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Convergence and Stability of a Split-Step Exponential Scheme Based on the Milstein Methods

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Abstract: We introduce two approaches by modifying split-step exponential schemes to study stochastic differential equations. Under the Lipschitz condition and linear-growth bounds, it is shown that our explicit schemes converge to the solution of the corresponding stochastic differential equations with the order 1.0 in the mean-square sense. The mean-square stability of our methods is investigated through some linear stochastic test systems. Additionally, asymptotic mean-square stability is analyzed for the two-dimensional system with symmetric and asymmetric coefficients and driven by two commutative noise terms. In particular, we prove that our methods are mean-square stable for any step-size. Finally, some numerical experiments are carried out to confirm the theoretical results.

Keywords: mean-square stability; stochastic differential equations; strong convergence; ODE solver; Milstein method; split-step schemes

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1. Introduction

Due to the significant role played by stochastic differential equations (SDEs) in describing different phenomena, studying their solutions' behavior is an important topic for researchers [1–5]. Since there is no analytical form solution for most SDEs, a critical issue in their study is the design of efficient schemes for numerical solutions. Thus, many numerical schemes are used to approximate their solution, for example, see [4,6–13].

In recent years, split-step techniques have been widely used to solve various SDEs. For example, Haghighi and Hosseini [7] analyzed the mean-square (MS) convergence of Rosenbrock stochastic balanced methods, which are a combination of Milstein methods [12] and Rosenbrock ordinary differential equation (ODE) solvers [14]. Additionally, using the Rosenbrock ODE solver, the split-step double-balanced scheme was constructed for solving SDEs [15]. In [16], Nouri et al. gave the split-step Rosenbrock scheme for SDEs. By the split Adams–Moulton ODE solver [17], new stochastic schemes are proposed to solve SDEs [18,19]. Recently, an attempt has been made to provide new explicit schemes with a wide stability region using ODE solvers [20]. In [21–23], the authors established new types of Euler–Maruyama and Milstein schemes by the exponential function for the solution of stiff SDEs arising in physical and chemical models.

In the last decades, using stability properties has led to the creation of better numerical algorithms. Saito and Mitsui [24] investigate the MS stability behavior of several numerical schemes for linear scalar SDE. Additionally, in [25], the MS stability condition of the Euler–Maruyama approach for a two-dimensional SDE with symmetric coefficients driven by one noise term is investigated. Buckwar and Sickenberger [26] studied the MS stability of the θ-Euler–Maruyama and θ-Milstein schemes for scalar SDEs driven by an s-dimensional Wiener process. Additionally, the same authors analyzed, in [27], the MS stability properties of the θ-Euler–Maruyama and θ-Milstein schemes for two-dimensional SDEs driven by commutative and non-commutative noise terms with symmetric and asymmetric coefficients. Following this line, the MS stability of many numerical schemes was studied, for example, see [11,12,16,18,19,26,28,29].

We modify the split-step forward methods based on the exponential Milstein scheme for solving stiff SDEs. In the following section, we make some necessary notations and assumptions of SDE (1). We also construct our schemes in this section. In Section 3, we study the convergence theorem and show that the proposed schemes converge to accurate solutions under the Lipschitz condition and linear-growth bounds of order 1.0 in the MS sense. Furthermore, the stability behavior of our schemes on some test systems of SDEs is considered in Section 4. Next, several numerical experiments are presented in Section 5. Some concluding remarks are given in the final section.

2. Hybrid Approach of Split-Step and Milstein Schemes

We consider d-dimensional Itô SDEs [11,16],

$$\mathrm{d}P(t) = f(P(t))\mathrm{d}t + \sum_{r=1}^{s} g_j(P(t))\mathrm{d}Z^j(t), \quad P(t_0) = P_0 \in \mathbb{R}^d, \quad t \in [t_0, T]. \tag{1}$$

For solving (1), many authors have suggested the split-step approach [8,28,30], in the following general form:

$$\begin{cases} \overline{Q}_l = Q_l + \Delta \mathrm{Y}(\overline{Q}_l, Q_l) \\ Q_{l+1} = \overline{Q}_l + \sum\limits_{r=1}^{s} g_j(\overline{Q}_l)\Delta Z_l^j, \end{cases}$$

where $\mathrm{Y}(\overline{Q}_l, Q_l)$ is the increment function of an appropriate ODE solver. To establish our schemes, we consider

$$\begin{aligned} \mathrm{Y}(\overline{Q}_l, Q_l) &= f(Q_l) + \frac{\Delta}{2} J_f(Q_l) f(Q_l) + \mathcal{O}(\Delta^2) \\ &= f(Q_l)\frac{\exp(\Delta J_f(Q_l)) - I}{\Delta J_f(Q_l)} + \mathcal{O}(\Delta^2), \end{aligned} \tag{2}$$

where J_f denotes a Jacobian Matrix. Now, we derive the following explicit approaches by the Milstein method, the first drifting split-step exponential modified Milstein (DRSSEMM) scheme

$$\begin{cases} \overline{Q}_l = Q_l + \Delta f(Q_l)\frac{\exp(\Delta J_f(Q_l)) - I}{\Delta J_f(Q_l)} \\ Q_{l+1} = \overline{Q}_l + \sum\limits_{r=1}^{s} g_r(\overline{Q}_l)\Delta Z_l^r + \sum\limits_{r_1, r_2=1}^{s} L^{r_1} g_{r_2}(\overline{Q}_l)\mathcal{I}_{(r_1, r_2)}, \end{cases} \tag{3}$$

and the second diffused split-step exponential modified Milstein (DISSEMM) scheme

$$\begin{cases} \overline{Q}_l = Q_l + \sum\limits_{r=1}^{s} g_r(Q_l)\Delta Z_l^r + \sum\limits_{r_1, r_2=1}^{s} L^{r_1} g_{r_2}(Q_l)\mathcal{I}_{(r_1, r_2)} \\ Q_{l+1} = \overline{Q}_l + \Delta f(\overline{Q}_l)\frac{\exp(\Delta J_f(Q_l)) - I}{\Delta J_f(Q_l)}, \end{cases} \tag{4}$$

where

$$L^{r_1} = \sum_{i=1}^{d} g_{r_1}^i \frac{\partial}{\partial x_l^i}, \quad \mathcal{I}_{(r_1, r_2)} = \int_{t_l}^{t_{l+1}} \int_{t_l}^{\tau_2} \mathrm{d}Z_{r_1}(\tau_1)\mathrm{d}Z_{r_2}(\tau_2).$$

Additionally, Q_l is the approximation to $P(t_l)$ for $t_l = l\Delta$, $l = 0, 1, \ldots, N$, $N = 1, 2, \ldots$, $\Delta = t_l - t_{l-1}$, and $\Delta Z_l = Z_{t_l} - Z_{t_{l-1}}$ is an independent variable with distribution $\mathcal{N}(0, \Delta)$.

