symmetry

MDPI

Article

Using Double Integral Transform (Laplace-ARA Transform) in Solving Partial Differential Equations

Abdelilah Kamal Sedeeg [1,2,*], Zahra. I. Mahamoud [3,4] and Rania Saadeh [5]

[1] Department of Mathematics, Faculty of Education, Holy Quran and Islamic Sciences University, Omdurman P.O. Box 14411, Sudan
[2] Department of Mathematics, Faculty of Sciences and Arts-Almikwah, Albaha University, Al Bahah 65528, Saudi Arabia
[3] Department of Mathematics, Faculty of Sciences and Arts-Bukirayh, Alqassim University, Buraydah 52571, Saudi Arabia
[4] Department of Mathematics, Faculty of Education, Omdurman Islamic University, Omdurman P.O. Box 382, Sudan
[5] Department of Mathematics, Faculty of Science, Zarqa University, Zarqa P.O. Box 2000, Jordan
* Correspondence: aelilah63@hotmail.com

Abstract: The main goal of this research is to present a new approach to double transforms called the double Laplace–ARA transform (DL-ARAT). This new double transform is a novel combination of Laplace and ARA transforms. We present the basic properties of the new approach including existence, linearity and some results related to partial derivatives and the double convolution theorem. To obtain exact solutions, the new double transform is applied to several partial differential equations such as the Klein–Gordon equation, heat equation, wave equation and telegraph equation; each of these equations has great utility in physical applications. In symmetry to other symmetric transforms, we conclude that our new approach is simpler and needs less calculations.

Keywords: Laplace transform; ARA transform; double Laplace–ARA transform; partial differential equations

Citation: Sedeeg, A.K.; Mahamoud, Z.I.; Saadeh, R. Using Double Integral Transform (Laplace-ARA Transform) in Solving Partial Differential Equations. *Symmetry* **2022**, *14*, 2418. https://doi.org/10.3390/sym14112418

Academic Editors: Manwai Yuen, Imre Ferenc Barna, Baofeng Feng, Biao Li, Li Jun Zhang and Calogero Vetro

Received: 23 September 2022
Accepted: 2 November 2022
Published: 15 November 2022

1. Introduction

For solving partial differential equations (PDE)s, integral transforms are considered the most effective technique. Their significance stems from the fact that PDEs can be used to mathematically explain a variety of occurrences in mathematical physics and some other scientific fields [1–13]. These equations can also be transformed in order to find the exact solutions to PDEs using integral transforms. Because of the strength and simplicity of the transform techniques, scientists and researchers have worked very hard at studying and improving them.

Many integral transforms have been established and implemented to solve partial and integral differential equations; these transforms allow us to obtain the exact solutions of the target equations without needing linearization or discretization; they are applied to transform partial differential equations into ordinary ones, in the case of using single transformation, or into algebraic ones if we use double integral transformation. As examples, we mention Laplace transform [14], Fourier transform [15], Novel transform [16], M-transform [17], Sumudu transform [18], Natural transform [19], Elzaki transform [20], Kamal transform [21], the Aboodh transform [22], ARA transform [23] and ZZ transform [24].

The double transforms have also been widely applied to solving PDEs with unknown functions of two variables, and as a result, double transforms are considered very effective in handling PDEs compared to other numerical approaches [25–28]. In addition, extensions of the double transform have been developed in the relevant literature, such as the double Laplace transform, double Shehu transform [29], double Kamal transform [30], double

Sumudu transform [31–36], double Elzaki transform [37], double Laplace–Sumudu transform [38] and ARA–Sumudu transform [39,40]. All these double transforms cited above can be considered special cases of the general double transform by Meddahi et al. [41], but sometimes we need to study special kinds of double transforms and compare them to consider the properties of each one, and decide on the best ones to use in handling new applications.

Recently, Saadeh and others introduced a novel integral transform known as the ARA transform. ARA is a name of the presented transform and it is not an abbreviation, it has novel properties; that is, it can generate many transforms by changing the value of the index n, also as introduced in [23], it has a duality to the Laplace transform and has an advantage that allows it to overcome the singularity at $t = 0$. For all of these merits, we decided to construct a new combination between the Laplace and ARA transforms, so that we could reap the benefits of these two powerful transforms. We called this new approach the double Laplace–ARA transform.

The main goal of this paper is to introduce a new double transform, namely DL-ARAT. Fundamental properties and theorems of DL-ARAT are presented and proven, and we also compute the values of DL-ARAT for some functions. New relations related to partial derivatives and the double convolution theorem are established and implemented to solve PDEs. The novelty of this work appears in the new combinations between Laplace and ARA transforms, in which the new DL-ARATs have the advantages of the two transforms, the simplicity of Laplace and the applicability of ARA, in handling some singular points that appeared in the equations.

In this research, we studied the nonhomogeneous linear PDE of the following form

$$Au_{xx}(x,\ t) + Bu_{xt}(x,\ t) + Cu_{tt}(x,\ t) + Du_x(x,\ t) + Eu_t(x,\ t) + Fu(x,\ t) = w(x,t),$$

with the initial conditions

$$u(x,\ 0) = f_1(x),\ u_t(x,\ 0) = f_2(x),$$

and the boundary conditions

$$u(0,\ t) = g_1(t),\ u_x(0,\ t) = g_2(t)),$$

and $u(0,\ 0) = \psi$, where $u(x,\ t)$ is an unknown function, $w(x,t)$ is the source term and A, B, C, D, E, F and ψ are constants.

A simple formula for the solution of the above equation was established and employed to solve some applications in order to display the efficiency and strength of this new approach.

This article is organized as follows: In Section 2, fundamental concepts and properties of Laplace and ARA transformations are introduced. In Section 3, we introduce the new double transform DL-ARAT, that combines the Laplace and ARA transforms; we also present some properties of the new transform. In Section 4, we introduce the application of DL-ARAT to solve some types of PDEs. In Section 5, some examples are presented and solved using DL-ARAT. Finally, In Section 6, a conclusion is provided.

2. Basic Definitions and Theorems for Laplace and ARA Transforms

In this section, we introduce the basic properties of Laplace and ARA transforms.

2.1. Laplace Transform [14]

Definition 1. Let $f(x)$ be a function of x specified for $x > 0$. Then, Laplace transform of $f(x)$, denoted by $\mathcal{L}[f(x)]$, is defined by

$$\mathcal{L}[f(x)] = F(v) = \int_0^\infty e^{-vx} f(x)dx,\ v > 0. \quad (1)$$

The inverse Laplace transform is provided by

$$\mathcal{L}^{-1}[F(v)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{vx}F(v)dv = f(x)\ ,\ x > 0\,. \tag{2}$$

Theorem 1. (Existence conditions). If $f(x)$ is a piecewise continuous function on the interval $[0,\infty)$ and of exponential order α, then $\mathcal{L}[f(x)]$ exists for $Re(v) > \alpha$ and satisfies

$$|f(x)| \leq Me^{\alpha x},$$

where $Re(v) > \alpha$. is positive constant. Then, Laplace transform integral converges absolutely for $Re(v) > \alpha$.

Proof of Theorem 1. Using the definition of Laplace transform, we find

$$\begin{aligned} |F(v)| = \left|\int_0^\infty e^{-vx}[f(x)]dx\right| \leq \int_0^\infty e^{-vx}|f(x)|dx \leq M\int_0^\infty e^{-(v-\alpha)x}dx \\ = \tfrac{M}{v-\alpha},\ Re(v) > \alpha. \end{aligned}$$

Thus, Laplace transform integral converges absolutely for $Re(v) > \alpha$.
In the following arguments, we present some properties of the Laplace transform.
Assume that $G(v) = \mathcal{L}[g(x)]$, $G(v) = \mathcal{L}[g(x)]$ and $a, b \in \mathcal{R}$. Then

$$\mathcal{L}[a\, f(x) + b\, g(x)] = a\,\mathcal{L}[f(x)] + b\,\mathcal{L}[g(x)]. \tag{3}$$

$$\mathcal{L}^{-1}[a\, F(v) + b\, G(v)] = a\,\mathcal{L}^{-1}[F(v)] + b\,\mathcal{L}^{-1}[G(v)]. \tag{4}$$

$$\mathcal{L}[x^\alpha] = \frac{\Gamma(\alpha+1)}{v^{n+1}},\ \ \alpha \geq 0. \tag{5}$$

$$\mathcal{L}[e^{ax}] = \frac{1}{v-a}\ , a \in \mathcal{R} \tag{6}$$

$$\mathcal{L}\left[f'(x)\right] = vF(v) - f(0). \tag{7}$$

$$\mathcal{L}\left[f^{(n)}(x)\right] = v^n F(v) - \sum_{k=1}^{n} v^{n-k} f^{(k-1)}(0). \tag{8}$$

□

2.2. ARA Transform [23]

Definition 2. The ARA integral transform of order n of a continuous function $f(t)$ on the interval $(0,\ \infty)$ is defined as

$$\mathcal{G}_n[f(t)](\mathrm{s}) = Q(n,s) = s\int_0^\infty t^{n-1}e^{-\mathrm{s}t}f(t)dt,\ s > 0. \tag{9}$$

Theorem 2. (Existence conditions). If the function $f(t)$ is piecewise continuous in every finite interval $0 \leq t \leq \alpha$ and satisfies

$$\left|t^{n-1}f(t)\right| \leq Me^{\alpha t} \tag{10}$$

where M is positive constant, then the ARA transform exists for all $s > \alpha$.

Proof of Theorem 2. Using the definition of ARA transform, we obtain

$$|Q(n,s)| = \left| s \int_0^\infty t^{n-1} e^{-\mathrm{st}} f(t) dt \right|.$$

Using the property of improper integral, we find

$$\begin{aligned} |Q(n,s)| &= \left| s \int_0^\beta t^{n-1} e^{-\mathrm{st}} f(t) dt + s \int_\beta^\infty t^{n-1} e^{-\mathrm{st}} f(t) dt \right| \leq s \left| \int_\beta^\infty t^{n-1} e^{-\mathrm{st}} f(t) dt \right| \\ &\leq s \int_\beta^\infty e^{-\mathrm{st}} \left| t^{n-1} f(t) \right| dt \leq s \int_\beta^\infty e^{-\mathrm{st}} M e^{\alpha t} dt \\ &= sM \int_\beta^\infty e^{-(s-\alpha)t} dt = \tfrac{sM}{s-\alpha} e^{-\beta(s-\alpha)}. \end{aligned}$$

This improper integral converges for all $s > \alpha$. Thus, $\mathcal{G}_{n+1}[f(t)]$ exists.

