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Singularities of Osculating Developable Surfaces of Timelike Surfaces along Curves

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Abstract: In this paper, we focus on a developable surface tangent to a timelike surface along a curve in Minkowski 3-space, which is called the *osculating developable surface* of the timelike surface along the curve. The ruling of the osculating developable surface is parallel to the *osculating Darboux vector field*. The main goal of this paper is to classify the singularities of the osculating developable surface. To this end, two new invariants of curves are defined to characterize these singularities. Meanwhile, we also research the singular properties of osculating developable surfaces near their lightlike points. Moreover, we give a relation between osculating Darboux vector fields and normal vector fields of timelike surfaces along curves from the viewpoint of Legendrian dualities. Finally, some examples with symmetrical structures are presented to illustrate the main results.

Keywords: osculating developable surfaces; Lorentzian support functions; singularities; Legendrian dualities

MSC: 57R45; 58K05

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1. Introduction

Minkowski space, which is regarded as the mathematical setting for the theory of relativity, has been studied by both physicists and differential geometers in large amounts; see, for example, [1–7]. As is known to all, there exist spacelike surfaces, timelike surfaces and lightlike surfaces in Minkowski 3-space. Timelike surfaces have a vital role in theoretical physics, which is usually called *world sheets*. In string theory, the world sheet is generated by a string, which moves through space–time. Recently, some new results concerning world sheets were obtained by physicists. For instance, Rojas introduced a covariant framework to research the stability of small perturbations on the gonihedric string model by variational techniques. A general expression of the world sheet perturbations is displayed in [8]. Singularity theory, on the other hand, which is a direct descendant of differential calculus, appeals to the research about geometry, equations and other disciplines (see [9–37]). A singularity is a point such that a function reaches a maximum/minimum or a submanifold is no longer smooth and regular. In this paper, we focus on a non-lightlike curve on a timelike surface and a developable surface tangent to the timelike surface along the curve in Minkowski 3-space. We focus on the investigation of the singularities of such a developable surface here.

Darboux frames along curves on surfaces in Euclidean 3-space are classical and famous. By using Darboux frames, Hananoi and Izumiya introduced a normal developable surface of a surface along a curve in [38]. At this point, the developable surface is orthogonal to the surface along the curve. Moreover, there exists a Lorentzian version of Darboux frames along curves on surfaces [39]. Inspired by the above work, we define a special direction in the Darboux frame at each point of the non-lightlike curve, which is directed by a vector in the tangent plane to the timelike surface. In this case, the vector field is called an osculating Darboux vector field along the non-lightlike curve. There exist three invariants with respect to the Darboux frame. Under a certain condition of these invariants, we define a ruled surface along the non-lightlike curve, which is called an osculating developable surface. The rulings of the osculating developable surface are directed by the osculating Darboux vector field. We show the relation between *normalized osculating Darboux vector fields* and normal vector fields of timelike surfaces along curves from the viewpoint of Legendrian dualities in Section 3. Moreover, the osculating developable surface is also shown as the envelope of the tangent planes of the timelike surface along the curve. By using the three invariants above, we introduce two new invariants, which are closely related to the singularities of osculating developable surfaces. In fact, one of these invariants equals zero constantly if and only if the osculating developable surface is a cylindrical surface. At this time, the non-lightlike curve is a contour generator associated with an orthogonal projection (Theorem 2, (A)). In the case that the first invariant never vanished, the other invariant equals zero constantly if and only if the osculating developable surface is a conical surface. Meanwhile, the non-lightlike curve is a contour generator associated with a central projection (Theorem 2, (B)). The concept of contour generators plays a significant role in computer vision theory [40]. By using these two invariants, we also show the classification of the singularities of the osculating developable surface (Theorem 3). Lightlike submanifolds are degenerate submanifolds and they were systematically studied in [41]. Here, we consider the singularities of the osculating developable surface near its lightlike rulings (Corollary 2). In Section 6, the geometric meaning of the second invariant is further discussed.

In Section 7, we consider curves on special timelike surfaces. Since de Sitter space is a classical model for studying Lorentzian spherical geometry and de Sitter 2-space is a timelike surface in Minkowski 3-space, we consider osculating developable surfaces of de Sitter 2-space along curves. If we consider the small spacelike circle or the great timelike hyperbolic curve, the osculating developable surface along the curve is a cylindrical surface. If we consider the great spacelike circle or the small timelike hyperbolic curve, the osculating developable surface along the curve is a conical surface (Propositions 5 and 6). In order to illustrate Theorem 6, we also display an example of a timelike curve on de Sitter 2-space so that the osculating developable surface along the curve has swallowtail singularities. At last, we consider non-lightlike curves on timelike surfaces of revolution. We show that the osculating developable surface along a timelike meridian curve is a cylinder, while the osculating developable surface along a spacelike circle is a cylinder or a cone.

We assume that all manifolds and maps are C^∞ throughout the paper, unless contrary statements are given.

2. Basic Notions

We introduce some basic notions in this section. Let $\mathbb{R}^3$ be a 3-dimensional vector space. For any two vectors $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$, the pseudo-scalar product of them is defined by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_1y_1 + x_2y_2 + x_3y_3.$$

We call the pair $(\mathbb{R}^3, \langle , \rangle)$ a *Minkowski* 3-*space* and denote it as $\mathbb{R}^3_1$.

For any two vectors $\boldsymbol{x} = (x_1, x_2, x_3)$, $\boldsymbol{y} = (y_1, y_2, y_3) \in \mathbb{R}^3_1$, we obtain a vector $\boldsymbol{x} \wedge \boldsymbol{y}$ that is defined by

$$\boldsymbol{x} \wedge \boldsymbol{y} = \begin{vmatrix} -\boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix},$$

where $\{\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}\}$ is the canonical basis of $\mathbb{R}^3_1$. We say that a non-zero vector $\boldsymbol{x} \in \mathbb{R}^3_1$ is *timelike*, *spacelike* or *lightlike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$, respectively. The norm of $\boldsymbol{x}$ is defined by $\|\boldsymbol{x}\| = (\mathrm{sign}(\boldsymbol{x})\langle \boldsymbol{x}, \boldsymbol{x} \rangle)^{1/2}$, in which $\mathrm{sign}(\boldsymbol{x})$ denotes the signature of $\boldsymbol{x}$, which is given by $\mathrm{sign}(\boldsymbol{x}) = -1, 0,$ or 1 when $\boldsymbol{x}$ is timelike, lightlike or spacelike, respectively. Moreover, for a vector $\boldsymbol{v} \in \mathbb{R}^3_1$ and a real number $c \in \mathbb{R}$, we define a plane whose normal vector is $\boldsymbol{v}$ as $HP(\boldsymbol{v},c) = \{\boldsymbol{x} \in \mathbb{R}^3_1 | \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c\}$. Then, we call $HP(\boldsymbol{v},c)$ a *spacelike plane*, a *timelike plane* or a *lightlike plane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike, respectively.

Let $\gamma : I \to \mathbb{R}^3_1$ be a regular curve (i.e., $\dot{\gamma}(t) = d\gamma/dt \neq 0$), where I is an open interval. For any $t \in I$, the curve γ is called *timelike, lightlike or spacelike* if $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle < 0$, $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle = 0$ or $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle > 0$, respectively. We say that γ is a *non-lightlike curve* if γ is a spacelike curve or a timelike curve. On the other hand, the arc-length of a non-lightlike curve γ measured from $\gamma(t_0)(t_0 \in I)$ is $s(t) = \int_{t_0}^{t} \|\dot{\gamma}(t)\| dt$. It is obvious that the parameter s is determined such that $\|\gamma'(s)\|$=1 for a non-lightlike curve. Then, $\gamma'(s) = d\gamma/ds$ is called the unit tangent vector of γ at s. We now define the *hyperbolic space* by

$$H^2(-1) = \{\boldsymbol{x} \in \mathbb{R}^3_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\},$$

the *de Sitter 2-space* by

$$S^2_1 = \{\boldsymbol{x} \in \mathbb{R}^3_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\},$$

and the *close lightcone* by

$$LC = \{\boldsymbol{x} \in \mathbb{R}^3_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}.$$

We set a timelike embedding $X : U \to \mathbb{R}^3_1$ from an open subset $U \subset \mathbb{R}^2$. We denote $M = X(U)$ and identify M and U according to the embedding X. Then, we say that X is a *timelike embedding* if its tangent space T_pM is a timelike plane at any point $p = X(u)$. Moreover, let $\overline{\gamma} : I \to U$ be a regular curve. Then, another curve $\gamma : I \to M \subset \mathbb{R}^3_1$ is defined by $\gamma(t) = X(\overline{\gamma}(t))$. At this time, we say that γ is a curve on the timelike surface M.

In this paper, we consider γ as a non-lightlike curve; then, we can reparametrize it by the arc-length s. Therefore, we can obtain the unit tangent vector $\boldsymbol{t}(s) = \gamma'(s)$ of $\gamma(s)$. Taking into consideration that X is a timelike embeding, then we can acquire a spacelike normal vector field $\boldsymbol{n}_\gamma$ along γ. Therefore, we construct a vector $\boldsymbol{b}(s) = \boldsymbol{n}_\gamma(s) \wedge \boldsymbol{t}(s)$. Thus, we have a pseudo-orthonormal frame $\{\boldsymbol{t}(s), \boldsymbol{n}_\gamma(s), \boldsymbol{b}(s)\}$ along γ. Moreover, we also have the following Frenet–Serret-type formulae:

$$\begin{cases} \boldsymbol{t}'(s) = \kappa_n(s)\boldsymbol{n}_\gamma(s) - \delta(s)\kappa_g(s)\boldsymbol{b}(s), \\ \boldsymbol{n}'_\gamma(s) = -\delta(s)\kappa_n(s)\boldsymbol{t}(s) + \delta(s)\tau_g(s)\boldsymbol{b}(s), \\ \boldsymbol{b}'(s) = -\delta(s)\kappa_g(s)\boldsymbol{t}(s) + \tau_g(s)\boldsymbol{n}_\gamma(s), \end{cases}$$

where $\delta(s) = \mathrm{sign}(\boldsymbol{t}(s))$, $\kappa_n(s) = \langle \boldsymbol{t}'(s), \boldsymbol{n}_\gamma(s) \rangle$, $\kappa_g(s) = \langle \boldsymbol{t}'(s), \boldsymbol{b}(s) \rangle$ and $\tau_g(s) = \langle \boldsymbol{n}_\gamma(s), \boldsymbol{b}'(s) \rangle$. We say that $\kappa_n(s)$ is the *normal curvature*, $\kappa_g(s)$ is the *geodesic curvature* and $\tau_g(s)$ is the *geodesic torsion* of γ, respectively. Meanwhile, we say that
(1) γ is an asymptotic curve of M if and only if $\kappa_n = 0$,
(2) γ is a geodesic curve of M if and only if $\kappa_g = 0$,
(3) γ is a principal curve of M if and only if $\tau_g = 0$.

