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The Submodular Inequality of Aggregation Operators

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Abstract: Aggregation operators have become an essential tool in many applications. The functional equations related to aggregation operators play an important role in fuzzy sets and fuzzy logic theory. The modular equation is strongly connected with the distributivity equation and can be considered as a constrained associative equation. In this paper, we consider the submodular inequality, which can be viewed as a generalization of the modular equation. First, we discuss the submodular inequality of two general aggregation operators under duality and isomorphism. Moreover, one result of the submodular inequality is presented for the ordinal sum aggregation operators. In the cases of triangular norms and triangular conorms, we present the solutions and validate the symmetry in the related results for some classes of aggregation operators.

Keywords: submodular inequality; aggregation operator; t-norm; t-conorm; symmetry and asymmetry; fuzzy logic

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1. Introduction

In a wide range of practical problems, the question of merging several pieces of input information into a simple one arises in a natural way. The fusion methods based on aggregation operators are very useful in this kind of problem. Recently, the theory of aggregation operators has become an essential tool in various fields of applied sciences [1–9]. Aggregation operators are interesting not only from a theoretical point of view, but also for their applications since they have been proved to be useful in several fields such as fuzzy logic, expert systems, neural networks, pattern recognition, and fuzzy decision. In particular, the functional equations of aggregation operators are important research directions that have been attracting the interest of researchers. There are two main reasons for this:

(1) Functional equations can be used to choose an appropriate aggregation operator;
(2) Functional equations can be used to study the properties of fusion methods since they can characterize the corresponding aggregation operators.

Note that these functional equations are usually related to some properties of aggregation operators, which often come from some concrete applications. The distributivity equation is strongly connected with the pseudo-operators in fuzzy measures and fuzzy integrals. In [10], the distributivity equations of triangular norms and conorms were discussed. The distributivity equations of uninorms and nullnorms (or t-operators) were considered in [11–13]. Moreover, the modular equation is also an important one. On the one hand, it is closely connected with the distributivity equation, which is usually required in fuzzy logic. On the other hand, it can be considered as a constrained associative equation, which is very useful in fuzzy set theory. In [14], modular equations of triangular norms and conorms were discussed. Moreover, the solutions of modular equations for uninorms and nullnorms were given in [15,16].

Distributivity inequalities between two aggregation operators were recently discussed in [17–19]. The submodular inequalities between the triangular norms and conorms of a De

Morgan triplet were studied in [20], and some solutions were presented. Thus, the study of submodular inequalities for general aggregation operators will be interesting. Some new results for submodular inequalities will be presented in this paper, including some general properties of duality and isomorphism as well as the resolution of submodular inequalities for general triangular norms and conorms.

In this paper, we focus on the submodular inequality between two general aggregation operators. Section 2 provides some essential notions concerning the aggregation operator, conjunctor, t-norm, and the submodular inequality. Section 3 discusses the submodular inequality of two aggregation operators under duality and isomorphism. In section 4, we provide one result on submodular inequalities for the ordinal sum conjunctors. Section 5 is devoted to the submodular inequalities between triangular norms and conorms. We end the paper with conclusions and some directions for future work.

2. Preliminaries

In this section, we will give some basic definitions and results associated with aggregation operators.

Definition 1. *[21] A binary aggregation operator is a mapping $A : [0,1]^2 \to [0,1]$, which is increasing in both arguments and fulfills the boundary conditions of $A(0,0) = 0$ and $A(1,1) = 1$.*

Definition 2. *[22] An aggregation operator $A : [0,1]^2 \to [0,1]$ is called a conjunctor if it is increasing with respect to both variables and has a neutral element 1, i.e., $A(x,1) = A(1,x) = x$ for any $x \in [0,1]$. Note that the conjunctor is also known as the semi-copula [22].*

Definition 3. *[23] A unary operator N: $[0,1] \to [0,1]$ is called a negation if it is decreasing and satisfies $N(0) = 1$ and $N(1) = 0$. A negation is called strict if it is strictly decreasing and continuous. A negation is called strong if it is involutive, i.e., $N(N(x)) = x$ for all $x \in [0,1]$.*

Definition 4. *[23] Let $A : [0,1]^2 \to [0,1]$ be a binary aggregation operator.*

(1) Let N be a strong negation. The binary operator $A^* : [0,1]^2 \to [0,1]$ defined by

$$A^*(x,y) = N(A(N(x),N(y)))$$

for all $x,y \in [0,1]$ is called the N-dual of A.

(2) Let φ be a bijection on $[0,1]$. The binary operator $A_\varphi : [0,1]^2 \to [0,1]$ defined by

$$A_\varphi(x,y) = \varphi^{-1}(A(\varphi(x),\varphi(y)))$$

for all $x,y \in [0,1]$ is called the isomorphism of A.

Remark 1. *It is easy to see that if A is an aggregation operator, then A^* and A_φ are also aggregation operators.*

Definition 5. *[23] A binary operator $T : [0,1]^2 \to [0,1]$ is called a triangular norm (t-norm) if it is associative, commutative, with 1 as a neutral element, and increasing in each place. Similarly, a binary operator $S : [0,1]^2 \to [0,1]$ is called a triangular conorm (t-conorm) if it is associative, commutative, with 0 as a neutral element, and increasing in each place.*

It is obvious that the t-norm is a special conjunctor. A t-norm is continuous if it is continuous as a binary function. A t-norm T is called Archimedean if it is continuous and $T(x,x) < x$ for all $x \in]0,1[$. A t-norm is called strict if it is continuous and strictly monotonic on $]0,1]^2$. A continuous t-norm is nilpotent if for each $x \in]0,1[$, there exists some positive integer n with $x_T^{(n)} = 0$, where $x_T^{(n)} := T(\underbrace{x,...,x}_{n})$.