In the following, we state some basic properties of the ARA transform

Assume that $F(n,s) = \mathcal{G}_n[f(t)]$ and $G(n,s) = \mathcal{G}_n[g(t)]$ and $a, b \in \mathcal{R}$. Then

$$\mathcal{G}_n[a\, f(t) + b\, g(t)] = a\, \mathcal{G}_n[f(t)] + b\, \mathcal{G}_n[g(t)]. \tag{11}$$

$$\mathcal{G}_n{}^{-1}[a\, F(n,s) + b\, G(n,s)] = a\, \mathcal{G}_n{}^{-1}[F(n,s)] + b\, \mathcal{G}_n{}^{-1}[G(n,s)]. \tag{12}$$

$$\mathcal{G}_n[t^\alpha] = \frac{\Gamma(\alpha+n)}{s^{\alpha+n-1}},\ \alpha > 0. \tag{13}$$

$$\mathcal{G}_n\left[e^{at}\right] = \frac{s\Gamma(n)}{(s-a)^n},\ a \in \mathcal{R} \tag{14}$$

$$\mathcal{G}_n\left[f^{(n)}(t)\right] = (-1)^{n-1} s \frac{d^{n-1}}{ds^{n-1}} \left(s^{n-1} \mathcal{G}_1[f(t)] - \textstyle\sum_{k=1}^n s^{n-k} f^{(k-1)}(0) \right). \tag{15}$$

where $\mathcal{G}_1[f(t)]$ is the ARA transform of order one of a piecewise continuous function $f(t)$ on $[0,\ \infty)$ and it is defined as

$$\mathcal{G}_1[f(t)](\mathrm{s}) = F(s) = s \int_0^\infty e^{-\mathrm{st}} f(t) dt, s > 0. \tag{16}$$

For simplicity, let us denote $\mathcal{G}_1[f(t)]$ using $\mathcal{G}[f(t)]$.

The above results can be obtained from the definition of Laplace and ARA transforms with simple calculations. □

3. Double Laplace–ARA Transform of Order One (DL-ARAT)

This section introduces a new integral transform, which is a novel combination between the famous Laplace transform and ARA transform denoted by DL-ARAT. We provide the fundamental properties and characteristics including the existence conditions, linearity and the inverse of the proposed new double transform. Moreover, some important properties and results are provided and used to compute the DL-ARAT for some elementary functions. The double convolution theorem and the derivatives properties of the new transform are also presented and illustrated.

Definition 3. Let $u(x,t)$ be continuous function of two positive variables x and t. Then the DL-ARAT of $u(x,t)$ is defined as

$$\mathcal{L}_x \mathcal{G}_t[u(x,t)] = Q(v,s) = s \int_0^\infty \int_0^\infty e^{-(vx+st)} u(x,t) dx dt,\ v,s > 0 \tag{17}$$

provided the integral exists.

Clearly, the DL-ARAT is a linear integral transformation as shown below,

$$\begin{aligned}\mathcal{L}_x\mathcal{G}_t[A\ u(x,t)+B\ w(x,t)] &= s\int_0^\infty\int_0^\infty e^{-(vx+st)}[A\ u(x,t)+B\ w(x,t)]dxdt \\ &= sA\int_0^\infty\int_0^\infty e^{-(vx+st)}[u(x,t)]dxdt+sB\int_0^\infty\int_0^\infty e^{-(vx+st)}[w(x,t)]dxdt \\ &= A\ \mathcal{L}_x\mathcal{G}_t[\ u(x,t)]+B\ \mathcal{L}_x\mathcal{G}_t[\ w(x,t)].\end{aligned}$$

where A and B are constants and $\mathcal{L}_x\mathcal{G}_t[u(x,t)], \mathcal{L}_x\mathcal{G}_t[\ w(x,t)]$ exists.

Additionally, the inverse of the DL-ARAT is found using

$$\mathcal{L}_x^{-1}\left[\mathcal{G}_t^{-1}[Q(v,s)]\right]=\left(\frac{1}{2\pi i}\right)\int_{c-i\infty}^{c+i\infty}e^{vx}dv\left(\frac{1}{2\pi i}\right)\int_{r-i\infty}^{r+i\infty}\frac{e^{st}}{s}Q(v,s)ds=u(x,t). \tag{18}$$

Property 1. Let $u(x,t)=f(x)g(t), x>0, t>0$. Then

$$\mathcal{L}_x\mathcal{G}_t[u(x,t)]=\mathcal{L}_x[f(x)]\mathcal{G}_t[g(t)].$$

Proof of Property 1.

$$\begin{aligned}\mathcal{L}_x\mathcal{G}_t[u(x,t)]=\mathcal{L}_x\mathcal{G}_t[f(x)g(t)] &= s\int_0^\infty\int_0^\infty e^{-(vx+st)}[f(x)g(t)]dxdt \\ &= \int_0^\infty f(x)e^{-vx}dx\cdot s\int_0^\infty g(t)e^{-st}dt \\ &= \mathcal{L}_x[f(x)]\mathcal{G}_t[g(t)].\end{aligned}$$

□

3.1. DL-ARAT of Some Basic Functions

i. Let $u(x,t)=1, x>0, t>0$. Then,

$$\mathcal{L}_x\mathcal{G}_t[1]=s\int_0^\infty\int_0^\infty e^{-(vx+st)}dxdt=\int_0^\infty e^{-vx}dx\cdot s\int_0^\infty e^{-st}dt=\mathcal{L}_x[1]\mathcal{G}_t[1]=\frac{1}{v},\ Re(s)>0.$$

ii. Let $u(x,t)=x^\alpha t^\beta, x>0, t>0$ and α,β are constants. Then,

$$\mathcal{L}_x\mathcal{G}_t\left[x^\alpha t^\beta\right]=s\int_0^\infty\int_0^\infty e^{-(vx+st)}\left[x^\alpha t^\beta\right]dxdt=\int_0^\infty e^{-vx}[x^\alpha]dx\cdot s\int_0^\infty e^{-st}\left[t^\beta\right]dt=\mathcal{L}_x[x^\alpha]\mathcal{G}_t\left[t^\beta\right]$$

From Equations (5) and (13), we find

$$\mathcal{L}_x\mathcal{G}_t\left[x^\alpha t^\beta\right]=\mathcal{L}_x[x^\alpha]\mathcal{G}_t\left[t^\beta\right]=\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{v^{\alpha+1}\left(s^\beta\right)}, Re(\alpha)>-1, Re(\beta)>-1$$

iii. Let $u(x,t)=e^{\alpha x+\beta t}, x>0, t>0$ and α,β are constants. Then,

$$\mathcal{L}_x\mathcal{G}_t\left[e^{\alpha x+\beta t}\right]=s\int_0^\infty\int_0^\infty e^{-(vx+st)}\left[e^{\alpha x+\beta t}\right]dxdt=\int_0^\infty e^{-vx}[e^{\alpha x}]dx\cdot s\int_0^\infty e^{-st}\left[e^{\beta t}\right]dt=\mathcal{L}_x[e^{\alpha x}]\mathcal{G}_t\left[e^{\beta t}\right].$$

From Equations (6) and (14), we find

$$\mathcal{L}_x\mathcal{G}_t\left[e^{\alpha x+\beta t}\right]=\frac{s}{(v-\alpha)(s-\beta)}.$$

Similarly,

$$\mathcal{L}_x\mathcal{G}_t\left[e^{i(\alpha x+\beta t)}\right]=\frac{s}{(v-i\alpha)(s-i\beta)}.$$

Using the property of complex analysis, we have

$$\mathcal{L}_x\mathcal{G}_t\left[e^{i(\alpha x+\beta t)}\right] = \frac{s(sv-\alpha\beta)+is(v\beta+s\alpha)}{(v^2+\alpha^2)(s^2+\beta^2)}.$$

Using Euler's formulas $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$, $\cos x = \frac{e^{ix}+e^{-ix}}{2}$. And the formulas $\sinh x = \frac{e^{x}-e^{-x}}{2}$, $\cosh x = \frac{e^{x}+e^{-x}}{2}$. Now, we find the DL-ARAT of the following functions

$$\mathcal{L}_x\mathcal{G}_t[\sin(\alpha x+\beta t)] = \frac{s(v\beta+s\alpha)}{(v^2+\alpha^2)(s^2+\beta^2)},$$

$$\mathcal{L}_x\mathcal{G}_t[\cos(\alpha x+\beta t)] = \frac{s(sv-\alpha\beta)}{(v^2+\alpha^2)(s^2+\beta^2)},$$

$$\mathcal{L}_x\mathcal{G}_t[\sinh(\alpha x+\beta t)] = \frac{s(v\beta+s\alpha)}{(v^2-\alpha^2)(s^2-\beta^2)},$$

$$\mathcal{L}_x\mathcal{G}_t[\cosh(\alpha x+\beta t)] = \frac{s(sv+\alpha\beta)}{(v^2-\alpha^2)(s^2-\beta^2)},$$

iv. Let $u(x,t) = J_0\left(\gimel\sqrt{xt}\right)$, then $\mathcal{L}_x\mathcal{G}_t\left[J_0\left(\gimel\sqrt{xt}\right)\right] = s\int_0^\infty\int_0^\infty e^{-(vx+st)}\left[J_0\left(\gimel\sqrt{xt}\right)\right]dxdt$ $= \int_0^\infty e^{-vx}\left[J_0\left(\gimel\sqrt{xt}\right)\right]dx\cdot s\int_0^\infty e^{-st}dt = s\int_0^\infty e^{-\frac{\gimel^2}{4s}t}e^{-st}dt$. From Equation (14), we get $\mathcal{L}_x\mathcal{G}_t\left[J_0\left(\gimel\sqrt{xt}\right)\right] = \frac{4s}{4vs+\gimel^2}$.

3.2. Existence Conditions for DL-ARAT

Let $u(x,t)$ be function of exponential order α and β as $x\to\infty$ and $t\to\infty$. If there exists a positive N such that $\forall x > X$ and $t > T$, we have

$$|u(x,t)| \le Ne^{\alpha x+\beta t}.$$

We can write $u(x,t) = O\left(e^{\alpha x+\beta t}\right)$ as $x\to\infty$ and $t\to\infty$, $v>\alpha$ and $s>\beta$.

Theorem 3. Let $u(x,t)$ be a continuous function on the region $[0,X)\times[0,T)$ of exponential orders α and β.Then $\mathcal{L}_x\mathcal{G}_t[u(x,t)]$ exists for v and s provided $Re(v)>\alpha$ and $Re(s)>\beta$.

