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On r-Regular Integers (mod n^r)

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Abstract: Let $\rho_r(n^r)$ denote the number of positive r-regular integers $\pmod{n^r}$ that are less than or equal to n^r; in this paper, we investigate some arithmetic properties of certain functions related to r-regular integers $\pmod{n^r}$. Then, we study the average orders and the extremal orders of $\rho_r(n^r)$ in connection with the divisor function and the generalized Dedekind function. Moreover, we also introduce an analogue of Cohen–Ramanujan's sum with respect to r-regular integers $\pmod{n^r}$ and show some basic properties of this function.

Keywords: r-regular integers; Euler product; extremal order; generalized Ramanujan's sums

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1. Introduction

Let k, n be two integers, and (k, n) is the greatest common divisor of k and n. Morgado [1] first introduced the concept of regular integers $\pmod{n}$, that is, if there exists an integer x such that $k^2x \equiv k \pmod{n}$, then the integer k is called regular $\pmod{n}$. It was also observed that k is regular $\pmod{n}$ if and only if (k, n) is a unitary divisor of n. We recall that d is said to be a unitary divisor of n if $d|n$ and $(d, n/d) = 1$, notation $d\|n$. As the background of this property, an element a of a ring R is said to be regular if there is an $x \in \mathbb{R}$ such that $a = axa$. Moreover, it is obvious that the regular integers $\pmod{n}$ are an extension of the multiplicative inverse elements $\pmod{n}$. In fact, we can get $kx \equiv 1 \pmod{n}$ if $(k, n) = 1$, so that $x = \bar{k}$ is the multiplicative inverse of $k \pmod{n}$, and $k = \bar{x}$ is the multiplicative inverse of $x \pmod{n}$, which are symmetric. On the other hand, there are still some regular integers $k \pmod{n}$ with $(k, n) > 1$, such that $k^2x \equiv k \pmod{n}$. However, a regular integer $k \pmod{n}$ also has a unique inverse element $\tilde{k}$ symmetric to it if and only if both $k^2\tilde{k} \equiv k \pmod{n}$ and $\tilde{k}^2k \equiv \tilde{k} \pmod{n}$ hold [2]. Hence, it would be interesting to better understand their behavior by studying their various arithmetic properties. For example, see references [3–6].

More generally, Prasad, Reddy, and Rao [7] introduced generalized r-regular integers $\pmod{n^r}$. Similar to the definition of regular integers $\pmod{n}$: let r be an integer; an integer k is said to be r-regular $\pmod{n^r}$ if there is an integer x such that $k^{r+1}x \equiv k^r \pmod{n^r}$. In addition, Rao [8] proved that k is r-regular $\pmod{n^r}$ if and only if $(k, n^r)_r$ is a unitary divisor of n^r, where $(k, n^r)_r$ is the greatest r^{th} power common divisor of k and n^r. The related research on the greatest r^{th} power common divisor can be found in [9].

Let $Reg_r(n^r)$={k: $1 \leq k \leq n^r$; k be an r-regular mod n^r}; $Reg(n)$={k: $1 \leq k \leq n$, k be a regular mod n}; $\rho_r(n^r) = \#Reg_r(n^r)$ be the number of elements in the set $Reg_r(n^r)$; $\rho(n) = \#Reg(n)$ be the number of elements in the set $Reg(n)$. Obviously, for every $n > 1$, $r \geq 1$, we know that $\varphi(n) < \rho(n) \leq n$ and $\phi_r(n^r) < \rho_r(n^r) \leq n^r$, where $\varphi(n)$ is a Euler φ-function, and ϕ_r is a generalization of the Euler φ-function defined by $\phi_r(n^r) = \#\{k\colon 1 \leq k \leq n^r, (k, n^r)_r = 1\}$.

In the past several years, a number of scholars have performed research on the basic properties, arithmetic function and other issues of r-regular integers $\pmod{n^r}$; see the papers [7,8]. Based on previous studies, we further investigate here some arithmetic properties of certain functions related to r-regular integers $\pmod{n^r}$. Furthermore, motivated and inspired by the work of Tóth and Apostol [3–5], we also study the average orders and the extremal orders of $\rho_r(n^r)$ in connection with the divisor function and the generalized Dedekind function. Then, we introduce an analogue of Cohen–Ramanujan's sum with respect to r-regular integers $(\mathrm{mod}\ n^r)$ and show some basic properties of this function.