It is well-known that a binary function $T : [0,1]^2 \to [0,1]$ is a continuous Archimedean t-norm if and only if there is a strictly decreasing and continuous function $t : [0,1] \to [0,+\infty]$ with $t(1) = 0$ such that

$$T(x,y) = t^{(-1)}(t(x) + t(y)),$$

where $t^{(-1)}$ is the pseudo-inverse of t defined by

$$t^{(-1)}(x) = \begin{cases} t^{-1}(x) & 0 \le x \le t(0), \\ 0 & x > t(0). \end{cases}$$

t is then said to be an additive generator of T. If $t(0) = +\infty$, then the pseudo-inverse $t^{(-1)} = t^{-1}$ is the inverse of t, and T is strict and its additive generator is unique up to a positive multiplicative constant. For the nilpotent t-norm T, the unique additive generator with $t(0) = 1$ is called the normalized additive generator, and $N_T(x) = t^{-1}(1 - t(x)) : [0,1] \to [0,1]$ is called the associated strong negation of T.

In the literature, the basic t-norms T_M, T_P, T_L, and T_D are given by

$$\begin{aligned} T_M(x,y) &= min(x,y), \\ T_P(x,y) &= x \cdot y, \\ T_L(x,y) &= max(x+y-1,0), \\ T_D(x,y) &= \begin{cases} 0 & (x,y) \in [0,1[^2, \\ min(x,y) & otherwise. \end{cases} \end{aligned}$$

It is well-known that $T \le T_M$ for an arbitrary t-norm T. In the case of the t-conorm, which is the N-dual of the t-norm, some symmetric results are similarly presented in [23].

Definition 6. *[22] If $(A_i)_{i\in I}$ is a family of conjunctors and $(]a_i, b_i[)_{i\in I}$ is a family of non-empty, pairwise disjoint open sub-intervals of $[0,1]$, then $A = (\langle a_i, b_i, A_i\rangle)_{i\in I} : [0,1]^2 \to [0,1]$ is defined by*

$$A(x,y) = \begin{cases} a_i + (b_i - a_i)A_i\left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}\right) & (x,y) \in [a_i,b_i]^2, \\ \min(x,y) & otherwise, \end{cases}$$

which is called the ordinal sum of conjunctors $(A_i)_{i\in I}$.

Definition 7. *[14] Let $A, B : [0,1]^2 \to [0,1]$ be commutative aggregation operators. A is modular over B if*

$$A(x, B(y,z)) = B(A(x,y), z)$$

for $x, y, z \in [0,1]$ and $z \le x$.

Definition 8. *[14] Let $A, B : [0,1]^2 \to [0,1]$ be commutative aggregation operators. A is submodular over B if*

$$A(x, B(y,z)) \le B(A(x,y), z) \tag{1}$$

for $x, y, z \in [0,1]$ and $z \le x$, and is denoted by $A \preceq_{sm} B$.

The submodular inequalities for the t-norms over t-conorms of a De Morgan triplet were discussed in [14]. Therefore, it is interesting to deal with the submodular inequalities for general t-norms and t-conorms, including aggregation operators.

3. Submodular Inequality of Aggregation Operators Under Duality and Isomorphism

This section is devoted to the general properties of submodular inequalities between binary aggregation operators. The following theorem shows the asymmetry of the submodular inequality of two aggregation operators under the N-dual.

Theorem 1. *Let A and B be two aggregation operators, and let A^* and B^* be their N-duals. Then, the following statements are equivalent:*

(1) $A \preceq_{sm} B$;
(2) Their N-duals satisfy $B^* \preceq_{sm} A^*$.

Proof. Suppose that $A \preceq_{sm} B$, i.e., for all $x, y, z \in [0,1]$ and $z \leq x$,

$$A(x, B(y,z)) \leq B(A(x,y), z).$$

It is obvious that $N(x) \leq N(z)$. Then, we obtain the following:

$$\begin{aligned} B^*(x, A^*(y,z)) &= N(B(N(x), N \circ N(A(N(y), N(z))))) \\ &= N(B(N(x), A(N(y), N(z)))) \\ &= N(B(A(N(y), N(z)), N(x))) \\ &= N(B(A(N(z), N(y)), N(x))) \\ &\leq N(A(N(z), B(N(y), N(x)))) \\ &= N(A(B(N(y), N(x)), N(z))) \\ &= N(A(N \circ N(B(N(y), N(x))), N(z))) \\ &= A^*(B^*(x,y), z). \end{aligned}$$

Thus, $B^* \preceq_{sm} A^*$. Since $(A^*)^* = A$ and $(B^*)^* = B$, the converse implication is obvious. □

Submodular Inequality of Aggregation Operators Under Isomorphism

The following theorem provides the symmetry (asymmetry) of the comparison of two binary aggregation operators under the increasing (decreasing) bijection.

Theorem 2. *Let A and B be two binary commutative aggregation operators, assuming that φ is a bijection and A_φ, B_φ are their isomorphisms.*

(1) If φ is increasing, then $A \preceq_{sm} B$ if and only if $A_\varphi \preceq_{sm} B_\varphi$;
(2) If φ is decreasing, then $A \preceq_{sm} B$ if and only if $B_\varphi \preceq_{sm} A_\varphi$.

Proof. Suppose that $A \preceq_{sm} B$. It thus holds that

$$A(x, B(y,z)) \leq B(A(x,y), z)$$

for all $x, y, z \in [0,1]$ and $z \leq x$.

(1) If φ is increasing, then $\varphi(z) \leq \varphi(x)$ and

$$\begin{aligned} A_\varphi\big(x, B_\varphi(y,z)\big) &= \varphi^{-1}(A(\varphi(x), \varphi \circ \varphi^{-1}(B(\varphi(y), \varphi(z))))) \\ &= \varphi^{-1}(A(\varphi(x), B(\varphi(y), \varphi(z)))) \\ &\leq \varphi^{-1}(B(A(\varphi(x), \varphi(y)), \varphi(z))) \\ &= \varphi^{-1}(B(\varphi \circ \varphi^{-1}(A(\varphi(x), \varphi(y))), \varphi(z))) \\ &= B_\varphi\big(A_\varphi(x,y), z\big). \end{aligned}$$

Thus, $A_\varphi \preceq_{sm} B_\varphi$.