Proof of Theorem 3. Using the definition of DL-ARAT, we get

$$\begin{aligned}|Q(v,s)| = \left|s\int_0^\infty\int_0^\infty e^{-(vx+st)}[u(x,t)]dxdt\right| &\le s\int_0^\infty\int_0^\infty e^{-(vx+st)}|u(x,t)|dxdt\\ &\le N\int_0^\infty e^{-(v-\alpha)x}dx\cdot s\int_0^\infty e^{-(s-\beta)t}dt\\ &= \tfrac{Ns}{(v-\alpha)(s-\beta)},\ Re(v)>\alpha \text{ and } Re(s)>\beta.\end{aligned}$$

Thus, $\mathcal{L}_x\mathcal{G}_t[u(x,t)]$ exists for v and s provided $Re(v)>\alpha$ and $Re(s)>\beta$. □

3.3. Some Theorems of DL-ARAT

Theorem 4. (Shifting Property). Let $u(x,t)$ be a continuous function and $\mathcal{L}_x\mathcal{G}_t[u(x,t)] = Q(v,s)$. Then

$$\mathcal{L}_x\mathcal{G}_t\left[e^{\alpha x+\beta t}u(x,t)\right] = \frac{s}{(s-\beta)}Q(v-\alpha,s-\beta). \tag{19}$$

Proof of Theorem 4.

$$\begin{aligned}\mathcal{L}_x\mathcal{G}_t\left[e^{\alpha x+\beta t}u(x,t)\right] &= s\int_0^\infty\int_0^\infty e^{-(v-\alpha)x-(s-\beta)t}[u(x,t)]dxdt \\ &= \frac{s}{(s-\beta)}(s-\beta)\int_0^\infty\int_0^\infty e^{-(v-\alpha)x}e^{-(s-\beta)t}[u(x,t)]dxdt \\ &= \frac{s}{(s-\beta)}Q(v-\alpha,s-\beta).\end{aligned}$$

□

Theorem 5. (Periodic Function). Let $\mathcal{L}_x\mathcal{G}_t[u(x,t)]$ exists, where $u(x,t)$ periodic function of periods α and β such that

$$u(x+\alpha,t+\beta) = u(x,t),\ \forall x,y.$$

Then

$$\mathcal{L}_x\mathcal{G}_t[u(x,t)] = \frac{1}{\left(1-e^{-(v\alpha+s\beta)}\right)}\left(s\int_0^\alpha\int_0^\beta e^{-(vx+st)}(u(x,t))dxdt\right) \tag{20}$$

Proof of Theorem 5. Using the definition of DL-ARAT, we find

$$\mathcal{L}_x\mathcal{G}_t[u(x,t)] = s\int_0^\infty\int_0^\infty e^{-(vx+st)}[u(x,t)]dxdt. \tag{21}$$

Using the property of improper integral, Equation (21) can be written as

$$\mathcal{L}_x\mathcal{G}_t[u(x,t)] = s\int_0^\alpha\int_0^\beta e^{-(vx+st)}(u(x,t))dxdt + s\int_\alpha^\infty\int_\beta^\infty e^{-(vx+st)}(u(x,t))dxdt \tag{22}$$

Putting $x=\alpha+\rho$ and $t=\beta+\tau$ on second integral in Equation (22), we obtain

$$Q(v,s) = s\int_0^\alpha\int_0^\beta e^{-(vx+st)}(u(x,t))dxdt + s\int_0^\infty\int_0^\infty e^{-(v(\alpha+\rho)+s(\beta+\tau))}(u(\alpha+\rho,\beta+\tau))d\rho d\tau. \tag{23}$$

Using the periodicity of the function $u(x,t)$, Equation (23) can be written as

$$Q(v,s) = s\int_0^\alpha\int_0^\beta e^{-(vx+st)}(u(x,t))dxdt + e^{-(v\alpha+s\beta)}s\int_0^\infty\int_0^\infty e^{-(v\rho+s\tau)}(u(\rho,\tau))d\rho d\tau. \tag{24}$$

Using the definition of DL-ARAT, we obtain

$$Q(v,s) = s\int_0^\alpha\int_0^\beta e^{-(vx+st)}(u(x,t))dxdt + e^{-(v\alpha+s\beta)}Q(v,s). \tag{25}$$

Thus, Equation (25) can be simplified into

$$Q(v,s) = \frac{1}{\left(1-e^{-(v\alpha+s\beta)}\right)}\left(s\int_0^\alpha\int_0^\beta e^{-(vx+st)}(u(x,t))dxdt\right).$$

□

Theorem 6. (Heaviside Function)**.** Let $\mathcal{L}_x\mathcal{G}_t[u(x,t)]$ exists and $\mathcal{L}_x\mathcal{G}_t[u(x,t)] = Q(v,s)$, then

$$\mathcal{L}_x\mathcal{G}_t[u(x-\delta,t-\varepsilon)H(x-\delta,t-\varepsilon)] = e^{-v\delta-s\varepsilon}Q(v,s). \tag{26}$$

where $H(x-\delta,t-\varepsilon)$ is the Heaviside unit step function defined as

$$H(x-\delta,t-\varepsilon)=\begin{cases}1, x>\delta, t>\varepsilon\\ 0,\ Ohtherwise\end{cases}$$

Proof of Theorem 6. Using the definition of DL-ARAT, we find

$$\begin{aligned}\mathcal{L}_x\mathcal{G}_t[u(x-\delta,t-\varepsilon)H(x-\delta,t-\varepsilon)] &= s\int_0^\infty\int_0^\infty e^{-(vx+st)}(u(x-\delta,t-\varepsilon)H(x-\delta,t-\varepsilon))dxdt\\ &= s\int_0^\infty\int_0^\infty e^{-(vx+st)}(u(x-\delta,t-\varepsilon))dxdt.\end{aligned} \tag{27}$$

Putting $x-\delta=\rho$ and $t-\varepsilon=\tau$ in Equation (27). We obtain

$$\mathcal{L}_x\mathcal{G}_t[u(x-\delta,t-\varepsilon)H(x-\delta,t-\varepsilon)] = s\int_0^\infty\int_0^\infty e^{-v(\delta+\rho)-s(\varepsilon+\tau)}(u(\rho,\tau))d\rho d\tau. \tag{28}$$

Thus, Equation (28) can be simplified into

$$\mathcal{L}_x\mathcal{G}_t[u(x-\delta,t-\varepsilon)H(x-\delta,t-\varepsilon)] = e^{-v\delta-s\varepsilon}\left(s\int_0^\infty\int_0^\infty e^{-v\rho-s\tau}(u(\rho,\tau))d\rho d\tau\right) = e^{-v\delta-s\varepsilon}Q(v,s). \tag{29}$$

□

Theorem 7. (Convolution Theorem). Let $\mathcal{L}_x\mathcal{G}_t[u(x,t)]$ and $\mathcal{L}_x\mathcal{G}_t[w(x,t)]$ exists and $\mathcal{L}_x\mathcal{G}_t[u(x,t)]=Q(v,s)$, $\mathcal{L}_x\mathcal{G}_t[w(x,t)]=W(v,s)$, then

$$\mathcal{L}_x\mathcal{G}_t[u(x,t)**w(x,t)] = \frac{1}{s}Q(v,s)W(v,s). \tag{30}$$

where $u(x,t)**w(x,t)=\int_0^x\int_0^t u(x-\rho,t-\tau)w(\rho,\tau)d\rho d\tau$ and the symbol $**$ is denotes the double convolution with respect to x and t.

Proof of Theorem 7. Using the definition of DL-ARAT, we obtain

$$\begin{aligned}\mathcal{L}_x\mathcal{G}_t[u(x,t)**w(x,t)] &= s\int_0^\infty\int_0^\infty e^{-(vx+st)}[u(x,t)**w(x,t)]dxdt\\ &= s\int_0^\infty\int_0^\infty e^{-(vx+st)}\left(\int_0^x\int_0^t u(x-\rho,t-\tau)w(\rho,\tau)d\rho d\tau\right)dxdt.\end{aligned} \tag{31}$$

Using the Heaviside unit step function, Equation (31) can be written as

$$\mathcal{L}_x\mathcal{G}_t[u**w(x,t)] = s\int_0^\infty\int_0^\infty e^{-(vx+st)}\left(\int_0^\infty\int_0^\infty u(x-\rho,t-\tau)H(x-\rho,t-\tau)w(\rho,\tau)d\rho d\tau\right)dxdt. \tag{32}$$

Thus, Equation (32) can be written as

$$\begin{aligned}\mathcal{L}_x\mathcal{G}_t[u**w(x,t)] &= \int_0^\infty\int_0^\infty w(\rho,\tau)d\rho d\tau\left(s\int_0^\infty\int_0^\infty e^{-v(x+\rho)-s(t+\tau)}u(x-\rho,t-\tau)H(x-\rho,t-\tau)\right)dxdt\\ &= \int_0^\infty\int_0^\infty w(\rho,\tau)d\rho d\tau(e^{-v\rho-s\tau}Q(v,s))\\ &= Q(v,s)\int_0^\infty\int_0^\infty e^{-v\rho-s\tau}w(\rho,\tau)d\rho d\tau = \tfrac{1}{s}Q(v,s)W(v,s)\end{aligned}$$

□

Theorem 8. (Derivatives' Properties). Let $u(x,t)$ be a continuous function and $\mathcal{L}_x\mathcal{G}_t[u(x,t)] = Q(v,s)$. Then, we get the following derivatives properties

(a)

$$\mathcal{L}_x\mathcal{G}_t\left[\frac{\partial u(x,t)}{\partial t}\right] = \mathrm{s}Q(v,s)-\mathrm{s}\mathcal{L}[u(x,0)]$$

(b)
$$\mathcal{L}\ _x\mathcal{G}_t\left[\frac{\partial u(x,t)}{\partial x}\right] = vQ(v,s) - \mathcal{G}[u(0,t)]$$

(c)
$$\mathcal{L}_x\mathcal{G}_t\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right] = s^2Q(v,s) - s^2\mathcal{L}[u(x,0)] - s\mathcal{L}\left[\frac{\partial u(x,0)}{\partial t}\right]$$

(d)
$$\mathcal{L}_x\mathcal{G}_t\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] = v^2Q(v,s) - v\mathcal{G}[u(0,t)] - \mathcal{G}\left[\frac{\partial u(0,t)}{\partial x}\right]$$

(e)
$$\mathcal{L}_x\mathcal{G}_t\left[\frac{\partial^2 u(x,t)}{\partial x\partial t}\right] = vsQ(v,s) - vs\mathcal{L}[u(x,0)] - \mathrm{s}\mathcal{G}[u(0,t)] + su(0,0)$$

Proof of Theorem 8.

