symmetry

MDPI

Article

Multiplicative Brownian Motion Stabilizes the Exact Stochastic Solutions of the Davey–Stewartson Equations

Farah M. Al-Askar [1], Clemente Cesarano [2] and Wael W. Mohammed [3,4,*]

[1] Department of Mathematical Science, Collage of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
[2] Section of Mathematics, International Telematic University Uninettuno, Corso, Vittorio Emanuele II, 39, 00186 Roma, Italy
[3] Department of Mathematics, Collage of Science, University of Ha'il, Ha'il 2440, Saudi Arabia
[4] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
* Correspondence: wael.mohammed@mans.edu.eg

Abstract: In this article, the stochastic Davey–Stewartson equations (SDSEs) forced by multiplicative noise are addressed. We use the mapping method to find new rational, elliptic, hyperbolic and trigonometric functions. In addition, we generalize some previously obtained results. Due to the significance of the Davey–Stewartson equations in plasma physics, nonlinear optics, hydrodynamics and other fields, the discovered solutions are useful in explaining a number of intriguing physical phenomena. By using MATLAB tools to simulate our results and display some of 3D graphs, we show how the multiplicative Brownian motion impacts the analytical solutions of the SDSEs. Finally, we demonstrate the effect of multiplicative Brownian motion on the stability and the symmetry of the achieved solutions of the SDSEs.

Keywords: stochastic Davey–Stewartson equations; multiplicative noise; the mapping method

MSC: 35Q51; 83C15; 35A20; 60H10; 60H15

check for updates

Citation: Al-Askar, F.M.; Cesarano, C.; Mohammed, W.W. Multiplicative Brownian Motion Stabilizes the Exact Stochastic Solutions of the Davey–Stewartson Equations. *Symmetry* **2022**, *14*, 2176. https://doi.org/10.3390/sym14102176

Academic Editors: Danny Arrigo and Sergei D. Odintsov

Received: 8 September 2022
Accepted: 14 October 2022
Published: 17 October 2022

1. Introduction

The majority of nonlinear physical phenomena that occur in a variety of scientific areas, including chemical kinetics, optical fibers, fluid dynamics, solid-state physics and mathematical biology, may be modeled via nonlinear partial differential equations (PDEs). The investigate of the analytical solutions of these PDEs is critical for comprehension most nonlinear physical phenomena and their applications. Various approaches for discovering the analytical solutions of PDEs have been presented to overcome this issue. Some examples of the most significant methods are tanh-sech [1–3], extended tanh-function [4], the Sine-Gordon expansion [5], the trial function [6], the Darboux transformation [7], the Jacobi elliptic function [8,9], the sine-cosine [10,11], (G'/G)-expansion [12,13], Hirota's function [14], $exp(-\phi(\varsigma))$-expansion [15], perturbation [16,17], the qualitative theory of dynamical systems [18–21], the direct method [22], the Riccati–Bernoulli sub-ODE [23], and the F-expansion method [24].

On the other side, the advantages of taking random influences into account in the analysis, simulation, prediction and modeling of complex processes have been highlighted in several fields including chemistry, geophysics, fluid mechanics, biology, atmosphere, physics, climate dynamics, engineering and other fields [25–28]. Since noise may produce statistical features and significant phenomena, it cannot be ignored. In general, it is more difficult to obtain exact solutions to PDEs forced by stochastic terms than to obtain those to classical ones.

The Davey–Stewartson equations affected by multiplicative noise in the Stratonovich sense are taken into consideration as follows:

$$iu_t + \frac{1}{2}\delta^2(u_{xx} + \delta^2 u_{yy}) + \kappa|u|^2 u - uv + i\sigma u \circ B_t = 0, \tag{1}$$

$$v_{xx} - \delta^2 v_{yy} - 2\kappa(|u|^2)_{xx} = 0, \tag{2}$$

where $u(x,y,t) \in \mathbb{C}$, $v(x,y,t) \in \mathbb{R}$, $\kappa = \pm 1$ and $\delta^2 = \pm 1$. The constant κ measures the cubic nonlinearity. The case $\delta = 1$ is known as the DS-I equation, while $\delta = i$ is known as the DS-II equation. They occur in a variety of applications, including the description of gravity–capillary surface wave packets in shallow water. $B(t)$ is Brownian motion, and σ is the strength of the noise.