(2) If φ is decreasing, then the proof is similar to that of item (1).

□

4. Submodular Inequality of the Ordinal Sum of Conjunctors

In this section, we deal with submodular inequalities between two conjunctors with similar ordinal sum structures. The following theorem reveals the symmetry of the comparison of two ordinal sum conjunctors.

Theorem 3. *Let $A = (\langle a_i, b_i, A_i \rangle)_{i \in I}$ and $B = (\langle a_i, b_i, B_i \rangle)_{i \in I}$ be ordinal sum conjunctors. Then, $A \preceq_{sm} B$ if and only if $A_i \preceq_{sm} B_i$ for each $i \in I$.*

Proof. Suppose that $A \preceq_{sm} B$, i.e., for all $x, y, z \in [0,1]$ and $z \le x$, it holds that

$$A(x, B(y,z)) \le B(A(x,y), z).$$

For the increasing bijection $\varphi_i : [a_i, b_i] \to [0,1]$, $x \to \frac{x - a_i}{b_i - a_i}$, there exist the unique $x', y', z' \in [a_i, b_i]$ such that $\varphi_i(x') = x, \varphi_i(y') = y, \varphi_i(z') = z$. From the ordinal sum structures of A and B, the above inequality can be written as

$$\varphi_i^{-1} \circ A_i(\varphi_i(x'), B_i(\varphi_i(y'), \varphi_i(z'))) \le \varphi_i^{-1} \circ B_i(A_i(\varphi_i(x'), \varphi_i(y')), \varphi_i(z')),$$

$$\varphi_i^{-1} \circ A_i(x, B_i(y,z)) \le \varphi_i^{-1} \circ B_i(A_i(x,y), z).$$

Both sides comprise φ_i, i.e., $A_i(x, B_i(y,z)) \le B_i(A_i(x,y), z)$ for $z \le x$.

Conversely, let us suppose that $A_i \preceq_{sm} B_i$ for each $i \in I$. Consider that for any $x, y, z \in [0,1]$, $z \le x$, we need to prove that $A \preceq_{sm} B$ in the cases below:

(1) $x, y, z \in [a_i, b_i], z \le x, i \in I$. For the increasing bijection $\varphi_i : [a_i, b_i] \to [0,1]$, $x \to \frac{x - a_i}{b_i - a_i}$, and according to the ordinal sum of A, B and Theorem 2, we have

$$\begin{aligned} A(x, B(y,z)) &= \varphi_i^{-1} \circ A_i(\varphi_i(x'), B_i(\varphi_i(y'), \varphi_i(z'))) \\ &\le \varphi_i^{-1} \circ B_i(A_i(\varphi_i(x'), \varphi_i(y')), \varphi_i(z')) = B(A(x,y), z). \end{aligned}$$

(2) $y \le z \le x$.

- $y \notin [a_i, b_i]$ for any $i \in I$. Hence, $B(y,z) = y, A(x,y) = y$, and

$$A(x, B(y,z)) = A(x,y) = y = B(y,z) = B(A(x,y), z).$$

- $y \in [a_i, b_i], z \in [a_i, b_i], x \notin [a_i, b_i]$ for some $i \in I$. Hence, $x > b_i, A(x,y) = y$ and

$$A(x, B(y,z)) = min(x, B(y,z)) = B(y,z) = B(A(x,y), z).$$

- $y \in [a_i, b_i], z \notin [a_i, b_i], x \notin [a_i, b_i]$ for some $i \in I$. Hence, $B(y,z) = y, A(x,y) = y$ and

$$A(x, B(y,z)) = A(x,y) = y = B(y,z) = B(A(x,y), z).$$

(3) $z \le y \le x$.

- $z \notin [a_i, b_i]$ for any $i \in I$. Hence, $B(y,z) = z, A(x,z) = z$ and

$$A(x, B(y,z)) = A(x,z) = z = min(A(x,y), z) = B(A(x,y), z).$$

- $z \in [a_i, b_i], y \in [a_i, b_i], x \notin [a_i, b_i]$ for some $i \in I$. Hence, $x > b_i, A(x,y) = y$ and

$$A(x, B(y,z)) = min(x, B(y,z)) = B(y,z) = B(A(x,y), z).$$

- $z \in [a_i, b_i], y \notin [a_i, b_i], x \notin [a_i, b_i]$ for some $i \in I$. Hence, $B(y,z) = z, A(x,z) = z, A(x,y) \geq b_i$ and
$$A(x, B(y,z)) = A(x,z) \leq z = B(A(x,y),z).$$

(4) $z \leq x \leq y$.

- $z \notin [a_i, b_i]$ for any $i \in I$. Hence, $B(y,z) = z, A(x,z) = z$ and
$$A(x, B(y,z)) = A(x,z) = z = min(A(x,y),z) = B(A(x,y),z).$$
- $z \in [a_i, b_i], x \in [a_i, b_i], y \notin [a_i, b_i]$ for some $i \in I$. Hence, $y > b_i, B(y,z) = B(b_i, z)$, $A(x,y) = A(x, b_i)$ and
$$A(x, B(y,z)) = A(x, B(b_i, z)) \leq B(A(x, b_i), z) = B(A(x,y),z).$$
- $z \in [a_i, b_i], x \notin [a_i, b_i], y \notin [a_i, b_i]$ for some $i \in I$. Hence, $x, y > b_i, A(x,y) \geq b_i$ and
$$A(x, B(y,z)) = A(x,z) = z \leq B(A(x,y),z).$$

This completes the proof that $A \preceq_{sm} B$. □

Remark 2. *With Theorem 2 and Theorem 3, we can obtain the symmetric results for the ordinal sum of t-norms or t-conorms [23,24].*

Example 1. *Let $A = (\langle 0, 0.5, T_D \rangle, \langle 0.5, 1, T_P \rangle)$, and $B = (\langle 0, 0.5, T_P \rangle, \langle 0.5, 1, T_M \rangle)$ be ordinal sums of t-norms. Through computation, we can demonstrate that $A \preceq_{sm} B$. Moreover, it is obvious that $T_D \preceq_{sm} T_P$ and $T_P \preceq_{sm} T_M$.*

For Remark 2, we only need to focus on the submodular inequality of the Archimedean t-norm and t-conorm in order to study the submodular inequality of the continuous t-norm and t-conorm.