(a)
$$\mathcal{L}_x\mathcal{G}_t\left[\frac{\partial u(x,t)}{\partial t}\right] = s\int_0^\infty\int_0^\infty e^{-(st+vx)}\left[\frac{\partial u(x,t)}{\partial t}\right]dxdt = \int_0^\infty e^{-vx}dx\cdot s\int_0^\infty e^{-st}\left(\frac{\partial u(x,t)}{\partial t}\right)dt.$$

Using the integration by part, we obtain
Let $u = e^{-st} \Rightarrow du = -se^{-st}dt$,

$$dv = \frac{\partial u(x,t)}{\partial t}dt \Rightarrow v = u(x,t).$$

Thus,

$$s\int_0^\infty e^{-st}\left(\frac{\partial u(x,t)}{\partial t}\right)dt = s\left(-u(x,0) + s\int_0^\infty e^{-st}u(x,t)dt\right).$$

$$\therefore\ \mathcal{L}_x\mathcal{G}_t\left[\frac{\partial u(x,t)}{\partial t}\right] = \mathrm{s}Q(v,s) - \mathrm{s}\mathcal{L}[u(x,0)] \tag{33}$$

(b)
$$\mathcal{L}_x\mathcal{G}_t\left[\frac{\partial u(x,t)}{\partial x}\right] = s\int_0^\infty\int_0^\infty e^{-(st+vx)}\left[\frac{\partial u(x,t)}{\partial x}\right]dxdt = s\int_0^\infty e^{-st}dt\cdot\int_0^\infty e^{-vx}\left(\frac{\partial u(x,t)}{\partial x}\right)dx.$$

Using the integration by part, we obtain
Let $u = e^{-vx} \Rightarrow du = -ve^{-vx}dx$,

$$dv = \frac{\partial u(x,t)}{\partial x}dx \Rightarrow v = u(x,t).$$

Thus,
$$\int_0^\infty e^{-vx}\left(\frac{\partial u(x,t)}{\partial x}\right)dx = \left(-u(0,t) + v\int_0^\infty e^{-vx}u(x,t)dx\right).$$

$$\therefore\ \mathcal{L}_x\mathcal{G}_t\left[\frac{\partial u(x,t)}{\partial x}\right] = vQ(v,s) - \mathcal{G}[u(0,t)]. \tag{34}$$

(c)
$$\mathcal{L}_x\mathcal{G}_t\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right] = s\int_0^\infty\int_0^\infty e^{-(st+vx)}\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right]dxdt = \int_0^\infty e^{-vx}dx\cdot s\int_0^\infty e^{-st}\left(\frac{\partial^2 u(x,t)}{\partial t^2}\right)dt.$$

Using the integration by part, we obtain
Let $u = e^{-st} \Rightarrow du = -se^{-st}dt$,

$$dv = \frac{\partial^2 u(x,t)}{\partial t^2}dt \Rightarrow v = \frac{\partial u(x,t).}{\partial t}$$

Thus,

$$s\int_0^\infty e^{-st}\left(\frac{\partial^2 u(x,t)}{\partial t^2}\right)dt = s\left(-\frac{\partial u(x,0)}{\partial t} + s\int_0^\infty e^{-st}\left(\frac{\partial u(x,t)}{\partial t}\right)dt\right).$$

Using Equation (33), we have

$$\mathcal{L}_x\mathcal{G}_t\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right] = s^2 Q(v,s) - s^2\mathcal{L}[u(x,0)] - s\mathcal{L}\left[\frac{\partial u(x,0)}{\partial x}\right]. \tag{35}$$

(d)

$$\mathcal{L}_x\mathcal{G}_t\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] = s\int_0^\infty\int_0^\infty e^{-(st+vx)}\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right]dxdt = s\int_0^\infty e^{-st}dt\cdot\int_0^\infty e^{-vx}\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)dx.$$

Using the integration by part, we obtain
Let $u = e^{-vx} \Rightarrow du = -ve^{-vx}dx$,

$$dv = \frac{\partial^2 u(x,t)}{\partial x^2}dx \Rightarrow v = \frac{\partial u(x,t)}{\partial x}.$$

Thus,

$$\int_0^\infty e^{-vx}\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)dx = \left(-\frac{\partial u(0,t)}{\partial x} + v\int_0^\infty e^{-vx}\left(\frac{\partial u(x,t)}{\partial x}\right)dx\right).$$

Using Equation (34), we have

$$\mathcal{L}_x\mathcal{G}_t\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] = v^2 Q(v,s) - v\mathcal{G}[u(0,t)] - \mathcal{G}\left[\frac{\partial u(0,t)}{\partial x}\right]. \tag{36}$$

(e)

$$\mathcal{L}_x\mathcal{G}_t\left[\frac{\partial^2 u(x,t)}{\partial x\partial t}\right] = s\int_0^\infty\int_0^\infty e^{-(st+vx)}\left[\frac{\partial^2 u(x,t)}{\partial x\partial t}\right]dxdt = s\int_0^\infty e^{-st}\left(\frac{\partial^2 u(x,t)}{\partial x\partial t}\right)dt\cdot\int_0^\infty e^{-vx}dx.$$

Using the integration by part, we obtain.

$$s\int_0^\infty\int_0^\infty e^{-(st+vx)}\left(\frac{\partial^2 u(x,t)}{\partial x\partial t}\right)dxdt = \left(-\int_0^\infty e^{-st}\left(\frac{\partial u(0,t)}{\partial t}\right)dt + vs\int_0^\infty\int_0^\infty e^{-(st+vx)}\left(\frac{\partial u(x,t)}{\partial t}\right)dxdt\right)$$

And, using Equations (7) and (33), we have

$$\mathcal{L}_x\mathcal{G}_t\left[\frac{\partial^2 u(x,t)}{\partial x\partial t}\right] = vsQ(v,s) - \mathrm{s}\mathcal{G}[u(0,t)] - vs\mathcal{L}[u(x,0)] + su(0,0).$$

□

The previous results of DL-ARAT to some basic functions, some theorems and basic derivatives are summed up in the in Table 1 below:

Table 1. DL-ARAT to some basic functions.

$u(x,t)$	$\mathcal{L}_x\mathcal{G}_t[u(x,t)] = Q(v,s)$
1	$\frac{1}{v}$
$x^\alpha t^\beta$	$\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{v^{\alpha+1}s^\beta}$, $\alpha,\ \beta > -1$
$e^{\alpha x+\beta t}$	$\frac{s}{(v-\alpha)(s-\beta)}$
$\sin(\alpha x+\beta t)$	$\frac{s(v\beta+s\alpha)}{(v^2+\alpha^2)(s^2+\beta^2)}$

Table 1. *Cont.*

$u(x,t)$	$\mathcal{L}_x\mathcal{G}_t[u(x,t)] = Q(v,s)$
$\cos(\alpha x+\beta t)$	$\dfrac{s(sv-\alpha\beta)}{(v^2+\alpha^2)(s^2+\beta^2)}$
$\sinh(\alpha x+\beta t)$	$\dfrac{s(v\beta+s\alpha)}{(v^2-\alpha^2)(s^2-\beta^2)}$
$\cosh(\alpha x+\beta t)$	$\dfrac{s(sv+\alpha\beta)}{(v^2-\alpha^2)(s^2-\beta^2)}$
$e^{\alpha x+\beta t}u(x,t)$	$\dfrac{s}{(s-\beta)}Q(v-\alpha,s-\beta)$
$u(x-\delta,t-\varepsilon)H(x-\delta,t-\varepsilon)$	$e^{-v\delta-s\varepsilon}Q(v,s)$
$u(x,t)**w(x,t)$	$\dfrac{1}{s}Q(v,s)W(v,s)$
$u_t(x,t)$	$sQ(v,s)-s\mathcal{L}[u(x,0)]$
$u_x(x,t)$	$vQ(v,s)-\mathcal{G}[u(0,t)]$
$u_{tt}(x,t)$	$s^2Q(v,s)-s^2\mathcal{L}[u(x,0)]-s\mathcal{L}[u_t(x,0)]$
$u_{xx}(x,t)$	$v^2Q(v,s)-v\mathcal{G}[u(0,t)]-\mathcal{G}[u_x(0,t)]$
$u_{xt}(x,t)$	$vsQ(v,s)-s\mathcal{G}[u(0,t)]-vs\mathcal{L}[u(x,0)]+su(0,0)$

4. Basic Idea of Double Laplace-ARA Transform Method

To illustrate the basic idea of this method for solving partial differential equations, we consider a second order linear partial differential equation in two independent variables x and t in its general form given by

$$Au_{xx}(x,\ t)+Bu_{xt}(x,\ t)+Cu_{tt}(x,\ t)+Du_x(x,\ t)+Eu_t(x,\ t)+Fu(x,\ t)=w(x,t). \tag{37}$$

with the initial conditions

$$u(x,\ 0)=f_1(x),\ u_t(x,\ 0)=f_2(x), \tag{38}$$

And the boundary conditions

$$u(0,\ t)=g_1(t),\ u_x(0,\ t)=g_2(t). \tag{39}$$

And assume that $u(0,\ 0)=\psi$, where $u(x,\ t)$ is unknown function, $w(x,t)$ is the source term and A,B,C,D,E,F and ψ are constants.

The main idea of this method is to apply the DL-ARAT to Equation (37), Laplace transform to the initial conditions in Equation (38) and ARA transform to the boundary conditions in Equation (39) as the following:

Operating Laplace transform to the initial conditions given in Equation (38) yields

$$\mathcal{L}[u(x,\ 0)]=\mathcal{L}[f_1(x)]=F_1(v)=F_1,\ \mathcal{L}[u_t(x,\ 0)]=\mathcal{L}[f_2(x))]=F_2(v)=F_2$$

ARA transform of the boundary conditions in Equation (39) is given by

$$\mathcal{G}[u(0,\ t)]=\mathcal{G}[g_1(t)]=G_1(s)=G_1,\ \mathcal{G}[u_x(0,\ t)]=\mathcal{G}[g_2(t)]=G_2(s)=G_2$$

And $W(v,s)=W=\mathcal{L}_x\mathcal{G}_t[w(x,t)]$.

Now, applying the DL-ARAT on both sides of Equation (37), we find that

$$\mathcal{L}_x\mathcal{G}_t[Au_{xx}(x,\ t)+Bu_{xt}(x,\ t)+Cu_{tt}(x,\ t)+Du_x(x,\ t)+Eu_t(x,\ t)+Fu(x,\ t)]=\mathcal{L}_x\mathcal{G}_t[w(x,t)].$$

Using the differentiation property of the DL-ARAT and above conditions, we have

$$\begin{gathered}A\left[v^2Q(v,\ s)-vG_1-G_2\right]+B[vsQ(v,\ s)-svF_1-sG_1+s\psi]+C\left[s^2Q(v,\ s)-s^2F_1-sF_2\right]+\\ D[vQ(v,\ s)-G_1]+E[sQ(v,\ s)-sF_1]+F[Q(v,\ s)]=W(v,\ s)\end{gathered} \tag{40}$$

Equation (40) can be simplified as follows-

$$Q(v,\ s) = \frac{(Av + Bs + D)G_1 + AG_2 - Bs\psi + \left(Bvs + Cs^2 + Es\right)F_1 + CsF_2 + W}{Av^2 + Bvs + Cs^2 + Dv + Es + F}. \quad (41)$$

Operating with the inverse DL-ARAT on both sides of Equation (41), we obtain

$$u(x,\ t) = \mathcal{L}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{(Av + Bs + D)G_1 + AG_2 - Bs\psi + \left(Bvs + Cs^2 + Es\right)F_1 + CsF_2 + W}{Av^2 + Bvs + Cs^2 + Dv + Es + F}\right] \quad (42)$$

where $u(x,\ t)$ represents the term arising from the known function $w(x,t)$ and all conditions.

