Res}_H(Z) \subseteq Z(o) \cup Z(o_1)$. Since $\Sigma \ne \emptyset$, $\mathrm{Res}_J(Z)$ contains at least one among $Z(o)$ and $Z(o_1)$, say $Z(o)$. If $\mathrm{Res}_H(Z) = Z(o)$ we use that $\deg(\pi_6(Z(o))) = 2$ and hence $h^1(\mathcal{I}_{Z(o)}(\hat{\varepsilon}_1)) = 0$. Now assume $\mathrm{Res}_H(Z) = Z(o) \cup Z'(o_1)$ with either $Z'(o_1) = \{o_1\}$ or $Z'(o_1) = Z(o_1)$. If $J \ne \{1,5,6\}$, i.e., there is $i \in J$ with $i \in \{2,3,4\}$, it is sufficient to use $H \cup N_i$ and that $\deg(\pi_5(Z(o_1))) = 2$. Now assume $J = \{1,5,6\}$ and hence $Z'(o_1) = Z(o_1)$. There are $p_i \in \mathbb{P}^1$, $i = 2,3,4$, such that $\{o, o_1\} \subset \Delta := \langle \pi_1(Z') \rangle \times \{p_2\} \times$

$\{p_3\} \times \{p_4\} \times \mathbb{P}^1 \times \mathbb{P}^1$. Since $\pi_1(Z(o) \cup Z(o_1) \subset \langle \pi_1(Z') \rangle$ and $\deg(\pi_i(Z(o))) = \deg(\pi_i(Z(o_1))) = 1$ for $i = 2,3,4\}$, we get $Z(o) \cup Z(o_1) \subset \Delta$. Set $\{T_2\} := |\mathcal{I}_{p_2}(\varepsilon_2)|$ and $\{T_4\} := |\mathcal{I}_{o_2}(\varepsilon_4)|$. Since $\mathrm{Res}_{T_2 \cup T_4}(Z) = Z(d)$ (by the assumption $\pi_i(d) \notin \pi_i(S')$ for $i = 2,4$) and $\deg(\pi_1(Z(d))) = 2$, we conclude.

(c3.4.2.3) Assume the existence of exactly one $i \in \{2,3,4\}$ such that $\pi_i(d) \in \pi_i(S')$. With no loss of generality we may assume $i = 4$. As in Claim 6, we get the existence of $u \in S'$ such that $\deg(\eta_i(Z(u))) = 1$. Since $i \notin \{5,6\}$, $u = o_2$. By assumption, $\pi_i(d) \notin \pi_i(S')$ for $i = 2,3$. Fix $u \in \Sigma$. Assume for the moment $\pi_i(u) \neq \pi_i(d)$ for at least one $i \in \{4,5,6\}$. Write $S' = \{u, u_1, u_2\}$. We use the divisor $D(2) \cup D(3) \cup D(i)$ with $\{D(i)\} := |\mathcal{I}_v(\varepsilon_i)|$, $\{D_2\} := |\mathcal{I}_{u_1}(\varepsilon_2)|$ and $\{D_3\} := |\mathcal{I}_{u_2}(\varepsilon_3)|$. We conclude, because $Z(d) \subseteq \mathrm{Res}_{D(2)\cup D(3) \cup D(i)}(Z) \subseteq Z(d) \cup \{u\}$. Now assume $\pi_i(x) = \pi_i(d)$ for all $i = 4,5,6$ and all $x \in \Sigma$. Remark 4 and [1] (Th. 4.12) applied to $\Sigma \cup \{d\}$ give $\#\Sigma = 1$. Thus, $\pi_1(u_1) = \pi_1(u_2)$. Set $H := |\mathcal{I}_{Z(d)}(\varepsilon_1)|$. In this case, we have $\deg(\pi_1(Z(w))) = 1$ for all $w \in S'$. Thus, $\mathrm{Res}_H(Z) = Z(u)$. Since there is $i \in \{4,5,6\}$ with $\deg(\pi_i(Z(u))) = 2$, we conclude.