We notice that there are various ways such as the Itô and Stratonovich calculus to interpret the stochastic integral $\int_0^t YdB$. The stochastic integral is Itô (indicated by $\int_0^t YdB$) when it is assessed at the left-end as opposed to a Stratonovich stochastic integral (indicated by $\int_0^t Y \circ dB$), which is computed in the center [29]. The following equation is how the Itô integral and Stratonovich integral are related:

$$\int_0^t Y(Z_\tau)dB(\tau) = \int_0^t Y(Z_\tau) \circ dB(\tau) - \frac{1}{2}\int_0^t Y(Z_\tau)\frac{\partial Y(Z_\tau)}{\partial z}d\tau, \tag{3}$$

where Y is supposed to be sufficiently regular and $\{Z_t, t \geq 0\}$ is a stochastic process.

The Davey–Stewartson equation was created in 1974 by Davey and Stewartson [30]. This equation is employed to demonstrate how a three-dimensional wave packet evolves over time in a restricted depth of water. The solutions of the deterministic Davey–Stewartson equations (DDSEs) (1) and (2), i.e., $\sigma = 0$, have been used in hydrodynamics, nonlinear optics, plasma physics and other fields. For example, the solutions of the DDSEs might explain the interaction of a properly matched spatiotemporal optical pattern and microwaves. As a result, many authors have investigated the analytical solutions for this equation by using different methods such as the extended Jacobi's elliptic function [31], the first integral method [32], the trial equation method [33], the uniform algebraic method [34], the double exp-function [35], generalized (G'/G)-expansion [36], (G'/G)-expansion [37] and sine–cosine [38].

Our motivation in this paper is to attain the analytical solutions of the stochastic Davey–Stewartson equations (SDSEs). This work is the first to obtain the analytical solutions of SDSEs (1) and (2). We employ the mapping method to obtain a wide range of stochastic solutions, such as rational, elliptic, trigonometric and hyperbolic functions. Due to the significance of the Davey–Stewartson equations in nonlinear optics, plasma physics, hydrodynamics and other areas, the discovered solutions are useful in explaining a number of intriguing physical phenomena. In addition, we extend previously obtained results, such as the one described in [31,37,38]. Moreover, to study the impacts of Brownian motion on the stability and symmetry of the obtained solutions of SDSEs (1) and (2), we built 3D graphs for some of the developed solutions by using MATLAB tools.

This is how the paper is organized: We use a suitable wave transformation in Section 3 to provide the wave equation of SDSEs. We employ the mapping method in Section 4 to obtain the analytical solutions of the SDSEs (1) and (2). In Section 5, we examine the impact of Brownian motion on the derived solutions. Finally, we state the conclusions of this paper.

2. Wave Equation for SDSEs

The following wave transformation is utilized to obtain the wave equation for the SDSEs (1) and (2):

$$u(x,y,t) = \varphi(\zeta)e^{(i\rho - \sigma B(t) - \sigma^2 t)}, \; v(x,y,t) = \psi(\zeta)e^{(-2\sigma B(t) - 2\sigma^2 t)}, \tag{4}$$

with
$$\zeta = \zeta_1 x + \zeta_2 y - \zeta_3 t \text{ and } \rho = \rho_1 x + \rho_2 y + \rho_3 t,$$
where ψ and φ are deterministic functions and $\{\zeta_j\}_{j=1}^{3}$, $\{\rho_j\}_{j=1}^{3}$ are nonzero constants. Putting Equation (4) into Equation (1) and utilizing

$$\begin{aligned} u_t &= (-\zeta_3\varphi' + i\rho_3\varphi - \sigma\varphi B_t - \frac{\sigma^2}{2}\varphi)e^{(i\rho-\sigma B(t)-\sigma^2 t)}, \\ &= (-\zeta_3\varphi' + i\rho_3\varphi - \sigma\varphi \circ B_t)e^{(i\rho-\sigma B(t)-\sigma^2 t)}, \end{aligned}$$

where we used (3), and

$$\begin{aligned} u_{xx} &= (\zeta_1^2\varphi'' + 2i\rho_1\zeta_1\varphi' - \rho_1^2\varphi)e^{(i\rho-\sigma B(t)-\sigma^2 t)}, \\ u_{yy} &= (\zeta_2^2\varphi'' + 2i\rho_2\zeta_2\varphi' - \rho_2^2\varphi)e^{(i\rho-\sigma B(t)-\sigma^2 t)}, \\ (|u|^2)_{xx} &= \zeta_1^2(\varphi^2)'e^{(-2\sigma B(t)-2\sigma^2 t)}, \\ v_x &= \zeta_1\psi' e^{(-2\sigma B(t)-2\sigma^2 t)},\ v_{xx} = \zeta_1^2\psi'' e^{(-2\sigma B(t)-2\sigma^2 t)}, \end{aligned}$$