5. Applications of Double Laplace–ARA Transform in Solving Partial Differential Equations

Many physical phenomena can be modeled by a set of governing equations, several of them begin as partial differential equations. One may encounter PDEs in many branches of sciences such as:

- Quantum mechanics.
- Particle physic.
- Astrophysics.
- Chemistry.
- Biology.
- Environmental science.

The list goes on. Solving these partial differential equations is another challenge. Current mathematics fail to provide a closed solution and more advances are yet to come. Meanwhile, many numerical techniques have been developed for solving PDEs. In this section we introduce the solution of some familiar PDEs such as the wave equation, heat equation, telegraph equation and others. All the following figures of the selected examples were obtained using Mathematica software 13.

Example 1. Let us consider the homogeneous wave equation

$$u_{xx}(x,t) - u_{tt}(x,t) = 0, \text{ where } x \text{ and } t \geq 0, \quad (43)$$

with the initial conditions $u(x,0) = \sin x, u_t(x,0) = 2$, and the boundary conditions $u(0,t) = 2t, u_x(0,t) = \cos t$.

Applying Laplace transform to the initial conditions and ARA transform to the boundary conditions, we find

$$F_1 = \frac{1}{v^2+1},\ F_2 = \frac{2}{v},\ G_1 = \frac{2}{s},\ G_2 = \frac{s^2}{s^2+1}.$$

By substituting the values of the functions $F_1,\ F_2,\ G_1,\ G_2$ and $A = 1,\ C = -1$, $B = D = E = F = W = \psi = 0$ in the general formula in Equation (41), we obtain

$$Q(v,\ s) = \frac{\frac{2v}{s} + \frac{s^2}{s^2+1} - \frac{s^2}{v^2+1} - \frac{s}{v}}{v^2 - s^2} = \frac{2\left(\frac{v^2-s^2}{vs}\right) + s^2\left(\frac{v^2-s^2}{(s^2+1)(v^2+1)}\right)}{v^2 - s^2} = \frac{2}{vs} + \frac{s^2}{(s^2+1)(v^2+1)}. \quad (44)$$

Applying the inverse DL-ARAT to Equation (44), then the solution to Equation (43) is

$$u(x,\ t) = \mathcal{L}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{2}{vs} + \frac{s^2}{(s^2+1)(v^2+1)}\right] = 2t + \cos t \sin x.$$

The following figure, Figure 1, illustrates the 3D graph of the exact solution of Example 1.

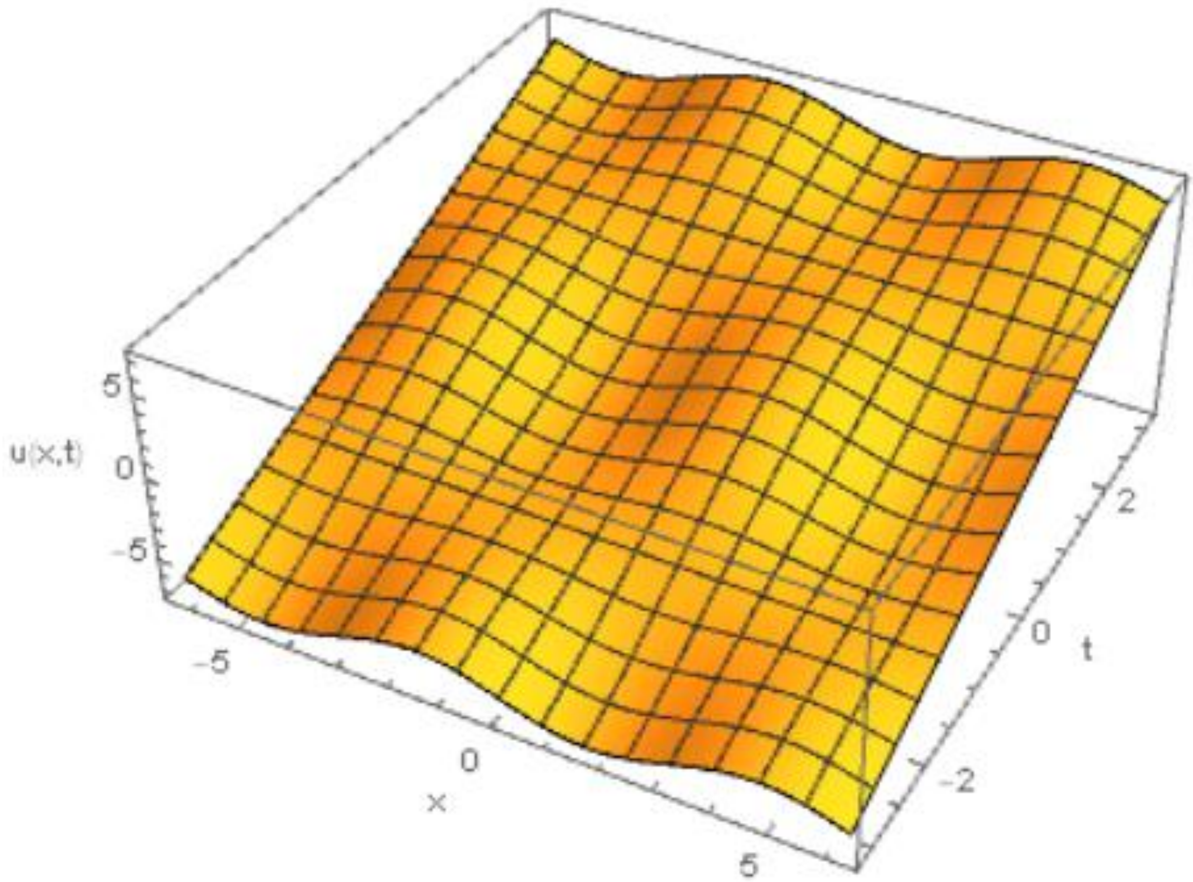

Figure 1. The exact solution of Example 1.

Example 2. Let us consider the homogeneous Laplace equation

$$u_{xx}(x,t) + u_{tt}(x,t) = 0, \; x \text{ and } t > 0, \tag{45}$$

with the initial conditions $u(x,0) = 0, \; u_t(x,0) = \cos x$, and the boundary conditions $u(0,t) = sinht, \; u_x(0,t) = 0$.

Applying Laplace transform to the initial conditions and ARA transform to the boundary conditions, we find

$$F_1 = 0, \; F_2 = \frac{v}{v^2+1}, \; G_1 = \frac{s}{s^2-1}, \; G_2 = 0$$

By substituting the values of the functions $F_1, \; F_2, \; G_1, G_2$and $A = 1, C = 1, B = D = E = F = \psi = W = 0$ in the general formula in Equation (41), we obtain

$$Q(v, \; s) = \frac{\frac{vs}{s^2-1} + \frac{sv}{v^2+1}}{v^2+s^2} = \frac{vs\left(\frac{v^2+s^2}{(s^2-1)(v^2+1)}\right)}{v^2+s^2} = \frac{vs}{(s^2-1)(v^2+1)}. \tag{46}$$

Now, applying the inverse DL-ARAT to Equation (46), then the solution to Equation (45) is

$$u(x, \; t) = \mathcal{L}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{vs}{(s^2-1)(v^2+1)}\right] = \cos x \sinh t.$$

The following figure, Figure 2, illustrates the 3D graph of the exact solution of Example 2.

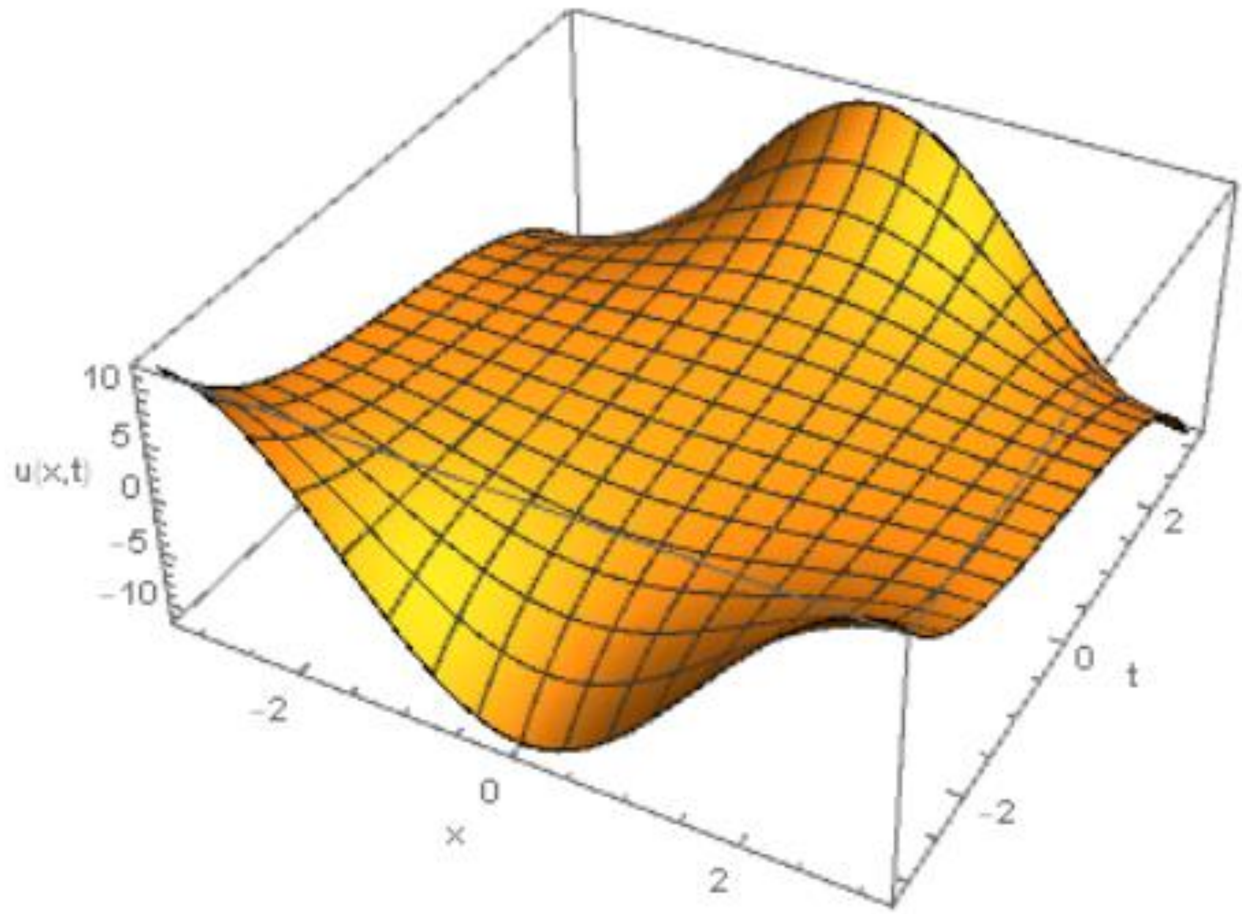

Figure 2. The exact solution of Example 2.

