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On New Hamiltonian Structures of Two Integrable Couplings

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Abstract: In this paper, we present new Hamiltonian operators for the integrable couplings of the Ablowitz–Kaup–Newell–Segur hierarchy and the Kaup–Newell hierarchy. The corresponding Hamiltonians allow nontrivial degeneration. Multi-Hamiltonian structures are investigated. The involutive property is proven for the new and known Hamiltonians with respect to the two Poisson brackets defined by the new and known Hamiltonian operators.

Keywords: integrable coupling; Hamiltonian structure; Hamiltonian; Ablowitz–Kaup–Newell–Segur hierarchy; Kaup–Newell hierarchy

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1. Introduction

The theory of the bi-Hamiltonian structure was introduced in [1,2] in the late 1970s by Magri, Gel'fand and Dorfman. For a (1 + 1)-dimensional evolution equation

$$u_t = K(u), \tag{1}$$

if it can be written as

$$u_t = \theta_1 \frac{\delta H_1}{\delta u} = \theta_2 \frac{\delta H_2}{\delta u},$$

where the two independent Hamiltonian operators θ_1 and θ_2 are compatible, then one can generate infinitely many involutive conserved quantities $\{H_j\}$ for Equation (1) by making use of θ_1 and θ_2 [1]. The compatibility of θ_1 and θ_2 requires their linear combination that is still a Hamiltonian operator. Later, it was proven in [3] by Fuchssteiner and Fokas that for a hierarchy

$$u_{t_j} = K_j(u) = L^j K(u), \;\; j = 0, 1, \cdots, \tag{2}$$

with a recursion operator L, if L allows an implectic–symplectic factorization $L = \theta J$, then θ and $\theta J \theta$ being compatible is equivalent to L being hereditary (such notions can be found in Section 2). Thus, the theory of multi-Hamiltonian structures of (1 + 1)-dimensional integrable hierarchies with recursion operators is established.

Integrable couplings are considered as enlarged integrable systems. According to the review paper [4], an integrable coupling associated with Equation (1) is a nontrivial system of evolution equations that is still integrable and includes (1) as a subsystem, e.g., in the form

$$\begin{cases} u_t = K(u), \\ v_t = S(u, v). \end{cases} \tag{3}$$

One way to obtain such an integrable coupling is to consider the first-order perturbation $u \to u + \epsilon v + o(\epsilon)$ in Equation (1). The resulting system reads

$$\begin{cases} u_t = K(u), \\ v_t = K'[v], \end{cases} \tag{4}$$

where the second equation is the linearized form of the first equation and v is known as a symmetry of the first equation. Early studies of integrable couplings by perturbations were due to [5,6], which investigated their symmetries and Lax pairs. Later, the multi-Hamiltonian structure of integrable couplings by perturbations was studied in [7,8]. For the (1 + 1)-dimensional evolution Equation (1) that has a Lax pair, its integrable coupling (4) can be obtained by enlarging either the original loop algebra or spectral problem [9,10]. There are many examples of obtaining integrable couplings by these two approaches, e.g., [11–20].

If Equation (1) has a Hamiltonian structure

$$u_t = K(u) = \theta \frac{\delta H}{\delta u}, \tag{5}$$

and if the Hamiltonian operator θ is independent of u, then, in light of Theorem 2.8 in [4], its integrable coupling (4) allows a Hamiltonian structure

$$\hat{u}_t = \begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix} \frac{\delta \hat{H}}{\delta \hat{u}}, \quad \hat{u} = \begin{pmatrix} u \\ v \end{pmatrix}. \tag{6}$$

In this paper, we will present two examples, namely the integrable coupling (4) of the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy and the Kaup–Newell (KN) hierarchy. We will see that the Hamiltonian $\hat{H}$ is trivial in degeneration in the sense that $\hat{H}$ reduces to zero (rather than H) when $v = 0$. We will provide a different class of Hamiltonian operators for the integrable coupling (4) such that the new Hamiltonian reduces to H when $v = 0$. Multi-Hamiltonian structures will be investigated along the lines of [3]. We will also prove the involutive property of the new Hamiltonian $\hat{H}$ and H with respect to the two Poisson brackets defined by the new and old Hamiltonian operators.

The paper is organized as follows. In Section 2, we recall some notions of Hamiltonian structures. Then, in Section 3, we present different Hamiltonian structures for the integrable couplings of the AKNS and KN hierarchies and investigate the involutive property of the Hamiltonians. Finally, concluding remarks are given in Section 4.

2. Basic Notions

Let us briefly recall some basic notions of Hamiltonian structures of the $(1 + 1)$-dimensional Equation (1), i.e.,

$$u_t = K(u). \tag{7}$$

One can refer to [3] for more details. In the $(1 + 1)$-dimensional case, (t, x) belongs to $\mathbb{R}^2$. We suppose $u = u(x, t) = (u_1, u_2, \cdots, u_n)^T$, where all functions $\{u_j \doteq u_j(t, x)\}$ and their derivatives with respect to t and x are smooth enough and decrease rapidly as $|x| \to \infty$. By V_n, we denote a function space consisting of vector fields of the form $f(t, x, u) = (f_1, f_2, \cdots, f_n)^T$, where each f_i is a scalar function and C^∞ differentiable with respect to t and x. The scalar product in V_n is defined as

$$\langle f, g\rangle = \int_{-\infty}^{\infty} \sum_{j=1}^{n} f_j g_j \, \mathrm{d}x, \quad f, g \in V_n. \tag{8}$$

For an operator $\Phi : V_n \to V_n$, its adjoint operator Φ^* is defined through

$$\langle f, \Phi g\rangle = \langle \Phi^* f, g\rangle.$$

Φ is self-adjoint (or symmetric) if $\Phi = \Phi^*$, and Φ is skew-symmetric if $\Phi = -\Phi^*$. For a vector field $f \in V_n$, its Gâteaux derivative with respect to u along a given direction g is defined as

$$f'(u)[g] = \frac{\mathrm{d} f(t, x, u + \varepsilon g)}{\mathrm{d}\, \varepsilon}\Big|_{\varepsilon=0}, \quad g \in V_n. \tag{9}$$

f' is also known as the linearized operator of f. If f depends on $u^{(j)}$, where $u^{(j)} = \partial_x^j u$, then f' can be written as

$$f' = \sum_j \frac{\partial f}{\partial u^{(j)}} \partial_x^j.$$

Without causing confusion, we usually write $f'(u)[g]$ as $f'[g]$, by dropping u. For the operator $\Phi(u) : V_n \to V_n$, its Gâteaux derivative can be defined in a similar way,

$$\Phi'(u)[g] = \frac{\mathrm{d}\,\Phi(u + \varepsilon g)}{\mathrm{d}\,\varepsilon}\Big|_{\varepsilon=0}, \quad g \in V_n, \tag{10}$$

and usually we write it as $\Phi'[g]$, which is still an operator. In the following, we list some notions involved in the Hamiltonian structures of Equation (7). One can also refer to [3] or [21].

Definition 1. *An operator $J : V_n \to V_n$ is symplectic if it is skew-symmetric with respect to the scalar product* (8) *and satisfies the Jacobi identity*

$$\langle f, J'[g]h \rangle + \langle g, J'[h]f \rangle + \langle h, J'[f]g \rangle = 0, \quad \forall f, g, h \in V_n. \tag{11}$$

Definition 2. *An operator $\theta : V_n \to V_n$ is implectic (The word "implectic" was introduced in* [3]*, which means "inverse symplectic".) if it is skew-symmetric with respect to the scalar product* (8) *and satisfies the Jacobi identity*

$$\langle f, \theta'[\theta g]h \rangle + \langle g, \theta'[\theta h]f \rangle + \langle h, \theta'[\theta f]g \rangle = 0, \quad \forall f, g, h \in V_n. \tag{12}$$

Definition 3. *An operator $L : V_n \to V_n$ is hereditary if it satisfies the relation*

$$L'[Lf]g - L'[Lg]f = L(L'[f]g - L'[g]f), \quad \forall f, g \in V_n. \tag{13}$$

Definition 4. *For Equation* (7)*, where $K(u) \in V_n$, we say that a function $\omega = \omega(u) \in V_n$ is a symmetry of* (7) *if for all solutions u of* (7) *such that*

$$\omega_t = K'[\omega]. \tag{14}$$

Here, ω_t means taking the total derivative with respect to t, e.g., if $\omega = 2tu + u_x$, we have $\omega_t = \frac{\partial \omega}{\partial t} + \frac{\partial \omega}{\partial u} u_t + \frac{\partial \omega}{\partial u_x} u_{xt} = \frac{\partial \omega}{\partial t} + \omega'[u_t] = 2u + 2tu_t + u_{xt}$.