In addition, a vector field $\boldsymbol{D}(s)$ along γ, which is defined by

$$\boldsymbol{D}(s) = \tau_g(s)\boldsymbol{t}(s) - \kappa_n(s)\boldsymbol{b}(s)$$

is called an *osculating Darboux vector* along γ. If $\tau_g^2 + \kappa_n^2 \neq 0$ and $\tau_g^2 \neq \kappa_n^2$, we have the following expressions of *normalized osculating Darboux vector fields*:

$$\boldsymbol{D}_s(s) = \frac{\tau_g(s)\boldsymbol{t}(s) - \kappa_n(s)\boldsymbol{b}(s)}{\sqrt{\delta(s)(\tau_g^2(s) - \kappa_n^2(s))}} \text{ if } \delta(s)\tau_g^2(s) > \delta(s)\kappa_n^2(s),$$

$$\boldsymbol{D}_t(s) = \frac{\tau_g(s)\boldsymbol{t}(s) - \kappa_n(s)\boldsymbol{b}(s)}{\sqrt{\delta(s)(\kappa_n^2(s) - \tau_g^2(s))}} \text{ if } \delta(s)\tau_g^2(s) < \delta(s)\kappa_n^2(s).$$

On the other hand, we list some basic notions and important properties of ruled surfaces and developable surfaces here. Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3_1$ and $\boldsymbol{\rho} : I \to \mathbb{R}^3_1 \backslash \{\mathbf{0}\}$ be C^∞-mappings. Then, we define a mapping $F(\gamma, \rho) : I \times \mathbb{R}^3_1 \to \mathbb{R}^3_1$ by

$$F_{(\gamma,\rho)}(u, v) = \boldsymbol{\gamma}(u) + v\boldsymbol{\rho}(u),$$

which is called a *ruled surface* in $\mathbb{R}^3_1$. At the same time, we call $\boldsymbol{\gamma}$ a *base curve* and $\boldsymbol{\rho}$ a *director curve*. Moreover, the straight line $\boldsymbol{\gamma}(u) + v\boldsymbol{\rho}(u)$ defined for a fixed $u \in I$ is called a *ruling*. Moreover, we know that the ruled surface $F_{(\gamma,\rho)}$ is developable if and only if

$$\langle \dot{\boldsymbol{\gamma}}(u), (\boldsymbol{\rho}(u) \wedge \dot{\boldsymbol{\rho}}(u)) \rangle = 0.$$

If the director curve $\boldsymbol{\rho}$ satisfies $\boldsymbol{\rho}(u) \wedge \dot{\boldsymbol{\rho}}(u) = \mathbf{0}$, then we say that $F_{(\gamma,\rho)}$ is a *cylinder*. If the singularity of the developable surface $F_{(\gamma,\rho)}$ is a constant, then we say $F_{(\gamma,\rho)}$ is a *cone*.

Finally, we recall relevant notions of contour generators, briefly. Let $S \subset \mathbb{R}^3_1$ be a surface and $\boldsymbol{n}$ be the unit normal vector field. Then, for a fixed vector $\boldsymbol{d} \in \mathbb{R}^3_1$, the *contour generator* of the orthogonal projection with respect to the direction $\boldsymbol{d}$ is defined by

$$\{\omega \in S | \langle \boldsymbol{n}(\omega), \boldsymbol{d} \rangle = 0\}.$$

Actually, the set above is the singular set of the orthogonal projection with respect to the direction $\boldsymbol{d}$. Furthermore, for a fixed point $\boldsymbol{c} \in \mathbb{R}^3_1$, the definition of the *contour generator* of the *central projection* with the center $\boldsymbol{c}$ is given by

$$\{\omega \in S | \langle \boldsymbol{n}(\omega), \omega - \boldsymbol{c} \rangle = 0\}.$$

It can be found that the set is the singular set of the central projection with the center $\boldsymbol{c}$. The concept of contour generators plays a significant role in computer vision theory [40].

3. Legendrian Dualities

In this section, we recall some properties of Legendrian submanifolds and contact manifolds [36].

Let M be a $(2m+1)$-dimensional smooth manifold and W be a tangent hyperplane field on M. Such a field is defined as the field of zeros of a 1-form $\wp$ locally. We say that the tangent hyperplane field W is *non-degenerate* if $\wp \wedge (d\wp)^m \neq 0$ at any point of M. Then, we say that the pair (M, W) is a *contact manifold* if W is a non-degenerate hyperplane field. In this case, $\wp$ and W are called the *contact form* and the *contact structure*, respectively. Suppose: $M \to M'$ is a diffeomorphism between contact manifolds (M, W) and (M', W'). Then, is called a *contact diffeomorphism* if $d(W) = W'$. Meanwhile, contact manifolds (M, W) and (M', W') are *contact diffeomorphic* if there exists the contact diffeomorphism: $M \to M'$. Moreover, a submanifold $i : L \subset M$ of a contact manifold (M, W) is *Legendrian* if $dim\ L = m$ and $di_x(T_xL) \subset W_{i(x)}$ hold at any $x \in L$. In addition, the mapping i is called an *isotropic mapping* if $di_x(T_xL) \subset W_{i(x)}$ at any $x \in L$. A smooth fiber bundle $\pi : E \to N$ is a *Legendrian fibration* if its total space E is provided with a contact structure and its fibers are Legendrian submanifolds. Suppose $\pi : E \to N$ to be a Legendrian fibration. Then, for a Legendrian submanifold $i : L \subset E$, the map $\pi \circ i : L \to N$ is called a *Legendrian map*. Meanwhile, the image of a Legendrian map $\pi \circ i$ is called a *wavefront set* of i. For any $y \in E$, as is known to all, there exists a local coordinate system $(x_1, \ldots, x_m, y_1, \ldots, y_m, z)$ near y such that

$$\pi(x_1, \ldots, x_m, y_1, \ldots, y_m, z) = (x_1, \ldots, x_m, z).$$

Simultaneously, the contact structure is given by the 1-form

$$\wp = dz - \sum_{i=1}^{m} y_i dx_i.$$

In [42], the Legendrian dualities between pseudo-spheres in Minkowski space are introduced, which become basic tools for studying submanifolds in pseudo-spheres. Firstly, we define 1-forms $\langle d\boldsymbol{v}, \boldsymbol{w}\rangle = -w_1 dv_1 + \sum_{i=2}^{3} w_i dv_i$, $\langle \boldsymbol{v}, d\boldsymbol{w}\rangle = -v_1 dw_1 + \sum_{i=2}^{3} v_i dw_i$ in $\mathbb{R}^3_1 \times \mathbb{R}^3_1$. Then, we consider the following:

(1) (a) $H^2(-1) \times S^2_1 \supset \Delta_1 = \{(\boldsymbol{v}, \boldsymbol{w}) | \langle \boldsymbol{v}, \boldsymbol{w}\rangle = 0\}$,
(b) $\pi_{11} : \Delta_1 \to H^2(-1)$, $\pi_{12} : \Delta_1 \to S^2_1$,
(c) $\theta_{11} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle | \Delta_1$, $\theta_{12} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle | \Delta_1$.

(2) (a) $S^2_1 \times S^2_1 \supset \Delta_5 = \{(\boldsymbol{v}, \boldsymbol{w}) | \langle \boldsymbol{v}, \boldsymbol{w}\rangle = 0\}$,
(b) $\pi_{51} : \Delta_5 \to S^2_1$, $\pi_{52} : \Delta_5 \to S^2_1$,
(c) $\theta_{51} = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle | \Delta_5$, $\theta_{52} = \langle \boldsymbol{v}, d\boldsymbol{w}\rangle | \Delta_5$.

Here, $\pi_{i1}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{v}$, $\pi_{i2}(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{w}$. Moreover, we remark that $\theta^{-1}_{i1}(0)$ and $\theta^{-1}_{i2}(0)$ define the same tangent hyperplane field over Δ_i, which are denoted by W_i $(i = 1, 5)$. It has been shown that (Δ_i, W_i) is a contact manifold and π_{ij} $(j = 1, 2)$ are Legendrian fibrations. Then, if $(\boldsymbol{v}, \boldsymbol{w}) \subset (\Delta_i, K_i)$, we say that $\boldsymbol{v}$ is Δ_i-*dual* to $\boldsymbol{w}$. Details of Legendrian fibrations can be found in [43]. Then, we have the following duality theorem.

Theorem 1. *Let $\boldsymbol{\gamma} : I \to M \subset \mathbb{R}^3_1$ be a non-lightlike curve with $\tau_g^2 + \kappa_n^2 \neq 0$ and $\tau_g^2 \neq \kappa_n^2$.*

(1) If $\delta\tau_g^2 > \delta\kappa_n^2$, then $\boldsymbol{D}_s(s)$ is a Δ_5-dual of $\boldsymbol{n}_\gamma(s)$.
(2) If $\delta\tau_g^2 < \delta\kappa_n^2$, then $\boldsymbol{D}_t(s)$ is a Δ_1-dual of $\boldsymbol{n}_\gamma(s)$.

Proof. We define a mapping $\mathcal{L}_5 : I \to \Delta_5$ by $\mathcal{L}_5(s) = (\boldsymbol{n}_\gamma(s), \boldsymbol{D}_s(s))$. Then, we have $\langle \boldsymbol{n}_\gamma(s), \boldsymbol{D}_s(s)\rangle = 0$ and $\mathcal{L}_5^* \theta_{51} = \langle \boldsymbol{n}'_\gamma(s), \boldsymbol{D}_s(s)\rangle = 0$. Thus, $\mathcal{L}_5$ is an isotropic mapping, so that $\boldsymbol{D}_s(s)$ is a Δ_5-dual of $\boldsymbol{n}_\gamma(s)$. We define another mapping

$$\mathcal{L}_1 : I \to \Delta_1;\ \mathcal{L}_1(s) = (\boldsymbol{n}_\gamma(s), \boldsymbol{D}_t(s)).$$

Then, we can also show that $\mathcal{L}_1$ is an isotropic mapping. This means that (2) holds. □

4. Osculating Developable Surfaces

We investigate a special surface of a given timelike surface M along a non-lightlike curve in this section.

For a non-lightlike curve $\boldsymbol{\gamma} : I \to M \subset \mathbb{R}^3_1$ with $\tau_g^2 + \kappa_n^2 \neq 0$, we define a mapping $\boldsymbol{OD} : I \times \mathbb{R} \to \mathbb{R}^3_1$ as

$$\boldsymbol{OD}(s, u) = \boldsymbol{\gamma}(s) + u\boldsymbol{D}(s) = \boldsymbol{\gamma}(s) + u(\tau_g(s)\boldsymbol{t}(s) - \kappa_n(s)\boldsymbol{b}(s)).$$

This is a ruled surface. Then, we have

$$\boldsymbol{D}' = (\tau_g' + \delta\kappa_n\kappa_g)\boldsymbol{t} - (\delta\kappa_g\kappa_g + \kappa_n')\boldsymbol{b},$$

Thus, we obtain

$$\begin{aligned}\langle \boldsymbol{\gamma}', \boldsymbol{D} \wedge \boldsymbol{D}'\rangle =& \det(\boldsymbol{t}, \tau_g\boldsymbol{t} - \kappa_n\boldsymbol{b}, (\tau_g' + \delta\kappa_n\kappa_g)\boldsymbol{t} - (\delta\kappa_g\kappa_g + \kappa_n')\boldsymbol{b}) \\ =& 0.\end{aligned}$$

This means that $\boldsymbol{OD}$ is a developable surface. In this case, we call $\boldsymbol{OD}$ an *osculating developable surface* of M along $\boldsymbol{\gamma}$. Moreover, we show two invariants $\varepsilon(s)$, $\sigma(s)$ of $\boldsymbol{\gamma}$ as follows:

$$\begin{aligned}\varepsilon(s) =&\delta(s)\big(\tau_g^2(s)-\kappa_n^2(s)\big)\kappa_g(s)+\kappa_n'(s)\tau_g(s)-\kappa_n(s)\tau_g'(s),\\ \sigma(s) =&\Big(\frac{\kappa_n(s)}{\varepsilon(s)}\Big)'+\frac{\delta(s)\kappa_g(s)\tau_g(s)}{\varepsilon(s)}, \quad \text{when } \varepsilon(s)\neq 0.\end{aligned}$$