Assumption 1. *Functions $f, g_r, r = 1, \ldots, s$ in SDE (1), J_f and fJ_f satisfy Lipschitz condition and linear-growth bounds with constants $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. Additionally,*

$$\sum_{r_1, r_2 = 1}^{s} |L^{r_1} g_{r_2}(x) - L^{r_1} g_{r_2}(y)|^2 \leq \mathcal{K}_1 |x - y|^2.$$

3. Mean-Square Convergence

We derive MS convergence results for the DRSSEMM (3) and DRSSEMM (4) schemes. Before proving the main theorem, we state the following lemma.

Lemma 1 ([9])**.** *For $l = 0, 1, \ldots, N-1$ and $N = 1, 2, \ldots$, assume that*

$$|\mathbb{E}[(Q_{l+1} - Q(t_{l+1}))|Q_l = P(t_l)]| \leq \mathcal{K}(1 + |Q_l|^2)^{1/2} \Delta^{m_1}, \quad (\textit{local mean error}) \tag{5}$$

$$\left|\mathbb{E}\left[|Q_{l+1} - P(t_{l+1})|^2 | Q_l = P(t_l)\right]\right|^{1/2} \leq \mathcal{K}(1 + |Q_l|^2)^{1/2} \Delta^{m_2}, \quad (\textit{MS error}) \tag{6}$$

with $m_2 \geq \frac{1}{2}$ and $m_1 \geq m_2 + \frac{1}{2}$. Then,

$$\left|\mathbb{E}\left[|Q_k - P(t_k)|^2 | Q_0 = P(t_0)\right]\right|^{1/2} \leq \mathcal{K}(1 + |Q_0|^2)^{1/2} \Delta^{m_2 - 1/2}, \quad k = 0, 1, \ldots, N.$$

Lemma 2. *Assume the linear-growth bounds hold. Then,*

$$\begin{aligned}
\left| \frac{\exp\left(\Delta J_f(Q_l)\right) - I}{\Delta J_f(Q_l)} \right| &\leq 1 + \Delta \sqrt{\mathcal{K}_2} \left(1 + |Q_l|^2\right)^{1/2}, \\
\left| \frac{\exp\left(\Delta J_f(Q_l)\right) - I}{\Delta J_f(Q_l)} \right|^2 &\leq 2 + \frac{1}{2} \Delta^2 \mathcal{K}_2 \left(1 + |Q_l|^2\right), \\
\left| f(Q_l) \left(\frac{\exp\left(\Delta J_f(Q_l)\right) - I}{\Delta J_f(Q_l)} - I \right) \right| &\leq \Delta \sqrt{\mathcal{K}_2} \left(1 + |Q_l|^2\right)^{1/2}, \\
\left| f(Q_l) \left(\frac{\exp\left(\Delta J_f(Q_l)\right) - I}{\Delta J_f(Q_l)} - I \right) \right|^2 &\leq \frac{1}{2} \Delta^2 \mathcal{K}_2 \left(1 + |Q_l|^2\right).
\end{aligned} \tag{7}$$

Proof. It is easy to write

$$\frac{\exp\left(\Delta J_f(Q_l)\right) - I}{\Delta J_f(Q_l)} = I + \frac{1}{2} \Delta J_f(Q_l) + \mathcal{O}(\Delta^2)$$

and

$$f(Q_l) \left(\frac{\exp\left(\Delta J_f(Q_l)\right) - I}{\Delta J_f(Q_l)} - I \right) = \frac{1}{2} \Delta f(Q_l) J_f(Q_l) + \mathcal{O}(\Delta^2).$$

So, we obtain the desired result by using $(a+b)^2 \leq 2(a^2 + b^2)$ and the linear-growth bounds. □

In what follows, we will employ Assumption 1 and Lemmas 1 and 2 to show MS convergence of DRSSEMM (3) and DRSSEMM (4) methods.

Theorem 1. *If, under Assumption 1, we apply the DRSSEMM (3) and DISSEMM (4) schemes with $\Delta = (T - t_0)/N$ to SDE (1), then*

$$\left|\mathbb{E}\left[|Q_k - P(t_k)|^2 \middle| Q_0 = P(t_0)\right]\right|^{1/2} = \mathcal{O}(\Delta), \quad k = 0, 1, \dots, N.$$

Proof. We show that the DRSSEMM and DISSEMM methods convergence to SDE (1), according to Lemma 1, by the local Milstein approximation step

$$Q_{l+1}^{\mathrm{Mil}} = Q_l^{\mathrm{Mil}} + \Delta f(Q_l^{\mathrm{Mil}}) + \sum_{r=1}^{s} g_r(Q_l^{\mathrm{Mil}})\Delta Z_l^r + \sum_{r_1,r_2=1}^{s} L^{r_1} g_{r_2}(Q_l^{\mathrm{Mil}})\mathcal{I}_{(r_1,r_2)}, \tag{8}$$

with [9]

$$\left|\mathbb{E}\left[(Q_{l+1}^{\mathrm{Mil}} - P(t_{l+1}))\middle| Q_l^{\mathrm{Mil}} = P(t_l)\right]\right| \le \Delta^2 K\left(1 + |Q_l|^2\right)^{1/2}, \tag{9a}$$

$$\left(\mathbb{E}\left[|Q_{l+1}^{\mathrm{Mil}} - P(t_{l+1})|^2 \middle| Q_l^{\mathrm{Mil}} = P(t_l)\right]\right)^{\frac{1}{2}} \le \Delta^{3/2} K\left(1 + |Q_l|^2\right)^{1/2}. \tag{9b}$$

First, we prove that relation (5) holds for our schemes with $m_1 = \frac{3}{2}$. Thus, by (9a) we can write

$$\begin{aligned}
\mathcal{E}_1 =& \left|\mathbb{E}\left[(Q_{l+1} - P(t_{l+1}))\middle| Q_l = P(t_l)\right]\right| \\
\le& \left|\mathbb{E}\left[\left(Q_{l+1}^{\mathrm{Mil}} - P(t_{l+1})\right)\middle| Q_l^{\mathrm{Mil}} = P(t_l)\right]\right| + \left|\mathbb{E}\left[\left(Q_{l+1} - Q_{l+1}^{\mathrm{Mil}}\right)\middle| Q_l = Q_l^{\mathrm{Mil}}\right]\right| \\
\le& \Delta^2 K\left(1 + |Q_l|^2\right)^{1/2} + \mathcal{E}_2,
\end{aligned} \tag{10}$$

with

$$\begin{aligned}
\mathcal{E}_2 =& \left|\mathbb{E}\left[\left(Q_{l+1} - Q_{l+1}^{\mathrm{Mil}}\right)\middle| Q_l = Q_l^{\mathrm{Mil}}\right]\right| \\
\le& \begin{cases} \Delta\left|f(Q_l)\left(\frac{\exp(\Delta J_f(Q_l))-I}{\Delta J_f(Q_l)} - I\right)\right|, & \text{(DRSSEMM method)}, \\ \Delta\left|\frac{\exp(\Delta J_f(Q_l))-I}{\Delta J_f(Q_l)}\right| |\mathbb{E}(f(\overline{Q}_l) - f(Q_l))| + \Delta\left|f(Q_l)\left(\frac{\exp(\Delta J_f(Q_k))-I}{\Delta J_f(Q_l)} - I\right)\right|, \\ \text{(DISSEMM method)}. \end{cases}
\end{aligned}$$