Definition 5. *An operator $L : V_n \to V_n$ is called a strong symmetry of Equation* (7) *if, for any symmetry ω of* (7)*, the function $L\omega$ is its symmetry too. This equivalently requires*

$$L_t = [K', L] \doteq K'L - LK'.$$

Note here that L_t is the total derivative with respect to t, i.e., $L_t = \frac{\partial L}{\partial t} + L'[u_t]$, and in particular, when L does not contain t explicitly, we have $\frac{\partial L}{\partial t} = 0$ and

$$L'[K] = [K', L], \tag{15}$$

where we have replaced u_t with K since u satisfies Equation (7).

Definition 6. *For a real-valued functional $H(u)$, the vector function $f \in V_n$ is called its (functional) gradient if the following holds ($H'[g]$ is defined along the lines of* (10)*),*

$$\langle f, g \rangle = H'[g], \quad \forall g \in V_n,$$

and we denote $f = \frac{\delta H}{\delta u}$.

If $f \in V_n$ is a gradient of some functional, then the linearized operator f' is self-adjoint, i.e., $f' = f'^*$; on the other hand, once we have a gradient f, the corresponding functional H can be recovered from f by

$$H = \int_0^1 \langle f(\lambda u), u \rangle \mathrm{d}\lambda. \tag{16}$$

Now, we come to introduce the Hamiltonian structure. We say that Equation (7) is a Hamiltonian system (or has a Hamiltonian structure) if it can be written in the form

$$u_t = K(u) = \theta \frac{\delta H}{\delta u}, \tag{17}$$

where θ is an implectic operator (see Definition 2), which is also called a Hamiltonian operator, and H is called the Hamiltonian (or Hamiltonian functional). If there are two independent Hamiltonian operators such that

$$u_t = K(u) = \theta_1 \frac{\delta H_1}{\delta u} = \theta_2 \frac{\delta H_2}{\delta u}, \tag{18}$$

where θ_1 and θ_2 are compatible (i.e., the linear combination $a\theta_1 + b\theta_2$, $a, b \in \mathbb{R}$, is still an implectic operator), we say that the equation $u_t = K(u)$ has a bi-Hamiltonian structure [1]. With a Hamiltonian operator θ, one can define the Poisson bracket of two Hamiltonians,

$$\{W, H\}_\theta = \left\langle \frac{\delta W}{\delta u}, \theta \frac{\delta H}{\delta u} \right\rangle. \tag{19}$$

For two independent Hamiltonians, they are involutive if their Poisson bracket is zero.

The Hamiltonian structure of Equation (7) can also be described using geometric terminology; see [3,4,22,23] for more details. In this setting, the aforementioned function u belongs to some manifold M, t is considered as a parameter and, consequently, $K(u)$ in Equation (7) is viewed as a tangent vector when u evolves on M along the parameter t. Thus, symmetries and gradients are considered as vector fields in the tangent space S at point u and cotangent space S^*, respectively. In this paper, we formally unite S and S^* to be V_n for convenience.

With respect to the Hamiltonian structures of the equations related to (7), we have the following (refer to Theorem 2 and Theorem 3 in [21]).

Theorem 1. *If L is a hereditary operator and a strong symmetry of Equation (7), and if L allows an implectic–symplectic decomposition,*

$$L = \theta J, \tag{20}$$

where θ and J are implectic and symplectic operators, respectively, and if Equation (7) has a Hamiltonian structure (17), then the equation $u_{t_j} = L^j K(u)$, for $j = 0, 1, 2, \cdots$ has a multi-Hamiltonian structure

$$u_{t_j} = L^j K(u) = \theta \frac{\delta H_j}{\delta u} = \theta L^* \frac{\delta H_{j-1}}{\delta u} = \theta (L^*)^2 \frac{\delta H_{j-2}}{\delta u} = \cdots = \theta (L^*)^j \frac{\delta H_0}{\delta u}, \tag{21}$$

where all $\{\theta(L^*)^i\}$, for $i = 0, 1, \cdots, j$, are compatible Hamiltonian operators; $H_0 = H$, $H_i = (L^*)^i H_0$, all $\{H_i\}$ are involutive in the sense

$$\{H_i, H_k\}_{\theta(L^*)^s} = 0, \ \ i, k, s = 0, 1, \cdots, j,$$

and any H_i is a conserved quantity of any equation in the hierarchy $u_{t_j} = L^j K(u)$ for $j = 0, 1, 2, \cdots$.

The above notions and results are also applicable to the coupled system

$$\begin{cases} u_t = K(u), \\ v_t = S(u, v). \end{cases} \tag{22}$$

after we write it in the form $\hat{u}_t = \hat{K}(\hat{u})$, where $\hat{u} = (u^T, v^T)^T$.

3. New Hamiltonian Structures of Two Integrable Couplings

In this section, we introduce new Hamiltonian structures of the integrable couplings of the AKNS hierarchy and the KN hierarchy and investigate the involutive property of the Hamiltonians.

3.1. The AKNS

The AKNS hierarchy is associated with the well-known AKNS spectral problem [24]

$$\varphi_x = M\varphi, \;\; M(u) = \begin{pmatrix} -\eta & q \\ r & \eta \end{pmatrix}, \tag{23}$$

where $u = (q, r)^T$, $q = q(t, x), r = r(t, x)$ are potentials; η is a spectrum parameter; $\varphi = (\phi_1, \phi_2)^T$ is the eigenfunction. (23) is a generalization of the Zakharov–Shabat spectral problem for the nonlinear Schrödinger equation [25]. The AKNS hierarchy is written as (see [26,27])

$$u_{t_n} = K_n(u) = L^n K_0(u), \;\; (n = 0, 1, 2, \cdots), \;\; K_0 = (-q, r)^T, \tag{24}$$

where L is the recursion operator defined as

$$L = \sigma\partial - 2\sigma u \partial^{-1} u^T \gamma, \tag{25}$$

where $\partial = \partial_x$, ∂^{-1} denotes the inverse of ∂_x, i.e., an integration operator, satisfying $\partial^{-1}\partial = \partial\partial^{-1} = 1$,

$$\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \;\; \gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{26}$$

The first three equations in the AKNS hierarchy are

$$u_{t_0} = K_0 = \sigma u, \tag{27a}$$
$$u_{t_1} = K_1 = u_x, \tag{27b}$$
$$u_{t_2} = K_2 = \begin{pmatrix} -q_{xx} + 2q^2 r \\ r_{xx} - 2r^2 q \end{pmatrix}. \tag{27c}$$

The recursion operator L is a hereditary operator and a strong symmetry of Equation (27a) (see [27,28]). It allows an implectic–symplectic factorization [27],

$$L = \theta J, \;\; \theta = \sigma\gamma, \;\; J = \gamma\partial - 2\gamma u \partial^{-1} u^T \gamma. \tag{28}$$