On the other hand, by calculation, we obtain $\boldsymbol{D}\wedge\boldsymbol{D}'=\boldsymbol{0}$ if and only if

$$\tau_g(\delta\tau_g\kappa_g+\kappa_n')=\kappa_n(\tau_g'+\delta\kappa_n\kappa_g),$$

which is equivalent to $\varepsilon(s)=0$. We also calculate that

$$\begin{aligned}\frac{\partial \boldsymbol{OD}}{\partial u}\wedge\frac{\partial \boldsymbol{OD}}{\partial s} =&\tau_g(u\delta\tau_g\kappa_g+u\kappa_n')\boldsymbol{n}_\gamma-\kappa_n(1+u\tau_g'+u\delta\kappa_n\kappa_g)\boldsymbol{n}_\gamma\\ =&(u\varepsilon-\kappa_n)\boldsymbol{n}_\gamma.\end{aligned}$$

Therefore, $(s_0,u_0)\in I\times\mathbb{R}$ is a singular point of $\boldsymbol{OD}$ if and only if $\varepsilon(s_0)\neq 0$ and $u_0=\frac{\kappa_n(s_0)}{\varepsilon(s_0)}$. If $(s_0,0)$ is a regular point (namely, $\kappa_n(s_0)\neq 0$), then the normal vector of $\boldsymbol{OD}$ at $\boldsymbol{OD}(s_0,0)=\boldsymbol{\gamma}(s_0)$ has the same direction of the normal vector of M at $\boldsymbol{\gamma}(s_0)$. Therefore, it is reasonable that we call $\boldsymbol{OD}$ the osculating developable surface of M along $\boldsymbol{\gamma}$. On the other hand, we use these two invariants to characterize the contour generators of M as the following.

Theorem 2. *Let $\boldsymbol{\gamma}: I\to M\subset\mathbb{R}^3_1$ be a non-lightlike curve with $\tau_g^2+\kappa_n^2\neq 0$. Then, we have the following:*

(A) The following are equivalent:
- *(1) $\boldsymbol{OD}$ is a cylinder,*
- *(2) $\varepsilon(s)\equiv 0$,*
- *(3) $\boldsymbol{\gamma}$ is a non-lightlike contour generator with respect to an orthogonal projection.*

(B) If $\varepsilon(s)\neq 0$, then the following are equivalent:
- *(1) $\boldsymbol{OD}$ is a cone,*
- *(2) $\sigma(s)\equiv 0$,*
- *(3) $\boldsymbol{\gamma}$ is a non-lightlike contour generator with respect to a central projection.*

Proof. (A) By definition, $\boldsymbol{OD}$ is a cylinder if and only if $\boldsymbol{D}\wedge\boldsymbol{D}'=\boldsymbol{0}$. Because $\boldsymbol{D}\wedge\boldsymbol{D}'=\boldsymbol{0}$ if and only if $\varepsilon(s)\equiv 0$, it means that (1) is equivalent to (2). Suppose that (3) holds; there exists a vector $\boldsymbol{d}\in\mathbb{R}^3_1$ such that $\langle\boldsymbol{n}_\gamma(s),\boldsymbol{d}\rangle\equiv 0$. Then, $\boldsymbol{d}=\lambda\boldsymbol{t}(s)+\mu\boldsymbol{b}(s)$ for some real numbers λ,μ. Since $\langle\boldsymbol{n}_\gamma'(s),\boldsymbol{d}\rangle\equiv 0$, we have $-\lambda\kappa_n(s)-\mu\tau_g(s)=0$, so that $\boldsymbol{D}(s)$ is parallel to $\boldsymbol{d}$. Condition (1) holds. It is obvious that (1) implies (3).

(B) If the condition (1) is satisfied, then the singular value set of $\boldsymbol{OD}$ is a point. We consider the following vector-valued function $\boldsymbol{f}(s)$ defined by

$$\boldsymbol{f}(s)=\boldsymbol{\gamma}(s)+\frac{\kappa_n(s)}{\varepsilon(s)}\boldsymbol{D}(s).$$

Therefore, if the condition (1) holds, it is equivalent to saying that the condition $\boldsymbol{f}'(s) \equiv \mathbf{0}$ holds. By a straightforward calculation, we obtain

$$\begin{aligned}\boldsymbol{f}'(s) =& \boldsymbol{t} + (\frac{\kappa_n}{\varepsilon})'(\tau_g \boldsymbol{t} - \kappa_n \boldsymbol{b}) + \frac{\kappa_n}{\varepsilon}[(\tau_g' + \delta\kappa_n\kappa_g)\boldsymbol{t} - (\delta\tau_g\kappa_g + \kappa_n')\boldsymbol{b}] \\ =& [(\frac{\kappa_n}{\varepsilon})'\tau_g + \frac{\delta\tau_g^2\kappa_g + \kappa_n'\tau_g}{\varepsilon}]\boldsymbol{t} - [(\frac{\kappa_n}{\varepsilon})'\kappa_n + \frac{\delta\tau_g\kappa_n\kappa_g + \kappa_n'\kappa_n}{\varepsilon}]\boldsymbol{b} \\ =& [(\frac{\kappa_n}{\varepsilon})' + \frac{\delta\kappa_g\tau_g}{\varepsilon}](\tau_g \boldsymbol{t} - \kappa_n \boldsymbol{b}).\end{aligned}$$

This means that the conditions (1) and (2) are equivalent. According to the definition of the contour generator with respect to a central projection, condition (3) implies that there exists $\boldsymbol{c} \in \mathbb{R}^3_1$ such that $\langle \boldsymbol{\gamma}(s) - \boldsymbol{c}, \boldsymbol{n}_\gamma(s)\rangle \equiv 0$. If condition (1) holds, then we know that $\boldsymbol{f}(s)$ is constant. Therefore, for the constant vector $\boldsymbol{c} = \boldsymbol{f}(s) \in \mathbb{R}^3_1$, we have

$$\begin{aligned}\langle \boldsymbol{\gamma}(s) - \boldsymbol{c}, \boldsymbol{n}_\gamma(s)\rangle =& \langle \boldsymbol{\gamma}(s) - \boldsymbol{f}(s), \boldsymbol{n}_\gamma(s)\rangle \\ =& \langle -\frac{\kappa_n(s)}{\varepsilon(s)}\boldsymbol{D}(s), \boldsymbol{n}_\gamma(s)\rangle \\ =& 0.\end{aligned}$$

This means that condition (3) is satisfied. Conversely, by condition (3), there exists a constant vector $\boldsymbol{c} \in \mathbb{R}^3_1$ such that $\langle \boldsymbol{\gamma}(s) - \boldsymbol{c}, \boldsymbol{n}_\gamma(s)\rangle = 0$. By taking the derivative at both sides, we have

$$\langle \boldsymbol{\gamma}(s) - \boldsymbol{c}, \boldsymbol{n}_\gamma(s)\rangle' = \langle \boldsymbol{\gamma}(s) - \boldsymbol{c}, -\delta(s)\kappa_n(s)\boldsymbol{t}(s) + \delta(s)\tau_g(s)\boldsymbol{b}(s)\rangle = 0.$$

Then, there exists $\lambda \in \mathbb{R}$ such that $\boldsymbol{\gamma}(s) - \boldsymbol{c} = \lambda\boldsymbol{D}(s)$. By taking the derivative again, we obtain

$$\begin{aligned}\langle \boldsymbol{\gamma} - \boldsymbol{c}, \boldsymbol{n}_\gamma\rangle'' =& \langle \boldsymbol{t}, -\delta\kappa_n \boldsymbol{t} + \delta\tau_g \boldsymbol{b}\rangle + \langle \boldsymbol{\gamma} - \boldsymbol{c}, (-\delta\kappa_n \boldsymbol{t} + \delta\tau_g \boldsymbol{b})'\rangle \\ =& \kappa_n + \lambda\varepsilon = 0.\end{aligned}$$

Then, we obtain

$$\boldsymbol{f}(s) = \boldsymbol{\gamma}(s) + \frac{\kappa_n(s)}{\varepsilon(s)}\boldsymbol{D}(s) = \boldsymbol{\gamma}(s) - \lambda\boldsymbol{D}(s) = \boldsymbol{c}.$$

Hence, $\boldsymbol{f}(s)$ is constant; namely, condition (1) holds. □

Corollary 1. *The osculating developable surface* $\boldsymbol{OD}$ *is non-cylindrical if and only if* $\varepsilon(s) \neq 0$.

According to the conclusions in Theorem 2, the invariants $\varepsilon(s)$ and $\sigma(s)$ might be closely related to the singularities of osculating developable surfaces. In fact, by using these two invariants, we can obtain the classification for the singularities of osculating developable surfaces of M along non-lightlike curves. The main result of this paper is as follows.

Theorem 3. *Let* $\boldsymbol{\gamma} : I \to M \subset \mathbb{R}^3_1$ *be a non-lightlike curve with* $\tau_g^2 + \kappa_n^2 \neq 0$. *Then, we have the following.*

(A) *The osculating developable surface* $\boldsymbol{OD}$ *of* M *along non-lightlike curve* $\boldsymbol{\gamma}$ *is not singular at* (s_0, u_0) *if and only if* $u_0\varepsilon(s_0) - \kappa_n(s_0) \neq 0$.

(B) *The osculating developable surface* $\boldsymbol{OD}$ *of* M *along non-lightlike curve* $\boldsymbol{\gamma}$ *is locally diffeomorphic to the cuspidal edge* $C \times \mathbb{R}$ *at* (s_0, u_0) *if*

(i) $\varepsilon(s_0) \neq 0$, $\sigma(s_0) \neq 0$ *and* $u_0 = \frac{\kappa_n(s_0)}{\varepsilon(s_0)}$, *or*

(ii) $\varepsilon(s_0) = \kappa_n(s_0) = 0$, $\varepsilon'(s_0) \neq 0$ *and*

$$u_0 \neq \frac{\kappa_n'(s_0)}{-\kappa_n''(s_0) + 2\delta(s_0)\kappa_g(s_0)\tau_g'(s_0) + \delta(s_0)\tau_g(s_0)\kappa_g'(s_0)},$$

or

(iii) $\varepsilon(s_0) = \varepsilon'(s_0) = \kappa_n(s_0) = 0$ *and* $\kappa_n'(s_0) \neq 0$.

(C) The osculating developable surface ***OD*** *of M along non-lightlike curve* γ *is locally diffeomorphic to the swallowtail SW at* (s_0, u_0) *if* $\varepsilon(s_0) \neq 0$, $\sigma(s_0) = 0$, $\sigma'(s_0) \neq 0$ *and* $u_0 = \frac{\kappa_n(s_0)}{\varepsilon(s_0)}$.