we obtain, for the real part,

$$(\frac{1}{2}\zeta_1^2\delta^2 + \frac{1}{2}\zeta_2^2\delta^4)\varphi'' - (\rho_3 + \frac{1}{2}\delta^2\rho_1^2 + \frac{1}{2}\delta^4\rho_2^2)\varphi + (\kappa\varphi^3 - \varphi\psi)e^{(-2\sigma B(t)-2\sigma^2 t)} = 0, \quad (5)$$

$$(\zeta_1^2 - \delta^2\zeta_2^2)\psi'' - 2\zeta_1^2\kappa(\varphi^2)'' = 0 \quad (6)$$

and, for the imaginary part,

$$(-\zeta_3 + 2\zeta_1\rho_1 + 2\zeta_2\rho_2)\varphi' = 0, \quad (7)$$

From Equation (7), we obtain

$$\zeta_3 = 2\zeta_1\rho_1 + 2\zeta_2\rho_2. \quad (8)$$

Now, integrating Equation (6) once, we attain

$$\psi = \frac{2\zeta_1^2\kappa}{(\zeta_1^2 - \delta^2\zeta_2^2)}\varphi^2. \quad (9)$$

Substituting Equation (9) into Equation (5), we obtain

$$\varphi'' - \gamma_2\varphi + \gamma_1\varphi^3 e^{(-2\sigma B(t)-2\sigma^2 t)} = 0, \quad (10)$$

where

$$\gamma_1 = \frac{2\kappa}{\delta^2(\zeta_1^2 - \delta^2\zeta_2^2)} \quad \text{and} \quad \gamma_2 = \frac{2\rho_3 + \delta^2\rho_1^2 + \delta^4\rho_2^2}{\zeta_1^2\delta^2 + \zeta_2^2\delta^4}. \quad (11)$$

We take the expectation $E(\cdot)$ on both sides

$$\varphi'' - \gamma_2\varphi - \gamma_1\varphi^3 e^{-2\sigma^2 t}\mathbb{E}(e^{-2\sigma B(t)}) = 0. \quad (12)$$

Since $B(t)$ is normally distributed , $E(e^{-2\sigma B(t)}) = e^{2\sigma^2 t}$. Hence, Equation (12) becomes

$$\varphi'' - \gamma_1\varphi^3 - \gamma_2\varphi = 0. \quad (13)$$

3. The Analytical Solutions of the SDSEs

We use here the mapping method [39] to obtain the solutions to Equation (13). Consequently, we obtain the analytical solutions of SDSEs (1) and (2).

3.1. Method Description

Let the solutions of Equation (13) have the form

$$\varphi(\zeta) = \sum_{i=1}^{N} a_i \mathcal{Z}^i, \tag{14}$$

where Z solves

$$\mathcal{Z}' = \sqrt{\frac{1}{2}\ell_1 \mathcal{Z}^4 + \ell_2 \mathcal{Z}^2 + \ell_3}, \tag{15}$$

where ℓ_1, ℓ_2 and ℓ_3 are real parameters.

We see that there are several different solutions, relying on ℓ_1, ℓ_2 and ℓ_3, of Equation (15) as follows (Table 1):

Table 1. All the solutions of Equation (15) for various values of ℓ_1, ℓ_2 and ℓ_3.

Case	ℓ_1	ℓ_2	ℓ_3	$Z(\zeta)$
1	$2m^2$	$-(1+m^2)$	1	$sn(\zeta)$
2	2	$2m^2-1$	$-m^2(1-m^2)$	$ds(\zeta)$
3	2	$2-m^2$	$(1-m^2)$	$cs(\zeta)$
4	$-2m^2$	$2m^2-1$	$(1-m^2)$	$cn(\zeta)$
5	-2	$2-m^2$	(m^2-1)	$dn(\zeta)$
6	$\frac{\mathbf{m}^2}{2}$	$\frac{(\mathbf{m}^2-2)}{2}$	$\frac{1}{4}$	$\frac{sn(\zeta)}{1\pm dn(\zeta)}$
7	$\frac{\mathbf{m}^2}{2}$	$\frac{(\mathbf{m}^2-2)}{2}$	$\frac{\mathbf{m}^2}{4}$	$\frac{sn(\zeta)}{1\pm dn(\zeta)}$
8	$\frac{-1}{2}$	$\frac{(\mathbf{m}^2+1)}{2}$	$\frac{-(1-\mathbf{m}^2)^2}{4}$	$mcn(\zeta) \pm dn(\zeta)$
9	$\frac{\mathbf{m}^2-1}{2}$	$\frac{(\mathbf{m}^2+1)}{2}$	$\frac{(\mathbf{m}^2-1)}{4}$	$\frac{dn(\zeta)}{1\pm sn(\zeta)}$
10	$\frac{1-\mathbf{m}^2}{2}$	$\frac{(1-\mathbf{m}^2)}{2}$	$\frac{(1-\mathbf{m}^2)}{4}$	$\frac{cn(\zeta)}{1\pm sn(\zeta)}$
11	$\frac{(1-\mathbf{m}^2)^2}{2}$	$\frac{(1-\mathbf{m}^2)^2}{2}$	$\frac{1}{4}$	$\frac{sn(\zeta)}{dn\pm cn(\zeta)}$
12	2	0	0	$\frac{c}{\zeta}$
13	0	1	0	ce^{ζ}