Each equation in the AKNS hierarchy (24) has a multi-Hamiltonian structure as in (21), i.e.,

$$u_{t_j} = K_j(u) = \theta \frac{\delta H_j}{\delta u} = \theta L^* \frac{\delta H_{j-1}}{\delta u} = \theta (L^*)^2 \frac{\delta H_{j-2}}{\delta u} = \cdots = \theta (L^*)^j \frac{\delta H_0}{\delta u}, \tag{29}$$

where the first Hamiltonian operator is $\theta = \sigma\gamma$ and the first few gradients are

$$f_0 = (r, q)^T = \gamma u,$$
$$f_1 = L^* f_0 = (r_x, -q_x)^T = -\sigma\gamma u_x,$$
$$f_2 = L^* f_1 = \begin{pmatrix} r_{xx} - 2r^2 q \\ q_{xx} - 2q^2 r \end{pmatrix}.$$

One can compute Hamiltonians using Formula (16), the first few of which are

$$H_0 = \int_{-\infty}^{+\infty} qr \, \mathrm{d}x, \tag{30a}$$

$$H_1 = \frac{1}{2} \int_{-\infty}^{+\infty} (r_x q - q_x r) \, \mathrm{d}x, \tag{30b}$$

$$H_2 = \frac{1}{2} \int_{-\infty}^{+\infty} (r_{xx} q + q_{xx} r - 2q^2 r^2) \, \mathrm{d}x. \tag{30c}$$

For the AKNS hierarchy (24), its integrable coupling obtained by perturbation, i.e.,

$$\begin{cases} u_{t_j} = K_j(u), \\ v_{t_j} = K_j'(u)[v], \end{cases} \tag{31}$$

is related to the enlarged 4×4 spectral problem [10]

$$\hat{\varphi}_x = \hat{M}\hat{\varphi}, \;\; \hat{M}(\hat{u}) = \begin{pmatrix} M(u) & M'(u)[v] \\ 0 & M(u) \end{pmatrix}, \tag{32}$$

where

$$M'(u)[v] = \begin{pmatrix} 0 & p \\ s & 0 \end{pmatrix}, \;\; v = \begin{pmatrix} p \\ s \end{pmatrix}, \;\; \hat{u} = \begin{pmatrix} u \\ v \end{pmatrix}. \tag{33}$$

Note that the spectral problem is gauge-equivalent to the form

$$\hat{\psi}_x = \hat{M}\hat{\psi}, \;\; \hat{M} = \begin{pmatrix} -\eta I_2 & Q \\ R & \eta I_2 \end{pmatrix}, \tag{34}$$

where I_k is the $k \times k$ unit matrix, and Q and R are triangular Toeplitz matrices

$$Q = \begin{pmatrix} q & p \\ 0 & q \end{pmatrix}, \;\; R = \begin{pmatrix} r & s \\ 0 & r \end{pmatrix}. \tag{35}$$

Since Q and R commute, the integrable coupling (31) can also be alternatively and more easily obtained from (24) by replacing q and r with the above Q and R. After some calculation (see Appendix A), the integrable coupling (31) is written as

$$\hat{u}_{t_j} = \hat{K}(\hat{u}) = \hat{L}^j \hat{K}_0(\hat{u}), \;\; j = 0, 1, 2, \cdots, \tag{36}$$

where

$$\hat{K}_0(\hat{u}) = (-q, r, -p, s)^T, \tag{37}$$

and

$$\hat{L} = \begin{pmatrix} L & 0 \\ L'[v] & L \end{pmatrix}, \tag{38}$$

with $L'[v] = -2\sigma u \partial^{-1} v^T \gamma - 2\sigma v \partial^{-1} u^T \gamma$, which is the Gâteaux derivative of $L(u)$ with respect to u in the direction v. Direct verification (see Appendix B) shows that $\hat{L}$ is hereditary and is a strong symmetry for $\hat{u}_{t_0} = \hat{K}_0(\hat{u})$. The first three equations in (36) are

$$\hat{u}_{t_0} = \hat{K}_0 = \begin{pmatrix} \sigma u \\ \sigma v \end{pmatrix}, \tag{39a}$$

$$\hat{u}_{t_1} = \hat{K}_1 = \hat{u}_x, \tag{39b}$$

$$\hat{u}_{t_2} = \hat{K}_2 = \begin{pmatrix} -q_{xx} + 2q^2 r \\ r_{xx} - 2qr^2 \\ -p_{xx} + 4pqr + 2sq^2 \\ s_{xx} - 4rsq - 2r^2 p \end{pmatrix}. \tag{39c}$$

Now, we introduce

$$\theta_1 = \begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix}, \tag{40}$$

where $\theta = \sigma\gamma$ is the first Hamiltonian operator of the AKNS hierarchy. Since θ is independent of u, in light of Theorem 2.8 of [4] (see Appendix C), the integrable coupling (36) has a Hamiltonian structure

$$\hat{u}_{t_j} = \hat{K}_j(\hat{u}) = \theta_1 \frac{\delta \hat{H}_j^{(1)}}{\delta \hat{u}}. \tag{41}$$

In addition, it can be verified that $\hat{L}$ allows an implectic–symplectic factorization (see Appendix D),

$$\hat{L} = \theta_1 J_1, \;\; J_1 = \begin{pmatrix} -2\gamma u \partial^{-1} v^T \gamma - 2\gamma v \partial^{-1} u^T \gamma & \gamma\partial - 2\gamma u \partial^{-1} u^T \gamma \\ \gamma\partial - 2\gamma u \partial^{-1} u^T \gamma & 0 \end{pmatrix}, \tag{42}$$

and, consequently, Equation (36) has a multi-Hamiltonian structure

$$\hat{u}_{t_j} = \hat{K}_j(\hat{u}) = \theta_1 \frac{\delta \hat{H}_j^{(1)}}{\delta \hat{u}} = \theta_1 \hat{L}^* \frac{\delta \hat{H}_{j-1}^{(1)}}{\delta \hat{u}} = \cdots = \theta_1 (\hat{L}^*)^j \frac{\delta \hat{H}_0^{(1)}}{\delta \hat{u}}, \tag{43}$$

where

$$\frac{\delta \hat{H}_0^{(1)}}{\delta \hat{u}} = \hat{f}_0^{(1)} = \begin{pmatrix} \gamma v \\ \gamma u \end{pmatrix}. \tag{44}$$

The first few Hamiltonians are

$$\hat{H}_0^{(1)} = \int_{-\infty}^{+\infty} (sq + pr)\, \mathrm{d}x, \tag{45a}$$

$$\hat{H}_1^{(1)} = \frac{1}{2} \int_{-\infty}^{+\infty} (s_x q - q_x s + r_x p - p_x r)\, \mathrm{d}x, \tag{45b}$$

$$\hat{H}_2^{(1)} = \frac{1}{2} \int_{-\infty}^{+\infty} (s_{xx} q + q_{xx} s + r_{xx} p + p_{xx} r - 4q^2 sr - 4r^2 pq)\, \mathrm{d}x, \tag{45c}$$

from which one can see that these Hamiltonians do not reduce to (30) but vanish when $v = 0$. In this sense, they are trivial in the degeneration of $v = 0$. Such a triviality extends to all Hamiltonians defined by (43).