Example 3. Let us consider the homogeneous telegraph equation

$$u_{xx}(x,t) = u_{tt}(x,t) + 4u_t(x,t) + 4u(x,t),\ x,t \geq 0 \tag{47}$$

with the initial conditions $u(x,0) = 1 + e^{2x}$, $u_t(x,0) = -2$, and the boundary conditions $u(0,t) = 1 + e^{-2t}$, $u_x(0,t) = 2$.

Applying Laplace transform to the initial conditions and ARA transform to the boundary conditions, we find

$$F_1 = \frac{1}{v} + \frac{1}{v-2},\ F_2 = \frac{-2}{v},\ G_1 = 1 + \frac{s}{s+2},\ G_2 = 2.$$

By substituting the values of the functions $F_1,\ F_2,\ G_1, G_2$ and $A = 1, C = -1, E = F = -4, B = D = W = \psi = 0$, in the general formula in Equation (41), we obtain

$$Q(v,\ s) = \frac{v + \frac{vs}{s+2} + 2 - \frac{s^2}{v} - \frac{s^2}{v-2} - \frac{2s}{v} - \frac{4s}{v-2}}{v^2 - s^2 - 4s - 4} = \frac{\left(v + 2 - \frac{s^2}{v-2} - \frac{4s}{v-2}\right) + \left(\frac{vs}{s+2} - \frac{s^2}{v} - -\frac{2s}{v}\right)}{v^2 - s^2 - 4s - 4} \tag{48}$$

Now, applying inverse DL-ARAT to Equation (48), then the solution to Equation (47) is

$$u(x,\ t) = \mathcal{L}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{1}{v-2} + \frac{s}{v(s+2)}\right] = e^{2x} + e^{-2t}.$$

The following figure, Figure 3, illustrates the 3D graph of the exact solution of Example 3.

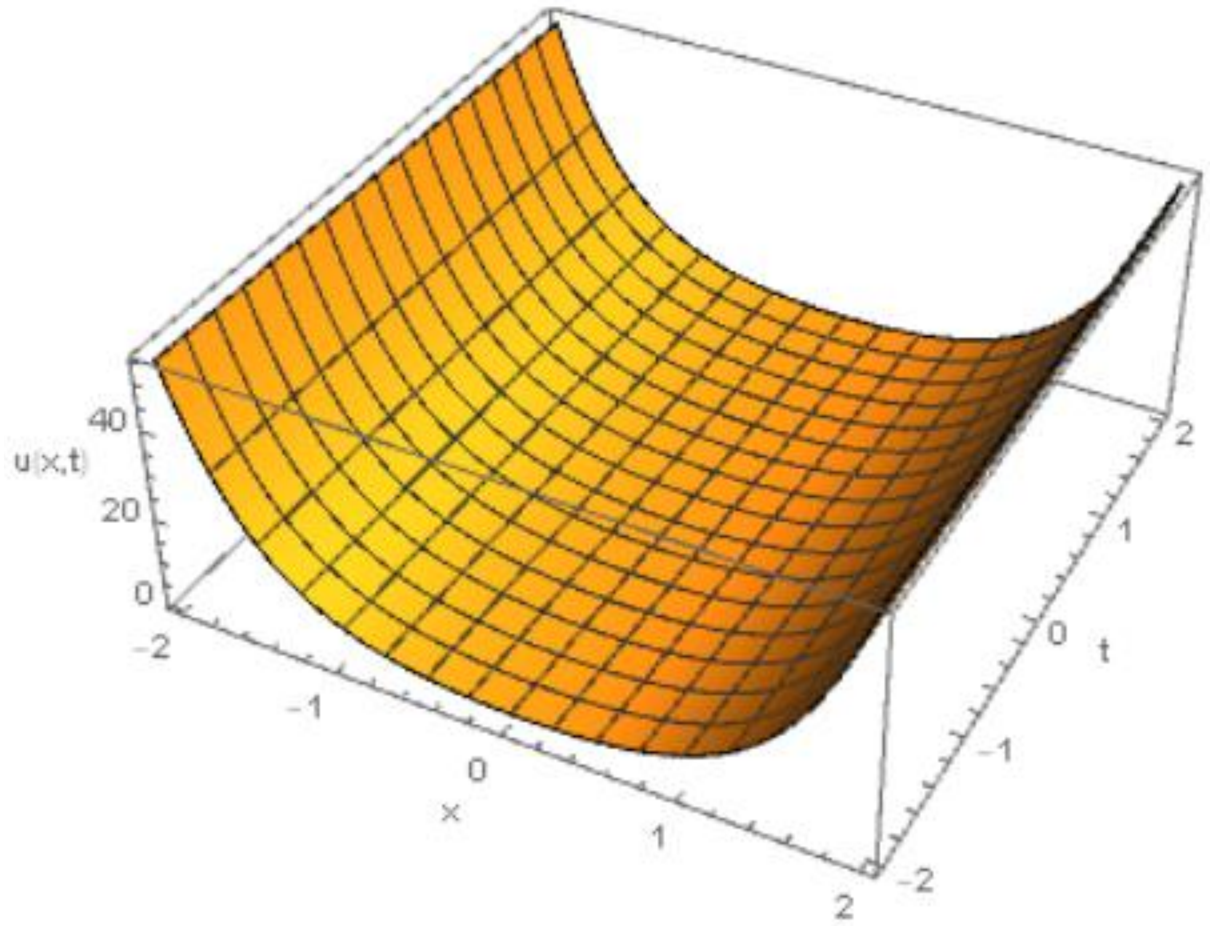

Figure 3. The exact solution of Example 3.

Example 4. Let us consider the nonhomogeneous heat equation

$$u_{xx}(x,t) - u_t(x,t) - 3u(x,t) = -3,\ x \text{ and } t \geq 0 \tag{49}$$

with the initial conditions $u(x,0) = 1 + \sin x$, and the boundary conditions $u(0,t) = 1,\ u_x(0,t) = e^{-4t}$.

Applying Laplace transform to the initial conditions and ARA transform to the boundary conditions, we find

$$F_1 = \frac{1}{v} + \frac{1}{v^2+1},\ G_1 = 1,\ G_2 = \frac{s}{s+4}.$$

By substituting the values of the functions $F_1,\ G_1,\ G_2$ and $A = 1, E = -1, F = -3$, $B = C = D = \psi = 0, W = \frac{3}{v}$ in the general formula in Equation (41), we obtain

$$Q(v,\ s) = \frac{v + \frac{s}{s+4} - s\left(\frac{1}{v} + \frac{1}{v^2+1}\right) - \frac{3}{v}}{v^2 - s - 3} = \frac{\left(v - \frac{s}{v} - \frac{3}{v}\right) + \left(\frac{s}{s+4} - \frac{s}{v^2+1}\right)}{v^2 - s - 3} = \frac{1}{v} + \frac{s}{(s+4)(v^2+1)} \tag{50}$$

Now, applying the inverse of DL-ARAT to Equation (50), then the solution to Equation (49) is

$$u(x,\ t) = \mathcal{L}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{1}{v} + \frac{s}{(s+4)(v^2+1)}\right] = 1 + e^{-4t}\sin x.$$

The following figure, Figure 4, illustrates the 3D graph of the exact solution of Example 4.

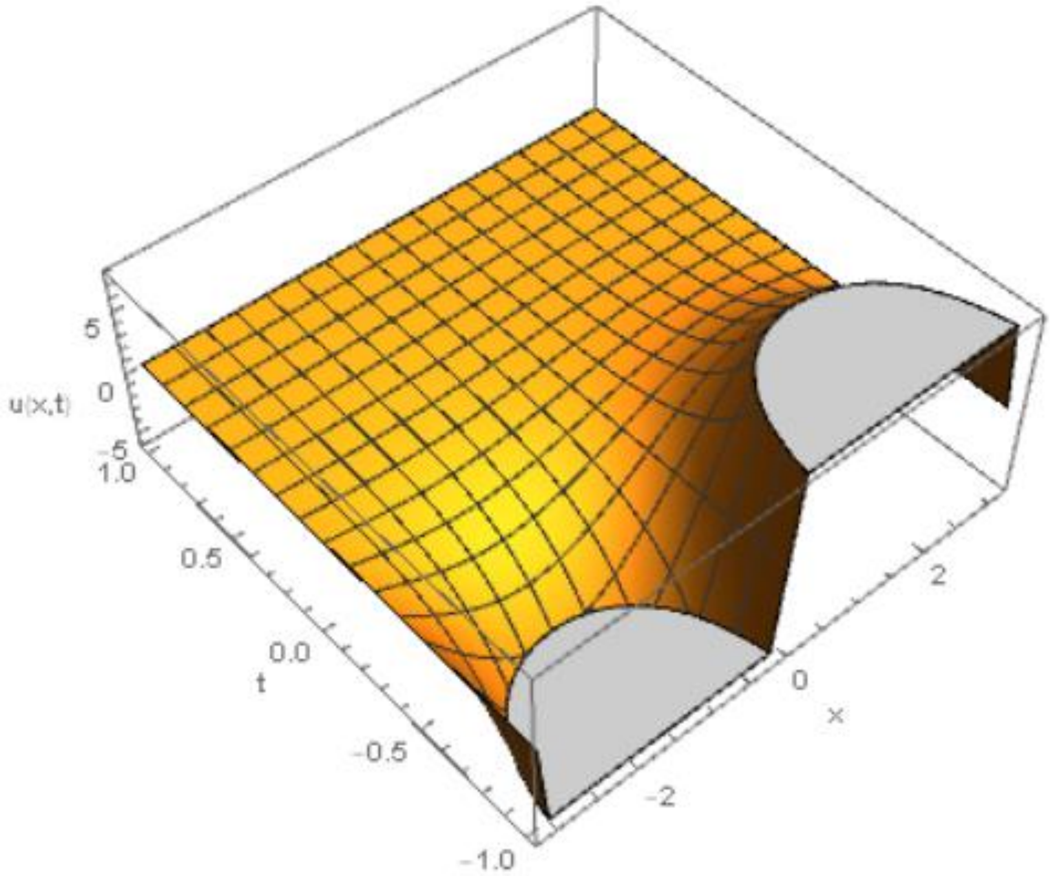

Figure 4. The exact solution of Example 4.

Example 5. Let us consider the Klein–Gordon equation

$$u_{tt}(x,t) - u(x,t) = u_{xx}(x,t) - \cos x \cos t \tag{51}$$

with the initial conditions $u(x,0) = \cos x,\ u_t(x,0) = 0$, and the boundary conditions $u(0,t) = \cos \mathrm{t}\ ,\ u_x(0,t) = 0$.