2. r-Regular Integers (mod n^r) and Function $\rho_r(n^r)$

In his early work, Tóth [3] first summarized some properties of regular integers $(\mathrm{mod}\ n)$, then he and Apostol [6] introduced the multidimensional generalization of $\rho(n)$ and established identities for the power sums of regular integers $(\mathrm{mod}\ n)$ and for other finite sums and products over regular integers $(\mathrm{mod}\ n)$. After that, with the help of [3,9], Rao [8] introduced the notion of r-regular integers $(\mathrm{mod}\ n^r)$, obtained some basic properties of such integers as well as the arithmetic properties of certain functions related to them.

Therefore, firstly, we state here the characterization of r-regular integers $(\mathrm{mod}\ n^r)$. In all that follows, $n > 1$ is of the canonical form $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_t^{\alpha_t}$, where $p_1 < p_2 < \ldots < p_t$ are primes and α_i are integers ≥ 1.

Theorem 1 (Rao, [8])**.** *For an integer $a \geq 1$, the following are equivalent:*

(i) $a \in Reg_r(n^r)$;
(ii) *For every $i \in 1, 2, \ldots, t$, we have either $p_i \nmid a$ or $p_i^{\alpha_i r} | a^{\alpha}$;*
(iii) $(a, n^r)_r \| n^r$;
(iv) $a^{\phi_r(n^r)+r} \equiv a^r (\mathrm{mod}\ n^r)$;
(v) *There exists an integer $k \geq 1$, such that $a^{k+r} \equiv a^r (\mathrm{mod}\ n^r)$.*

Here, $\phi_r(n^r) = \#R_{n,r} = \#\{k \colon 1 \leq k \leq n^r, (k, n^r)_r = 1\}$ denote the number of elements in $R_{n,r}$.

Then, based on the research above, we next give the sums of the s-th powers ($s \in \mathbb{N}$) of r-regular integers $(\mathrm{mod}\ n^r)$ and investigate the average orders of the functions $\sum_{n \leq x} \rho_r(n^r)$ and $\sum_{n \leq x} \frac{\rho_r(n^r)}{\phi_r(n^r)}$.

Theorem 2. *For every $s \in \mathbb{N}$ and real $x > 1$, we have the following asymptotic formula:*

$$\sum_{\substack{k \leq x \\ k \in Reg_r(n^r)}} k^s = \frac{x^{s+1} \rho_r(n^r)}{(s+1) n^r} + O(x^s n^{\varepsilon}).$$

Proof. For $s \geq 0$, notice that

$$\sum_{n \leq x} n^s = \frac{x^{s+1}}{s+1} + O(x^s),$$

which means that we can deduce the following:

$$\begin{aligned}
\sum_{\substack{n \leq x \\ (k,n)_r = 1}} n^s &= \sum_{n \leq x} n^s \sum_{d | (k,n)_r} \mu^r(d) = \sum_{d|k} d^s \mu^r(d) \sum_{e \leq x/d} e^s \\
&= \sum_{d|k} d^s \mu^r(d) \left(\frac{x^{s+1}}{(s+1) d^{s+1}} + O(x/d)^s \right) \\
&= \frac{x^{s+1}}{s+1} \sum_{d|k} \frac{\mu^r(d)}{d} + O\left(x^s \sum_{d|k} |\mu^r(d)| \right) \\
&= \frac{x^{s+1}}{s+1} \cdot \frac{\phi_r(k)}{k} + O(x^s \tau(k)).
\end{aligned}$$

Moreover, the integer k is r-regular $\pmod{n^r}$ if and only if $(k, n^r)_r$ is a unitary divisor of n^r. So, from the definition of $\rho_r(n^r)$, we have the following:

$$\begin{aligned}\rho_r(n^r) &= \sum_{\substack{k=1\\k\in Reg_r(n^r)}}^{n^r} 1 = \sum_{d^r\|n^r}\sum_{\substack{k=1\\(k,n^r)_r=d^r}}^{n^r} 1 = \sum_{d^r\|n^r}\sum_{\substack{j=1\\(j,n^r/d^r)_r=1}}^{n^r/d^r} 1\\ &= \sum_{d^r\|n^r}\phi_r(n^r/d^r) = \sum_{d\|n}\phi_r(d^r).\end{aligned}$$