Proposition 1. *The Hamiltonian $\hat{H}_j^{(1)}$ vanishes when $v = 0$.*

Proof. According to Theorem 2.8 in [4], $\hat{H}_j^{(1)} = H_j(u)'[v]$, where $H_j(u)$ is the Hamiltonian given in (30). It then immediately follows that $\hat{H}_j^{(1)} = 0$ when $v = 0$ because $H_j(u)'[0] = 0$. □

In what follows, we introduce a new Hamiltonian operator for the integrable coupling (36) and the corresponding Hamiltonians will reduce to those of the AKNS equations when $v = 0$. Let us consider

$$\theta_2 = \begin{pmatrix} 0 & \theta \\ \theta & c_0\theta \end{pmatrix}, \tag{46}$$

where $\theta = \sigma\gamma$ and c_0 is a nonzero constant. Noting that

$$\theta_2 = \theta_1 \begin{pmatrix} I_2 & c_0 I_2 \\ 0 & I_2 \end{pmatrix},$$

where I_2 stands for the 2×2 identity matrix, we introduce

$$J_2 = \begin{pmatrix} I_2 & 0 \\ -c_0 I_2 & I_2 \end{pmatrix} J_1 = \begin{pmatrix} J_{21} & \gamma\partial - 2\gamma u\partial^{-1}u^T\gamma \\ \gamma\partial - 2\gamma u\partial^{-1}u^T\gamma & 0 \end{pmatrix}, \tag{47}$$

where $J_{21} = -2\gamma u\partial^{-1}v^T\gamma - 2\gamma v\partial^{-1}u^T\gamma + c_0(2\gamma u\partial^{-1}u^T\gamma - \gamma\partial)$. Thus, we obtain another factorization

$$\hat{L} = \theta_2 J_2, \tag{48}$$

Both θ_2 and J_2 are skew-symmetric. θ_2 is obviously implectic since it is independent of $\hat{u}$. In addition, for J_2, we can verify that it is symplectic along the lines of Appendix D. Thus, $\hat{L}$ allows another implectic–symplectic factorization. The first Equation (39a) has a Hamiltonian structure

$$\hat{u}_{t_0} = \hat{K}_0 = \theta_2 \frac{\delta \hat{H}_0^{(2)}}{\delta \hat{u}} \tag{49}$$

with

$$\frac{\delta \hat{H}_0^{(2)}}{\delta \hat{u}} = \hat{f}_0^{(2)} = (s - c_0 r, p - c_0 q, r, q)^T. \tag{50}$$

Thus, in light of Theorem 1, the integrable coupling (36) has a new multi-Hamiltonian structure (For Equation (7) with Hamiltonian structure (17), the operator θ_2 is also a Hamiltonian operator of the integrable coupling of the so-called "nonstandard perturbation system", where the initial equation is $u_t = K(u) + c_0\epsilon K(u)$ instead of (7). See Section 3.2 of [4])

$$\hat{u}_{t_j} = \hat{K}_j(\hat{u}) = \theta_2 \frac{\delta \hat{H}_j^{(2)}}{\delta \hat{u}} = \theta_2 \hat{L}^* \frac{\delta \hat{H}_{j-1}^{(2)}}{\delta \hat{u}} = \cdots = \theta_2 (\hat{L}^*)^j \frac{\delta \hat{H}_0^{(2)}}{\delta \hat{u}}. \tag{51}$$

Note that the Hamiltonian operator (46) with $c_0 = -1$ was also introduced in [15]. However, in this paper, we will focus more on the related multi-Hamiltonian structures and the involutive property of Hamiltonians. Using Formula (16), we can compute the Hamiltonians, of which the first few indicate relations

$$\hat{H}_i^{(2)} = \hat{H}_i^{(1)} - c_0 H_i, \ \ i = 0, 1, 2, \tag{52}$$

and they do reduce to the Hamiltonians (30) when $v = 0$ and $c_0 = -1$. Let us prove a more general result in the following.

Proposition 2. *The Hamiltonian $\hat{H}_j^{(2)}$ defined in (51) can be expressed as*

$$\hat{H}_j^{(2)} = \hat{H}_j^{(1)} - c_0 H_j, \ \ i = 0, 1, 2, \cdots, \tag{53}$$

where H_j is the Hamiltonian defined in (29). This indicates that $\hat{H}_j^{(2)}$ reduces to H_j when $v = 0$ and $c_0 = -1$.

Proof. Denote $\frac{\delta \hat{H}_j^{(2)}}{\delta \hat{u}} = f_j^{(2)}(\hat{u})$. Noting that

$$\theta_2 = \theta_1 S, \ \ S = \begin{pmatrix} I_2 & c_0 I_2 \\ 0 & I_2 \end{pmatrix}, \tag{54}$$

from which we have

$$f_j^{(1)} = S f_j^{(2)}. \tag{55}$$

Then, using the Formula (16), we have

$$\begin{aligned}
\hat{H}_j^{(2)} &= \int_0^1 \langle f_j^{(2)}(\lambda\hat{u}), \hat{u}\rangle \mathrm{d}\lambda = \int_0^1 \langle S^{-1} f_j^{(1)}(\lambda\hat{u}), \hat{u}\rangle \mathrm{d}\lambda \\
&= \int_0^1 \langle f_j^{(1)}(\lambda\hat{u}), (S^{-1})^T \hat{u}\rangle \mathrm{d}\lambda = \int_0^1 \langle f_j^{(1)}(\lambda\hat{u}), (I_4 - c_0 \begin{pmatrix} 0 & 0 \\ I_2 & 0 \end{pmatrix})\hat{u}\rangle \mathrm{d}\lambda \\
&= \int_{-\infty}^{\infty}\int_0^1 (u^T, v^T) f_j^{(1)}(\lambda\hat{u}) \mathrm{d}\lambda \mathrm{d}x - c_0 \int_{-\infty}^{\infty}\int_0^1 (0, u^T) f_j^{(1)}(\lambda\hat{u}) \mathrm{d}\lambda \mathrm{d}x \\
&= \hat{H}_j^{(1)} - c_0 \int_{-\infty}^{\infty}\int_0^1 (0, u^T) f_j^{(1)}(\lambda\hat{u}) \mathrm{d}\lambda \mathrm{d}x.
\end{aligned}$$

We are now going to prove that

$$\int_{-\infty}^{\infty}\int_0^1 (0, u^T) f_j^{(1)}(\lambda\hat{u}) \mathrm{d}\lambda \mathrm{d}x = H_j.$$

In fact, noting that $f_j^{(1)}(\hat{u}) = (\hat{L}^*)^j f_0^{(1)}(\hat{u})$, where $f_0^{(1)}(\hat{u})$ is given as (44), and $(\hat{L}^*)^j$ has the following form

$$(\hat{L}^*(\hat{u}))^j = \begin{pmatrix} (L^*(u))^j & B(\hat{u}) \\ 0 & (L^*(u))^j \end{pmatrix}, \tag{56}$$

it then follows that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_0^1 (0, u^T) f_j^{(1)}(\lambda\hat{u}) \mathrm{d}\lambda \mathrm{d}x \\
=& \int_{-\infty}^{\infty}\int_0^1 (0, u^T)(\hat{L}^*(\lambda\hat{u}))^j \begin{pmatrix} \lambda\gamma v \\ \lambda\gamma u \end{pmatrix} \mathrm{d}\lambda \mathrm{d}x \\
=& \int_{-\infty}^{\infty}\int_0^1 \lambda u^T (L^*(\lambda u))^j \gamma u \,\mathrm{d}\lambda \mathrm{d}x \\
=& H_j.
\end{aligned}$$

Thus, we obtain the expression (53), and in light of Proposition 1, one finds that $\hat{H}_j^{(2)}$ reduces to H_j when $v = 0$. The proof is completed. □

Now, for each coupled equation in the hierarchy (36), it has two simple Hamiltonian operators, θ_1 and θ_2. This means that we have more Hamiltonian structures in the integrable coupling case. Using these two Hamiltonian operators, we can define Poisson brackets $\{\cdot, \cdot\}_{\theta_k}$ for $k = 1, 2$ and investigate the involutive property of Hamiltonians $\{H_i^{(1)}\}$ and $\{H_j^{(2)}\}$ with respect to these two Poisson brackets.