Here, $C \times \mathbb{R} = \{(x_1, x_2, x_3) | x_1^2 = x_2^3\} \times \mathbb{R}$ *is the cuspidal edge (see Figure 1).* $SW = \{(x_1, x_2, x_3) | x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$ *is the swallowtail (see Figure 2).*

Let $\gamma : I \to M \subset \mathbb{R}_1^3$ be a non-lightlike curve. If $\tau_g^2(s_0) = \kappa_n^2(s_0) \neq 0$, then $\boldsymbol{D}(s_0) = \tau_g(s_0)\boldsymbol{t}(s_0) - \kappa_n(s_0)\boldsymbol{b}(s_0)$ is a lightlike vector, and we call $\gamma(s_0)$ a *lightlike point* of ***OD***. If $\gamma : I \to M \subset \mathbb{R}_1^3$ is a non-lightlike curve with $\tau_g^2(s) \equiv \kappa_n^2(s) \neq 0$, then $\boldsymbol{D}(s)$ along $\gamma(s)$ are lightlike vectors. In this case, we say that ***OD*** is a *lightlike osculating developable surface* of M along γ. Then, we have the following corollary.

Corollary 2. *Let* $\gamma : I \to M \subset \mathbb{R}_1^3$ *be a non-lightlike curve with* $\tau_g^2 + \kappa_n^2 \neq 0$. *Then, we have the following:*
(1) If $\tau_g^2(s) \equiv \kappa_n^2(s)$, *the lightlike osculating developable surface* ***OD*** *of M along* γ *has no singular points.*
(2) If ***OD*** *is not a lightlike osculating developable surface of M along* γ *and* $\gamma(s_0)$ *is a lightlike point of* ***OD***, *then the osculating developable surface* ***OD*** *is locally diffeomorphic to the cuspidal edge* $C \times \mathbb{R}$ *at* (s_0, u_0) *if* $\varepsilon(s_0) = \kappa_n'(s_0)\tau_g(s_0) - \kappa_n(s_0)\tau_g'(s_0) \neq 0$, $\sigma(s_0) \neq 0$ *and* $u_0 = \frac{\kappa_n(s_0)}{\kappa_n'(s_0)\tau_g(s_0) - \kappa_n(s_0)\tau_g'(s_0)}$. *The osculating developable surface* ***OD*** *of M along non-lightlike curve* γ *is locally diffeomorphic to the swallowtail SW at* (s_0, u_0) *if* $\varepsilon(s_0) = \kappa_n'(s_0)\tau_g(s_0) - \kappa_n(s_0)\tau_g'(s_0) \neq 0$, $\sigma(s_0) = 0$, $\sigma'(s_0) \neq 0$ *and* $u_0 = \frac{\kappa_n(s_0)}{\kappa_n'(s_0)\tau_g(s_0) - \kappa_n(s_0)\tau_g'(s_0)}$.

Figure 1. Cuspidal edge.

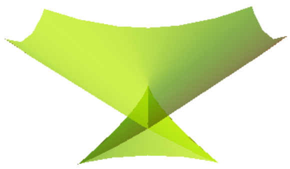

Figure 2. Swallowtail.

5. Lorentzian Support Functions

5.1. Unfoldings of Lorentzian Support Functions

We show a family of functions on a non-lightlike curve, which will be useful for studying invariants of curves on timelike surfaces in this section. Let $\gamma : I \to M \subset \mathbb{R}_1^3$ be a non-lightlike curve. Then, we define a function $G : I \times \mathbb{R}_1^3 \to \mathbb{R}$ by $G(s, \boldsymbol{x}) = \langle \boldsymbol{x} - \gamma(s), \boldsymbol{n}_\gamma(s) \rangle$. Here, G is called a *Lorentzian support function* on γ with respect to $\boldsymbol{n}_\gamma$. We denote $g_{x_0}(s) = G(s, \boldsymbol{x}_0)$ for any $\boldsymbol{x}_0 \in \mathbb{R}_1^3$. Then, we have the following proposition.

Proposition 1. *Let $\gamma : I \to M \subset \mathbb{R}^3_1$ be a non-lightlike curve with $\tau_g^2 + \kappa_n^2 \neq 0$. Then, we have*

(1) $g_{x_0}(s_0) = 0$ *if and only if there exist $\mu, \nu \in \mathbb{R}$ such that $\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = \mu \boldsymbol{t}(s_0) + \nu \boldsymbol{b}(s_0)$.*

(2) $g_{x_0}(s_0) = g'_{x_0}(s_0) = 0$ *if and only if there exists $\mu \in \mathbb{R}$ such that $\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = \mu(\tau_g(s_0)\boldsymbol{t}(s_0) - \kappa_n(s_0)\boldsymbol{b}(s_0))$.*

Suppose $\varepsilon(s_0) \neq 0$. Then, we have the following:

(3) $g_{x_0}(s_0) = g'_{x_0}(s_0) = g''_{x_0}(s_0) = 0$ *if and only if* $\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = \frac{\kappa_n(s_0)}{\varepsilon(s_0)}(\tau_g(s_0)\boldsymbol{t}(s_0) - \kappa_n(s_0)\boldsymbol{b}(s_0))$.

(4) $g_{x_0}(s_0) = g'_{x_0}(s_0) = g''_{x_0}(s_0) = g^{(3)}_{x_0}(s_0) = 0$ *if and only if* $\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = \frac{\kappa_n(s_0)}{\varepsilon(s_0)}(\tau_g(s_0)\boldsymbol{t}(s_0) - \kappa_n(s_0)\boldsymbol{b}(s_0))$ *and* $\sigma(s_0) = 0$.

(5) $g_{x_0}(s_0) = g'_{x_0}(s_0) = g''_{x_0}(s_0) = g^{(3)}_{x_0}(s_0) = g^{(4)}_{x_0}(s_0) = 0$ *if and only if* $\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = \frac{\kappa_n(s_0)}{\varepsilon(s_0)}(\tau_g(s_0)\boldsymbol{t}(s_0) - \kappa_n(s_0)\boldsymbol{b}(s_0))$, $\sigma(s_0) = 0$ *and* $\sigma'(s_0) = 0$.

Suppose $\varepsilon(s_0) = 0$. Then, we have

(6) $g_{x_0}(s_0) = g'_{x_0}(s_0) = g''_{x_0}(s_0) = 0$ *if and only if $\kappa_n(s_0) = 0$ (namely, $\kappa_n(s_0) = 0$, $\kappa'_n(s_0) = -\delta(s_0)\tau_g(s_0)\kappa_g(s_0)$) and there exists $\mu \in \mathbb{R}$ such that $\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = \mu \boldsymbol{t}(s_0)$.*

(7) $g_{x_0}(s_0) = g'_{x_0}(s_0) = g''_{x_0}(s_0) = g^{(3)}_{x_0}(s_0) = 0$ *if and only if one of the following equations holds:*

(a) $\varepsilon'(s_0) \neq 0$, $\kappa_n(s_0) = 0$, namely,

$$\begin{aligned} &\kappa_n(s_0) = 0, \ \ \kappa'_n(s_0) = -\delta(s_0)\tau_g(s_0)\kappa_g(s_0), \\ &-\kappa''_n(s_0) + 2\delta(s_0)\tau'_g(s_0)\kappa_g(s_0) + \delta(s_0)\tau_g(s_0)\kappa'_g(s_0) \neq 0 \end{aligned}$$

and

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = \frac{\kappa'_n(s_0)}{-\kappa''_n(s_0) + 2\delta(s_0)\tau'_g(s_0)\kappa_g(s_0) + \delta(s_0)\tau_g(s_0)\kappa'_g(s_0)}\boldsymbol{t}(s_0).$$

(b) $\varepsilon'(s_0) = 0$, $\kappa_n(s_0) = \kappa'_n(s_0) = 0$ (namely, $\kappa_g(s_0) = \kappa_n(s_0) = \kappa'_n(s_0) = 0$), and there exists $\mu \in \mathbb{R}$ such that $\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = \mu \boldsymbol{t}(s_0)$.

Proof. Since $g_{x_0}(s) = \langle \boldsymbol{x}_0 - \boldsymbol{\gamma}(s), \boldsymbol{n}_\gamma(s)\rangle$, we have the following:

$$\begin{aligned}
(i)\ g_{x_0} =& \langle \boldsymbol{x} - \boldsymbol{\gamma}, \boldsymbol{n}_\gamma \rangle, \\
(ii)\ g'_{x_0} =& \langle \boldsymbol{x} - \boldsymbol{\gamma}(s), -\delta\kappa_n \boldsymbol{t} + \delta\tau_g \boldsymbol{b}\rangle, \\
(iii)\ g''_{x_0} =& \kappa_n + \langle \boldsymbol{x} - \boldsymbol{\gamma}, -(\delta\kappa'_n + \tau_g\kappa_g)\boldsymbol{t} + \delta(\tau_g^2 - \kappa_n^2)\boldsymbol{n}_\gamma + (\delta\tau'_g + \kappa_n\kappa_g)\boldsymbol{b}\rangle, \\
(iv)\ g^{(3)}_{x_0} =& 2\kappa'_n + \delta\tau_g\kappa_g \\
&+ \langle \boldsymbol{x} - \boldsymbol{\gamma}, -[\delta\kappa''_n + 2\tau'_g\kappa_g + \tau_g\kappa'_g + \kappa_n(\tau_g^2 - \kappa_n^2 + \delta\kappa_g^2)]\boldsymbol{t} \\
&+ 3\delta(\tau_g\tau'_g - \kappa_n\kappa'_n)\boldsymbol{n}_\gamma + [\delta\tau''_g - 2\kappa'_n\kappa_g + \kappa_n\kappa'_g + \tau_g(\delta\kappa_g^2 + \tau_g^2 - \kappa_n^2)]\boldsymbol{b}\rangle, \\
(v)\ g^{(4)}_{x_0} =& 3\kappa''_n + 3\delta\kappa_g\tau'_g + 2\delta\kappa'_g\tau_g + \delta\kappa_n(\delta\kappa_g^2 + \tau_g^2 - \kappa_n^2) \\
&+ \langle \boldsymbol{x} - \boldsymbol{\gamma}, [-\delta\kappa'''_n - 3\kappa'_g\tau'_g - 3\kappa_g\tau''_g - \kappa''_n\tau_g + \kappa'_n(\tau_g^2 + 6\kappa_n^2 - 3\delta\kappa_g^2) \\
&+ \kappa_n(-5\tau_g\tau'_g - 3\delta\kappa_g\kappa'_g) + \kappa_g\tau_g(-\kappa_g^2 - \delta\tau_g^2 + \kappa_n^2)]\boldsymbol{t} \\
&+ [\delta\tau'''_g + 3\kappa'_n\kappa'_g + 3\kappa''_n\kappa_g + \kappa_n\kappa''_g + \tau'_g(3\delta\kappa_g^2 + 6\tau_g^2 - \kappa_n^2) \\
&+ \tau_g(3\delta\kappa_g\kappa'_g - 5\kappa_n\kappa'_n) + \kappa_n\kappa_g(\delta\tau_g^2 - \delta\kappa_n^2 + \kappa_g^2)]\boldsymbol{b} \\
&+ [(\kappa_n^2 - \tau_g^2)(-\tau_g^2 + \kappa_n^2 - \delta\kappa_g^2) + 2\kappa_g(\kappa'_n\tau_g - \kappa_n\tau'_g) + 3\delta((\tau'_g)^2 - (\kappa'_n)^2) \\
&- 4\delta\kappa_n\kappa''_n + 4\delta\tau_g\tau''_g]\boldsymbol{n}\rangle.
\end{aligned}$$

We know that $\{\boldsymbol{t}(s), \boldsymbol{n}_\gamma(s), \boldsymbol{b}(s)\}$ is a pseudo-orthonormal frame for the formula (i), so the assertion (1) holds.