Thus, we can calculate as follows:

$$\begin{aligned}\sum_{\substack{k\le x\\k\in Reg_r(n^r)}} k^s &= \sum_{d^r\|n^r}\sum_{\substack{k\le x\\(k,n^r)_r=d^r}} k^s = \sum_{d^r\|n^r} d^{rs}\sum_{\substack{j\le x/d^r\\(j,n^r/d^r)_r=1}} j^s\\ &= \sum_{d^r\|n^r} d^{rs}\left(\frac{1}{s+1}\left(\frac{x}{d^r}\right)^{s+1}\frac{\phi_r(n^r/d^r)}{n^r/d^r} + O\left(\left(\frac{x}{d^r}\right)^s\tau\left(\frac{n^r}{d^r}\right)\right)\right)\\ &= \frac{x^{s+1}}{(s+1)n^r}\sum_{d^r\|n^r}\phi_r(n^r/d^r) + O\left(x^s\sum_{d^r\|n^r}\tau\left(\frac{n^r}{d^r}\right)\right)\\ &= \frac{x^{s+1}\rho_r(n^r)}{(s+1)n^r} + O(x^s n^\varepsilon).\end{aligned}$$

This completes the proof of Theorem 2. □

Here, we give a necessary lemma to prove Theorem 3.

Lemma 1. *Let $t \in \mathbb{N}$ and real $x > 1$; then, we can obtain:*

$$\sum_{\substack{n\le x\\(n,t)=1}} \phi_r(n^r) = \frac{x^{r+1}t^r\varphi(t)}{(r+1)\zeta(r+1)J_{r+1}(t)} + O(x^r\tau(t)).$$

where $\zeta(n)$ is a Riemann ζ-function, $J_r(n)$ is a Jordan function, defined as $J_r(n) = n^r\prod_{p|n}\left(1-\frac{1}{p^r}\right)$.

Proof. For $s > 0$ and $t \in \mathbb{N}$, we know that

$$\sum_{\substack{n\le x\\(n,t)=1}} n^s = \frac{x^{s+1}\varphi(t)}{(s+1)t} + O(x^s\tau(t)),$$

and by using the properties of $\phi_r(n^r)$, we can obtain the following equation:

$$\begin{aligned}\sum_{\substack{n\le x\\(n,t)=1}} \phi_r(n^r) &= \sum_{\substack{de=n\le x\\(n,t)=1}} \mu(d)e^r = \sum_{\substack{d\le x\\(d,t)=1}} \mu(d)\sum_{\substack{e\le x/d\\(e,t)=1}} e^r\\ &= \sum_{\substack{d\le x\\(d,t)=1}} \mu(d)\left(\frac{\varphi(t)}{(r+1)t}\left(\frac{x}{d}\right)^{r+1} + O((x/d)^r\tau(t))\right)\\ &= \frac{x^{r+1}\varphi(t)}{(r+1)t}\sum_{\substack{d=1\\(d,t)=1}}^{\infty}\frac{\mu(d)}{d^{r+1}} + O\left(x^{r+1}\sum_{d>x}\frac{1}{d^{r+1}}\right) + O(x^r\tau(t)).\end{aligned}$$

Let $D(k) = 1$, if $(k, n^r)_r = 1$; 0, if $(k, n^r)_r = 1$. It is clear that for $R(s) > 0$ (see [10] (Lemma 5)),

$$\sum_{k=1}^{\infty} \frac{D(k)}{k^s} = \frac{\zeta(s)}{n^{rs}} J_{rs}(n).$$

Thus, let $r = 1$; it can easily be seen that

$$\sum_{\substack{k=1 \\ (k,n)=1}}^{\infty} \frac{\mu(k)}{k^s} = \frac{n^s}{\zeta(s) J_s(n)}.$$

Moreover, notice that $s > 1$, $\sum_{n>x} \frac{1}{n^s} = O(x^{1-s})$; therefore, we have the following formula:

$$\sum_{\substack{n \le x \\ (n,t)=1}} \phi_r(n^r) = \frac{x^{r+1} t^r \varphi(t)}{(r+1)\zeta(r+1) J_{r+1}(t)} + O(x^r \tau(t)).$$