Assumption 1, Lemmas 2, and mean value Theorem as

$$f(\overline{Q}_l) - f(Q_l) = J_f(\widehat{Q_l})(\overline{Q}_l - Q_l), \quad Q_l < \widehat{Q_l} < \overline{Q}_l, \tag{11}$$

lead to

$$\mathcal{E}_2 \le \Delta^2 \sqrt{\mathcal{K}_2}\left(1 + |Q_l|^2\right)^{1/2}. \tag{12}$$

From (10) and (12), we obtain

$$\mathcal{E}_1 \le \Delta^2 (K + \sqrt{\mathcal{K}_2})\left(1 + |Q_l|^2\right)^{1/2}. \tag{13}$$

If we divide (6) into two parts by standard arguments, using (9b) we can obtain local MS error for DRSSEMM (3) and DISSEMM (4) schemes as

$$\begin{aligned}\mathcal{E}_3 =&\left|\mathbb{E}\left[|Q_{l+1}-P(t_{l+1})|^2\middle|Q_l=P(t_l)\right]\right|^{1/2}\\ \leq&\left|\mathbb{E}\left[\left|Q^{\mathrm{Mil}}_{l+1}-P(t_{l+1})\right|^2\middle|Q^{\mathrm{Mil}}_l=P(t_l)\right]\right|^{1/2}+\left|\mathbb{E}\left[\left|Q_{l+1}-Q^{\mathrm{Mil}}_{l+1}\right|^2\middle|Q_l=Q^{\mathrm{Mil}}_l\right]\right|^{1/2}\\ \leq&\Delta^{3/2}K\left(1+|Q_l|^2\right)^{1/2}+\mathcal{E}_4.\end{aligned}\tag{14}$$

To continue, we use Assumption 1, Lemma 2, and (11) for the estimation of $\mathcal{E}_4$.

$$\begin{aligned}\mathcal{E}_4 =&\left|\mathbb{E}\left[\left|Q_{l+1}-Q^{\mathrm{Mil}}_{l+1}\right|^2\middle|Q_l=Q^{\mathrm{Mil}}_l\right]\right|^{1/2}\\ \leq&\begin{cases}\sqrt{1+3s}\left|\Delta^2\left|f(Q_l)\left(\frac{\exp(hJ_f(Q_l))-I}{\Delta J_f(Q_l)}-I\right)\right|^2+\mathbb{E}\left|\sum\limits_{r=1}^{s}(g_r(\overline{Q}_l)-g_r(Q_l))\Delta Z^r_l\right|^2\right.\\ \left.+\mathbb{E}\left|\sum\limits_{r_1,r_2=1}^{s}(L^{r_1}g_{r_2}(\overline{Q}_l)-L^{r_1}g_{r_2}(Q_l))\mathcal{I}_{(r_1,r_2)}\right|^2\right|^{1/2},\\ \text{(DRSSEMM method)},\\ \left|2\Delta^2\left|\frac{\exp(hJ_f(Q_l))-I}{\Delta J_f(Q_l)}\right|^2\mathbb{E}|f(\overline{Q}_l)-f(Q_l)|^2\right.\\ \left.+2\Delta^2\left|f(Q_l)\left(\frac{\exp(\Delta J_f(Q_l))-I}{\Delta J_f(Q_l)}-I\right)\right|^2\right|^{1/2},\\ \text{(DISSEMM method)},\end{cases}\\ \leq&\Delta^{3/2}\sqrt{1+3s}\sqrt{\mathcal{K}_2(\Delta(1+C\mathcal{K}_1)+\mathcal{K}_1)}\left(1+|Q_l|^2\right)^{1/2}.\end{aligned}\tag{15}$$

In the above relation, we used the fact that $\mathbb{E}[(\Delta Z^r_l)^2]=\Delta$ and $\mathbb{E}\left[I^2_{(r_1,r_2)}\right]\leq C\Delta^2$, $C>0, r, r_1, r_2=1,\ldots,s$. Finally, by substituting (15) in (14) and using (13) in Lemma 1, we can easily show that the MS convergence orders of the DRSSEMM and DISSEMM schemes are 1.0. □

4. Stability Properties

This section is devoted to the MS stability analysis of the DRSSEMM and DISSEMM schemes for scalar linear SDEs driven by a one-dimensional Wiener process and multi-dimensional commutative noise terms. Also, we study the asymptotic MS stability of our schemes for two-dimensional SDE with two commutative noise terms.

4.1. A Scalar Linear SDE Driven by a One-Dimensional Wiener Process

In this part of the paper, we study the MS stability of the DRSSEMM (3) and DISSEMM (4) schemes by considering the Itô test equation

$$\mathrm{d}P(t)=aP(t)\mathrm{d}t+bP(t)\mathrm{d}Z(t),\quad t\geq t_0,\quad P(t_0)=P_0,\tag{16}$$

where $a,\ b\in\mathbb{C}$. The actual value of (16) is

$$P(t)=P_0\exp\left(\left(a-\frac{1}{2}b^2\right)t+bZ(t)\right),\tag{17}$$

which is MS-stable if and only if [24]

$$\lim_{t\to\infty} \mathbb{E}\Big[|P(t)|^2\Big] = 0 \Leftrightarrow 2\Re(a) + |b|^2 < 0. \tag{18}$$

By applying the DRSSEMM (3) and DISSEMM (4) methods to (16), we obtain the following explicit difference equation:

$$Q_{l+1} = \Upsilon(a, b, \Delta, \xi_l) Q_l,$$

with

$$\Upsilon(a, b, \Delta, \xi_l) = e^{a\Delta}\left(1 + b\sqrt{\Delta}\xi_l + \frac{1}{2}b^2\Delta(\xi_k^2 - 1)\right) \tag{19}$$

where $\xi_l = \frac{\Delta Z_l}{\sqrt{\Delta}} \sim \mathcal{N}(0,1)$. Now, we compute stability domains of proposed methods by (19) and the $\mathbb{E}(|\Upsilon(a, b, \Delta, \xi_l)|^2) < 1$ condition,

$$\begin{aligned}
|\Upsilon(a, b, \Delta, \xi_l)|^2 &= \Upsilon(a, b, \Delta, \xi_l).\overline{\Upsilon(a, b, \Delta, \xi_l)} \\
&= e^{a\Delta}\left(1 + b\sqrt{\Delta}\xi_l + \frac{1}{2}|b|^2\Delta(\xi_l^2 - 1)\right) \\
&\quad \times e^{\bar{a}\Delta}\left(1 + \bar{b}\sqrt{\Delta}\xi_l + \frac{1}{2}|b|^2\Delta(\xi_l^2 - 1)\right) \\
&= e^{2\Re(a)\Delta}\Big(1 + 2\Re(b)\sqrt{\Delta}\xi_l + \Delta|\Delta|^2(2\xi_l^2 - 1) \\
&\quad + \Re(b)\Delta^{3/2}|b|^2\xi_l(\xi_l^2 - 1) + \frac{1}{4}\Delta^2|b|^4(\xi_l^2 - 1)^2\Big).
\end{aligned}$$