Theorem 2. *The Hamiltonians $\{H_i^{(1)}\}$ and $\{H_j^{(2)}\}$ are involutive with respect to Poisson brackets $\{\cdot, \cdot\}_{\theta_k}$ for $k = 1, 2$, i.e.,*

$$\{H_i^{(l)}, H_j^{(s)}\}_{\theta_k} = 0, \;\; l, s, k \in \{1, 2\}, \;\; i, j = 0, 1, 2, \cdots. \tag{57}$$

Moreover,

$$\{H_i, H_j^{(s)}\}_{\theta_k} = 0, \;\; s, k \in \{1, 2\}, \;\; i, j = 0, 1, 2, \cdots. \tag{58}$$

Proof. The cases of $l = s = 1$ and $l = s = 2$ can be understood in light of Section 2. We prove other cases in the following. Note that

$$\theta_2 = \theta_1 S, \;\; S = \begin{pmatrix} I_2 & c_0 I_2 \\ 0 & I_2 \end{pmatrix}, \tag{59}$$

from which we have

$$f_j^{(1)} = S f_j^{(2)}. \tag{60}$$

In addition, we also note that $\theta_2 \hat{L}^* = \hat{L}\theta_2$, S and $\hat{L}^*$ commute, and $\theta_2 S$ is skew-symmetric, i.e.,

$$\theta_2 S = -(\theta_2 S)^T.$$

Making use of these relations, we have

$$\begin{aligned}\{H_i^{(2)}, H_j^{(1)}\}_{\theta_2} =& \langle f_i^{(2)}, \theta_2 f_j^{(1)}\rangle = \langle f_i^{(2)}, \theta_2 S f_j^{(2)}\rangle = -\langle \theta_2 S f_i^{(2)}, f_j^{(2)}\rangle \\ =& -\langle \theta_2 f_i^{(1)}, f_j^{(2)}\rangle = \langle f_i^{(1)}, \theta_2 f_j^{(2)}\rangle = \{H_i^{(1)}, H_j^{(2)}\}_{\theta_2}.\end{aligned}$$

If $i = j$, then we have $\{H_i^{(1)}, H_j^{(2)}\}_{\theta_2} = 0$. If $i \neq j$, e.g., $i > j$, we further have

$$\begin{aligned}\{H_i^{(2)}, H_j^{(1)}\}_{\theta_2} =& \langle f_i^{(1)}, \theta_2 f_j^{(2)}\rangle = \langle (\hat{L}^*)^i f_0^{(1)}, \theta_2 (\hat{L}^*)^j f_0^{(2)}\rangle = \langle (\hat{L}^*)^j f_0^{(1)}, \hat{L}^{i-j}\theta_2 (\hat{L}^*)^j f_0^{(2)}\rangle \\ =& \langle (\hat{L}^*)^j f_0^{(1)}, \theta_2 (\hat{L}^*)^i f_0^{(2)}\rangle = \langle f_j^{(1)}, \theta_2 f_i^{(2)}\rangle = \{H_j^{(1)}, H_i^{(2)}\}_{\theta_2} = 0.\end{aligned}$$

Since $\{\cdot,\cdot\}_{\theta_2}$ reduces to $\{\cdot,\cdot\}_{\theta_1}$ when $c_0 = 0$, we immediately have $\{H_i^{(2)}, H_j^{(1)}\}_{\theta_1} = 0$. Relation (58) is the consequence of (57) and Proposition 2. Thus we complete the proof. □

3.2. The KN

Let us look at the KN hierarchy, which is related to the KN spectral problem [29,30]

$$\varphi_x = M\varphi,\ M = \begin{pmatrix} -\eta^2 & q\eta \\ r\eta & \eta^2 \end{pmatrix}. \tag{61}$$

This spectral problem is gauge-equivalent to the one for the massive Thirring model found by Mikhailov [31]. For the details of the gauge transformation, one may refer to Appendix A of [32]. The KN soliton hierarchy is written as (see, e.g., [27])

$$u_{t_n} = K_n(u) = L^{n-1}K_1(u),\ \ (n = 1, 2, \cdots),\ K_1 = u_x,\ u = (q, r)^T, \tag{62}$$

where L is the recursion operator defined as

$$L = \partial\sigma - \partial u \partial^{-1} u^T \gamma, \tag{63}$$

in which σ and γ are defined as before. The first three equations in the hierarchy are

$$u_{t_1} = K_1 = u_x, \tag{64a}$$

$$u_{t_2} = K_2 = \begin{pmatrix} -q_x - q^2 r \\ r_x - q r^2 \end{pmatrix}_x, \tag{64b}$$

$$u_{t_3} = K_3 = \begin{pmatrix} q_{xx} + 3qrq_x + \frac{3}{2}q^3r^2 \\ r_{xx} - 3qrr_x + \frac{3}{2}q^2r^3 \end{pmatrix}_x. \tag{64c}$$

One can verify that the recursion operator L is hereditary, a strong symmetry for the first Equation (64a), and allows an implectic–symplectic factorization,

$$L = \theta J,\ \ \theta = \partial\gamma,\ \ J = \gamma\sigma - \gamma u \partial^{-1} u^T \gamma. \tag{65}$$

Thus, the KN hierarchy (62) has the multi-Hamiltonian structure

$$u_{t_j} = K_j(u) = \theta \frac{\delta H_j}{\delta u} = \theta L^* \frac{\delta H_{j-1}}{\delta u} = \theta (L^*)^2 \frac{\delta H_{j-2}}{\delta u} = \cdots = \theta (L^*)^{j-1} \frac{\delta H_1}{\delta u}, \tag{66}$$

where the first Hamiltonian operator is $\theta = \partial\gamma$ and the first few gradients are

$$f_0 = (r, q)^T = \gamma u,$$
$$f_1 = L^* f_0 = \begin{pmatrix} r_x - r^2 q \\ -q_x - q^2 r \end{pmatrix},$$
$$f_2 = L^* f_1 = \begin{pmatrix} r_{xx} - 3qrr_x + \frac{3}{2}q^2r^3 \\ q_{xx} + 3rqq_x + \frac{3}{2}r^2q^3 \end{pmatrix}.$$

The corresponding Hamiltons are

$$H_1 = \int_{-\infty}^{+\infty} qr \, \mathrm{d}x, \tag{67a}$$
$$H_2 = \frac{1}{2}\int_{-\infty}^{+\infty} (qr_x - q_x r - (qr)^2) \, \mathrm{d}x, \tag{67b}$$
$$H_3 = \frac{1}{2}\int_{-\infty}^{+\infty} (qr_{xx} + q_{xx}r - \frac{3}{2}q^2rr_x + \frac{3}{2}qr^2q_x + (qr)^3) \, \mathrm{d}x. \tag{67c}$$

Note that the Hamiltonian operator $\theta = \partial\gamma$ is again independent of u; therefore, the integrable couplings of the KN hierarchy by perturbation can have Hamiltonian structures and properties similar to the case of the AKNS. In the following, we only list these results for the integrable couplings of the KN hierarchy. Note that all the following results can be proven. Since the proofs are long and the processes are similar to those for the AKNS given in Section 3.1 and Appendices A, B and D, we do not present the details of these proofs.

The KN hierarchy (62) gives rise to integrable couplings (cf. [33])

$$\begin{cases} u_{t_j} = K_j(u), \\ v_{t_j} = K_j'(u)[v], \end{cases} \tag{68}$$

where $u = (q, r)^T, v = (p, s)^T$, which can be derived from an enlarged 4×4 spectral problem (32) where $M(u)$ is taken as in (61). It can also be obtained from the KN hierarchy (62) by replacing q and r with the triangular Toeplitz matrices Q and R as given in (35). The integrable coupling (68) can be written as

$$\hat{u}_{t_j} = \hat{K}_j(\hat{u}) = \hat{L}^{j-1}\hat{K}_1(\hat{u}), \tag{69}$$

where $\hat{u} = (u^T, v^T)^T$,

$$\hat{K}_1(\hat{u}) = \hat{u}_x, \tag{70}$$

and

$$\hat{L} = \begin{pmatrix} L & 0 \\ L'[v] & L \end{pmatrix}, \tag{71}$$

with L defined in (63) and $L'[v] = -\partial u \partial^{-1} v^T \gamma - \partial v \partial^{-1} u^T \gamma$. The first two equations in (69) are

$$\hat{u}_{t_1} = \hat{K}_1 = \hat{u}_x, \tag{72a}$$
$$\hat{u}_{t_2} = \begin{pmatrix} -q_x - q^2 r \\ r_x - qr^2 \\ -p_x - sq^2 - 2pqr \\ s_x - pr^2 - 2sqr \end{pmatrix}_x. \tag{72b}$$