□

Theorem 3. *For real $x > 1$, we have the following asymptotic formula:*

$$\sum_{n \le x} \rho_r(n^r) = \frac{x^{r+1}}{(r+1)\zeta(r+1)} \prod_p \left(1 + \frac{p^{r+1} - p^r}{(p^{r+1}-1)^2}\right) + O(x^r). \tag{1}$$

$$\sum_{n \le x} \frac{\rho_r(n^r)}{\phi_r(n^r)} = x \prod_p \left(1 + \frac{p^r - 1(p-1)}{(p^r - 1)(p^{r+1}-1)}\right) + O(x^{1-r} \log x), \quad (r > 1). \tag{2}$$

Proof. Firstly, we give the proof for (1). It follows from Lemma 1 that

$$\begin{aligned} \sum_{n \le x} \rho_r(n^r) &= \sum_{n \le x} \sum_{\substack{k=1 \\ k \in Reg_r(n^r)}}^{n^r} 1 = \sum_{e \le x} \sum_{\substack{d \le x/e \\ (d,e)=1}} \phi_r(d^r) \\ &= \frac{x^{r+1}}{(r+1)\zeta(r+1)} \sum_{e=1}^{\infty} \frac{\varphi(e)}{e J_{r+1}(e)} + O\left(x^{r+1} \sum_{e>x} \frac{\varphi(e)}{e J_{r+1}(e)}\right) + O\left(x^r \sum_{e \le x} \frac{\tau(e)}{e^s}\right). \end{aligned}$$

Then, by using the Euler product, we obtain the following formula:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\varphi(n)}{n J_{r+1}(n)} &= \prod_p \left(1 + \frac{p(1-1/p)}{p p^{r+1}(1 - 1/p^{r+1})} + \frac{p^2(1-1/p)}{p^2 p^{2(r+1)}(1 - 1/p^{r+1})} + \dots\right) \\ &= \prod_p \left(1 + \frac{1 - 1/p}{1 - 1/p^{r+1}} \left(\frac{1}{p^{r+1}} + \frac{1}{p^{2(r+1)}} + \dots\right)\right) \\ &= \prod_p \left(1 + \frac{p^{r+1} - p^r}{(p^{r+1}-1)^2}\right). \end{aligned}$$

To sum up,

$$\sum_{n \le x} \rho_r(n^r) = \frac{x^{r+1}}{(r+1)\zeta(r+1)} \prod_p \left(1 + \frac{p^{r+1} - p^r}{(p^{r+1}-1)^2}\right) + O(x^r).$$

Next, we give the proof of (2). We know that the arithmetic function ρ_r is an analogue of ϕ_r with respect to r-regular integers $\pmod{n^r}$. Additionally, for every $n \ge 1$, the function

$\rho_r(n^r)$ is multiplicative and $\rho_r(n^r) = \sum_{d\|n} \phi_r(d^r) = \sum_{p^\alpha\|n} (\phi_r(p^{\alpha r}) + 1)$. In addition, for any prime power p^α $(\alpha \geq 1)$, there is

$$\rho_r(p^{\alpha r}) = p^{\alpha r} - p^{(\alpha-1)r} + 1, \rho_r(p^{\alpha r})/\phi_r(p^{\alpha r}) = 1 + 1/\phi_r(p^{\alpha r}).$$

Thus, we have

$$\begin{aligned}
\sum_{n\leq x} \frac{\rho_r(n^r)}{\phi_r(n^r)} &= \sum_{n\leq x}\sum_{d\|n} \frac{1}{\phi_r(d^r)} = \sum_{\substack{de\leq x\\(d,e)=1}} \frac{1}{\phi_r(d^r)} = \sum_{d\leq x} \frac{1}{\phi_r(d^r)} \sum_{\substack{e\leq x/d\\(d,e)=1}} 1 \\
&= \sum_{d\leq x} \frac{1}{\phi_r(d^r)} \left(\frac{\varphi(d)x}{d^2} + O(\tau(d)) \right) \\
&= x\sum_{d\leq x} \frac{\varphi(d)}{\phi_r(d^r)d^2} + O\left(\sum_{d\leq x} \frac{\tau(d)}{\phi_r(d^r)} \right) \\
&= x\sum_{d=1}^{\infty} \frac{\varphi(d)}{\phi_r(d^r)d^2} + O\left(x\sum_{d>x} \frac{1}{d^{r+1}} \right) + O\left(\sum_{d\leq x} \frac{\tau(d)}{\phi_r(d^r)} \right).
\end{aligned}$$