Using $\mathbb{E}(\xi_l) = 0$, $\mathbb{E}[\xi_l^2] = 1$ and $\mathbb{E}[\xi_l^4] = 3$, we can compute the MS stability function of the DRSSEMM (3) and DISSEMM (4) schemes, as

$$\begin{aligned}
\overline{\Upsilon}(a, b, \Delta) &= \mathbb{E}(|\Upsilon(a, b, \Delta, \xi_l)|^2) \\
&= e^{2\Re(a)\Delta}\left(1 + \Delta|b|^2 + \frac{1}{2}\Delta^2|b|^4\right) \\
&< 1.
\end{aligned} \tag{20}$$

Theorem 2. *For any values a, b satisfying relation (18) and any step-size $\Delta > 0$, the DRSSEMM (3) and DISSEMM (4) methods give numerical MS-stable solutions.*

Proof. It is easy to see that (20) is equivalent to

$$\begin{aligned}
\Psi(a, b, \Delta) = &-\sum_{n=3}^{\infty} \frac{(-2\Re(a)\Delta)^n}{n!} + \Delta\Big(2\Re(a) + |b|^2\Big) \\
&+ \frac{1}{2}\Delta^2\Big(|b|^2 + 2\Re(a)\Big)\Big(|b|^2 - 2\Re(a)\Big).
\end{aligned}$$

From $\Re(a) < 0$, $2|b|^2 - 2\Re(a) > 0$ and (18), we conclude that $\Psi(a, b, \Delta) < 0$. □

In Figure 1, the region of MS stability is presented for the DRSSEMM (3) and DISSEMM (4) schemes (shaded) when $a, b \in \mathbb{R}$. The MS stability areas belonging to the proposed schemes cover the area of scalar test Equation (16) (gridded). As a result, the DRSSEMM (3) and DISSEMM (4) approaches are appropriate in this respect.

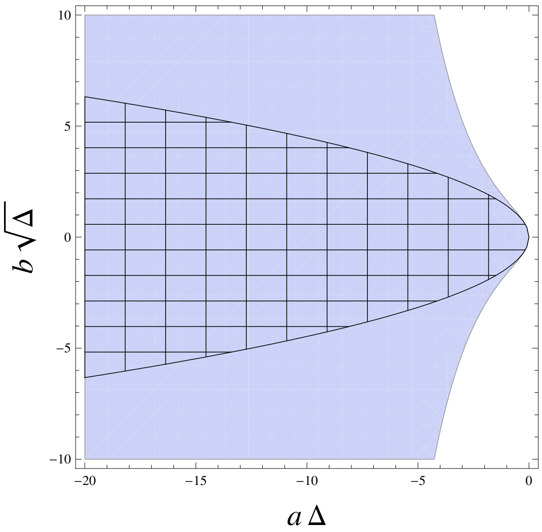


Figure 1. MS stability areas of (16) (gridded), DRSSEMM (3), and DISSEMM (4) schemes (shaded).

4.2. A Scalar Linear SDE with Multi-Dimensional Commutative Noise Terms

For the analysis of the MS stability of our scheme, we consider the following one-dimensional linear SDE driven by multi-dimensional commutative noise terms:

$$\mathrm{d}P(t) = aP(t)\mathrm{d}t + \sum_{r=1}^{s} b_r P(t)\mathrm{d}Z^r(t), \quad t \geq t_0, \quad P(t_0) = P_0, \tag{21}$$

with theoretical solution

$$P(t) = P_0 \exp\left(\left(a - \frac{1}{2}\sum_{r=1}^{s} b_r^2\right)t + \sum_{r=1}^{s} b_r Z^r(t)\right), \tag{22}$$

where $a,\ b_r \in \mathbb{C}$, $r = 1, \ldots, s$.

The SDE (21) is MS-stable if and only if [31–33]

$$2\Re(a) + \sum_{r=1}^{s} |b_r|^2 < 0. \tag{23}$$

Applying DRSSEMM (3) and DISSEMM (4) approaches to the (21) yields

$$Q_{l+1} = \Xi(a, \{b_r\}_{r=1}^{s}, \Delta, \{\xi_l^r\}_{r=1}^{s})Q_l, \tag{24}$$

with

$$\begin{aligned}\Xi(a, \{b_r\}_{r=1}^{s}, \Delta, \{\xi_l^r\}_{r=1}^{s}) =& \mathrm{e}^{2a\Delta}\Bigg(1 + \sqrt{\Delta}\sum_{r=1}^{s} b_r\xi_l^r + \frac{1}{2}\Delta\sum_{r=1}^{s} b_r^2\left((\xi_l^r)^2 - 1\right) \\ &+ \frac{1}{2}\Delta \sum_{\substack{r_1,r_2=1 \\ r_1 \neq r_2}}^{s} b_{r_1}b_{r_2}\xi_l^{r_1}\xi_l^{r_2}\Bigg).\end{aligned}$$

We note that the above explicit difference equation is the result of using the commutative property of noise terms, i.e., $\mathcal{I}_{(r_1,r_2)} + \mathcal{I}_{(r_2,r_1)} = \Delta\xi_l^{r_1}\xi_l^{r_2}$. Therefore, the stability domain of schemes is obtained as follows:

$$
\begin{aligned}
\overline{\Xi}(a,\{b_r\}_{r=1}^{s},\Delta) &= \mathbb{E}\Big[\Xi(a,\{b_r\}_{r=1}^{s},\Delta,\{\xi_l^r\}_{r=1}^{s}).\overline{\Xi(a,\{b_r\}_{r=1}^{s},\Delta,\{\xi_l^r\}_{r=1}^{s})}\Big] \\
&= \mathbb{E}\Bigg[\mathrm{e}^{a\Delta}\Bigg(1+\sqrt{\Delta}\sum_{r=1}^{s} b_r\xi_l^r+\frac{1}{2}\Delta\sum_{r=1}^{s}|b_r|^2\Big((\xi_l^r)^2-1\Big)+\frac{1}{2}\Delta\sum_{\substack{r_1,r_2=1\\ r_1\neq r_2}}^{s} b_{r_1}b_{r_2}\xi_l^{r_1}\xi_l^{r_2}\Bigg) \\
&\qquad\times \mathrm{e}^{\bar{a}\Delta}\Bigg(1+\sqrt{\Delta}\sum_{r=1}^{s} \bar{b}_r\xi_l^r+\frac{1}{2}\Delta\sum_{r=1}^{s}|b_r|^2\Big((\xi_l^r)^2-1\Big)+\frac{1}{2}\Delta\sum_{\substack{r_1,r_2=1\\ r_1\neq r_2}}^{s} \overline{b_{r_1}b_{r_2}}\xi_l^{r_1}\xi_l^{r_2}\Bigg)\Bigg] \qquad (25)\\
&= \mathrm{e}^{2\Re(a)\Delta}\Bigg(1+\Delta\sum_{r=1}^{s}|b_r|^2+\frac{1}{2}\Delta^2\sum_{r=1}^{s}|b_r|^4+\frac{1}{4}\Delta^2\Bigg|\sum_{\substack{r_1,r_2=1\\ r_1\neq r_2}}^{s} b_{r_1}b_{r_2}\Bigg|^2\Bigg).
\end{aligned}
$$