The recursion operator $\hat{L}$ is hereditary, a strong symmetry of the first Equation (72a), and allows an implectic–symplectic factorization,

$$\hat{L} = \theta_1 J_1, \ \theta_1 = \begin{pmatrix} 0 & \partial\gamma \\ \partial\gamma & 0 \end{pmatrix}, \ J_1 = \begin{pmatrix} -\gamma u\partial^{-1}v^T\gamma - \gamma v\partial^{-1}u^T\gamma & \gamma\sigma - \gamma u\partial^{-1}u^T\gamma \\ \gamma\sigma - \gamma u\partial^{-1}u^T\gamma & 0 \end{pmatrix}. \tag{73}$$

As a result, the integrable coupling (69) has multi-Hamiltonian structure

$$\hat{u}_{t_j} = \hat{K}_j(\hat{u}) = \theta_1 \frac{\delta \hat{H}_j^{(1)}}{\delta \hat{u}} = \theta_1 \hat{L}^* \frac{\delta \hat{H}_{j-1}^{(1)}}{\delta \hat{u}} = \cdots = \theta_1 (\hat{L}^*)^{j-1} \frac{\delta \hat{H}_1^{(1)}}{\delta \hat{u}}, \tag{74}$$

where the simplest gradient reads

$$\frac{\delta \hat{H}_1^{(1)}}{\delta \hat{u}} = \hat{f}_1^{(1)} = \begin{pmatrix} \gamma v \\ \gamma u \end{pmatrix}. \tag{75}$$

The first few Hamiltonians are

$$\hat{H}_1^{(1)} = \int_{-\infty}^{+\infty} (sq + rp)\, \mathrm{d}x, \tag{76a}$$

$$\hat{H}_2^{(1)} = \frac{1}{2} \int_{-\infty}^{+\infty} (s_x q - q_x s + r_x p - p_x r - 2q^2 rs - 2qr^2 p)\, \mathrm{d}x, \tag{76b}$$

$$\begin{aligned} \hat{H}_3^{(1)} =& \frac{1}{2} \int_{-\infty}^{+\infty} (s_{xx} q + q_{xx} s + p_{xx} r + r_{xx} p - 6pqrr_x + \frac{3}{2}(pqr^2)_x \\ & + 6sqq_x r - \frac{3}{2}(srq^2)_x + 3pq^2r^3 + 3sq^3r^2)\, \mathrm{d}x. \end{aligned} \tag{76c}$$

Similar to the AKNS case, these Hamiltonians are trivial in the degeneration of $v = 0$ in the sense that they reduce to zero when $v = 0$. A general statement is the following.

Proposition 3. *The Hamiltonians* $\{\hat{H}_j^{(1)}\}$ *defined in (74) vanish when* $v = 0$.

The recursion operator $\hat{L}$ (71) allows another implectic–symplectic factorization,

$$\hat{L} = \theta_2 J_2, \tag{77}$$

where

$$\theta_2 = \begin{pmatrix} 0 & \partial\gamma \\ \partial\gamma & c_0\partial\gamma \end{pmatrix} = \theta_1 S,$$

$$J_2 = S^{-1} J_1 = \begin{pmatrix} J_{21} & \gamma\sigma - \gamma u\partial^{-1}u^T\gamma \\ \gamma\sigma - \gamma u\partial^{-1}u^T\gamma & 0 \end{pmatrix},$$

S is the matrix given in (54), and $J_{21} = -\gamma u\partial^{-1}v^T\gamma - \gamma v\partial^{-1}u^T\gamma + c_0(\gamma u\partial^{-1}u^T\gamma - \gamma\sigma)$. This fact leads to a second multi-Hamiltonian structure of the integrable coupling (69):

$$\hat{u}_{t_j} = \hat{K}_j(\hat{u}) = \theta_2 \frac{\delta \hat{H}_j^{(2)}}{\delta \hat{u}} = \theta_2 \hat{L}^* \frac{\delta \hat{H}_{j-1}^{(2)}}{\delta \hat{u}} = \cdots = \theta_2 (\hat{L}^*)^{j-1} \frac{\delta \hat{H}_1^{(2)}}{\delta \hat{u}}, \tag{78}$$

where the simplest gradient is

$$\frac{\delta \hat{H}_1^{(2)}}{\delta \hat{u}} = \hat{f}_1^{(2)} = (s - c_0 r, p - c_0 q, r, q)^T. \tag{79}$$

The first three Hamiltonians corresponding to θ_2 are

$$\hat{H}_i^{(2)} = \hat{H}_i^{(1)} - c_0 H_i, \ \ i = 1, 2, 3, \tag{80}$$

which can reduce to the Hamiltonians (67) when $v = 0$ and $c_0 = -1$. A general result is described as follows.

Proposition 4. *The Hamiltonian $\hat{H}_j^{(2)}$ defined in (78) can be expressed as*

$$\hat{H}_j^{(2)} = \hat{H}_j^{(1)} - c_0 H_j, \quad i = 1, 2, \cdots, \tag{81}$$

where H_j is the Hamiltonian defined in (66). $\hat{H}_j^{(2)}$ reduces to H_j when $v = 0$ and $c_0 = -1$.

Thus, the coupled equations in the hierarchy (69) have the two simplest Hamiltonian operators, θ_1 and θ_2, from which we can define two Poisson brackets $\{\cdot, \cdot\}_{\theta_k}$ for $k = 1, 2$. The involutive property of the Hamiltonians $\{H_i^{(1)}\}$ and $\{H_j^{(2)}\}$ is described as the following.

Theorem 3. *The Hamiltonians $\{H_i^{(1)}\}$ and $\{H_j^{(2)}\}$ of the KN integrable couplings are involutive with respect to the Poisson brackets $\{\cdot, \cdot\}_{\theta_k}$ for $k = 1, 2$, i.e.,*

$$\{H_i^{(l)}, H_j^{(s)}\}_{\theta_k} = 0, \quad l, s, k \in \{1, 2\}, \quad i, j = 1, 2, \cdots. \tag{82}$$

In addition, the Hamiltonians $\{H_i\}$ of the KN hierarchy (66) are involutive with $H_j^{(s)}$, i.e.,

$$\{H_i, H_j^{(s)}\}_{\theta_k} = 0, \quad s, k \in \{1, 2\}, \quad i, j = 1, 2, \cdots. \tag{83}$$

4. Concluding Remarks

In this paper, we have provided more Hamiltonian structures for two integrable couplings. For the AKNS and KN hierarchies, of which the first Hamiltonian operators (denoted by θ) are independent of u, their integrable couplings by the first-order perturbation allow Hamiltonian operators with the form (40), i.e., $\theta_1 = \begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix}$ (cf. Theorem 2.8 in [4]). We have shown that the corresponding Hamiltonians $\{\hat{H}_j^{(1)}\}$ are trivial in the degeneration of $v = 0$ (see Proposition 1 and 3). We have introduced new Hamiltonian operators of the form (46), i.e., $\theta_2 = \begin{pmatrix} 0 & \theta \\ \theta & c_0\theta \end{pmatrix}$, and proven that the corresponding Hamiltonians $\{\hat{H}_j^{(2)}\}$ allow nontrivial degeneration (see Proposition 2 and 4). The involved Hamiltonians are involutive with respect to the two Poisson brackets.