5. Submodular Inequality of T-Norm and T-Conorm

In this section, we deal with the submodular inequalities of t-norms and t-conorms.

5.1. Submodular Inequality of T-Norm over T-Conorm

In this subsection, we discuss the submodular inequalities of t-norms over t-conorms.

Proposition 1. *For an arbitrary t-norm T, $T \preceq_{sm} S_M$.*

Proof. For arbitrary $x, y, z \in [0,1], z \leq x$, we can prove these in the following two cases.

- $y \leq z$.
$$T(x, S_M(y,z)) = T(x,z) \leq z \leq S_M(T(x,y),z).$$
- $y > z$.
$$T(x, S_M(y,z)) = T(x,y) \leq S_M(T(x,y),z).$$

Hence, $T(x, S_M(y,z)) \leq S_M(T(x,y),z)$. □

Example 2. *Let $A(x,y) = T_D(x,y)$ and $B(x,y) = S_L(x,y) = min(x+y, 1)$. By taking $x = 0.8, y = 0.6, z = 0.7$ in Equation (1), we have*
$$T_D(x, S_L(y,z)) = T_D(x, \min(y+z, 1)) = T_D(x, 1) = 0.8$$
and
$$S_L(T_D(x,y), z) = \min(T_D(x,y) + z, 1) = 0.7.$$

Thus, T_D is not submodular over S_L.

From Example 2, we know that there exist some t-norm T and t-conorm S such that T is not submodular over S. Subsequently, some sufficient and necessary conditions are presented for the submodular inequalities.

Lemma 1. *Let $A : [0,1]^2 \to [0,1]$ be a t-norm and $B : [0,1]^2 \to [0,1]$ be a t-conorm. Then, A is submodular over B if and only if*

$$A(x, B(y,z)) \leq B(A(x,y), z)$$

for any $x, y, z \in [0,1]$.

Proof. If A is submodular over B, i.e., $A(x, B(y,z)) \leq B(A(x,y), z)$ for $x, y, z \in [0,1], z \leq x$, and when $z > x$,

$$A(x, B(y,z)) \leq T_M(x, B(y,z)) \leq x < z \leq S_M(A(x,y), z) \leq B(A(x,y), z).$$

Conversely, the conclusion is obvious. □

Theorem 4. *Let A be a strict t-norm and B be a strict t-conorm. If $t, g : [0,1] \to [0,\infty]$ are additive generators of A, B, respectively, then $A \preceq_{sm} B$ if and only if the composition $h(x) = g \circ t^{-1}(x), x \in [0,\infty]$ is a convex function.*

Proof. Assume that A is submodular over B. Then, by Lemma 1, we have

$$A(x, B(y,z)) \leq B(A(x,y), z)$$

for $x, y, z \in [0,1]$. The above inequality can be stated as follows:

$$t^{-1}\Big(t(x) + t \circ g^{-1}(g(y) + g(z))\Big) \leq g^{-1}\Big(g \circ t^{-1}(t(x) + t(y)) + g(z)\Big),$$

or equivalently as

$$t(x) + t \circ g^{-1}(g(y) + g(z)) \geq t \circ g^{-1}\Big(g \circ t^{-1}(t(x) + t(y)) + g(z)\Big),$$

and by setting $t(x) = a, t(y) = b, t(z) = c$, and $h(x) = g \circ t^{-1}(x)$, the above inequality holds if and only if

$$a + h^{-1}(h(b) + h(c)) \geq h^{-1}(h(a+b) + h(c)) \tag{2}$$

for all $a, b, c \in [0,\infty]$. Equation (2) holds if and only if the function $h(x) = g \circ t^{-1}(x)$ is convex.

Indeed, if h is convex, then h^{-1} is convex because h is decreasing. Let us consider the function $H(x) = h^{-1}(h(b) + h(c)) + x - h^{-1}(h(b+x) + h(c))$. If $0 \leq x < y$, then

$$H(x) - H(y) = h^{-1}(h(b+y) + h(c)) - y - h^{-1}(h(b+x) + h(c)) + x$$

$$= h^{-1}(h(b+y) + h(c)) - h^{-1}(h(b+y)) - \Big[h^{-1}(h(b+x) + h(c)) - h^{-1}(h(b+x))\Big]$$

$$= \frac{h^{-1}(h(b+y) + h(c)) - h^{-1}(h(b+y))}{h(c)} - \frac{\big[h^{-1}(h(b+x) + h(c)) - h^{-1}(h(b+x))\big]}{h(c)}.$$

Since h is decreasing, h^{-1} is convex, and with Lemma 6.1.1 in [25], we have $h(b+x) \geq h(b+y)$,

$$\frac{h^{-1}(h(b+y)+h(c))-h^{-1}(h(b+y))}{h(c)} \leq \frac{h^{-1}(h(b+x)+h(c))-h^{-1}(h(b+y))}{h(b+x)+h(c)-h(b+y)}$$

and

$$\frac{h^{-1}(h(b+x)+h(c))-h^{-1}(h(b+y))}{h(b+x)+h(c)-h(b+y)} \leq \frac{h^{-1}(h(b+x)+h(c))-h^{-1}(h(b+x))}{h(c)}.$$

Hence, $H(x)-H(y) \leq 0$ and H is increasing. Due to the fact that $H(0)=0$, Equation (2) holds.

Since h is decreasing and h^{-1} is convex, we have $H(x)-H(y) \leq 0$ by Lemma 6.1.1 in [25]. Thus, H is increasing. Because $H(0)=0$, Equation (2) holds.