Theorem 3. *For any values a, $\{b_{r_1}\}_{r=1}^{s}$ satisfying condition (23) and any step size $\Delta > 0$, the DRSSEMM (3) and DISSEMM (4) methods, give numerical MS-stable solutions.*

Proof. The schemes are MS-stable if and only if $\overline{\Xi}(a,\{b_r\}_{r=1}^{s},\Delta) < 1$, which is equivalent to $\Gamma(a,\{b_r\}_{r=1}^{s},\Delta) < 0$, where

$$
\begin{aligned}
\Gamma(a,\{b_r\}_{r=1}^{s},\Delta) = &-\sum_{n=3}^{\infty}\frac{(-2\Re(a)\Delta)^n}{n!}+\Delta\Bigg(2\Re(a)+\sum_{r=1}^{s}|b_r|^2\Bigg) \\
&+\frac{1}{2}\Delta^2\Bigg(\sum_{r=1}^{s}|b_r|^2+2\Re(a)\Bigg)\Bigg(\sum_{r=1}^{s}|b_r|^2-2\Re(a)\Bigg).
\end{aligned}
$$

By using $\Re(a) < 0$, $\sum_{r=1}^{s}|b_r|^2-2\Re(a) > 0$ and (23), the desired result is easily obtained. □

In Figure 2, we show the real MS stability regions, i.e., $a,\ b_1,\ b_2 \in \mathbb{R}$, of our schemes (25) (shaded) and SDE test Equation (23) (gridded) for two commutative noise terms. It is obvious that the MS stability areas belonging to the proposed schemes cover the area of SDE Equation (21). Therefore, once again the results of Theorem 3 are confirmed.

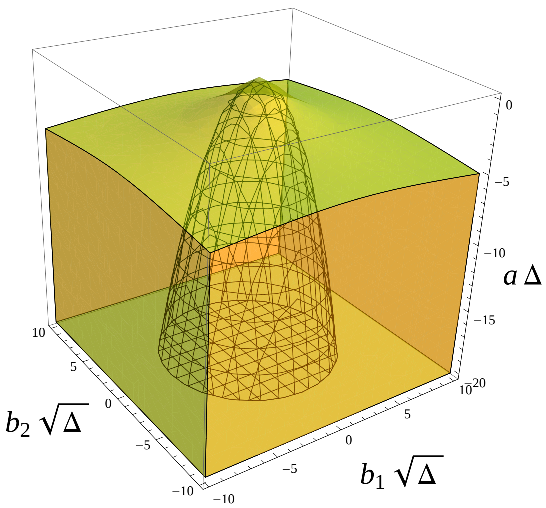


Figure 2. MS stability areas of (21) (gridded), DRSSEMM (3), and DISSEMM (4) schemes (shaded).

4.3. A Two-Dimensional System with Two Commutative Noise Terms

Let us consider

$$\mathrm{d}P(t) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} P(t)\mathrm{d}t + \begin{pmatrix} b_1 & 0 \\ 0 & b_1 \end{pmatrix} P(t)\mathrm{d}Z^1(t) + \begin{pmatrix} 0 & -b_2 \\ b_2 & 0 \end{pmatrix} P(t)\mathrm{d}Z^2(t), \tag{26}$$

where $a, b_1, b_2 \in \mathbb{R}$ and $Z^1(t)$, $Z^2(t)$ are two commutative noise terms. If we apply our schemes to system (26), we have

$$Q_{l+1} = \Theta\Big(a, b_1, b_2, \Delta, \xi_l^1, \xi_l^2\Big) Q_l, \tag{27}$$

where

$$\begin{aligned}\Theta\Big(a, b_1, b_2, \Delta, \xi_l^1, \xi_l^2\Big) = {} & \mathrm{e}^{a\Delta} I_{2\times 2} + \sqrt{\Delta}\mathrm{e}^{a\Delta} b_1 I_{2\times 2}\xi_l^1 + \sqrt{\Delta}\mathrm{e}^{a\Delta}\begin{pmatrix} 0 & -b_2 \\ b_2 & 0 \end{pmatrix}\xi_l^2 \\ & + \frac{1}{2}\Delta\mathrm{e}^{a\Delta}\Big((\xi_l^1)^2 - 1\Big) b_1^2 I_{2\times 2} - \frac{1}{2}\Delta\mathrm{e}^{a\Delta}\Big((\xi_l^2)^2 - 1\Big) b_2^2 I_{2\times 2} \\ & + \Delta\mathrm{e}^{a\Delta}\begin{pmatrix} 0 & -b_1 b_2 \\ b_1 b_2 & 0 \end{pmatrix}.\end{aligned}$$

In order to investigate the asymptotic MS stability of the equilibrium position for proposed schemes, we use the findings of Buckwar and Sickenberger [27]. Therefore, from (27), we obtain

$$\begin{aligned}\overline{\Theta}(a, b_1, b_2, \Delta) &= \mathbb{E}\Big[\Theta\Big(a, b_1, b_2, \Delta, \xi_l^1, \xi_l^2\Big) \otimes \Theta\Big(a, b_1, b_2, \Delta, \xi_l^1, \xi_l^2\Big)\Big] \\ &= \mathrm{e}^{2a\Delta} I_{4\times 4} + \Delta\mathrm{e}^{2a\Delta} b_1^2 I_{4\times 4} + \Delta\mathrm{e}^{2a\Delta}\begin{pmatrix} 0 & -b_2 \\ b_2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -b_2 \\ b_2 & 0 \end{pmatrix} \\ &\quad + \frac{1}{2}\Delta^2\mathrm{e}^{2a\Delta} b_1^4 I_{4\times 4} + \frac{1}{2}\Delta^2\mathrm{e}^{2a\Delta} b_2^4 I_{4\times 4} \\ &\quad + \Delta^2\mathrm{e}^{2a\Delta}\begin{pmatrix} 0 & -b_1 b_2 \\ b_1 b_2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -b_1 b_2 \\ b_1 b_2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} d_1 & 0 & 0 & d_2 \\ 0 & d_1 & -d_2 & 0 \\ 0 & -d_2 & d_1 & 0 \\ d_2 & 0 & 0 & d_1 \end{pmatrix},\end{aligned} \tag{28}$$

where $\otimes$ is a Kronecker product, $d_1 = \mathrm{e}^{2a\Delta}\Big(1 + \Delta b_1^2 + \frac{1}{2}\Delta^4(b_1^4 + b_2^4)\Big)$, and $d_2 = b_2^2\mathrm{e}^{2a\Delta}\Delta$ $(1 + \Delta b_1^2)$. From [27], we know that the equilibrium position of difference Equation (27) is asymptotically MS-stable if and only if $\rho(\overline{\Theta}(a, b_1, b_2, \Delta)) < 1$, where $\rho(\overline{\Theta}(a, b_1, b_2, \Delta))$ is the spectral radius of $\overline{\Theta}(a, b_1, b_2, \Delta)$. So, the eigenvalues of the MS stability matrix (28) are

$$\begin{aligned}d_1 &= \mathrm{e}^{2a\Delta}\Big(1 + \Delta\Big(b_1^2 + b_2^2\Big) + \frac{1}{2}\Delta^2\Big(b_1^2 + b_2^2\Big)^2\Big), \\ d_2 &= \mathrm{e}^{2a\Delta}\Big(1 + \Delta\Big(b_1^2 - b_2^2\Big) + \frac{1}{2}\Delta^2\Big(b_1^2 - b_2^2\Big)^2\Big).\end{aligned}$$