(c3.4.2.4) Assume the existence of at least 2 indices $i \in \{2,3,4\}$ such that $\pi_i(d) \in \pi_i(S')$, say $i = 3$ and $i = 4$. The first part of step c34.2.2 gives the equality $\deg(\eta_4(Z(o_2))) = 1$. Using $i = 3$ instead of $i = 4$, we get $\deg(\eta_3(Z(o_2))) = 1$, and hence, $\deg(Z(o_2)) = 1$, a contradiction.

□

Proof of Theorem 5. Since we may assume $k \geq 3$ (Remark 9 and Theorem 8) and $n_1 \leq 3$, all cases are covered by Propositions 9, 11–13. □

Proof of Theorem 6. Assume $\mathbb{T}(Y,4) \neq \emptyset$. Remark 9 and Theorem 8 give $k \geq 3$. Theorem 5 gives $\dim Y \leq 6$. Theorem 10 excludes the case $Y = (\mathbb{P}^1)^3$. All cases with $\dim Y = 6$ are allowed by Theorem 3. The case $Y = \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $Y = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$ are excluded by Lemma 18. Proposition 9 gives the cases $Y \in \{\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1, \mathbb{P}^2 \times (\mathbb{P}^1)^3, (\mathbb{P}^1)^5\}$. Proposition 2 gives the case $Y = (\mathbb{P}^1)^4$. □

8. Examples

Proposition 15. *Fix an integer $n > 1$ and set $Y = \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^1 \times \mathbb{P}^1$. Then, a general $S \in S(Y, 2n+1)$ is an element of $\mathbb{T}(Y, 2n+1)' \cap \mathbb{S}(Y, 2n+1)$.*

Proof. A general $q \in \sigma_{2n+1}(\nu(Y))$ has rank exactly $2n+1$ and for a general q a general $A \in \mathcal{S}(Y,q)$ is a general element of $S(Y, n+1)$. By [3] (Prop. 4.7(i)), we have $A \in \mathbb{T}_1(Y, 2n+1)$. Since $A \in \mathcal{S}(Y,q)$, $A \in \mathbb{S}(Y, 2n+1)$. Since $\#A = 2n+1 > n$ and A is general, Y is the minimal multiprojective space containing A (Remark 14). Thus, $A \in \mathbb{T}(Y, 2n+1)$. Fix $E \subsetneq A$, $E \neq \emptyset$ and set $e := \#E$. Since A is general, E is a general element of $S(Y,e)$. Thus, to prove that $\delta(2E, Y) = 0$ it is sufficient to use that for each $e \leq 2n$ the e-th secant variety of Y is not defective ([3], Proposition 4.7(iii)). Thus, $A \in \mathbb{T}(Y, 2n+1)'$. □

Proposition 16. *Take either $Y = \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^2$ or $Y = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then, a general $S \in S(Y,5)$ is an element of $\mathbb{T}(Y,5)' \cap \mathbb{S}(Y,5)$.*

Proof. Take $k \geq 3$ an $Y := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with $n_1 \geq \cdots \geq n_k > 0$. The secant variety $\sigma_5(\nu(Y))$ is defective if and only if either $k = 3$ and $(n_1, n_2, n_3) \in \{(3,3,2), (a,2,1), (a,3,1)\}$ for some $a \geq 5$ or $k = 4$ and $(n_1, n_2, n_3, n_4) = (2,2,1,1)$ ([3], Th. 4.12). Since we are looking at elements of $S(Y,5)$ such that Y is the minimal multiprojective space containing S, we exclude to cases $(a,3,1)$ and $(a,2,1)$ with $a \geq 5$. If either $Y = \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^2$ or $Y = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$, a general $S \in S(Y,5)$ is an element of $\mathbb{T}(Y,5) \cap \mathbb{S}(Y,5)$. The set S is an element of $\mathbb{T}(Y,5)'$, because any $E \subset S$ may be seen as a general element of $S(Y, \#E)$ and no secant variety of order ≤ 4 of Y is defective (Remark 1). □

9. Conclusions and Further Open Problems

In this paper, we consider four notions of Terracini loci, two of which are introduced here, and provide several results for them with full proofs. Concerning the most interesting one, minimally Terracini sets, $\mathbb{T}(Y,x)'$, we raise the following two conjectures and the following question.