Moreover, the series $\sum_{d=1}^{\infty} \frac{\varphi(d)}{\phi_r(d^r)d^2}$ is absolutely convergent, so by applying the Euler product, we can obtain:

$$\begin{aligned}
\sum_{d=1}^{\infty} \frac{\varphi(d)}{\phi_r(d^r)d^2} &= \prod_p \left(1 + \frac{p-1}{p^2(p^r-1)} + \frac{p(p-1)}{p^4p^r(p^r-1)} + \frac{p^2(p-1)}{p^6p^{2r}(p^r-1)} + \dots \right) \\
&= \prod_p \left(1 + \frac{p^{r-1}(p-1)}{(p^r-1)(p^{r+1}-1)} \right),
\end{aligned}$$

and observe that

$$O\left(x\sum_{d>x} \frac{1}{d^{r+1}} \right) = O(x^{1-r}),$$

$$O\left(\sum_{d\leq x} \frac{\tau(d)}{\phi_r(d^r)} \right) \leq O\left(\sum_{d\leq x} \frac{\tau(d)}{d^r} \right) \ll O(x^{1-r}\log x).$$

To sum up, we can get:

$$\sum_{n\leq x} \frac{\rho_r(n^r)}{\phi_r(n^r)} = x\prod_p \left(1 + \frac{p^r - 1(p-1)}{(p^r-1)(p^{r+1}-1)} \right) + O(x^{1-r}\log x).$$

□

3. Extremal Orders

In many cases, it is difficult to determine the growth of some arithmetical functions, so mathematicians will often first explore the average orders and extremal orders of those functions to provide the bounds. For example, Tóth [3] investigated the average orders and extremal orders of the functions $\rho(n)/\varphi(n)$ and $\rho(n)$ to compare the rates of growth of the functions $\rho(n)$ and $\varphi(n)$. In [4], Apostol studied the extremal orders of the function $\rho(n)$ in connection with the divisor function $\sigma(n)$ and the Dedekind function $\psi(n)$. As we were so inspired by the work of Tóth and Apostol, we now study the extremal orders of $\rho_r(n^r)$. Before the proof, we introduce the following result:

Lemma 2 (Tóth and Wirsing, [11])**.** *If f is a non-negative real-valued multiplicative arithmetic function such that for each prime p,*

(i) $\eta(p) = \sup_{\alpha>0}(f(p^\alpha)) \leq \left(1 - \frac{1}{p}\right)^{-1}$,
(ii) there is an exponent $e_p = p^{o(1)} \in \mathbb{N}$ *satisfying* $f(p^{e_p}) > 1 + \frac{1}{p}$,
then $\limsup_{n\to\infty} \frac{f(n)}{\log\log n} = e^\gamma \prod_p \left(1 - \frac{1}{p}\right)\eta(p)$, *where* γ *is Euler's constant.*

This result can be used to obtain the maximal or minimal orders of a large class of multiplicative arithmetic functions. For its application, we have the following theorems:

Theorem 4. *For* $r \geq 1$, *we have*
$$\limsup_{n\to\infty} \frac{\rho_r(n^r)}{n^r} = 1,$$
$$\liminf_{n\to\infty} \frac{\rho_r(n^r)\log\log n}{n^r} = e^{-\gamma}.$$

Proof. Since $\rho_r(n^r) \leq n^r$ for every $n \geq 1$, and $\rho_r(p^r) = p^r$ for every prime p, it follows that $\limsup_{n\to\infty} \frac{\rho_r(n^r)}{n^r} = 1$.