where $dn(\zeta) = dn(\zeta, m)$, $sn(\zeta) = sn(\zeta, m)$, $cn(\zeta) = cn(\zeta, m)$ for $0 < m < 1$ are the Jacobi elliptic functions (JEFs). If $m \to 1$, then the following hyperbolic functions are created from JEFs:

$$\begin{aligned} cs(\zeta) &\to \operatorname{csch}(\zeta), \quad sn(\zeta) \to \tanh(\zeta), \quad cn(\zeta) \to \operatorname{sech}(\zeta), \\ dn(\zeta) &\to \operatorname{sech}(\zeta), \quad ds \to \operatorname{csch}(\zeta). \end{aligned}$$

When $m \to 0$, the following e triangular functions are created:

$$\begin{aligned} sn(\xi) &\to \sin(\xi), \quad cn(\xi) \to \cos(\xi), \quad dn(\xi) \to 1, \\ cs(\xi) &\to \cot(\xi), \quad ds \to \csc(\xi). \end{aligned}$$

3.2. Solutions of SDSEs

Let us balance φ'' with φ^3 in Equation (13) to determine the parameter M as follows:

$$M + 2 = 3M \Longrightarrow M = 1.$$

Equation (15) is rewritten with $M = 1$ as

$$\varphi = a_0 + a_1 \mathcal{Z}. \tag{16}$$

Upon differentiating Equation (16) twice, we have, by using (15),

$$\varphi'' = a_1\ell_2\mathcal{Z} + a_1\ell_1\mathcal{Z}^3. \tag{17}$$

Upon plugging Equation (16) and Equation (17) into Equation (13), we have

$$(a_1\ell_1 - \gamma_1 a_1^3)\mathcal{Z}^3 - 3a_0 a_1^2\gamma_1\mathcal{Z}^2 + (a_1\ell_2 - 3\gamma_1 a_0^2 a_1 + \gamma_2 a_1)\mathcal{Z} - (\gamma_1 a_0^3 - \gamma_2 a_0) = 0.$$

By setting each coefficient of Z^k for $k = 0, 1, 2, 3$ equal to zero, we attain

$$a_1\ell_1 - \gamma_1 a_1^3 = 0,$$

$$3a_0 a_1^2\gamma_1 = 0,$$

$$a_1\ell_2 - 3\gamma_1 a_0^2 a_1 + \gamma_2 a_1 = 0,$$

and

$$\gamma_1 a_0^3 - \gamma_2 a_0 = 0.$$

We obtain, by solving these equations,

$$a_0 = 0, \;\; a_1 = \pm\sqrt{\frac{\ell_1}{\gamma_1}}, \;\; \ell_2 = -\gamma_2,$$

Thus, Equation (13) has the following solution:

$$\varphi(\zeta) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}\mathcal{Z}(\zeta), \;\; \text{for} \;\; \frac{\ell_1}{\gamma_1} > 0. \tag{18}$$

The following are two sets that rely on ℓ_1 and γ_1 :

First set: If $\ell_1 > 0$ and $\gamma_1 > 0$, then there are many cases:

First case: If $\ell_1 = 2m^2$, $\ell_2 = -(m^2+1)$ and $\ell_3 = 1$, then $Z(\zeta) = sn(\zeta)$ and the solutions of the wave Equation (13) are

$$\varphi(\zeta) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}sn(\zeta).$$

Therefore, the solution of SDSEs (1) and (2) has the form

$$u(x,y,t) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}sn(\zeta_1 x + \zeta_2 y - (2\zeta_1\rho_1 + 2\zeta_2\rho_2)t)e^{(i\rho - \sigma\beta(t) - \sigma^2 t)}, \tag{19}$$

$$v(x,y,t) = \frac{2\zeta_1^2\kappa\ell_1}{(\zeta_1^2 - \delta^2\zeta_2^2)\gamma_1}sn^2(\zeta_1 x + \zeta_2 y - (2\zeta_1\rho_1 + 2\zeta_2\rho_2)t)e^{(-2\sigma\beta(t) - 2\sigma^2 t)}. \tag{20}$$