Conversely, assume that Equation (2) holds. Let $0 \leq x < y$. By taking $a = h^{-1}(x) - h^{-1}\left(\frac{x+y}{2}\right), b = h^{-1}\left(\frac{x+y}{2}\right), c = h^{-1}\left(\frac{y-x}{2}\right)$ in Equation (2), we have the following inequality:

$$h^{-1}(x)-h^{-1}\left(\frac{x+y}{2}\right)+h^{-1}(y) \geq h^{-1}\left(x+\frac{y-x}{2}\right),$$

$$\frac{(h^{-1}(x)+h^{-1}(y))}{2} \geq h^{-1}\left(\frac{x+y}{2}\right).$$

Hence, h^{-1} is convex. Since h^{-1} is decreasing, h is also convex. □

Example 3. *Let the strict t-norm $A(x,y) = T_P(x,y)$ and the strict t-conorm $B = S_P(x,y) = x+y-xy$ for all $x,y \in [0,1]$. Through computation, we know that $A \preceq_{sm} B$. Note that $t(x) = -lnx, g(x) = -ln(1-x)$ are the additive generators of A and B, respectively. It is obvious that $h(x) = g \circ t^{-1}(x) = -ln(1-e^{-x}), x \in [0,\infty]$, is a convex function.*

Theorem 5. *Let A be a nilpotent t-norm and B be a strict t-conorm. Let t be the normalized additive generator of A and g be the additive generator of B. Then, $A \preceq_{sm} B$ if and only if the composition $h(x) = g \circ t^{-1}(x) : [0,1] \to [0,\infty]$ is a convex function.*

Proof. Assume that N_A is the negation associated with the t-norm A. Let $h = g \circ t^{-1} : [0,1] \to [0,\infty]$. It is obvious that h is continuous, strictly decreasing, and that $h(1) = 0$, $h(0) = \infty$. Hence, h can be considered as an additive generator of one strict t-norm T_h, i.e., $T_h = h^{(-1)}(h(x)+h(y))$ for each $x,y \in [0,1]$.

According to Theorem 2.10 in [14], A is submodular over B if and only if the t-norm T_h verifies

$$T_h(a,c) - T_h(b,c) \leq a-b \tag{3}$$

for all $a,b,c \in [0,1]$ such that $a \geq b$.

First, let us suppose that A is submodular over B, that is,

$$A(x,B(y,z)) \leq B(A(x,y),z) \tag{4}$$

for $x,y,z \in [0,1]$. If $x \geq N_A(y)$, then $t(x)+t(y) \leq 1$. Moreover, since $B(y,z) \geq y$, by the monotonicity of t, we have $t(x)+t(B(y,z)) \leq 1$. Equation (4) can thus be stated as follows:

$$t^{-1}\Big(t(x)+t\circ g^{-1}(g(y)+g(z))\Big) \leq g^{-1}\Big(g\circ t^{-1}(t(x)+t(y))+g(z)\Big),$$

or equivalently as

$$t(x) + t \circ g^{-1}(g(y) + g(z)) \geq t \circ g^{-1}\Big(g \circ t^{-1}(t(x) + t(y)) + g(z)\Big),$$

and setting $t(x) = u, t(y) = v, t(z) = w$, and $h = g \circ t^{-1}$, we have

$$u + h^{-1}(h(v) + h(w)) \geq h^{-1}(h(u + v) + h(w))$$

for all $u, v, w \in [0, 1]$ such that $u + v \leq 1$. Now, taking $u + v = a, v = b, w = c$, we can obtain the following:

$$h^{-1}(h(a) + h(c)) - h^{-1}(h(b) + h(c)) \leq a - b$$

for all $a, b, c \in [0, 1]$ such that $a \geq b$. Hence, we confirm that T_h verifies Equation (3) for all $a, b, c \in [0, 1]$ such that $a \geq b$.

Conversely, let us suppose that T_h verifies Equation (3) for all $a, b, c \in [0, 1]$ such that $a \geq b$. We then need to prove that A is submodular over B in the following cases:

- $x \geq N_A(y)$.
  The proof is the inversion of the arguments above.
- $x < N_A(B(y, z)) \leq N_A(y)$.
  In this case, $A(x, B(y, z)) = 0$, $B(A(x, y), z) = B(0, z) = z \geq A(x, B(y, z))$.
- $N_A(B(y, z)) \leq x < N_A(y)$.
  In this case, $t(x) + t(y) > 1, A(x, y) = 0, t(x) + t(B(y, z)) \leq 1$. By setting $a = t(x), b = t(y), c = t(z)$ in Equation (3), we have

  $$T_h(1, c) - T_h(b, c) \leq 1 - b \leq a.$$

  Hence, $c - h^{-1}(h(b) + h(c)) \leq a$ and $t(z) - t \circ g^{-1}(g(y) + g(z)) \leq t(x)$. We then obtain

  $$t(z) \leq t(x) + t \circ g^{-1}(g(y) + g(z)),$$

  or equivalently as

  $$z \geq t^{-1}\Big(t(x) + t \circ g^{-1}(g(y) + g(z))\Big).$$

  Thus, we have

  $$B(A(x, y), z) = B(0, z) = z \geq A(x, B(y, z)).$$

  From above discussion, we know that A is submodular over B. □

With the similar proof, we have the following result for the submodular inequality between a strict t-norm A and a nilpotent t-conorm B.

Theorem 6. *Let A be a strict t-norm and B be a nilpotent t-conorm. If t is an additive generator of A and g is the normalized additive generator of B, then $A \preceq_{sm} B$ if and only if the composition $h : [0, 1] \to [0, \infty], h(x) = t \circ g^{-1}(x)$ is a convex function.*

Theorem 7. *Let A be a nilpotent t-norm and B be a nilpotent t-conorm. Let t and g be the normalized additive generators of A and B, respectively. Then, $A \preceq_{sm} B$ if and only if the composition $h = g \circ t^{-1} : [0, 1] \to [0, 1]$ is a convex function.*

Proof. The proof is similar to that of Theorem 3.7 in [14]. Assume that N_A and N_B are the negations associated with the t-norm A and the t-conorm B, respectively. Let $h = g \circ t^{-1} :$ $[0, 1] \to [0, 1]$. It is obvious that h is continuous, strictly decreasing and $h(1) = 0, h(0) = 1$. Hence, h can be considered as the normalized additive generator of one nilpotent t-norm T_h, i.e., $T_h(x,y) = h^{(-1)}(h(x)+h(y))$ for each $x,y \in [0,1]$. According to Theorem 2.10 in [14], A is submodular over B if and only if the t-norm T_h verifies

$$T_h(a,c) - T_h(b,c) \leq a - b \tag{5}$$

for all $a,b,c \in [0,1]$ such that $a \geq b$.