Moreover, it is clear that the function $\rho_r(n^r)$ is multiplicative and $\rho_r(p^{\alpha r}) = p^{\alpha r} - p^{(\alpha-1)r} + 1$ for any prime p and integer $\alpha \geq 1$. Hence, take $f(n) = \frac{n^r}{\rho_r(n^r)}$ in Lemma 2, which is a non-negative real-valued multiplicative arithmetic function. So we have:
$$f(p^\alpha) = \frac{p^{\alpha r}}{p^{\alpha r} - p^{(\alpha-1)r} + 1} = \left(1 - \frac{1}{p^r} + \frac{1}{p^{\alpha r}}\right)^{-1} < \left(1 - \frac{1}{p}\right)^{-1} = \eta(p),$$
and for $e_p = 2$,
$$f(p^2) = \frac{p^{2r}}{p^{2r} - p^r + 1} = 1 + \frac{p^r - 1}{p^{2r} - p^r + 1} > 1 + \frac{1}{p}$$
for every prime p, so that (ii) in Lemma 2 is satisfied. Then, we can get
$$\liminf_{n\to\infty} \frac{\rho_r(n^r)\log\log n}{n^r} = e^{-\gamma}.$$
□

Theorem 5. *For* $r \geq 2$, *we have*
$$\liminf_{n\to\infty} \frac{\sigma_r(n)}{\rho_r(n^r)} = \liminf_{n\to\infty} \frac{\psi_r(n)}{\rho_r(n^r)} = 1,$$
$$\limsup_{n\to\infty} \frac{\sigma_r(n)}{\rho_r(n^r)(\log\log n)^2} = \limsup_{n\to\infty} \frac{\psi_r(n)}{\rho_r(n^r)(\log\log n)^2} = \frac{6}{\pi^2}e^{2\gamma},$$
where $\sigma_r(n)$ *is the divisor function,* $\psi_r(n)$ *is the generalization of the Dedekind function, as defined by* $\psi_r(n) = n^r \prod_{p|n}\left(1 + \frac{1}{p^r}\right)$.

Proof. Firstly, we notice that $\rho_r(n^r) \leq n^r \leq \sigma_r(n)$ for every $n \geq 1$ since
$$\lim_{n\to\infty} \frac{\sigma_r(p)}{\rho_r(p^r)} = \lim_{n\to\infty} \frac{p^r + 1}{p^r} = 1,$$
for every prime p, and it is also clear that $\psi_r(n) \geq \rho_r(n^r)$, $\frac{\psi_r(n)}{\rho_r(n^r)} = \frac{p^r+1}{p^r}$ for every prime p; hence, we have
$$\liminf_{n\to\infty} \frac{\sigma_r(n)}{\rho_r(n^r)} = \liminf_{n\to\infty} \frac{\psi_r(n)}{\rho_r(n^r)} = 1.$$

Thus, the minimal order of $\frac{\sigma_r(n)}{\rho_r(n^r)}$ and $\frac{\psi_r(n)}{\rho_r(n^r)}$ is 1.

Next, we prove the maximal order of $\frac{\sigma_r(n)}{\rho_r(n^r)}$ and $\frac{\psi_r(n)}{\rho_r(n^r)}$. When $r = 1$, it [4] is proved that

$$\limsup_{n\to\infty} \frac{\sigma(n)}{\rho(n)(\log\log n)^2} = e^{2\gamma},$$

$$\limsup_{n\to\infty} \frac{\psi(n)}{\rho(n)(\log\log n)^2} = \frac{6}{\pi^2}e^{2\gamma}.$$

For $r \geq 2$, let $f(n) = \sqrt{\frac{\sigma_r(n)}{\rho_r(n^r)}}$ in Lemma 2. Then,

$$f(p^\alpha) = \sqrt{\frac{p^{r(\alpha+1)}-1}{(p^r-1)(p^{r\alpha}-p^{r(\alpha-1)}+1)}} \leq \sqrt{\frac{p+1}{p-1}} = \eta(p) < \left(1-\frac{1}{p}\right)^{-1}$$

and

$$f(p^2) = \sqrt{\frac{p^{3r}-1}{(p^r-1)(p^{r\alpha}-p^{2r}+1)}} > \left(1+\frac{1}{p}\right)$$

for every prime p, so (i) and (ii) in Lemma 2 are satisfied. Hence, we can obtain:

$$\limsup_{n\to\infty} \frac{\sqrt{\sigma_r(n)}}{\sqrt{\rho_r(n^r)}\log\log n} = e^\gamma \prod_p \left(1-\frac{1}{p}\right)\sqrt{\frac{p+1}{p-1}} = e^\gamma \prod_p \sqrt{\left(1-\frac{1}{p^2}\right)} = \sqrt{\frac{6}{\pi^2}}e^\gamma;$$

moreover, since $\psi_r(n) \leq \sigma_r(n)$ and $\psi_r(p) = \sigma_r(p) = p^r + 1$ for every prime p, it follows that