Conjecture 13. *Fix an integer $x \geq 5$ and set $Y := (\mathbb{P}^1)^k$. We conjecture that $\mathbb{T}(Y,x)' = \emptyset$ if $k \geq 2x-1$.*

Conjecture 14. *Fix integers $x \geq 5$, $m \geq 2$ and set $Y := (\mathbb{P}^m)^k$. We conjecture that $\mathbb{T}(Y,x)' = \emptyset$ if $km \geq 2x-1$.*

Question 15. *Fix an integer $x \geq 5$. Find a small integer $e_x \geq 0$ such that $\mathbb{T}(Y,x)' = \emptyset$ for all multiprojective spaces $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ such that $n_1 \geq \cdots \geq n_k > 0$ and $n_1 \leq n_k - e_x$.*

The multiprojective spaces in Conjectures 13 and 14 are balanced and the dimensions of their secant varieties are known, with one possible exception ([12]). Question 15 concerns the "almost balanced" ones.

Funding: The author received no funding. The author is a member of the GNSAGA of INdAM (Rome, Italy).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author has no conflict of interest.

References

1. Ballico, E.; Bernardi, A.; Santarsiero, P. Terracini locus for three points on a Segre variety. *arXiv* **2020**, arXiv:2012.00574.
2. Landsberg, J.M. *Tensors: Geometry and Applications*; American Mathematical Society: Providence, RI, USA, 2012; Volume 128.
3. Abo, H.; Ottaviani, G.; Peterson, C. Induction for secant varieties of Segre varieties. *Trans. Amer. Math. Soc.* **2009**, *361*, 767–792. [CrossRef]
4. Ballico, E. Linearly dependent subsets of Segre varieties. *J. Geom.* **2020**, *111*, 23. [CrossRef]
5. Ballico, E.; Bernardi, A. Stratification of the fourth secant variety of Veronese varieties via the symmetric rank. *Adv. Pure Appl. Math.* **2013**, *4*, 215–250. [CrossRef]
6. Chandler, K.A. Hilbert functions of dots in linearly general positions. In Proceedings of the Conference on Zero-Dimensional Schemes, Ravello, Italy, 7 June 1992; pp. 65–79.
7. Chandler, K. A brief proof of a maximal rank theorem for generic 2-points in projective space. *Trans. Amer. Math.* Soc. **2000**, *353*, 1907–1920. [CrossRef]
8. Ballico, E. Examples on the non-uniqueness of the rank 1 tensor decomposition of rank 4 tensors. *Symmetry* **2022**, *14*, 1889. [CrossRef]
9. Buczyńska, W.; Buczyński, J. Secant varieties to high degree veronese reembeddings, catalecticant matrices and smoothable Gorenstein schemes. *J. Algebraic Geom.* **2014** *23*, 63–90. [CrossRef]
10. Catalisano, M.V.; Geramita, A.V.; Gimigliano, A. Ranks of tensors, secant varieties of Segre varieties and fat points. *Linear Algebra Appl.* **2002**, *355*, 263–285. [CrossRef]
11. Abo, H.; Brambilla, M.C. On the dimensions of secant varieties of Segre-Veronese varieties. *Ann. Mat. Pura Appl.* **2013**, *192*, 61–92. [CrossRef]
12. Aladpoosh, T.; Haghighi, H. On the dimension of higher secant varieties of Segre varieties $\mathbb{P}^n \times \cdots \times \mathbb{P}^n$. *J. Pure Appl. Algebra* **2011**, *215*, 1040–1052. [CrossRef]