By the formula (ii), $g_{x_0}(s_0) = g'_{x_0}(s_0) = 0$ if and only if there exist $a, b \in \mathbb{R}$ such that $\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = a\boldsymbol{t}(s_0) + b\boldsymbol{b}(s_0)$ and $a\kappa_n(s_0) + b\tau_g(s_0) = 0$. Thus, there exists $\mu \in \mathbb{R}$ such that $a = \mu\tau_g(s_0)$ and $b = -\mu\kappa_n(s_0)$. The assertion (2) holds.

Moreover, by the formula (iii), $g_{x_0}(s_0) = g'_{x_0}(s_0) = g''_{x_0}(s_0) = 0$ if and only if

$$\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = \mu(\tau_g(s_0)\boldsymbol{t}(s_0) - \kappa_n(s_0)\boldsymbol{b}(s_0))$$

and

$$\begin{aligned}\kappa_n(s_0) + \langle \mu(\tau_g(s_0)\boldsymbol{t}(s_0) - \kappa_n(s_0)\boldsymbol{b}(s_0)), -(\delta(s_0)\kappa'_n(s_0) + \tau_g(s_0)\kappa_g(s_0))\boldsymbol{t}(s_0) \\ + \delta(s_0)(\tau_g^2(s_0) - \kappa_n^2(s_0))\boldsymbol{n}_\gamma(s_0) + (\delta(s_0)\tau'_g(s_0) + \kappa_n(s_0)\kappa_g(s_0))\boldsymbol{b}(s_0)\rangle = 0.\end{aligned}$$

Since $\varepsilon(s_0) = \delta(s_0)\big(\tau_g^2(s_0) - \kappa_n^2(s_0)\big)\kappa_g(s_0) + \kappa'_n(s_0)\tau_g(s_0) - \kappa_n(s_0)\tau'_g(s_0)$, then $\kappa_n(s_0) - \mu\varepsilon(s_0) = 0$. It follows that $\varepsilon(s_0) \neq 0$ and $\mu = \frac{\kappa_n(s_0)}{\varepsilon(s_0)}$, or $\varepsilon(s_0) = 0$ and $\kappa_n(s_0) = 0$. This means that the proof of the assertions (3) and (6) is complete.

Suppose that $\varepsilon(s_0) \neq 0$. Then, by the formula (iv), $g_{x_0}(s_0) = g'_{x_0}(s_0) = g''_{x_0}(s_0) = g^{(3)}_{x_0}(s_0) = 0$ if and only if

$$2\kappa'_n + \delta\tau_g\kappa_g + \frac{\kappa_n}{\varepsilon}[-\tau_g\kappa''_n + \tau''_g\kappa_n - 2\delta\tau_g\tau'_g\kappa_g + 2\delta\kappa_n\kappa'_n\kappa_g - \delta\tau_g^2\kappa'_g + \delta\kappa_n^2\kappa'_g] = 0$$

at $s = s_0$. Since

$$\sigma(s_0) = (\frac{\kappa_n(s_0)}{\varepsilon(s_0)})' + \frac{\delta(s_0)\kappa_g(s_0)\tau_g(s_0)}{\varepsilon(s_0)},$$

then the above equation is equivalent to $\varepsilon(s_0)\sigma(s_0) = 0$. It follows that $\sigma(s_0) = 0$. The assertion also holds in reverse.

Suppose that $\varepsilon(s_0) = 0$. Then, by the formulae (iv), $g_{x_0}(s_0) = g'_{x_0}(s_0) = g''_{x_0}(s_0) = g^{(3)}_{x_0}(s_0) = 0$ if and only if $\kappa_n(s_0) = 0$ (*i.e.*, $\kappa_n(s_0) = 0, \kappa'_n(s_0) = -\delta(s_0)\tau_g(s_0)\kappa_g(s_0)$), and there exists $\mu \in \mathbb{R}$ such that $\boldsymbol{x}_0 - \boldsymbol{\gamma}(s_0) = \mu\boldsymbol{t}(s_0)$ and

$$2\kappa'_n(s_0) + \delta(s_0)\tau_g(s_0)\kappa_g(s_0) - \mu(-\kappa''_n(s_0) + 2\delta(s_0)\kappa_g(s_0)\tau'_g(s_0) + \delta(s_0)\kappa'_g(s_0)\tau_g(s_0)) = 0.$$

It follows that

$$-\kappa''_n(s_0) + 2\delta(s_0)\kappa_g(s_0)\tau'_g(s_0) + \delta(s_0)\kappa'_g(s_0)\tau_g(s_0) \neq 0$$

and

$$\mu = \frac{\kappa'_n(s_0)}{-\kappa''_n(s_0) + 2\delta(s_0)\tau'_g(s_0)\kappa_g(s_0) + \delta(s_0)\tau_g(s_0)\kappa'_g(s_0)}$$

or

$$-\kappa''_n(s_0) + 2\delta(s_0)\kappa_g(s_0)\tau'_g(s_0) + \delta(s_0)\kappa'_g(s_0)\tau_g(s_0) = 0 \text{ and } \kappa'_n(s_0) = 0.$$

Moreover, $\varepsilon'(s_0) = 0$ is equivalent to $-\kappa''_n(s_0) + 2\delta(s_0)\kappa_g(s_0)\tau'_g(s_0) + \delta(s_0)\kappa'_g(s_0)\tau_g(s_0) = 0$. Then, we have (6) and (7).

By a similar discussion to the above, we have the assertion (5). This completes the proof. $\square$

For the sake of proving Theorem 3, we need some general results on the singularity theory for the germs of functions. For detailed descriptions, please refer to the book [36]. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \to \mathbb{R}$ be a function germ and $f(s) = F_{x_0}(s, \boldsymbol{x}_0)$. We say F is an *r-parameter unfolding* of f. If $f^{(l)}(s_0) = 0$ for all $1 \leq l \leq k$ and $f^{(k+1)}(s_0) \neq 0$, then we say f has *A_k-singularity* at s_0. We also say f has *$A_{\geq k}$-singularity* at s_0 if $f^{(l)}(s_0) = 0$ for all $1 \leq l \leq k$. Meanwhile, let F be an r-parameter unfolding of f and f has A_k-singularity

$(k \geq 1)$ at s_0; we define the $(k-1)$-jet of the partial derivative $\partial F/\partial x_i$ at s_0 as

$$j^{(k-1)} \frac{\partial F}{\partial x_i}(s, \boldsymbol{x}_0)(s_0) = \sum_{j=0}^{k-1} a_{ji}(s - s_0)^j, \quad (i = 1, \ldots, r).$$

If the rank of the $k \times r$ matrix (a_{ji}) is k $(k \leq r)$, where $a_{0i} = \partial F/\partial x_i(s_0, \boldsymbol{x}_0)$, then F is called an *R-versal unfolding* of f. The *discriminant set* of F is defined by

$$D_F = \{\boldsymbol{x} \in \mathbb{R}^r \mid \exists\, s \in \mathbb{R}, F(s, \boldsymbol{x}) = \frac{\partial F}{\partial s}(s, \boldsymbol{x}) = 0\}.$$

Then, there exists the following famous result (see [36]).

Theorem 4. *Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \to \mathbb{R}$ be an r-parameter unfolding of $f(s)$ that has A_k-singularity at s_0. Supposing that F is an R-versal unfolding of f, if $k = 2$, then the germ of D_F at $\boldsymbol{x}_0$ is diffeomorphic to $C \times \mathbb{R}^{r-1}$; if $k = 3$, then the germ of D_F at $\boldsymbol{x}_0$ is diffeomorphic to $SW \times \mathbb{R}^{r-2}$.*

For the sake of proving Theorem 3, we have the following.

Proposition 2. *Let $\boldsymbol{\gamma} : I \to M \subset \mathbb{R}^3_1$ be a non-lightlike curve with $\tau_g^2 + \kappa_n^2 \neq 0$ and $G : I \times \mathbb{R}^3_1 \to \mathbb{R}$ be the Lorentzian support function on $\boldsymbol{\gamma}$ with respect to $\boldsymbol{n}_\gamma$. If g_{x_0} has an A_k-singularity at s_0 $(k = 2, 3)$, then G is an R-versal unfolding of g_{x_0}. Here, we suppose $\varepsilon(s_0) \neq 0$ for $k = 3$.*

Proof. We denote that $\boldsymbol{x} = (x_1, x_2, x_3)$, $\boldsymbol{\gamma}(s) = (r_1(s), r_2(s), r_3(s))$ and $\boldsymbol{n}_\gamma(s) = (n_1(s), n_2(s), n_3(s))$. Then,

$$G(s, \boldsymbol{x}) = -n_1(s)(x_1 - r_1(s)) + n_2(s)(x_2 - r_2(s)) + n_3(s)(x_3 - r_3(s)),$$

so that

$$\frac{\partial G}{\partial x_1} = -n_1(s), \quad \frac{\partial G}{\partial x_2} = n_2(s), \quad \frac{\partial G}{\partial x_3} = n_3(s).$$

Therefore, the 2-jet is

$$\begin{aligned}
j^2 \frac{\partial G}{\partial x_1}(s_0, \boldsymbol{x}_0) &= -n_1(s_0) - n_1'(s_0)(s - s_0) - \frac{1}{2} n_1''(s_0)(s - s_0)^2, \\
j^2 \frac{\partial G}{\partial x_i}(s_0, \boldsymbol{x}_0) &= n_i(s_0) + n_i'(s_0)(s - s_0) + \frac{1}{2} n_i''(s_0)(s - s_0)^2, \quad (i = 2, 3).
\end{aligned}$$

We denote the following matrix:

$$A = \begin{pmatrix} -n_1(s_0) & n_2(s_0) & n_3(s_0) \\ -n_1'(s_0) & n_2'(s_0) & n_3'(s_0) \\ -n_1''(s_0) & n_2''(s_0) & n_3''(s_0) \end{pmatrix}.$$

According to the Frenet–Serret-type formulae, we obtain

$$\begin{aligned}
-\det A =& \langle \boldsymbol{n}_\gamma(s_0) \wedge \boldsymbol{n}_\gamma'(s_0), \boldsymbol{n}_\gamma''(s_0) \rangle \\
=& \langle \boldsymbol{n}_\gamma(s_0) \wedge (-\delta(s_0)\kappa_n(s_0)\boldsymbol{t}(s_0) + \delta(s_0)\tau_g(s_0)\boldsymbol{b}(s_0)), (-\delta(s_0)\kappa_n'(s_0) - \kappa_g(s_0)\tau_g(s_0))\boldsymbol{t}(s_0) \\
&+ (\delta(s_0)\tau_g^2(s_0) - \delta(s_0)\kappa_n^2(s_0))\boldsymbol{n}_\gamma(s_0) + (\delta(s_0)\tau_g'(s_0) + \kappa_n(s_0)\kappa_g(s_0)) \rangle \\
=& \langle \delta(s_0)\kappa_n(s_0)\boldsymbol{b}(s_0) - \delta(s_0)\tau_g(s_0), (-\delta(s_0)\kappa_n'(s_0) - \kappa_g(s_0)\tau_g(s_0))\boldsymbol{t}(s_0) \\
&+ (\delta(s_0)\tau_g^2(s_0) - \delta(s_0)\kappa_n^2(s_0))\boldsymbol{n}_\gamma(s_0) + (\delta(s_0)\tau_g'(s_0) + \kappa_n(s_0)\kappa_g(s_0)) \rangle \\
=& \kappa_g(s_0)(\tau_g^2(s_0) - \kappa_n^2(s_0)) + \delta(s_0)(\kappa_n'(s_0)\tau_g(s_0) - \kappa_n(s_0)\tau_g'(s_0)) \\
=& \delta(s_0)\varepsilon(s_0) \neq 0.
\end{aligned}$$

Therefore, rank $A = 3$. Moreover, the rank of

$$\widetilde{B} = \begin{pmatrix} n_1(s_0) & n_2(s_0) & n_3(s_0) \\ n'_1(s_0) & n'_2(s_0) & n'_3(s_0) \end{pmatrix} = \begin{pmatrix} \boldsymbol{n}_\gamma(s_0) \\ -\delta(s_0)\kappa_n(s_0)\boldsymbol{t}(s_0) + \delta(s_0)\tau_g(s_0)\boldsymbol{b}(s_0) \end{pmatrix}$$

is always 2. Then, the rank of

$$B = \begin{pmatrix} -n_1(s_0) & n_2(s_0) & n_3(s_0) \\ -n'_1(s_0) & n'_2(s_0) & n'_3(s_0) \end{pmatrix}$$

is also always 2.