Thus, the MS stability domain of our methods is $\{(a, b_1, b_2, \Delta) | d_1 < 1\}$. In Figure 3, the stability regions of our schemes (shaded) and SDE system (26) (gridded) are compared. The MS stability function of SDE test Equation (26) is $2a + b_1^2 + b_2^2 < 0$, obtained by Buckwar and Sickenberger [27].

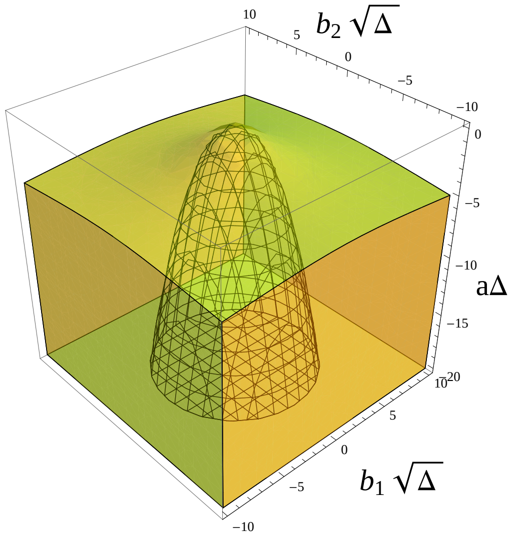


Figure 3. MS stability areas of (26) (gridded), DRSSEMM (3), and DISSEMM (4) schemes (shaded).

Corollary 1. *Since the MS stability function of our schemes applied to* (21) *for* $s = 2$ *and* (26), *and because the MS stability function of test equations* (21) *for* $s = 2$ *and* (26) *are equal, it follows that Theorem 3 is satisfied for the test system* (26).

5. Numerical Results

In the following, several examples are used to demonstrate the theoretical outcomes stated in Sections 3 and 4. The theoretical and numerical solutions are compared by considering the MS errors (MSEs), which are defined by

$$MSEs = \left(\frac{1}{M_0} \sum_{j=1}^{M_0} \left| P_{j,t_N} - Q_{j,N} \right|^2 \right)^{1/2},$$

where P_{j,t_N} and $Q_{j,N}$ are the calculated and theoretical values on the jth independent sample path evaluated at $t_N = T$ and the total number of samples $M_0 = 5000$.

5.1. A Scalar Linear SDE

Consider a scalar linear SDE (16) with real parameters a and b on $t \in [0,1]$ with initial value $P_0 = 1$. To validate the findings of the Theorem 1, we carry out a numerical experiment with eight different step-sizes $\Delta = 2^{-i}$, $i = 4, \dots, 11$ at $T = 1$.

For $a = 3$ and $b = 1$, Figure 4 shows the MSEs and a comparison of the DRSSEMM (3) and DISSEMM (4) schemes with the Milstein scheme. Additionally, Figure 4 reveals that the convergence order of our approaches is close to 1.0, and this is a confirmation of the theoretical conclusion of Theorem 1.

We investigate the numerical MS stability of the Milstein method and of our schemes, for the linear SDE (16) with coefficients $a = -13$ and $b = 5$. The coefficients satisfy the condition (18). Figure 5 shows that our proposed schemes preserve the MS stability of the exact solution for various step-sizes $\Delta = 2^{-i}$, $i = 0, \dots, 4$, while the numerical solution from the Milstein method is MS-stable only for step-size $\Delta = 2^{-4}$. To obtain the graphs of Figure 5, we simulated 50,000 sample paths of numerical solutions.

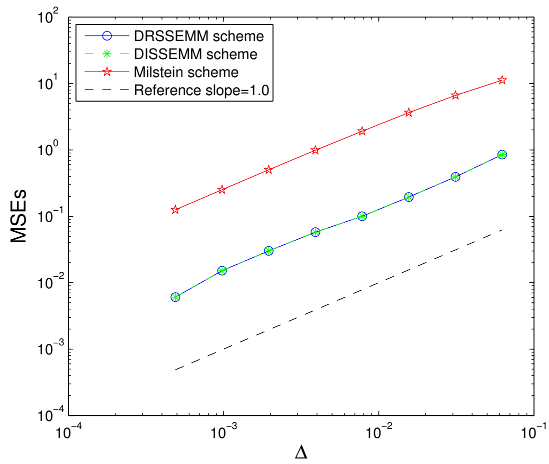


Figure 4. MSEs of the Milstein, DRSSEMM, and DISSEMM schemes applied to SDE (16).