$$\limsup_{n\to\infty} \frac{\sigma_r(n)}{\rho_r(n^r)(\log\log n)^2} = \limsup_{n\to\infty} \frac{\psi_r(n)}{\rho_r(n^r)(\log\log n)^2} = \frac{6}{\pi^2}e^{2\gamma},$$

and the proof is complete. □

4. Ramanujan's Sum with Respect to r-Regular Integers $(\bmod\ n^r)$

Ramanujan's sum $c_q(j)$ and Cohen–Ramanujan's sum $c_q^r(j)$ are defined as

$$c_q(j) = \sum_{\substack{a \bmod q \\ (a,q)=1}} \exp(2\pi i j a/q),$$

$$c_q^r(j) = \sum_{\substack{a \bmod q^r \\ (a,q^r)_r=1}} \exp(2\pi i j a/q^r),$$

where $r, q \in \mathbb{N}$ and $j \in \mathbb{Z}$. In [12], Tóth introduced an analogue of Ramanujan's sum with respect to regular integers $(\bmod\ q)$

$$\bar{c}_q(j) = \sum_{\substack{a \bmod q \\ a\in Reg(q)}} \exp(2\pi i j a/q)$$

and revealed that this analogue had properties similar to the usual Ramanujan's sum.

Let g_q denote the characteristic function of the unitary divisors of q; that is, $g_q(a) = 1$ if $a\|q$, and $g_q(a) = 0$ if otherwise. In addition, let $\bar{\mu}_q$ denote the function defined by $(\bar{\mu}_q * 1)(a) = g_q(a)$, where $*$ is the Dirichlet convolution and $1(a) = 1$ for all $a \in \mathbb{N}$. Then, $g_q(a)$ and $\bar{\mu}_q(a)$ are both multiplicative in a.

We know that Ramanujan's sums and their variations make appearances in the singular series of the Hardy–Littlewood asymptotic formula for Waring problems and in the asymptotic formula of Vinogradov on the sums of three primes. Furthermore, its generalizations have also been extensively studied by scholars. Therefore, we now define the analogue of Ramanujan's sum with respect to r-regular integers $(\bmod\ q^r)$ as follows:

$$\bar{c}_q^r(j) = \sum_{\substack{a \bmod q^r \\ a \in Reg_r(q^r)}} \exp(2\pi i j a / q^r).$$

Then, we have following results.

Theorem 6. *Let $r, q \in \mathbb{N}$ and $j \in \mathbb{Z}$, and we have*
(i) $\bar{c}_q^r(j) = \sum_{d|(j,q^r)_r} d^r \bar{\mu}_{q^r}\left(\frac{q}{d}\right)$;
(ii) $\bar{c}_q^r(j) = \sum_{d^r \| q^r} c_d^r(j)$;
(iii) $\bar{c}_q^r(j)$ *is multiplicative in q.*

Proof. (i). It is clear that a is r-regular $(\bmod\ q^r)$ if and only if $(a, q^r)_r$ is a unitary divisor of q^r; thus, we can deduce that:

$$\begin{aligned}
\bar{c}_q^r(j) &= \sum_{a \bmod q^r} \exp(2\pi i j a / q^r) g_{q^r}((a, q^r)_r) \\
&= \sum_{a=1}^{q^r} \exp(2\pi i j a / q^r) \sum_{d|(a,q^r)_r} \bar{\mu}_{q^r}(d) \\
&= \sum_{d|q} \bar{\mu}_{q^r}(d) \sum_{\substack{a=1 \\ d^r|a}}^{q^r} \exp(2\pi i j a / q^r) \\
&= \sum_{d|q} \bar{\mu}_{q^r}(d) \sum_{a=1}^{q^r/d^r} \exp(2\pi i j a / \frac{q^r}{d^r}) \\
&= \sum_{d|q} \bar{\mu}_{q^r}(\frac{q}{d}) \sum_{a=1}^{d^r} \exp(2\pi i j a / d^r) \\
&= \sum_{\substack{d|q \\ d^r|j}} d^r \bar{\mu}_{q^r}\left(\frac{q}{d}\right) \\
&= \sum_{d|(j,q^r)_r} d^r \bar{\mu}_{q^r}\left(\frac{q}{d}\right),
\end{aligned}$$

where we used

$$\sum_{a=1}^{q^r} \exp(2\pi i j a / q^r) = \begin{cases} q^r, & \text{if } q^r | j; \\ 0, & \text{if } q^r \nmid j. \end{cases}$$

It is well-known that the usual Cohen–Ramanujan sum can be written as follows:

$$c_q^r(j) = \sum_{d|(j,q^r)_r} d^r \mu\left(\frac{q}{d}\right).$$

Therefore here, formula (i) also gave such a convolutional expression of $\bar{c}_q^r(j)$, which is accordingly an analogous result of the Cohen–Ramanujan sum.