If g_{x_0} has an A_k-singularity at s_0 $(k = 2, 3)$, then G is an R-versal unfolding of g_{x_0}. This completes the proof. □

5.2. Proof of Theorem 3

Proof of Theorem 3. Now, we prove the main result of Theorem 3. By straightforward calculations, we obtain

$$\begin{aligned} \frac{\partial \boldsymbol{OD}}{\partial u} \wedge \frac{\partial \boldsymbol{OD}}{\partial s} &= \tau_g(u\delta\tau_g\kappa_g + u\kappa'_n)\boldsymbol{n}_\gamma - \kappa_n(1 + u\tau'_g + u\delta\kappa_n\kappa_g)\boldsymbol{n}_\gamma \\ &= (u\varepsilon - \kappa_n)\boldsymbol{n}_\gamma. \end{aligned}$$

We know that (s_0, u_0) is non-singular if and only if

$$\frac{\partial \boldsymbol{OD}}{\partial u} \wedge \frac{\partial \boldsymbol{OD}}{\partial s} \neq 0.$$

It is equivalent to $u_0\varepsilon(s_0) - \kappa_n(s_0) \neq 0$. Thus, we finish the proof of the assertion (1).

According to Proposition 1, the discriminant set D_G of the Lorentzian support functions G of $\boldsymbol{\gamma}$ with respect to $\boldsymbol{n}_\gamma$ is the osculating developable surface of M along $\boldsymbol{\gamma}$.

Suppose $\varepsilon(s_0) \neq 0$. By assertions (3), (4) and (5) in Proposition 1, g_{x_0} has the A_2-singularity (respectively, the A_3-singularity) at s_0 if and only if

$$u_0 = \frac{\kappa_n(s_0)}{\varepsilon(s_0)}$$

and $\sigma(s_0) \neq 0$ (respectively, $\sigma(s_0) = 0$ and $\sigma'(s_0) \neq 0$). Then, by Theorem 4 and Proposition 2, we know assertions (2), (i) and (3) hold.

Suppose $\varepsilon(s_0) = 0$. By assertions (6) and (7) of Proposition 1, g_{x_0} has the A_2-singularity at s_0 if and only if $\varepsilon(s_0) = 0$, $\kappa_n(s_0) = 0$ and

$$\kappa'_n(s_0) - u_0\big(-\kappa''_n(s_0) + 2\delta(s_0)\tau'_g(s_0)\kappa_g(s_0) + \delta(s_0)\tau_g(s_0)\kappa'_g(s_0)\big) \neq 0.$$

It means that

$$-\kappa''_n(s_0) + 2\delta(s_0)\tau'_g(s_0)\kappa_g(s_0) + \delta(s_0)\tau_g(s_0)\kappa'_g(s_0) \neq 0$$

and

$$u_0 \neq \frac{\kappa'_n(s_0)}{-\kappa''_n(s_0) + 2\delta(s_0)\tau'_g(s_0)\kappa_g(s_0) + \delta(s_0)\tau_g(s_0)\kappa'_g(s_0)},$$

or

$$-\kappa''_n(s_0) + 2\delta(s_0)\tau'_g(s_0)\kappa_g(s_0) + \delta(s_0)\tau_g(s_0)\kappa'_g(s_0) = 0 \ \text{ and } \ \kappa'_n(s_0) \neq 0.$$

Since $\varepsilon'(s_0) = 0$ is equivalent to $-\kappa''_n(s_0) + 2\delta(s_0)\kappa_g(s_0)\tau'_g(s_0) + \delta(s_0)\kappa'_g(s_0)\tau_g(s_0) = 0$. By Theorem 4 and Proposition 2, we obtain the assertions (2), (ii) and (iii). Therefore, we finish the proof. □

6. Invariants of Non-Lightlike Curves on Timelike Surfaces

In this section, we will consider geometric meanings of the invariant σ.

Let $\mathbf{\Gamma} : I \to \mathbb{R}^3_1 \times S^2_1$ be a curve and $F : \mathbb{R}^3_1 \times S^2_1 \to \mathbb{R}$ be a submersion. We say that $\mathbf{\Gamma}$ and $F^{-1}(0)$ have the contact of *at least order* k at $t = t_0$ if the function $g(t) = F \circ \mathbf{\Gamma}(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k)}(t_0) = 0$. Moreover, if $\mathbf{\Gamma}$ and $F^{-1}(0)$ have the contact of at least order k at $t = t_0$ and satisfy the condition $g^{(k+1)}(t_0) \neq 0$, then we say that $\mathbf{\Gamma}$ and $F^{-1}(0)$ have the *contact of order* k at $t = t_0$. Meanwhile, for any $\boldsymbol{x} \in \mathbb{R}^3_1$, we define the function $\mathfrak{g}_x : \mathbb{R}^3_1 \times S^2_1 \to \mathbb{R}$ as $\mathfrak{g}_x(\boldsymbol{a}, \boldsymbol{b}) = \langle \boldsymbol{x} - \boldsymbol{a}, \boldsymbol{b} \rangle$. Then, we have

$$\mathfrak{g}_x^{-1}(0) = \{(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{R}^3_1 \times S^2_1 | \langle \boldsymbol{a}, \boldsymbol{b} \rangle = \langle \boldsymbol{x}, \boldsymbol{b} \rangle\}.$$

For a fixed $\boldsymbol{b} \in S^2_1$, $\mathfrak{g}_x^{-1}(0) | \mathbb{R}^3_1 \times \{\boldsymbol{b}\}$ is a timelike plane that is defined by $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = c$, where $c = \langle \boldsymbol{x}, \boldsymbol{b} \rangle$. For the reason that this plane is pseudo-orthogonal to $\boldsymbol{b}$, it is parallel to the tangent plane $T_v S^2_1$ at $\boldsymbol{b}$. On the other hand, we can represent the tangent bundle of S^2_1 as follows:

$$TS^2_1 = \{(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{R}^3_1 \times S^2_1 | \langle \boldsymbol{a}, \boldsymbol{b} \rangle = 1\}.$$

Let $\pi_2 | \mathfrak{g}_x^{-1}(0) : \mathfrak{g}_x^{-1}(0) \to S^2_1$ be the canonical projection, where $\pi_2 : \mathbb{R}^3_1 \times S^2_1 \to S^2_1$. Then, $\pi_2 | \mathfrak{g}_x^{-1}(0) : \mathfrak{g}_x^{-1}(0) \to S^2_1$ is a bundle over S^2_1. Moreover, a map $\Phi : \mathfrak{g}_x^{-1}(0) \to TS^2_1$ is defined by $\Phi(\boldsymbol{a}, \boldsymbol{b}) = (\boldsymbol{a}/\langle \boldsymbol{x}, \boldsymbol{b} \rangle, \boldsymbol{b})$; then, Φ is a bundle isomorphism. Here, we denote $TS^2_1(\boldsymbol{x}) = \mathfrak{g}_x^{-1}(0)$. Meanwhile, we call it an *affine tangent bundle over* S^2_1 *through* $\boldsymbol{x}$.

Let $\boldsymbol{\gamma} : I \to M \subset \mathbb{R}^3_1$ be a non-lightlike curve with $\tau_g^2 + \kappa_n^2 \neq 0$. Assume that $\varepsilon(s) \neq 0$. According to the proof of Theorem 2 (B), we have the derivative of the vector-valued function $\boldsymbol{f}$, which is $\boldsymbol{f}'(s) = \sigma(s)\boldsymbol{D}(s)$. Thus, if we suppose that $\sigma(s) \equiv 0$, then $\boldsymbol{f}$ is a constant vector $\boldsymbol{x}_0$. We have

$$\boldsymbol{\gamma}(s) - \boldsymbol{x}_0 = -\frac{\kappa_n(s)}{\varepsilon(s)}\boldsymbol{D}(s).$$

Therefore, we obtain

$$\mathfrak{g}_{x_0}(\boldsymbol{\gamma}(s), \boldsymbol{n}_\gamma(s)) = g_{x_0}(s) = \langle \boldsymbol{\gamma}(s) - \boldsymbol{x}_0, \boldsymbol{n}_\gamma(s) \rangle = 0.$$

On the other hand, if there exists $\boldsymbol{x}_0 \in \mathbb{R}^3_1$ such that $\mathfrak{g}_{x_0}(\boldsymbol{\gamma}(s), \boldsymbol{n}_\gamma(s)) = 0$, then we can obtain

$$\boldsymbol{\gamma}(s) - \boldsymbol{x}_0 = -\frac{\kappa_n(s)}{\varepsilon(s)}\boldsymbol{D}(s)$$

and $\sigma(s) \equiv 0$. We consider such a curve $(\boldsymbol{\gamma}, \boldsymbol{n}_\gamma) : I \to \mathbb{R}^3_1 \times S^2_1$. Then, we have

Proposition 3. *Let $\boldsymbol{\gamma} : I \to M \subset \mathbb{R}^3_1$ be a non-lightlike curve with $\tau_g^2 + \kappa_n^2 \neq 0$ and $\varepsilon(s) \neq 0$. Then, there exists $\boldsymbol{x}_0 \in \mathbb{R}^3_1$ such that $(\boldsymbol{\gamma}, \boldsymbol{n}_\gamma)(I) \subset TS^2_1(\boldsymbol{x}_0)$ if and only if $\sigma(s) \equiv 0$.*

The result of the proposition above states that the geometric meaning of the singularities of $\boldsymbol{OD}$ is related to both the curve and the shape of the timelike surface along the non-lightlike curve. Let $\boldsymbol{\gamma} : I \to M \subset \mathbb{R}^3_1$ be a non-lightlike curve with $\tau_g^2 + \kappa_n^2 \neq 0$. Meanwhile, we consider the support function $g_{x_0}(s) = \mathfrak{g}_{x_0}(\boldsymbol{\gamma}(s), \boldsymbol{n}_\gamma(s))$. According to Proposition 1 (2) , one can find that $(\boldsymbol{\gamma}, \boldsymbol{n}_\gamma)$ is tangent to $TS^2_1(\boldsymbol{x}_0)$ at $s = s_0$ if and only if $\boldsymbol{x}_0 = \boldsymbol{OD}(s_0, u_0)$ for some $u_0 \in \mathbb{R}$. In addition, we have the following.