As remarks, first, in this paper, for the sake of comparison with the known Hamiltonian operator θ_1, we have introduced parameter c_0 in θ_2, cf. (40) and (46). In fact, one can also examine that, for the two integrable couplings investigated in this paper, both

$$\theta_1 = \begin{pmatrix} 0 & \theta \\ \theta & c_1\theta \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} 0 & \theta \\ \theta & c_2\theta \end{pmatrix} \tag{84}$$

with distinct parameters c_1 and c_2 can be Hamiltonian operators, and their corresponding Hamiltonians (still denoted as $\{\hat{H}_i^{(1)}\}$ and $\{\hat{H}_j^{(2)}\}$), together with the Hamiltonians $\{H_i\}$ of the original equations, are involutive with respect to the Poisson brackets $\{\cdot, \cdot\}_{\theta_k}$ for the above θ_1 and θ_2. Second, we believe that these results for the AKNS and KN hierarchies imply a more general theory for integrable couplings by perturbations. In other words, for a generic hierarchy $u_{t_j} = K_j(u) = L^j K(u)$, if the initial Equation (7) and the recursion operator L satisfy the assumption in Theorem 1, then it is possible to come up with certain settings such that its integrable couplings together with the recursion operator $\hat{L}$ with the form (38) can inherit the assumption of Theorem 1. Moreover, when the Hamiltonian operator θ is independent of u, it is possible to obtain general proof that the recursion operator $\hat{L}$ allows implectic–symplectic decomposition $\hat{L} = \theta_2 J_2$, where θ_2 takes the form (46) for arbitrary c_0. This general theory should also hold for the integrable couplings by high-order perturbations. We will explore such a general theory in a future investigation.

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Appendix A. Derivation of (36)

We start from the enlarged matrix spectral problem (32). We rewrite it as

$$\hat{\varphi}_x = \hat{M}\hat{\varphi}, \;\; \hat{M}(\hat{u}) = \begin{pmatrix} M_0 & M_1 \\ 0 & M_0 \end{pmatrix}, \tag{A1a}$$

where

$$M_0 = M(u) = \begin{pmatrix} -\eta & q \\ r & \eta \end{pmatrix}, \;\; M_1 = M'(u)[v] = \begin{pmatrix} 0 & p \\ s & 0 \end{pmatrix}.$$

The associated time part is

$$\hat{\varphi}_t = \hat{N}\hat{\varphi}, \;\; \hat{N} = \begin{pmatrix} N_0 & N_1 \\ 0 & N_0 \end{pmatrix}, \tag{A1b}$$

where

$$N_0 = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \; N_1 = \begin{pmatrix} E & F \\ G & -E \end{pmatrix}.$$

Here, $u = (q, r)^T$, $v = (p, s)^T$, $\hat{u} = (u^T, v^T)^T$, $\hat{\varphi} = (\phi_1, \phi_2, \phi_3, \phi_3)^T$ is the eigenfunction. The compatibility condition $(\hat{\varphi}_x)_t = (\hat{\varphi}_x)_t$ gives rise to the zero curvature equation

$$\hat{M}_t - \hat{N}_x + [\hat{M}, \hat{N}] = 0,$$

which is

$$\begin{cases} M_{0,t} - N_{0,x} + [M_0, N_0] = 0, & \text{(A2a)} \\ M_{1,t} - N_{1,x} + [M_0, N_1] + [M_1, N_0] = 0. & \text{(A2b)} \end{cases}$$

Note that the first Equation (A2a) is nothing but the zero curvature equation associated with the original AKNS spectral problem (23), which gives rise to the AKNS hierarchy (24), while the second one, (A2b), has a solution

$$N_1 = N_0'(u)[v]. \tag{A3}$$

In fact, taking the Gâteaux derivative of Equation (A2a) with respect to u in direction v immediately yields (A2b) with the above setting (A3). Since the zero curvature Equation (A2a) gives rise to the AKNS hierarchy (24), which can be alternatively written as

$$u_{t_{j+1}} = Lu_{t_j}, \tag{A4}$$

its Gâteaux derivative with respect to u in direction v yields

$$v_{t_{j+1}} = L'(u)[v]u_{t_j} + Lv_{t_j}. \tag{A5}$$

Thus, we have

$$\begin{pmatrix} u_{t_{j+1}} \\ u_{t_{j+1}} \end{pmatrix} = \begin{pmatrix} L(u) & 0 \\ L'(u)[v] & L(u) \end{pmatrix} \begin{pmatrix} u_{t_j} \\ u_{t_j} \end{pmatrix}, \tag{A6}$$

which yields the hierarchy (36).

One can also derive the hierarchy (36) in a direct way, cf. [34,35].

Appendix B. Property of $\hat{L}$ (38)

In the following, we prove that the operator $\hat{L}$ given in (38) is hereditary and a strong symmetry for Equation (39a).

Suppose that the vector fields $F_1, F_2, G_1, G_2 \in V_2$ and denote $F = (F_1^T, F_2^T)^T, G = (G_1^T, G_2^T)^T$. It is easy to obtain

$$\hat{L}F = \begin{pmatrix} \tilde{F}_1 \\ \tilde{F}_2 \end{pmatrix}, \ \hat{L}G = \begin{pmatrix} \tilde{G}_1 \\ \tilde{G}_2 \end{pmatrix} \tag{A7}$$

where

$$\begin{aligned} \tilde{F}_1 &= \sigma F_{1,x} - 2\sigma u \partial^{-1} u^T \gamma F_1, \\ \tilde{F}_2 &= -2\sigma u \partial^{-1} v^T \gamma F_1 - 2\sigma v \partial^{-1} u^T \gamma F_1 + \sigma F_{2,x} - 2\sigma u \partial^{-1} u^T \gamma F_2, \end{aligned}$$

and $\tilde{G}_1$ and $\tilde{G}_2$ are given by the formulae by replacing $\{F_1, F_2\}$ with $\{G_1, G_2\}$. Meanwhile, direct calculation yields

$$\hat{L}'[\hat{L}F]G - \hat{L}'[\hat{L}G]F = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \tag{A8}$$

and, on the other hand, we have

$$\hat{L}(\hat{L}'[F]G - \hat{L}'[G]F) = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \tag{A9}$$

where

$$\begin{aligned} A_1 =& -2\sigma \tilde{F}_1 \partial^{-1} u^T \gamma G_1 - 2\sigma u \partial^{-1} (\tilde{F}_1)^T \gamma G_1 + 2\sigma \tilde{G}_1 \partial^{-1} u^T \gamma F_1 + 2\sigma u \partial^{-1} (\tilde{G}_1)^T \gamma F_1, \\ B_1 =& (\sigma \partial - 2\sigma u \partial^{-1} u^T \gamma)(-2\sigma F_1 \partial^{-1} u^T \gamma G_1 + 2\sigma G_1 \partial^{-1} u^T \gamma F_1), \\ A_2 =& -2\sigma \tilde{F}_1 \partial^{-1} v^T \gamma G_1 - 2\sigma u \partial^{-1} (\tilde{F}_2)^T \gamma G_1 - 2\sigma \tilde{F}_2 \partial^{-1} u^T \gamma G_1 - 2\sigma v \partial^{-1} (\tilde{F}_1)^T \gamma G_1 \\ & - 2\sigma \tilde{F}_1 \partial^{-1} u^T \gamma G_2 - 2\sigma u \partial^{-1} (\tilde{F}_1)^T \gamma G_2 + 2\sigma \tilde{G}_1 \partial^{-1} v^T \gamma F_1 + 2\sigma u \partial^{-1} (\tilde{G}_2)^T \gamma F_1 \\ & + 2\sigma \tilde{G}_2 \partial^{-1} u^T \gamma F_1 + 2\sigma v \partial^{-1} (\tilde{G}_1)^T \gamma F_1 + 2\sigma \tilde{G}_1 \partial^{-1} u^T \gamma F_2 + 2\sigma u \partial^{-1} (\tilde{G}_1)^T \gamma F_2, \\ B_2 =& (-2\sigma u \partial^{-1} v^T \gamma - 2\sigma v \partial^{-1} u^T \gamma)(-2\sigma F_1 \partial^{-1} u^T \gamma G_1 + 2\sigma G_1 \partial^{-1} u^T \gamma F_1) \\ & + (\sigma \partial - 2\sigma u \partial^{-1} u^T \gamma)(-2\sigma F_1 \partial^{-1} v^T \gamma G_1 - 2\sigma F_2 \partial^{-1} u^T \gamma G_1 - 2\sigma F_1 \partial^{-1} u^T \gamma G_2 \\ & + 2\sigma G_1 \partial^{-1} v^T \gamma F_1 + 2\sigma G_2 \partial^{-1} u^T \gamma F_1 + 2\sigma G_1 \partial^{-1} u^T \gamma F_2). \end{aligned}$$

Substituting $\tilde{F}_j, \tilde{G}_j$ $(j = 1, 2)$ into the above, and by direct computation, one can find $A_1 = B_1$ and $A_2 = B_2$. This means that the recursion operator $\hat{L}$ satisfies

$$\hat{L}'[\hat{L}F]G - \hat{L}'[\hat{L}G]F = \hat{L}(\hat{L}'[F]G - \hat{L}'[G]F), \tag{A10}$$

i.e., $\hat{L}$ is a hereditary operator.