First, let us suppose that A is submodular over B, that is,

$$A(x, B(y,z)) \leq B(A(x,y),z) \tag{6}$$

for $x,y,z \in [0,1]$, by Lemma 1. Then, we prove Equation (5) in two different cases.

- $x \geq N_A(y), z \leq N_B(y)$. In this case, we have $t(x)+t(y) \leq 1$ and $g(y)+g(z) \leq 1$. Moreover, since $B(y,z) \geq y$ and $A(x,y) \leq y$, by the monotonicity of t and g, we have $t(B(y,z)) + t(x) \leq 1$ and $g(A(x,y)) + g(z) \leq 1$. Equation (6) can thus be stated as follows:

  $$t^{-1}\Big(t(x) + t \circ g^{-1}(g(y)+g(z))\Big) \leq g^{-1}\Big(g \circ t^{-1}(t(x)+t(y)) + g(z)\Big),$$

  or equivalently as

  $$t(x) + t \circ g^{-1}(g(y)+g(z)) \geq t \circ g^{-1}\Big(g \circ t^{-1}(t(x)+t(y)) + g(z)\Big),$$

  and by setting $t(x) = u, t(y) = v, t(z) = w$ and $h = g \circ t^{-1}$, we have

  $$u + h^{-1}(h(v)+h(w)) \geq h^{-1}(h(u+v)+h(w))$$

  for all $u,v,w \in [0,1]$ such that $u+v \leq 1$, and $h(v)+h(w) \leq 1$. Now, taking $u+v = a, v = b, w = c$, we obtain the following:

  $$h^{-1}(h(a)+h(c)) - h^{-1}(h(b)+h(c)) \leq a - b \tag{7}$$

  for all $a,b,c \in [0,1]$ such that $a \geq b$ and $h(b)+h(c) \leq 1$. From Equation (7), we confirm that T_h verifies Equation (5) for all $a,b,c \in [0,1]$ such that $a \geq b$ and $h(b)+h(c) \leq 1$.
- $x \geq N_A(y), N_B(y) < z \leq N_B(A(x,y))$. In this case, we have $t(x)+t(y) \leq 1$ and $g(y)+g(z) > 1$ and $g \circ t^{-1}(t(x)+t(y)) + g(z) \leq 1$. Hence, $B(y,z) = 1$, $A(x,B(y,z)) = x$. Equation (6) can thus be stated as follows:

  $$x \leq g^{-1}\Big(g \circ t^{-1}(t(x)+t(y)) + g(z)\Big),$$

  i.e.,

  $$t(x) \geq t \circ g^{-1}\Big(g \circ t^{-1}(t(x)+t(y)) + g(z)\Big),$$

  and by setting $t(x) = u, t(y) = v, t(z) = w$ and $h = g \circ t^{-1}$, we have

  $$u \geq h^{-1}(h(u+v)+h(w)) \tag{8}$$

  for all $u,v,w \in [0,1]$ such that $u+v \leq 1$ and $h(v)+h(w) > 1$ and $h(u+v)+h(w) \leq 1$. Now, by taking $u+v = a, v = b, w = c$ in Equation (8), we obtain the following:

  $$T_h(a,c) - T_h(b,c) \leq T_h(a,c) \leq a - b$$

  for all $a,b,c \in [0,1]$ such that $a \leq b$, $h(a)+h(c) > 1$ and $h(b)+h(c) \leq 1$. In particular, when $a \geq b$, $h(a)+h(c) > 1$ and $h(b)+h(c) > 1$, $T_h(b,c) - T_h(a,c) = 0 \leq a - b$, i.e., Equation (5) holds.
- $x \geq N_A(y), z > N_B(A(x,y))$. In this case, we have $B(A(x,y),z) = 1$ and the result.

Conversely, let us suppose that T_h verifies Equation (5) for all $a, b, c \in [0,1]$ such that $a \geq b$. We need to prove that A is submodular over B in the following cases:

- $x \geq N_A(y), z \leq N_B(y)$.
  The proof is the inversion of the arguments above.
- $x \geq N_A(y), N_B(y) < z \leq N_B(A(x,y))$.
  The proof is the inversion of the arguments above.
- $x \geq N_A(y), N_B(y) < N_B(A(x,y)) < z$.
  In this case, we have $B(A(x,y),z) = 1$, and the result is trivial.
- $x \leq N_A(B(y,z)) < N_A(y)$.
  In this case, we have $A(x, B(y,z)) = 0$, and the result is trivial.
- $N_A(B(y,z)) < x < N_A(y), z < N_B(y)$.
  In this case, the proof is dual for the second case.
- $N_A(B(y,z)) < x < N_A(y), z \geq N_B(y)$.
  In this case, we have $A(x,y) = 0, B(y,z) = 1$. Hence, $A(x, B(y,z)) = x$ and $B(A(x,y),z) = z$; therefore, we shall prove that $x \leq z$. Let $b \in [0,1]$ such that $t(b) = y$. Since h is a decreasing convex function with $h(0) = 1$ and $h(1) = 0$, $\frac{h(b)-h(0)}{b} \leq h(1) - h(0) \leq \frac{h(1)-h(1-b)}{b}$ by Lemma 6.1.1 in [25]. Thus, $h(1-b) \leq 1 - h(b)$, i.e., $g \circ t^{-1}(1 - t(y)) \leq 1 - g(y)$. Then, $t^{-1}(1 - t(y)) \leq g^{-1}(1 - g(y))$. Hence, we have

  $$x < N_A(y) \leq N_B(y) \leq z$$

  by assumption.