(ii). From (i) and (iii) of Theorem 1, we can easily obtain:

$$\bar{c}_q^r(j) = \sum_{\substack{a \bmod q^r \\ a \in Reg_r(q^r)}} \exp(2\pi i j a / q^r) = \sum_{d^r \| q^r} \sum_{\substack{a=1 \\ (a,q^r)_r = d^r}}^{q^r} \exp(2\pi i j a / q^r)$$

$$= \sum_{d^r \| q^r} \sum_{\substack{a=1 \\ (a,q^r/d^r)_r=1}}^{q^r/d^r} \exp(2\pi i j a / \frac{q^r}{d^r}) = \sum_{d^r \| q^r} c_d^r(j).$$

(iii). Let q, h be positive integers and $(q,h) = 1$; we know that for every positive integer j, Cohen-Ramanujan's sum $c_q^r(j)$ is multiplicative in q, that is, $c_{qh}^r(j) = c_q^r(j)c_h^r(j)$. Thus, from (ii), we can get the following equation:

$$\begin{aligned} \bar{c}_{qh}^r(j) &= \sum_{d^r \| q^r h^r} \bar{c}_d^r(j) = \sum_{d_1^r \| q^r} \sum_{d_2^r \| h^r} c_{d_1 d_2}^r(j) \\ &= \sum_{d_1^r \| q^r} c_{d_1}^r(j) \sum_{d_2^r \| h^r} c_{d_2}^r(j) = \bar{c}_q^r(j)\bar{c}_h^r(j). \end{aligned}$$

Hence, its values at prime powers $q = p^\alpha$ are given as follows:

$$\bar{c}_{p^\alpha}^r(j) = \sum_{d^r \| p^{\alpha r}} c_d^r(j) = 1 + c_{p^\alpha}^r(j) = \begin{cases} 1 + p^{\alpha r} - p^{(\alpha-1)r}, & \text{if } p^{\alpha r} | j, \\ 1 - p^{(\alpha-1)r}, & \text{if } p^{(\alpha-1)r} | j,\ p^{\alpha r} \nmid j, \\ 1, & \text{otherwise.} \end{cases}$$

Note that for $j = 0$, we have $\rho_r(p^\alpha) = \phi_r(p^{\alpha r}) + 1$. $\square$

5. Conclusions

In this article, the aim was to investigate various functions based on r-regular integers $(\mathrm{mod}\ n^r)$. We know that the arithmetic function ρ_r is an analogue of ϕ_r, and that the function $\rho_r(n^r)$ and $\phi_r(n^r)$ are both multiplicative for every $n \geq 1$. Hence, in order to compare the rates of growth of the functions, Theorem 3 investigated the average orders and extremal orders of the functions $\rho_r(n^r)$ in connection with $\phi_r(n^r)$, the divisor function and generalized Dedekind function in Theorems 3–5. These results add to the rapidly expanding field and provide a basis for deeper research into r-regular integers. Moreover, we introduced an analogue of Cohen–Ramanujan's sum with respect to r-regular integers $(\mathrm{mod}\ n^r)$ and showed some basic properties of this function in Theorem 6. This function is a natural generalization of the usual Ramanujan sum and its investigation provided a deeper insight into Ramanujan's sum. More broadly, research is also needed to determine the multidimensional generalization of the arithmetic function $\rho_r(n^r)$, which will be the focus of our upcoming research. Furthermore, a more natural progression of this work is to investigate the behavior of some arithmetic functions with respect to r-regular integers $(\mathrm{mod}\ n^r)$. These include the weighted average of the $\bar{c}_q^r(j)$, expansions of the arithmetic functions of several variables with respect to the $\bar{c}_q^r(j)$, and so on.

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