Proposition 4. *Let $\boldsymbol{\gamma} : I \to M \subset \mathbb{R}^3_1$ with $\tau_g^2 + \kappa_n^2 \neq 0$ be a non-lightlike curve and $\varepsilon(s) \neq 0$. For $\boldsymbol{x}_0 = \boldsymbol{OD}(s_0, u_0)$, we have the following:*
(1) $(\boldsymbol{\gamma}, \boldsymbol{n}_\gamma)$ and $TS^2_1(\boldsymbol{x}_0)$ have contact of order 2 at $s = s_0$ if and only if $u_0 = \frac{\kappa_n(s_0)}{\varepsilon(s_0)}$ and $\sigma(s_0) \neq 0$.

(2) $(\boldsymbol{\gamma}, \boldsymbol{n}_\gamma)$ *and* $TS^2_1(\boldsymbol{x}_0)$ *have contact of order* 3 *at* $s = s_0$ *if and only if* $u_0 = \frac{\kappa_n(s_0)}{\varepsilon(s_0)}$, $\sigma(s_0) = 0$ *and* $\sigma'(s_0) \neq 0$.

Proof. By Proposition 1, (3) and (4), one can obtain $g_{x_0}(s_0) = g'_{x_0}(s_0) = g''_{x_0}(s_0) = 0$ and $g^{(3)}_{x_0}(s_0) \neq 0$ if and only if $u_0 = \frac{\kappa_n(s_0)}{\varepsilon(s_0)}$ and $\sigma(s_0) \neq 0$. Since $\mathfrak{g}_{x_0}(\boldsymbol{\gamma}(s), \boldsymbol{n}_\gamma(s)) = g_{x_0}(s)$, the conditions above imply that $(\boldsymbol{\gamma}, \boldsymbol{n}_\gamma)$ and $TS^2_1(\boldsymbol{x}_0)$ have contact of order 2 at $s = s_0$. By using Proposition 1, (4) and (5), we can obtain the assertion (2) similar to the case above. □

Moreover, for the classification results of Theorem 3, we show the geometric meaning as follows.

Theorem 5. *Let* $\boldsymbol{\gamma} : I \to M \subset \mathbb{R}^3_1$ *be a non-lightlike curve with* $\tau_g^2 + \kappa_n^2 \neq 0$ *and* $\varepsilon(s) \neq 0$.

(1) $(\boldsymbol{\gamma}, \boldsymbol{n}_\gamma)$ *and* $TS^2_1(\boldsymbol{x}_0)$ *have contact of order* 2 *at* $s = s_0$ *if and only if* $u_0 = \frac{\kappa_n(s_0)}{\varepsilon(s_0)}$ *and* $\sigma(s_0) \neq 0$. *In this case, the image of the osculating developable surface* $\boldsymbol{OD}$ *of M along* $\boldsymbol{\gamma}$ *is locally diffeomorphic to the cuspidal edge* $C \times \mathbb{R}$ *at* (s_0, u_0).

(2) $(\boldsymbol{\gamma}, \boldsymbol{n}_\gamma)$ *and* $TS^2_1(\boldsymbol{x}_0)$ *have contact of order* 3 *at* $s = s_0$ *if and only if* $u_0 = \frac{\kappa_n(s_0)}{\varepsilon(s_0)}$, $\sigma(s_0) = 0$ *and* $\sigma'(s_0) \neq 0$. *In this case, the image of the osculating developable surface* $\boldsymbol{OD}$ *of M along* $\boldsymbol{\gamma}$ *is locally diffeomorphic to the swallowtail SW at* (s_0, u_0).

7. Curves on Special Timelike Surfaces

7.1. Curves on the de Sitter 2-Space

De Sitter 2-space $S^2_1 = \{\boldsymbol{x} \in \mathbb{R}^3_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}$ is a special timelike surface in $\mathbb{R}^3_1$. We consider the non-lightlike curves on S^2_1. Let $\boldsymbol{\gamma} : I \to S^2_1$ be a non-lightlike curve. In this case, the Darboux frame along $\boldsymbol{\gamma}$ is $\{\boldsymbol{t}, \boldsymbol{\gamma}, \boldsymbol{b}\}$. We have $\kappa_n(s) = -\delta(s)$ and $\tau_g(s) = 0$. The Frenet–Serret-type formula is as follows:

$$\begin{cases} \boldsymbol{t}'(s) = -\delta(s)\boldsymbol{\gamma}(s) - \delta(s)\kappa_g \boldsymbol{b}(s), \\ \boldsymbol{\gamma}'(s) = \boldsymbol{t}(s), \\ \boldsymbol{b}'(s) = -\delta(s)\kappa_g(s)\boldsymbol{t}(s). \end{cases}$$

It follows that $\boldsymbol{D}(s) = \delta(s)\boldsymbol{b}(s)$ and $\boldsymbol{OD}(s, u) = \boldsymbol{\gamma}(s) + u\delta(s)\boldsymbol{b}(s)$. Therefore, we have

$$\varepsilon(s) = -\delta(s)\kappa_g(s), \;\; \sigma(s) = -\frac{\kappa'_g(s)}{\kappa^2_g(s)}.$$

Then, as a corollary of Theorem 3, we have the following theorem.

Theorem 6. *Let* $\boldsymbol{\gamma} : I \to S^2_1$ *be a non-lightlike curve. Then,*
(1) $(\boldsymbol{OD}, (s_0, u_0))$ *is regular if and only if* $-u_0\kappa_g(s_0) + 1 \neq 0$.
(2) The image of $(\boldsymbol{OD}, (s_0, u_0))$ *is locally diffeomorphic to the cuspidal edge* $C \times \mathbb{R}$ *if* $\kappa_g(s_0) \neq 0$, $\kappa'_g(s_0) \neq 0$ *and* $u_0 = \frac{1}{\kappa_g(s_0)}$.
(3) The image of $(\boldsymbol{OD}, (s_0, u_0))$ *is locally diffeomorphic to the swallowtail SW if* $\kappa_g(s_0) \neq 0$, $\kappa'_g(s_0) = 0$, $\kappa''_g(s_0) \neq 0$ *and* $u_0 = \frac{1}{\kappa_g(s_0)}$.

Now, we consider some special curves on S^2_1, such as the spacelike circle, which is the intersection of the plane $x_1 = k$ with S^2_1. It is defined by

$$C_k = \{\boldsymbol{x} \in S^2_1 | x_1 = k\}.$$

We call it a *small circle* if $k = 0$, and call it a *great circle* if $k \neq 0$. C_k is a spacelike curve on S^2_1.

Proposition 5. *Let* $\boldsymbol{\gamma} : I \to S^2_1$ *be a non-lightlike curve and* $\boldsymbol{OD}$ *be the osculating developable surface of* S^2_1 *along* $\boldsymbol{\gamma}$. *Then, we have the assertions below.*

(1) *If γ is a small circle, then* $\boldsymbol{OD}$ *is a circular cylinder.*
(2) *If γ is a great circle, then* $\boldsymbol{OD}$ *is a circular cone.*

Proof. If γ is a small circle, then $\kappa_g(s) \equiv 0$ and $\boldsymbol{b}(s)$ is constant. Thus, $\boldsymbol{OD}(s,u) = \gamma(s) + u\boldsymbol{b}(s)$ is a circular cylinder that is tangential to S_1^2 along γ (see Figure 3). If γ is a great circle, then $\kappa_g(s) \equiv \frac{k}{\sqrt{k^2+1}}$ and $\kappa_g'(s) \equiv 0$, so that $\varepsilon(s) \neq 0$ and $\sigma(s) \equiv 0$. It means that $\boldsymbol{OD}(s,u) = \gamma(s) + u\delta(s)\boldsymbol{b}(s)$ is a cone tangent that is tangential to S_1^2 along γ (see Figure 4). □

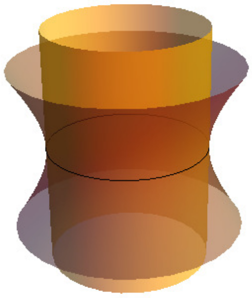

Figure 3. $\boldsymbol{OD}$ along the small circle.

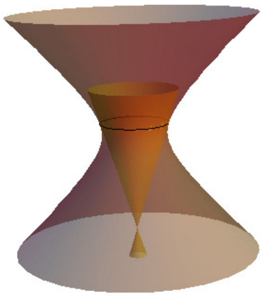

Figure 4. $\boldsymbol{OD}$ along the great circle.

On the other hand, we consider the timelike hyperbolic curve, which is the intersection of the plane $x_3 = k$ $(-1 < k < 1)$ with S_1^2. It is defined by

$$H_k = \{\boldsymbol{x} \in S_1^2 | x_3 = k\}.$$

We call it a *great hyperbolic curve* if $k = 0$, and call it a *small hyperbolic curve* if $k \neq 0$. H_k is a timelike curve on S_1^2, and we have the proposition below.

Proposition 6. *Let $\gamma : I \to S_1^2$ be a non-lightlike curve and* $\boldsymbol{OD}$ *the osculating developable surface of S_1^2 along γ. Then, we have the assertions below.*
(1) *If γ is a great hyperbolic curve, then* $\boldsymbol{OD}$ *is a cylinder.*
(2) *If γ is a small hyperbolic curve, then* $\boldsymbol{OD}$ *is a cone.*

Proof. Let γ be a timelike hyperbolic curve H_k; then, one can define γ by

$$\gamma(s) = (\sqrt{1-k^2}\sinh\frac{s}{\sqrt{1-k^2}}, \sqrt{1-k^2}\cosh\frac{s}{\sqrt{1-k^2}}, k).$$

Then, we have

$$\boldsymbol{t}(s) = (\cosh\frac{s}{\sqrt{1-k^2}}, \sinh\frac{s}{\sqrt{1-k^2}}, 0),$$

and $\kappa_g(s) = 2 - 2\sqrt{1-k^2}$. If γ is a great hyperbolic curve, then $\kappa_g(s) \equiv 0$ and $\boldsymbol{b}$ is constant. Hence, $\boldsymbol{OD}(s,u) = \gamma(s) - u\boldsymbol{b}(s)$ is a cylinder that is tangential to S_1^2 along γ (see

Figure 5). If γ is a small hyperbolic curve, then $\kappa_g(s) \neq 0$ and $\kappa'_g(s) \equiv 0$. It follows that $\boldsymbol{OD}(s,u) = \gamma(s) + u\delta(s)\boldsymbol{b}(s)$ is a cone that is tangential to S^2_1 along γ (see Figure 6). $\square$

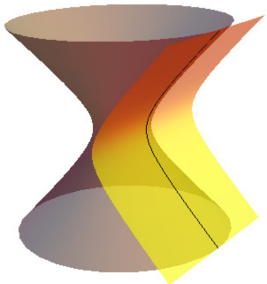

Figure 5. ***OD*** along the great hyperbolic curve.

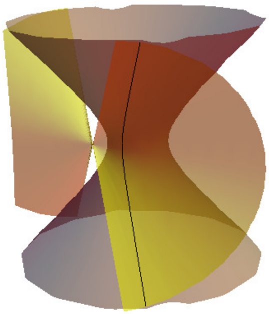

Figure 6. ***OD*** along the small hyperbolic curve.

The following example of a timelike curve on S^2_1 shown below serves to illustrate Theorem 6.