From the discussion above, we know that A is submodular over B. □

Example 4. *Let the nilpotent t-norm $A(x,y) = T_L(x,y)$ and the nilpotent t-conorm $B = S_L(x,y) = \min(x+y,1)$ for all $x, y \in [0,1]$. By computation, we know that $A \preceq_{sm} B$. Note that $t(x) = 1 - x, g(x) = x$ are the additive generators of A and B, respectively. It is obvious that $h(x) = g \circ t^{-1}(x) = 1 - x, x \in [0,1]$, is a convex function.*

5.2. Submodular Inequality of T-Conorm over T-Norm

In this subsection, we discuss the submodular inequalities of the t-conorm over the t-norm.

Theorem 8. *Let A and B be an arbitrary t-conorm and t-norm, respectively. Then, A is not submodular over B.*

Proof. On the contrary, suppose that there exist a t-conorm A and a t-norm B such that $A \preceq_{sm} B$, i.e., for $x, y, z \in [0,1]$ and $z \leq x$,

$$A(x, B(y,z)) \leq B(A(x,y),z).$$

Taking $y = 1$ and $z < x$, we have

$$A(x,z) \leq B(1,z) = z.$$

However, since $S_M \leq A$, $x = S_M(x,z) \leq A(x,z)$, there is a contradiction with the assumption. □

It is easy to see that there exists an asymmetry between Sections 5.1 and 5.2.

5.3. Submodular Inequality of T-Norm over T-Norm

In this subsection, we deal with the submodular inequality of the t-norm over the t-norm.

Example 5. *$T_L \preceq_{sm} T_M$. Indeed, for an arbitrary $x, y, z \in [0,1], z \leq x$, we provide proofs in the following four cases.*

- $x + y \leq 1$

$$T_L(x, T_M(y,z)) = \max(x + \min(y,z) - 1, 0) = 0$$

and

$$T_M(T_L(x,y), z) = \min(\max(x + y - 1, 0), z) = 0;$$

- $x + y > 1$ *and* $z \geq y$

$$T_L(x, T_M(y,z)) = \max(x + \min(y,z) - 1, 0) = x + y - 1$$

and

$$T_M(T_L(x,y), z) = \min(\max(x + y - 1, 0), z) = x + y - 1;$$

- $x + y > 1$ *and* $z < y$ *and* $x + z > 1$

$$T_L(x, T_M(y,z)) = \max(x + \min(y,z) - 1, 0) = x + z - 1$$

and

$$T_M(T_L(x,y), z) = \max(x + \min(y,z) - 1, 0) = \min(x + y - 1, z);$$

- $x + y > 1$ *and* $z < y$ *and* $x + z \leq 1$

$$T_L(x, T_M(y,z)) = \max(x + \min(y,z) - 1, 0) = 0$$

and

$$T_M(T_L(x,y), z) = \min(\max(x + y - 1), z) = \min(x + y - 1, z) \geq 0.$$

Hence, $T_L(x, T_M(y,z)) \leq T_M(T_L(x,y), z)$.
Similarly, $T_D \preceq_{sm} T_M$ and $T_P \preceq_{sm} T_M$.

From Example 5, we have T_D, T_L, T_P being submodular over T_M. Indeed, a similar result holds for an arbitrary t-norm.

Proposition 2. *For every t-norm T, $T \preceq_{sm} T_M$.*

Proof. By the monotonicity of the t-norm and the fact that $T_M \geq T$, we have

$$T(x,y) \geq T(x, T_M(y,z))$$

$$z \geq T(x,z) \geq T(x, T_M(y,z))$$

for all $x, y, z \in [0,1]$ and $z \leq x$. Hence, $T(x, T_M(y,z)) \leq min(T(x,y), z) = T_M(T(x,y), z)$, i.e., $T \preceq_{sm} T_M$. □

Theorem 9. *Let A and B be strict t-norms. Let t_1 and t_2 be additive generators of A and B, respectively. Then, $A \preceq_{sm} B$ if and only if the composition $h(x) = t_1 \circ t_2^{-1}(x) : [0,\infty] \to [0,\infty]$ satisfies $h^{-1}(h(a) + h(b+c)) \geq h^{-1}(h(a) + h(b)) + c$ for all $a, b, c \in [0,\infty]$ and $a \leq c$.*

Proof. A is submodular over B, i.e., for $x, y, z \in [0,1]$ and $z \leq x$,

$$A(x, B(y,z)) \leq B(A(x,y), z).$$

The above inequality can be stated as follows:

$$t_1^{-1}\Big(t_1(x) + t_1 \circ t_2^{-1}(t_2(y) + t_2(z))\Big) \leq t_2^{-1}\Big(t_2 \circ t_1^{-1}(t_1(x) + t_1(y)) + t_2(z)\Big),$$

or equivalently as

$$t_2 \circ t_1^{-1}\Big(t_1(x) + t_1 \circ t_2^{-1}(t_2(y) + t_2(z))\Big) \geq t_2 \circ t_1^{-1}(t_1(x) + t_1(y)) + t_2(z),$$

and by setting $t_2(x) = a, t_2(y) = b, t_2(z) = c$ and $h(x) = t_1 \circ t_2^{-1}(x)$, we have

$$h^{-1}(h(a) + h(b + c)) \geq h^{-1}(h(a) + h(b)) + c \tag{9}$$

for all $a, b, c \in [0, \infty]$ and $a \leq c$. □

Proposition 3. *Let A and B be t-norms. If $A \preceq_{sm} B$, then $A \leq B$.*

Proof. If $A \preceq_{sm} B$, then for $x, y, z \in [0, 1]$ and $z \leq x$,

$$A(x, B(y, z)) \leq B(A(x, y), z).$$

Taking $y = 1$, we have $A(x, B(y, z)) = A(x, z)$ and $B(A(x, y), z) = B(x, z)$, i.e.,

$$A(x, z) \leq B(x, z).$$

Hence, $A \leq B$ by the commutativity of A, B. □

Example 6. *By computation, we know that $T_L \preceq_{sm} T_P$. It is obvious that $T_L \leq T_P$.*