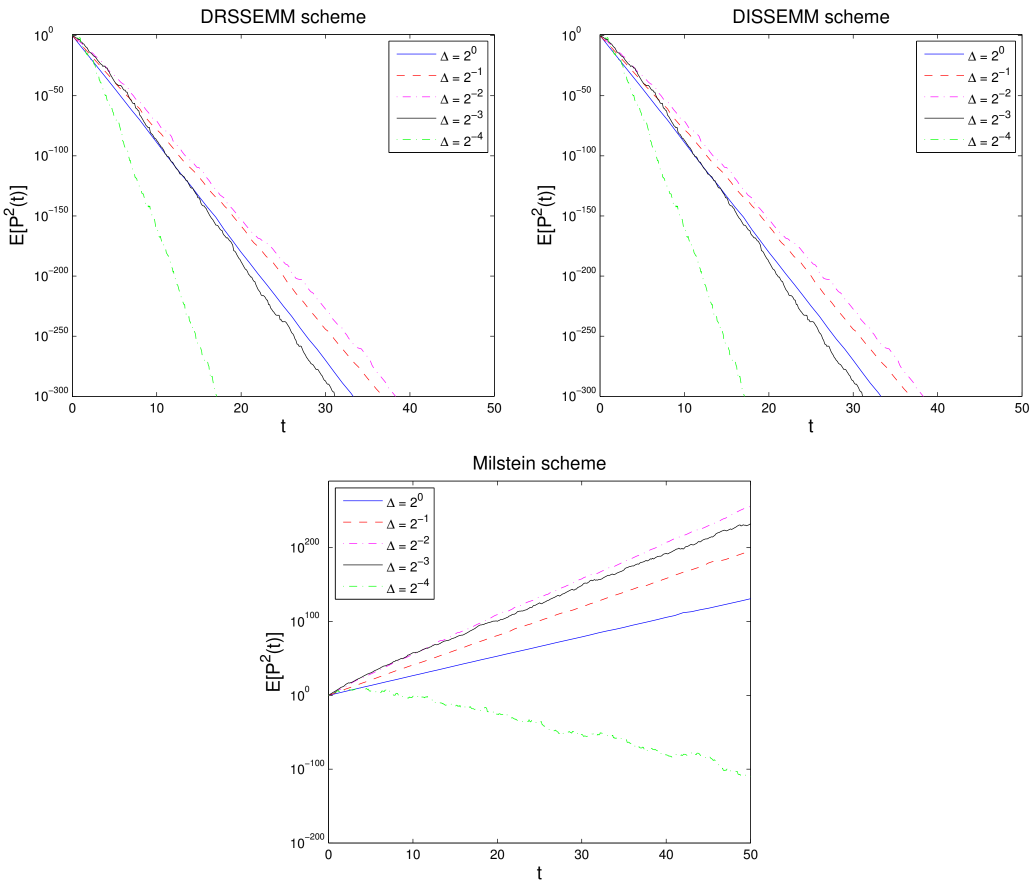


Figure 5. MS stability of the Milstein, DRSSEMM, and DISSEMM applied to SDE (16).

5.2. *A One-Dimensional Linear SDE with Two Commutative Noise Terms*

Consider a linear SDE (21) with real parameters a, b_1 and b_2 on $t \in [0, 1]$ with initial value $P_0 = 1$. We choose $a = 1$, $b_1 = b_2 = 1$ and eight different step-sizes $\Delta = 2^{-i}$, $i = 4, \ldots, 11$, for computer simulation, then we compare the MSEs of the DRSSEMM (3) and DISSEMM (4) schemes with the Milstein scheme in Figure 6.

In Figure 7, we plotted the numerical MS stability of the Milstein method and of our schemes, for the linear SDE (21) with coefficients $a = -10$ and $b_1 = b_2 = 3$. We note that these coefficients satisfy condition (18). The numerical solutions are evaluated by 50,000 sample paths. According to Figure 7, our methods are MS-stable for all step-sizes

$\Delta = 2^{-i}, i = 0, \ldots, 4$, while the numerical solution from the Milstein method preserves the MS stability of the exact solution only for step-size $\Delta = 2^{-4}$.

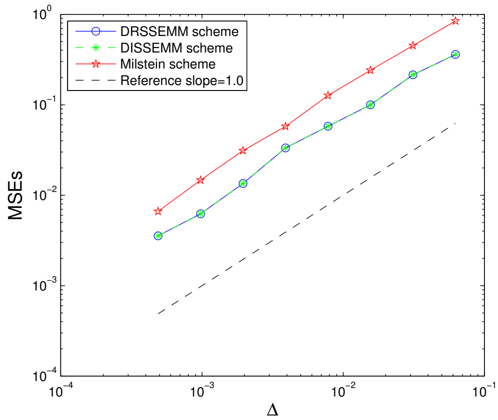


Figure 6. MSEs of the Milstein, DRSSEMM, and DISSEMM schemes applied to SDE (21).

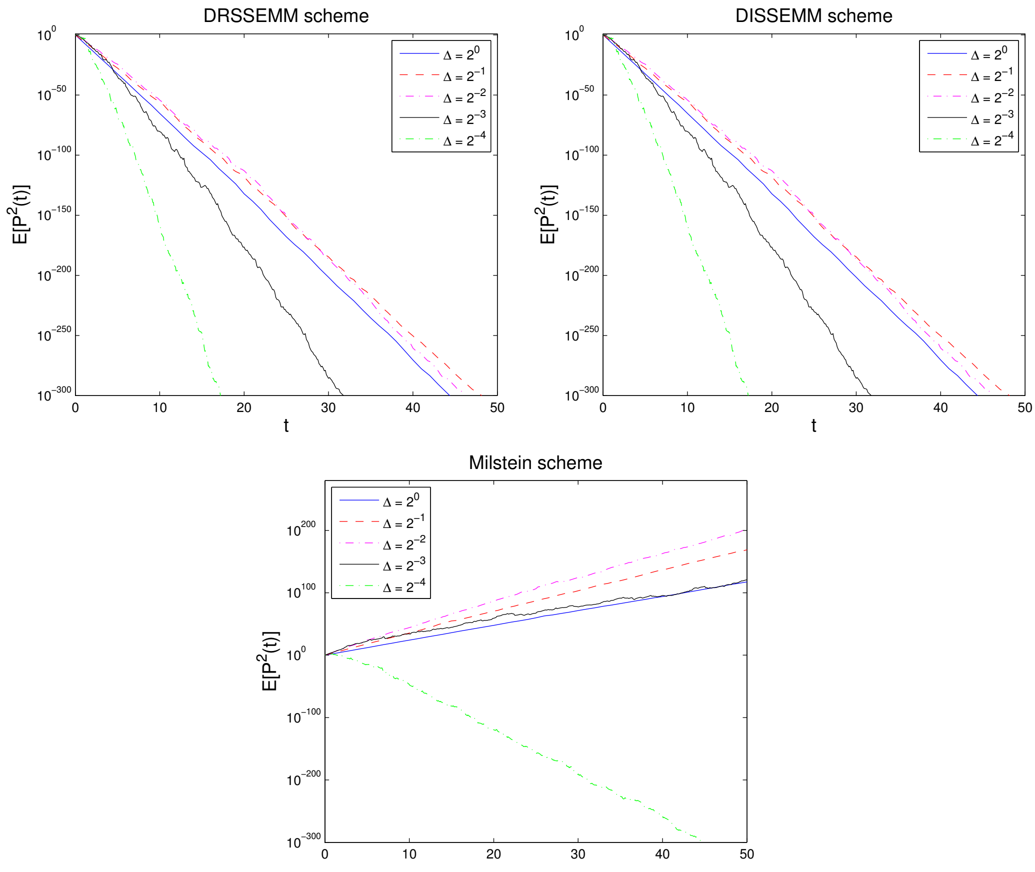


Figure 7. MS stability of the Milstein, DRSSEMM, and DISSEMM applied to SDE (21).