Example 1. *Let $\gamma : I \to S^2_1$ be a timelike curve defined by*

$$\gamma(t) = (t, t^2, \sqrt{1 - t^4 + t^2}), \ \ t \in (-0.41, 0.41).$$

The Darboux frame along γ is $\{\gamma(t), \boldsymbol{t}(t), \boldsymbol{b}(t)\}$. Then, by a straightforward calculation, we obtain

$$\boldsymbol{t}(t) = \Big(\frac{\sqrt{1-t^4+t^2}}{\sqrt{1-4t^2-t^4}}, \frac{2t\sqrt{1-t^4+t^2}}{\sqrt{1-4t^2-t^4}}, \frac{t-2t^3}{\sqrt{1-4t^2-t^4}}\Big),$$
$$\boldsymbol{b}(t) = \Big(\frac{t^3+2t}{\sqrt{1-4t^2-t^4}}, \frac{t^4+1}{\sqrt{1-4t^2-t^4}}, \frac{t^2\sqrt{1-t^4+t^2}}{\sqrt{1-4t^2-t^4}}\Big).$$

The derivative of $\boldsymbol{b}(t)$ is given by

$$\boldsymbol{b}'(t) = \Bigg(-\frac{-2-3t^2+6t^4+t^6}{(1-4t^2-t^4)^{\frac{3}{2}}}, -\frac{2t(-2-3t^2+6t^4+t^6)}{(1-4t^2-t^4)^{\frac{3}{2}}}, \frac{t(2-t^2-12t^4+11t^6+2t^8)}{(1-4t^2-t^4)^{\frac{3}{2}}\sqrt{1-t^4+t^2}}\Bigg).$$

By the Frenet-type formulae, one can obtain $\kappa_g(t) = \frac{2+3t^2-6t^4-t^6}{(1-4t^2-t^4)^{\frac{3}{2}}}$. It follows that

$$\kappa'_g(t) = \frac{30(t+t^5)}{(1-4t^2-t^4)^{\frac{5}{2}}}, \ \ \kappa''_g(t) = \frac{30(1+16t^2+14t^4+5t^8)}{(1-4t^2-t^4)^{\frac{7}{2}}}.$$

We have $\kappa_g'(t) = 0$ if and only if $t = 0$, at this moment, $\kappa_g(0) = 2 \neq 0$ and $\kappa_g''(0) = 30 \neq 0$. By Theorem 6, if $t = 0$, the osculating developable surface $\boldsymbol{OD}$ along γ has the swallowtail singularities (see Figure 7).

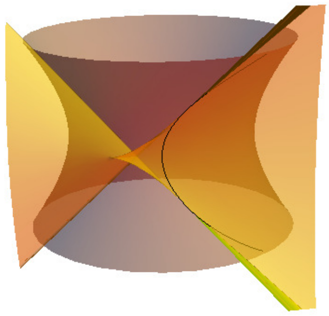

Figure 7. ***OD*** along the timelike curve with swallowtail singularities.

7.2. Curves on a Timelike Surface of Revolution

We focus on non-lightlike curves on a timelike surface of revolution in this subsection. A *timelike surface of revolution* is defined by

$$\boldsymbol{X}(u,v) = (u, f(u)\cos v, f(u)\sin v)$$

for $(u,v) \in U \subset \mathbb{R}^2$, where $f(u) \neq 0$ and $(f'(u))^2 < 1$. It is easy to show that

$$\boldsymbol{X}_u = (1, f'(u)\cos v, f'(u)\sin v), \quad \boldsymbol{X}_v = (0, -f(u)\sin v, f(u)\cos v).$$

Then, the unit spacelike normal vector field along $M = \boldsymbol{X}(U)$ is

$$\boldsymbol{n}(u,v) = \left(-\frac{f(u)f'(u)}{\sqrt{f^2(u) - f^2(u)(f'(u))^2}}, -\frac{f(u)\cos v}{\sqrt{f^2(u) - f^2(u)(f'(u))^2}}, -\frac{f(u)\sin v}{\sqrt{f^2(u) - f^2(u)(f'(u))^2}} \right).$$

Then, for a non-lightlike curve

$$\gamma(t) = \big(u(t), f(u(t))\cos v(t), f(u(t))\sin v(t)\big)$$

on M, we obtain the Darboux frame as follows.

$$\begin{aligned}
\boldsymbol{n}_\gamma(t) &= -\frac{f}{\sqrt{f^2(1-f'^2)}}(f', \cos v, \sin v), \\
\boldsymbol{t}(t) &= \frac{1}{\sqrt{\delta(t)\left(f'^2\dot{u}^2 - \dot{u}^2 + f^2\dot{v}^2\right)}}(\dot{u}, f'\dot{u}\cos v - f\dot{v}\sin v, f'\dot{u}\sin v + f\dot{v}\cos v), \\
\boldsymbol{b}(t) &= \boldsymbol{n}_\gamma(t) \wedge \boldsymbol{t}(t) \\
&= \frac{1}{\sqrt{\delta(t) f^2\left(1-f'^2\right)\left(f'^2\dot{u}^2 - \dot{u}^2 + f^2\dot{v}^2\right)}}(f^2\dot{v}, -f\dot{u}\sin v + ff'^2\dot{u}\sin v + f^2 f'\dot{v}\cos v, \\
&\quad - ff'^2\dot{u}\cos v + f^2 f'\dot{v}\sin v + f\dot{u}\cos v),
\end{aligned}$$

where

$$\begin{aligned}
\delta(t) = \operatorname{sign}(\boldsymbol{t}(t)), \ \ u &= u(t), \ \ v = v(t), \ \ f' = \frac{df}{du}, \\
\dot{u} &= \frac{du}{dt}, \ \ \dot{v} = \frac{dv}{dt}.
\end{aligned}$$

We can calculate that

$$
\begin{aligned}
\dot{\gamma}(t) =& \big(\dot{u}, f'\dot{u}\cos v - f\dot{v}\sin v, f'\dot{u}\sin v + f\dot{v}\cos v\big),\\
\ddot{\gamma}(t) =& \big(\ddot{u}, f''\dot{u}^2\cos v + f'\ddot{u}\cos v - 2f'\dot{u}\dot{v}\sin v - f\ddot{v}\sin v - f\dot{v}^2\cos v,\\
& f''\dot{u}^2\sin v + f'\ddot{u}\sin v + 2f'\dot{u}\dot{v}\cos v + f\ddot{v}\cos v - f\dot{v}^2\sin v\big),\\
\dot{\boldsymbol{n}}_\gamma(t) =& \frac{f^2f'f''\dot{u} + ff'^3\dot{u} - ff'\dot{u}}{[f^2(1-f'^2)]^{\frac{3}{2}}}\big(-ff', -f\cos v, -f\sin v\big)\\
&+ \frac{1}{\sqrt{f^2(1-f'^2)}}\big(-f'^2\dot{u} - ff''\dot{u}, -f'\dot{u}\cos v + f\dot{v}\sin v, -f'\dot{u}\sin v - f\dot{v}\cos v\big).
\end{aligned}
$$

Moreover, one can obtain

$$
\begin{aligned}
\kappa_g(t) =& \frac{\langle \boldsymbol{n}_\gamma(t)\wedge\dot{\gamma}(t), \ddot{\gamma}(t)\rangle}{\|\dot{\gamma}(t)\|^3}\\
=& \frac{1}{\sqrt{\delta(t)[f'^2\dot{u}^2 - \dot{u}^2 + f^2\dot{v}^2]^3}\sqrt{f^2(1-f'^2)}}\big(f^2\ddot{u}\dot{v}(f'^2-1) + f^2\dot{u}\ddot{v}(1-f'^2) + 2(1-f'^2)ff'\dot{u}^2\dot{v}\\
&+ f^2f'f''\dot{u}^2\dot{v} - f^3f'\dot{v}^3\big),\\
\kappa_n(t) =& \frac{\langle\ddot{\gamma}(t), \boldsymbol{n}_\gamma(t)\rangle}{\|\dot{\gamma}(t)\|^2}\\
=& \frac{-ff''\dot{u}^2 + f^2\dot{v}^2}{\delta(t)(f'^2\dot{u}^2 - \dot{u}^2 + f^2\dot{v}^2)\sqrt{f^2(1-f'^2)}},\\
\tau_g(t) =& -\frac{\langle\dot{\boldsymbol{n}}_\gamma(t), \boldsymbol{b}(t)\rangle}{\|\dot{\gamma}(t)\|}\\
=& \frac{f^2\dot{u}\dot{v} - f^2f'^2\dot{u}\dot{v} - f^3f''\dot{u}\dot{v}}{\delta(t)f^2(1-f'^2)(f'^2\dot{u}^2 - \dot{u}^2 + f^2\dot{v}^2)}.
\end{aligned}
$$

For a timelike meridian curve $\gamma(u) = \boldsymbol{X}(u, v_0) = \big(u, f(u)\cos v_0, f(u)\sin v_0\big)$, we have $\dot{v} = \frac{dv_0}{dt} = 0$. Then,

$$
\kappa_g(u) \equiv 0,\ \ \kappa_n(u) = \frac{-ff''}{(1-f'^2)\sqrt{f^2(1-f'^2)}},\ \ \tau_g(u) \equiv 0.
$$

In this case $\varepsilon \equiv 0$, the osculating developable surface $\boldsymbol{OD}$ along γ is a cylinder (see Figure 8).

For a spacelike circle $\gamma(v) = \boldsymbol{X}(u_0, v) = \big(u_0, f(u_0)\cos v, f(u_0)\sin v\big)$, we have $\dot{u} = \frac{du_0}{dt} = 0$. Then,

$$
\kappa_g(v) = \frac{-f'}{\sqrt{f^2(1-f'^2)}},\ \ \kappa_n(v) = \frac{1}{\sqrt{f^2(1-f'^2)}},\ \ \tau_g(v) \equiv 0.
$$

Since $|f'(u)| < 1$, if $f'(u_0) = 0$, we have $\varepsilon = -\kappa_n^2\kappa_g \equiv 0$. At this time, the osculating developable surface $\boldsymbol{OD}$ along γ is a cylinder (see Figure 9). If $f'(u_0) \neq 0$, then ε is a nonzero constant and $\sigma \equiv 0$. At this time, the osculating developable surface $\boldsymbol{OD}$ along γ is a cone (see Figure 10).

Figure 8. $\boldsymbol{OD}$ along the timelike meridian curve.

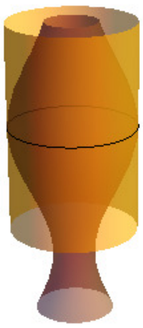

Figure 9. If $f'(u_0) = 0$, $\boldsymbol{OD}$ along the spacelike circle.

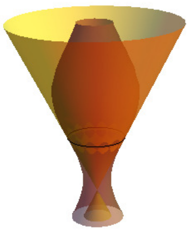

Figure 10. If $f'(u_0) \neq 0$, $\boldsymbol{OD}$ along the spacelike circle.

8. Conclusions

By choosing the three-dimensional Minkowski space as a background in space–time, we define the osculating developable surface of a timelike surface along a curve, whose ruling is parallel to the osculating Darboux vector field. Our main purpose is to study the singularities of such a surface. For this, by using the singularity theory, we classify the generic singularities of osculating developable surfaces that are cuspidal edges and swallowtails. In particular, these types of singularities are characterized by the invariants $\varepsilon(s)$ and $\sigma(s)$. In fact, the osculating developable surface is a cylinder if and only if $\varepsilon(s) \equiv 0$; the osculating developable surface is a cone if and only if $\varepsilon(s) \neq 0$ and $\sigma(s) \equiv 0$. We also show some special geometric properties of the singularities of osculating developable surfaces from the viewpoint of contact geometry. Moreover, we obtain the dual relationship between the rulings and the normals of osculating developable surfaces.

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