Remark 3. *In general, $A \leq B$ does not imply $A \preceq_{sm} B$. For example, let A and B be the Hamacher t-norms [23] on the unit interval $[0, 1]$ with parameter $\lambda = \frac{1}{6}, \frac{1}{3}$, respectively; that is, $A(x, y) = \frac{xy}{\frac{1}{6} + \frac{1}{6}(x+y-xy)}$ and $B(x, y) = \frac{xy}{\frac{1}{3} + \frac{1}{3}(x+y-xy)}$. It is obvious that $A \geq B$. However, by taking $x = y = z = \frac{1}{3}$ in (1), we have $A(x, B(y, z)) = \frac{9}{31} > \frac{9}{34} = B(A(x, y), z)$. Thus, A is not submodular over B.*

5.4. Submodular Inequality of T-Conorm over T-Conorm

In this subsection, we deal with the submodular inequality of the t-conorm over the t-conorm.

Example 7. *$S_M \preceq_{sm} S_P$. Indeed, for an arbitrary $x, y, z \in [0, 1], z \leq x$, we can prove it in the following two cases.*

- *$y > x$*

$$S_M(x, S_P(y, z)) = \max(x, y + z - y \cdot z) = y + z - y \cdot z$$

and

$$S_P(S_M(x, y), z) = \max(x, y) + z - \max(x, y) \cdot z = y + z - y \cdot z;$$

- *$y \leq x$*

$$S_M(x, S_P(y, z)) = \max(x, y + z - y \cdot z)$$

and

$$S_P(S_M(x, y), z) = \max(x, y) + z - \max(x, y) \cdot z = x + z - x \cdot z.$$

Since $x \leqslant x + z(1 - x)$ and $y(1 - z) + z \leqslant x(1 - z) + z$, we have

$$S_M(x, S_P(y, z)) \leq S_P(S_M(x, y), z).$$

By the example above, we have the following general result.

Proposition 4. *For every t-conorm S, $S_M \preceq_{sm} S$.*

Proof. By the monotonicity of the t-conorm and the fact that $S_M \leq S$, we have

$$S(S_M(x,y),z) \geq S(x,z) \geq x$$

$$S(S_M(x,y),z) \geq S(y,z)$$

for all $x,y,z \in [0,1]$ and $z \leq x$. Hence, $S_M(x,S(y,z)) \leq S(S_M(x,y),z)$, i.e., $S_M \preceq_{sm} S$. □

Theorem 10. *Let A and B be strict t-conorms. Let g_1 and g_2 be additive generators of A and B, respectively. Then, $A \preceq_{sm} B$ if and only if the composition $h(x) = g_2 \circ g_1^{-1}(x) : [0,\infty] \to [0,\infty]$ satisfies $a + h^{-1}(h(b) + h(c)) \leq h^{-1}(h(a+b) + h(c))$ for all $a,b,c \in [0,\infty]$ and $a \geq c$.*

Proof. The proof is similar to that of Theorem 9. Let us suppose that A is submodular over B, i.e., for $x,y,z \in [0,1]$, and $z \leq x$,

$$A(x,B(y,z)) \leq B(A(x,y),z).$$

The above inequality can be stated as follows:

$$g_1^{-1}\Big(g_1(x) + g_1 \circ g_2^{-1}(g_2(y) + g_2(z))\Big) \leq g_2^{-1}\Big(g_2 \circ g_1^{-1}(g_1(x) + g_1(y)) + g_2(z)\Big),$$

or equivalently as

$$g_1(x) + g_1 \circ g_2^{-1}(g_2(y) + g_2(z)) \leq g_2 {}^\circ g_1^{-1}\Big(g_2 \circ g_1^{-1}(g_1(x) + g_1(y)) + g_2(z)\Big),$$

and by setting $g_1(x) = a, g_1(y) = b, g_1(z) = c$ and $h(x) = g_2 \circ g_1^{-1}(x)$, we have

$$a + h^{-1}(h(b) + h(c)) \leq h^{-1}(h(a+b) + h(c)) \tag{10}$$

for all $a,b,c \in [0,\infty]$ and $a \geq c$. □

Proposition 5. *Let A and B be t-conorms. If $A \preceq_{sm} B$, then $A \leq B$.*

Proof. Assume that $A \preceq_{sm} B$. Then, for $x,y,z \in [0,1]$ and $z \leq x$,

$$A(x,B(y,z)) \leq B(A(x,y),z)$$

when $y = 0$, $A(x,B(y,z)) = A(x,z)$ and $B(A(x,y),z) = B(x,z)$; i.e.,

$$A(x,z) \leq B(x,z).$$

Hence, $A \leq B$ by the commutativity of A, B. □

From the results above, we can see the symmetry between Sections 5.3 and 5.4.

6. Conclusions

In this paper, we mainly studied the submodular inequality for two aggregation operators. The main results of this paper include the following:

(1) Some general properties of submodular inequalities in the sense of duality and isomorphism were discussed. The submodular inequality was preserved under the isomorphism of the aggregation operators, while it was reversed under the duality of aggregation operators.
(2) The submodular inequality between the ordinal sum of conjunctors with the same sum and carriers was determined by the submodular inequality between all corresponding sums and conjunctors [23].
(3) The characterization of t-norms and t-conorms in submodular inequalities were presented in terms of the composition of their additive generators. More specifically,

in the cases where the Archimedean t-norm was submodular over the Archimedean t-conorm, we offered a characterization based on the convexity of the composition of their additive generators.

In the future, we will focus on the submodular inequalities of other classes of aggregation operators, such as uninorms and nullnorms [23]. Moreover, the relationship between different inequalities, such as the distributivity inequality, submodular inequality, and super-migrativity, will also be a topic of interest.

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