To check $\hat{L}$ as the strong symmetry of Equation (39a), we need to verify

$$\hat{L}'[\hat{K}_0] = [\hat{K}_0', \hat{L}]. \tag{A11}$$

Direct computation yields

$$\hat{L}'[\hat{K}_0] = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \tag{A12}$$

where

$$C_{11} = -2\sigma(\sigma u)\partial^{-1}u^T\gamma - 2\sigma u\partial^{-1}(\sigma u)^T\gamma,\ \ C_{12} = 0,$$
$$C_{21} = -2\sigma(\sigma u)\partial^{-1}v^T\gamma - 2\sigma u\partial^{-1}(\sigma v)^T\gamma - 2\sigma(\sigma v)\partial^{-1}u^T\gamma - 2\sigma v\partial^{-1}(\sigma u)^T\gamma,\ \ C_{22} = C_{11},$$

and

$$\hat{K}_0'\hat{L} - \hat{L}\hat{K}_0' = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}, \tag{A13}$$

where

$$D_{11} = -2\sigma^2 u\partial^{-1}u^T\gamma + 2\sigma u\partial^{-1}u^T\gamma\sigma,\ \ D_{12} = 0,$$
$$D_{21} = -2v\partial^{-1}u^T\gamma - 2u\partial^{-1}v^T\gamma + 2\sigma v\partial^{-1}u^T\gamma\sigma + 2\sigma u\partial^{-1}v^T\gamma\sigma,\ \ D_{22} = D_{11}.$$

It is easy to find that (A11) holds, i.e., $\hat{L}$ is a strong symmetry of Equation (39a).

Appendix C. Theorem 2.8 in [4]

In the following, we present Theorem 2.8 in [4], subject to the notations of this paper.

Theorem A1. *For Equation (7), the perturbation*

$$u \to \tilde{u} = \eta_0 + \varepsilon\eta_1 + \varepsilon^2\eta_2 + \cdots + \varepsilon^N\eta_N + o(\varepsilon^N),\ \ \eta_0 = u,\ \eta_j \in V_n$$

yields an integrable coupling

$$\hat{u}_t = \hat{K}(\hat{u}) = \left.\begin{pmatrix} K(\tilde{u}) \\ \frac{\mathrm{d}}{\mathrm{d}\varepsilon}K(\tilde{u}) \\ \vdots \\ \frac{1}{N!}\frac{\mathrm{d}^N}{\mathrm{d}\varepsilon^N}K(\tilde{u}) \end{pmatrix}\right|_{\varepsilon=0}, \tag{A14}$$

where $\hat{u} = (u^T, \eta_1^T, \cdots, \eta_N^T)^T$. If Equation (7) admits a Hamiltonian structure (17), then the system (A14) has a Hamiltonian formulation

$$\hat{u}_t = \hat{K}(\hat{u}) = \hat{\theta}\frac{\delta\hat{H}(\hat{u})}{\delta\hat{u}}, \tag{A15}$$

where

$$\hat{\theta} = \left.\begin{pmatrix} 0 & 0 & \cdots & 0 & \theta(\tilde{u}) \\ 0 & 0 & \cdots & \theta(\tilde{u}) & \frac{1}{1!}\frac{\mathrm{d}\theta(\tilde{u})}{\mathrm{d}\varepsilon} \\ \vdots & \vdots & \iddots & \iddots & \vdots \\ 0 & \theta(\tilde{u}) & \iddots & \frac{1}{(N-2)!}\frac{\mathrm{d}^{N-2}\theta(\tilde{u})}{\mathrm{d}\varepsilon^{N-2}} & \frac{1}{(N-1)!}\frac{\mathrm{d}^{N-1}\theta(\tilde{u})}{\mathrm{d}\varepsilon^{N-1}} \\ \theta(\tilde{u}) & \frac{1}{1!}\frac{\mathrm{d}\theta(\tilde{u})}{\mathrm{d}\varepsilon} & \cdots & \frac{1}{(N-1)!}\frac{\mathrm{d}^{N-1}\theta(\tilde{u})}{\mathrm{d}\varepsilon^{N-1}} & \frac{1}{N!}\frac{\mathrm{d}^N\theta(\tilde{u})}{\mathrm{d}\varepsilon^N} \end{pmatrix}\right|_{\varepsilon=0} \tag{A16}$$

and

$$\hat{H}(\hat{u}) = \left.\frac{1}{N!}\frac{\mathrm{d}^N H(\tilde{u})}{\mathrm{d}\varepsilon^N}\right|_{\varepsilon=0}. \tag{A17}$$

If Equation (7) admits a bi-Hamiltonian structure (18), then the system (A14) has a bi-Hamiltonian formulation

$$\hat{u}_t = \hat{\theta}_1\frac{\delta\hat{H}_1(\hat{u})}{\delta\hat{u}} = \hat{\theta}_2\frac{\delta\hat{H}_2(\hat{u})}{\delta\hat{u}}, \tag{A18}$$

where $\hat{\theta}_j$ and $\hat{H}_j$ are generated from θ_j and H_j along the Formulas (A16) and (A17), respectively.

Appendix D. The Implectic–Symplectic Factorization (42)

Recalling (42), we have

$$\theta_1 = \begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix}, \quad \theta = \sigma\gamma \tag{A19}$$

and

$$J_1 = \begin{pmatrix} -2\gamma u\partial^{-1}v^T\gamma - 2\gamma v\partial^{-1}u^T\gamma & \gamma\partial - 2\gamma u\partial^{-1}u^T\gamma \\ \gamma\partial - 2\gamma u\partial^{-1}u^T\gamma & 0 \end{pmatrix}. \tag{A20}$$

Both θ_1 and J_1 are skew-symmetric operators. Since θ_1 is independent of $\hat{u}$, we have $\theta_1' = 0$ and therefore it automatically satisfies the Jacobi identity (11). θ_1 is implectic according to Definition 2.

Next, we prove that J_1 satisfies (12), i.e.,

$$\langle F, J_1'[G]H\rangle + \langle G, J_1'[H]F\rangle + \langle H, J_1'[F]G\rangle = 0, \tag{A21}$$

where $F, G \in V_4$ are defined as in Appendix B and $H \in V_4$ is defined along the same lines. Direct computation yields

$$\begin{aligned}
&\langle F, J_1'[G]H\rangle \\
=&2\langle \gamma v\partial^{-1}G_1^T\gamma F_1, H_1\rangle + 2\langle \gamma u\partial^{-1}G_2^T\gamma F_1, H_1\rangle + 2\langle \gamma u\partial^{-1}G_1^T\gamma F_1, H_2\rangle \\
&- 2\langle F_1, \gamma u\partial^{-1}G_2^T\gamma H_1\rangle - 2\langle F_1, \gamma v\partial^{-1}G_1^T\gamma H_1\rangle - 2\langle F_1, \gamma u\partial^{-1}G_1^T\gamma H_2\rangle \\
&+ 2\langle \gamma u\partial^{-1}G_1^T\gamma F_2, H_1\rangle - 2\langle F_2, \gamma u\partial^{-1}G_1^T\gamma H_1\rangle.
\end{aligned}$$

Note that $J_1'[G] = J_1'(\hat{u})[G]$. Similarly, we can have expressions for $(G, J_1'[H]F)$ and $(H, J_1'[F]G)$, and finally we arrive at the Jacobi identity (A21). Thus, J_1 is a symplectic operator according to Definition 1.

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