Applying Laplace transform to the initial conditions and ARA transform to the boundary conditions, we find

$$F_1 = \frac{v}{v^2+1}, F_2 = 0, G_1 = \frac{s^2}{s^2+1},\ G_2 = 0.$$

By substituting the values of the functions $F_1,\ F_2\ , G_1$ and $G_2$and $A = 1, C = -1$, $F = 1, B = D = E = \psi = 0, W = \frac{vs^2}{(s^2+1)(v^2+1)}$ in the general formula in Equation (41), we obtain

$$Q(v,\ s) = \frac{\frac{vs^2}{s^2+1} - \frac{vs^2}{v^2+1} - \frac{vs^2}{(s^2+1)(v^2+1)}}{v^2 - s^2 + 1} = \frac{vs^2}{(s^2+1)(v^2+1)}. \tag{52}$$

Now, applying the inverse DL-ARAT to Equation (52), then the solution to Equation (51) is

$$u(x,\ t) = \mathcal{L}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{vs^2}{(s^2+1)(v^2+1)}\right] = \cos x \cos t.$$

The following figure, Figure 5, illustrates the 3D graph of the exact solution of Example 5.

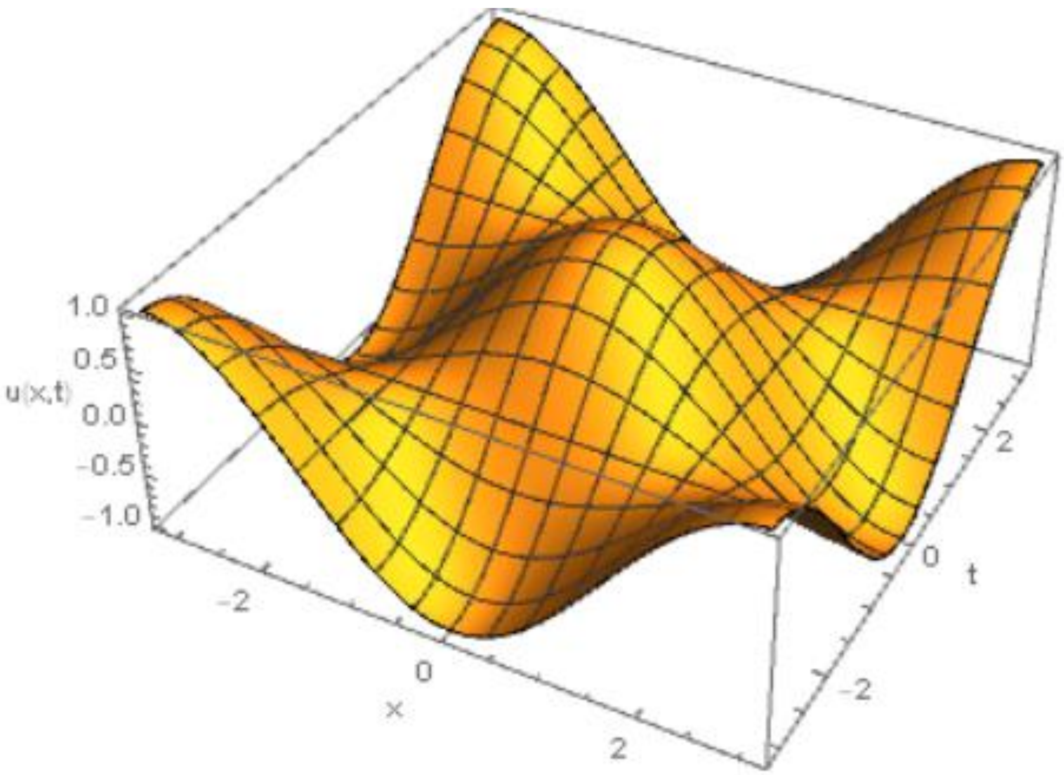

Figure 5. The exact solution of Example 5.

Example 6. Let us consider the advection–diffusion equation

$$u_t(x,t) = u_{xx}(x,t) - u_x(x,t),\ x \text{ and } t \geq 0 \tag{53}$$

with the initial conditions $u(x,0) = x + e^x$, and the boundary conditions $u(0,t) = 1 - t$, $u_x(0,t) = 2$.

Applying Laplace transform to the initial conditions and ARA transform to the boundary conditions, we find

$$F_1 = \frac{1}{v-1} + \frac{1}{v^2},\ G_1 = 1 - \frac{1}{s},\ G_2 = 2.$$

By substituting the values of functions $F_1,\ G_1, G_2$and $A = 1, D = E = -1, W = F = B = C = \psi = 0$, in the general formula in Equation (41), we obtain

$$Q(v,\ s) = \frac{(v-1)\left(1 - \frac{1}{s}\right) + 2 - s\left(\frac{1}{v-1} + \frac{1}{v^2}\right)}{v^2 - v - s} = \frac{1}{v-1} + \frac{1}{sv} - \frac{1}{v^2}. \tag{54}$$

Now, applying the inverse DL-ARAT to Equation (54), then the solution to Equation (53) is

$$u(x,\ t) = \mathcal{L}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{1}{v-1} + \frac{1}{sv} - \frac{1}{v^2}\right] = e^x + t - x.$$

The following figure,Figure 6, illustrates the 3D graph of the exact solution of Example 6.

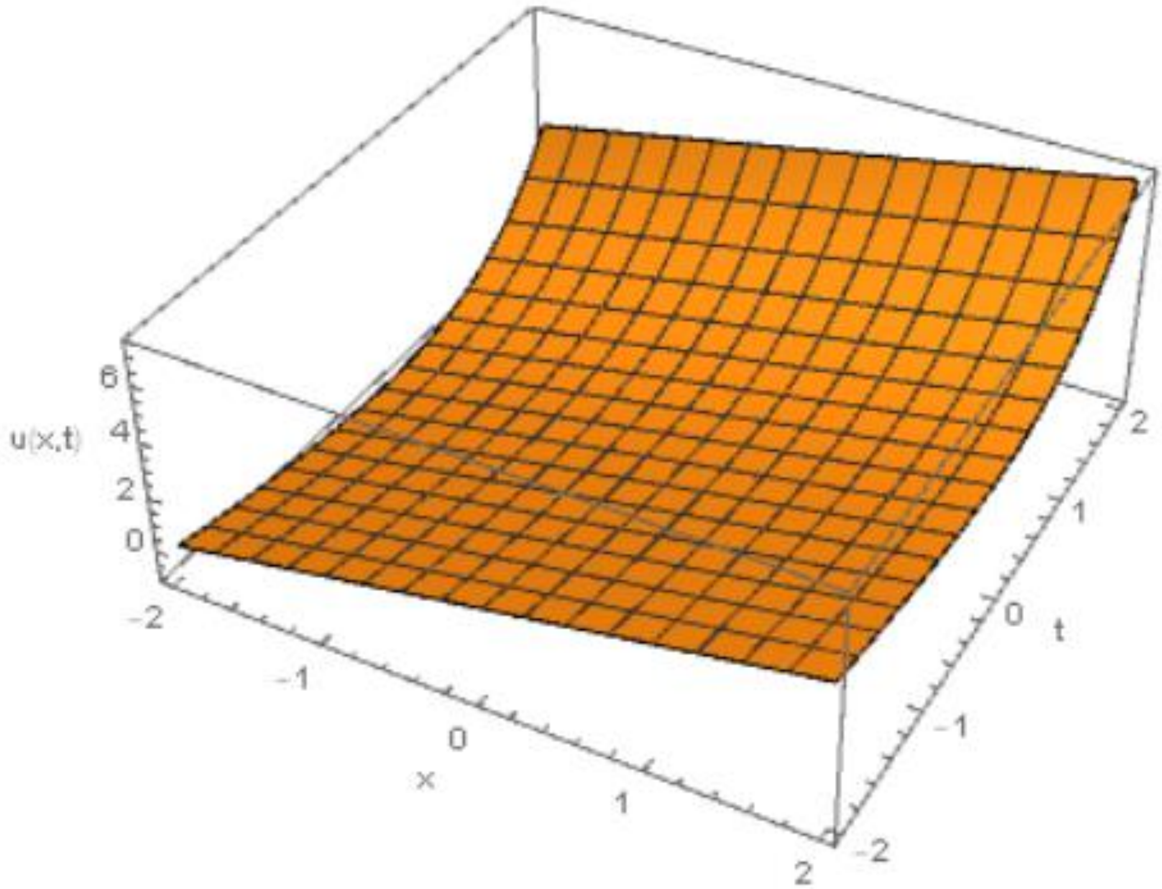

Figure 6. The exact solution of Example 6.

Example 7. Let us consider the Goursat equation

$$u_{xt}(x,t) = 4xt - x^2t^2 + u(x,t), \tag{55}$$

with the initial condition> $u(x,0) = e^x$, boundary the condition $u(0,t) = e^t$ and $u(0,0) = 1$.

Applying Laplace transform to the initial conditions and ARA transform to the boundary conditions, we find

$$F_1 = \frac{1}{v-1},\ G_1 = \frac{s}{s-1}.$$

By substituting the values of the functions $F_1,\ G_1$ and $B = 1, F = -1, W = \frac{4}{v^2 s} - \frac{4}{v^3 s^2},\ A = C = D = E = 0, \psi = 1$, in the general form in Equation (41), we obtain

$$Q(v,\ s) = \frac{s\left(\frac{s}{s-1}\right) - s - vs\left(\frac{1}{v-1}\right) + \frac{4}{v^2 s} - \frac{4}{v^3 s^2}}{sv-1} = \frac{\left(\frac{s^2}{s-1} - s - \frac{vs}{v-1}\right) + \left(\frac{4}{v^2 s} - \frac{4}{v^3 s^2}\right)}{sv-1}. \tag{56}$$

Now, applying the inverse DL-ARAT to Equation (56), then the solution to Equation (55) is

$$u(x,\ t) = \mathcal{L}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{s}{(v-1)(s-1)} + \frac{4}{v^3 s^2}\right] = e^{x+t} + 4x^2t^2.$$

The following figure, Figure 7, illustrates the 3D graph of the exact solution of Example 7.