5.3. A Scalar Nonlinear SDE

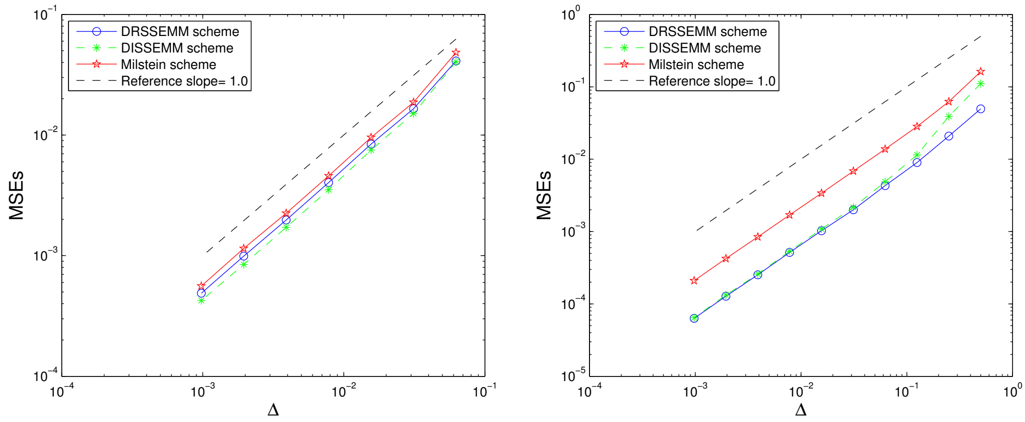
Consider the nonlinear SDE of the form

$$\mathrm{d}P(t) = -\Big(\alpha + \beta^2 P(t)\Big)\Big(1 - P^2(t)\Big)\mathrm{d}t + \beta\Big(1 - P^2(t)\Big)\mathrm{d}Z(t), \quad t \in [0, 1], \tag{29}$$

with the exact solution [4]

$$P(t) = \frac{(1 + P_0)\exp(-2\alpha t + 2\beta Z(t)) + P_0 - 1}{(1 + P_0)\exp(-2\alpha t + 2\beta Z(t)) - P_0 + 1}.$$

Figure 8 depicts the MSEs of methods for two cases, a stiff equation ($\alpha = \beta = 1$) and a non-stiff one ($\alpha = 1$, $\beta = \frac{1}{2}$), with different step-sizes $\Delta = 2^{-i}$, $i = 1, \ldots, 10$, and $P_0 = 0$.



(**a**) Stiff ($\alpha = \beta = 1$). (**b**) Non-stiff ($\alpha = 1$, $\beta = \frac{1}{2}$).

Figure 8. MSEs of the Milstein, DRSSEMM, and DISSEMM schemes applied to nonlinear SDE (29).

5.4. A Two-Dimensional Linear SDE with Two Commutative Noise Terms

To study the numerical MS stability of the stochastic system (26), we choose $a = -10$, $b_1 = b_2 = 3$, and $P_0 = [1, 1]^T$. Figure 9 indicates the numerical MS stability of the Milstein, DRSSEMM, and DISSEMM schemes. It is obvious that our methods preserve the MS stability of the exact solution for all step-sizes $\Delta = 2^{-i}$, $i = 0, \ldots, 4$, but the Milstein scheme is MS-stable only for step-size $\Delta = 2^{-4}$.

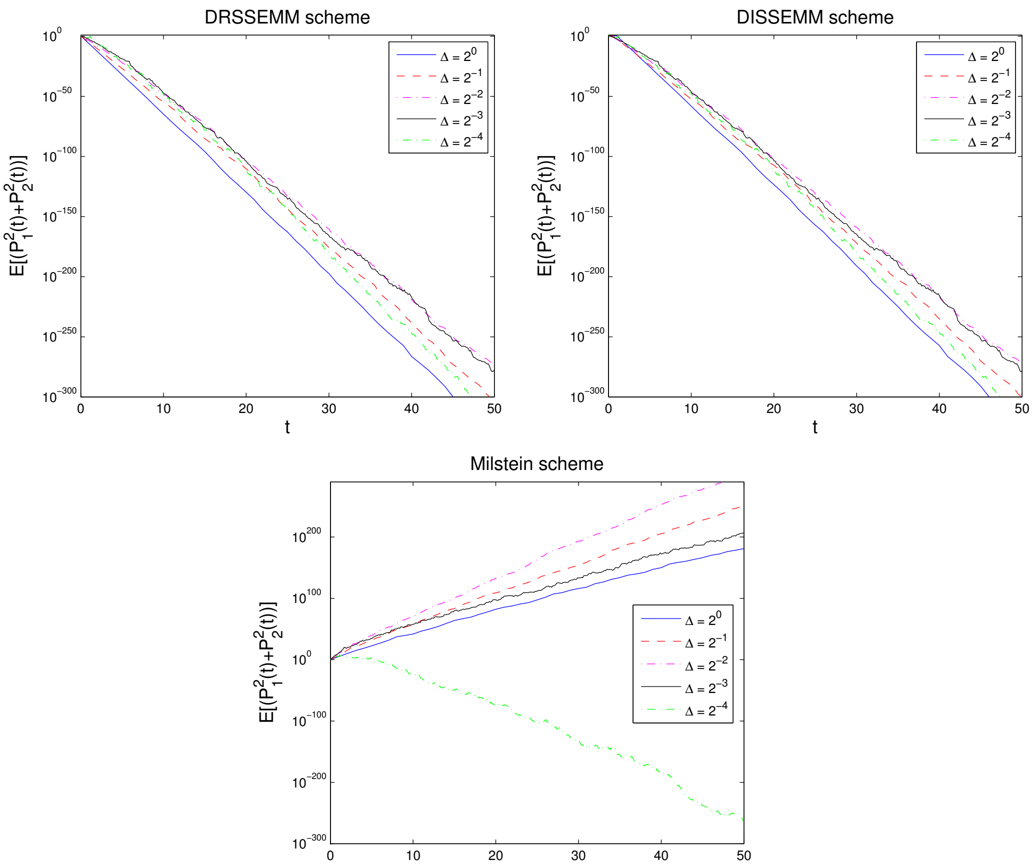


Figure 9. MS stability of the Milstein, DRSSEMM, and DISSEMM schemes applied to SDE (26).

6. Conclusions

In the last decade, developing explicit numerical methods with a wide stability region has become one of the topics of interest for researchers. In this paper, we proposed new numerical schemes, DRSSEMM and DISSEMM, by combining the ODE solver and the explicit Milstein method and proved that the proposed schemes convergence to the exact solution with order 1.0 in MS sense. Furthermore, we investigated the MS stability of the DRSSEMM and DISSEMM methods, for three different SDEs. It was shown that our approaches applied to the scalar linear SDE (16) are MS-stable for any step-size (see Theorem 2). Figure 1 confirms this. Additionally, Theorem 3 and Corollary 1 indicated that our methods are MS-stable for any step-size for scalar one-dimensional and two-dimensional SDEs driven by two commutative noise terms, respectively. Figures 2 and 3 support this.

In several examples, we tested the theoretical findings of this paper. For scalar linear and non-linear SDEs, we show that the convergence order of our schemes is 1.0, see Figures 4, 6 and 8. Additionally, we compare the MSEs of the presented methods and the Milstein scheme in these figures. The results show that the accuracy of the DRSSEMM and DISSEMM approaches is comparable to that of the Milstein scheme. Next, the MS stability theory of the linear scalar Equations (16), (21) and (26) was investigated and compared. The results show that our methods preserve the MS stability of the exact value, while the Milstein scheme is MS-stable for small step-sizes, see Figures 5, 7 and 9.

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