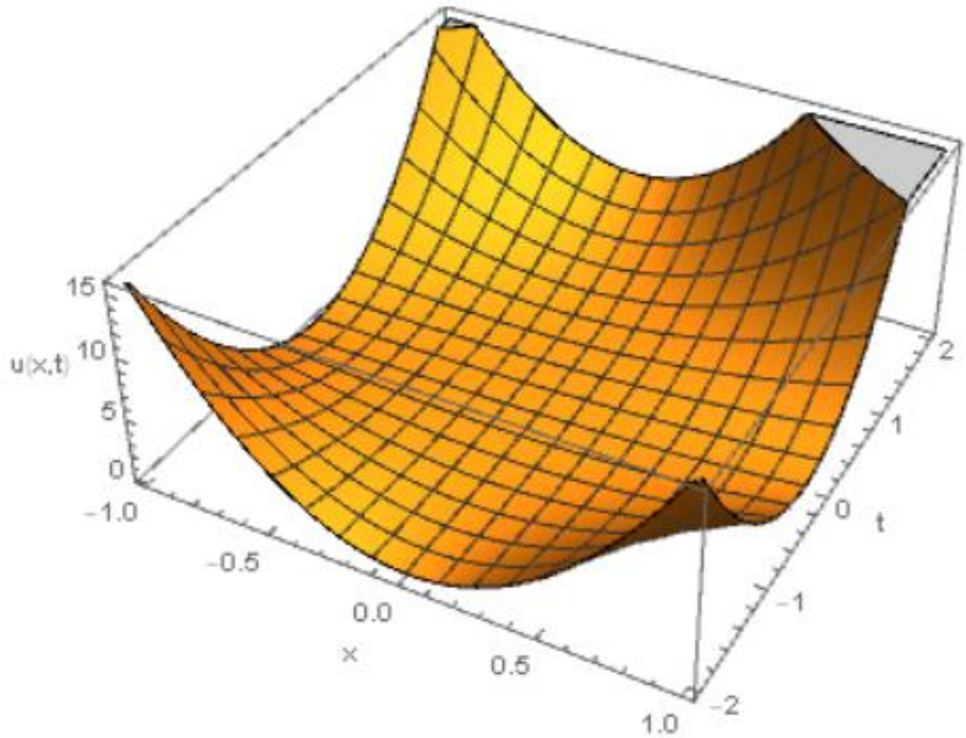

Figure 7. The exact solution of Example 7.

6. Conclusions

In this paper, a new double transform called the DL-ARAT were presented, several properties and theorems related to the linearity, existence, partial derivatives and the double convolution theorem were introduced. The new results were implemented to establish a new formula for solving PDEs, we discussed some numerical examples and get the exact solutions using the new double transform. Applications of the DL-ARAT will be developed in the future and utilized to solve integral equations, PDEs with variable coefficients and the equations appeared during the fluid flow [1,2]. Moreover, we have mentioned here that this new double transform can be combined with one of the iteration numerical methods to solve nonlinear PDEs and equations with variables coefficients, since all integral transforms cannot solve nonlinear problems directly, unless they are combined with iteration methods.

In a general view of what was discussed in [1,2] we find that this research focuses on the same direction with the possibility of using partial differential equations in explaining physical phenomena. Therefore, we recommend that this study be continued using the applications of this method as a future project for solving equations that appear during fluid flow.

Author Contributions: Conceptualization, A.K.S., Z.I.M. and R.S.; methodology, A.K.S., Z.I.M. and R.S.; software, R.S., A.K.S. and Z.I.M.; validation, A.K.S., Z.I.M. and R.S.; formal analysis, A.K.S., Z.I.M. and R.S.; investigation, A.K.S., Z.I.M. and R.S.; resources, R.S.; data curation, A.K.S.; writing—original draft preparation, Z.I.M.; writing—review and editing, A.K.S. and Z.I.M..; visualization, A.K.S. and Z.I.M.; supervision, R.S.; project administration, Z.I.M.; funding acquisition, Z.I.M. and R.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Ahmad, L.; Khan, M. Importance of activation energy in development of chemical covalent bounding in flow of Sisko magneto-Nano fluids over porous moving curved surface. *Int. J. Hydrogen Energy.* **2019**, *44*, 10197–10206. [CrossRef]
2. Ahmad, L.; Khan, M. Numerical simulation for MHD flow of Sisko nanofluid over a moving curved surface: A revised model. *Microsyst. Technol.* **2019**, *25*, 2411–2428. [CrossRef]
3. Constanda, C. *Solution Techniques for Elementary Partial Differential Equations*; Chapman and Hall/CRC: New York, NY, USA, 2002.
4. Debnath, L. The double Laplace transforms and their properties with applications to functional, integral and partial differential equations. *Int. J. Appl. Comput. Math.* **2016**, *2*, 223–241. [CrossRef]
5. Muatjetjeja, B. Group classification and conservation laws of the generalized Klein–Gordon–Fock equation. *Int. J. Mod. Phys. B* **2016**, *30*, 1640023. [CrossRef]
6. Wazwaz, A.M. *Partial Differential Equations and Solitary Waves Theory*; Springer: New York, NY, USA, 2009.
7. Eddine, N.C.; Ragusa, M.A. Generalized critical Kirchhoff-type potential systems with Neumann boundary conditions. *Appl. Anal.* **2022**, *101*, 3958–3988. [CrossRef]
8. Rashid, S.; Ashraf, R.; Bonyah, E. On analytical solution of time-fractional biological population model by means of generalized integral transform with their uniqueness and convergence analysis. *J. Funct. Spaces* **2022**, *2022*, 7021288. [CrossRef]
9. Zid, S.; Menkad, S. The lambda-Aluthge transform and its applications to some classes of operators. *Filomat* **2022**, *36*, 289–301. [CrossRef]
10. Debnath, L. *Nonlinear Partial Differential Equations for Scientists and Engineers*; Birkhäuser: Boston, MA, USA, 1997.
11. Qazza, A.; Hatamleh, R.; Alodat, N. About the solution stability of Volterra integral equation with random kernel. *Far East J. Math. Sci.* **2016**, *100*, 671–680. [CrossRef]
12. Gharib, G.; Saadeh, R. Reduction of the self-dual yang-mills equations to sinh-poisson equation and exact solutions. *WSEAS Trans. Math.* **2021**, *20*, 540–554. [CrossRef]
13. Saadeh, R.; Al-Smadi, M.; Gumah, G.; Khalil, H.; Khan, A. Numerical investigation for solving two-point fuzzy boundary value problems by reproducing kernel approach. *Appl. Math. Inf. Sci.* **2016**, *10*, 2117–2129. [CrossRef]
14. Widder, V. *The Laplace Transform*; Princeton University Press: Princeton, NJ, USA, 1941.
15. Bochner, S.; Chandrasekharan, K. *Fourier Transforms*; Princeton University Press: London, UK, 1949.
16. Atangana, A.; Kiliçman, A. A novel integral operator transform and its application to some FODE and FPDE with some kind of singularities. *Math. Probl. Eng.* **2013**, *2013*, 531984. [CrossRef]
17. Srivastava, H.; Luo, M.; Raina, R. A new integral transform and its applications. *Acta Math. Sci.* **2015**, *35*, 1386–1400. [CrossRef]
18. Watugula, G.K. Sumudu transform: A new integral transform to solve differential equations and control engineering problems. *Int. J. Math. Educ. Sci. Technol.* **1993**, *24*, 35–43. [CrossRef]
19. Khan, Z.H.; Khan, W.A. Natural-transform-properties and applications. *NUST J. Eng. Sci.* **2008**, *1*, 127–133.
20. Elzaki, T.M. The new integral transform "Elzaki transform". *Glob. J. Pure Appl. Math.* **2011**, *7*, 57–64.
21. Sedeeg, A.H. The New Integral Transform "Kamal Transform". *Adv. Theor. Appl. Math.* **2016**, *11*, 451–458.
22. Aboodh, K.; Idris, A.; Nuruddeen, R. On the Aboodh transform connections with some famous integral transforms. *Int. J. Eng. Inform. Syst.* **2017**, *1*, 143–151.
23. Saadeh, R.; Qazza, A.; Burqan, A. A new integral transform: ARA transform and its properties and applications. *Symmetry* **2020**, *12*, 925. [CrossRef]
24. Zafar, Z. ZZ transform method. *Int. J. Adv. Eng. Glob. Technol.* **2016**, *4*, 1605–1611.
25. Aghili, A.; Moghaddam, B.P. Certain theorems on two dimensional Laplace transform and non-homogeneous parabolic partial differential equations. *Surv. Math. Its Appl.* **2011**, *6*, 165–174.
26. Dhunde, R.R.; Bhondge, N.M.; Dhongle, P.R. Some remarks on the properties of double Laplace transforms. *Int. J. Appl. Phys. Math.* **2013**, *3*, 293–295. [CrossRef]
27. Dhunde, R.R.; Waghmare, G.L. Double Laplace transform method in mathematical physics. *Int. J. Theor. Math. Phys.* **2017**, *7*, 14–20.
28. Eltayeb, H.; Kiliçman, A. A note on double Laplace transform and telegraphic equations. *Abstr. Appl. Anal.* **2013**, *2013*, 932578. [CrossRef]
29. Alfaqeih, S.; Misirli, E. On double Shehu transform and its properties with applications. *Int. J. Anal. Appl.* **2020**, *18*, 381–395.
30. Sonawane, S.M.; Kiwne, S.B. Double Kamal transforms: Properties and Applications. *J. Appl. Sci. Comput.* **2019**, *4*, 1727–1739.
31. Ganie, J.A.; Ahmad, A.; Jain, R. Basic analogue of double Sumudu transform and its applicability in population dynamics. *Asian J. Math. Stat.* **2018**, *11*, 12–17. [CrossRef]
32. Eltayeb, H.; Kiliçman, A. On double Sumudu transform and double Laplace transform. *Malays. J. Math. Sci.* **2010**, *4*, 17–30.
33. Tchuenche, J.M.; Mbare, N.S. An application of the double Sumudu transform. *Appl. Math. Sci.* **2007**, *1*, 31–39.
34. Al-Omari, S.K.Q. Generalized functions for double Sumudu transformation. *Int. J. Algebra* **2012**, *6*, 139–146.
35. Eshag, M.O. On double Laplace transform and double Sumudu transform. *Am. J. Eng. Res.* **2017**, *6*, 312–317.
36. Ahmed, Z.; Idrees, M.I.; Belgacemc, F.B.M.; Perveen, Z. On the convergence of double Sumudu transform. *J. Nonlinear Sci. Appl.* **2020**, *13*, 154–162. [CrossRef]
37. Idrees, M.I.; Ahmed, Z.; Awais, M.; Perveen, Z. On the convergence of double Elzaki transform. *Int. J. Adv. Appl. Sci.* **2018**, *5*, 19–24. [CrossRef]

38. Ahmed, S.; Elzaki, T.; Elbadri, M.; Mohamed, M.Z. Solution of partial differential equations by new double integral transform (Laplace–Sumudu transform). *Ain Shams Eng. J.* **2021**, *12*, 4045–4049. [CrossRef]
39. Saadeh, R.; Qazza, A.; Burqan, A. On the Double ARA-Sumudu transform and its applications. *Mathematics* **2022**, *10*, 2581. [CrossRef]
40. Qazza, A.; Burqan, A.; Saadeh, R.; Khalil, R. Applications on Double ARA–Sumudu Transform in Solving Fractional Partial Differential Equations. *Symmetry* **2022**, *14*, 1817. [CrossRef]
41. Meddahi, M.; Jafari, H.; Yang, X.-J. Towards new general double integral transform and its applications to differential equations. *Math. Methods Appl. Sci.* **2021**, *45*, 1916–1933. [CrossRef]