If $m \to 1$, then Equations (19) and (20) become

$$u(x,y,t) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}\tanh(\zeta_1 x + \zeta_2 y - (2\zeta_1\rho_1 + 2\zeta_2\rho_2)t)e^{(i\rho - \sigma\beta(t) - \sigma^2 t)}, \tag{21}$$

$$v(x,y,t) = \frac{2\zeta_1^2\kappa\ell_1}{(\zeta_1^2 - \delta^2\zeta_2^2)\gamma_1}\tanh^2(\zeta_1 x + \zeta_2 y - (2\zeta_1\rho_1 + 2\zeta_2\rho_2)t)e^{(-2\sigma\beta(t) - 2\sigma^2 t)}. \tag{22}$$

Second case: If $\ell_1 = 2$, $\ell_2 = (2 - m^2)$ and $\ell_3 = 1 - m^2$, then $Z(\zeta) = cs(\zeta)$ and the solutions of the wave Equation (13) are

$$\varphi(\zeta) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}cs(\zeta).$$

Therefore, the solution of SDSEs (1) and (2) has the form

$$u(x,y,t) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}cs(\zeta_1 x + \zeta_2 y - (2\zeta_1\rho_1 + 2\zeta_2\rho_2)t)e^{(i\rho - \sigma\beta(t) - \sigma^2 t)}, \tag{23}$$

$$v(x,y,t) = \frac{2\zeta_1^2\kappa\ell_1}{(\zeta_1^2 - \delta^2\zeta_2^2)\gamma_1}cs^2(\zeta_1 x + \zeta_2 y - (2\zeta_1\rho_1 + 2\zeta_2\rho_2)t)e^{(-2\sigma\beta(t) - 2\sigma^2 t)}. \tag{24}$$

If $m \to 1$, then Equations (23) and (24) become

$$u(x,y,t) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}\tanh(\zeta_1 x + \zeta_2 y - (2\zeta_1\rho_1 + 2\zeta_2\rho_2)t)e^{(i\rho - \sigma\beta(t) - \sigma^2 t)}, \tag{25}$$

$$v(x,y,t) = \frac{2\zeta_1^2\kappa\ell_1}{(\zeta_1^2 - \delta^2\zeta_2^2)\gamma_1}\tanh^2(\zeta_1 x + \zeta_2 y - (2\zeta_1\rho_1 + 2\zeta_2\rho_2)t)e^{(-2\sigma\beta(t) - 2\sigma^2 t)}. \tag{26}$$

Third case: If $\ell_1 = 2$, $\ell_2 = 2m^2 - 1$ and $\ell_3 = -m^2(1 - m^2)$, then $Z(\zeta) = ds(\zeta)$ and the solutions of the wave Equation (13) are

$$\varphi(\zeta) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}ds(\zeta).$$

Therefore, the solution of SDSEs (1) and (2) has the form

$$u(x,y,t) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}ds(\zeta_1 x + \zeta_2 y - (2\zeta_1\rho_1 + 2\zeta_2\rho_2)t)e^{(i\rho - \sigma\beta(t) - \sigma^2 t)}, \tag{27}$$

$$v(x,y,t) = \frac{2\zeta_1^2\kappa\ell_1}{(\zeta_1^2 - \delta^2\zeta_2^2)\gamma_1}ds^2(\zeta_1 x + \zeta_2 y - (2\zeta_1\rho_1 + 2\zeta_2\rho_2)t)e^{(-2\sigma\beta(t) - 2\sigma^2 t)}. \tag{28}$$

If $m \to 1$, then Equations (27) and (28) become

$$u(x,y,t) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}\operatorname{csch}(\zeta_1 x + \zeta_2 y - (2\zeta_1\rho_1 + 2\zeta_2\rho_2)t)e^{(i\rho - \sigma\beta(t) - \sigma^2 t)}, \tag{29}$$

$$v(x,y,t) = \frac{2\zeta_1^2\kappa\ell_1}{(\zeta_1^2 - \delta^2\zeta_2^2)\gamma_1}\operatorname{csch}^2(\zeta_1 x + \zeta_2 y - (2\zeta_1\rho_1 + 2\zeta_2\rho_2)t)e^{(-2\sigma\beta(t) - 2\sigma^2 t)}. \tag{30}$$

If $m \to 0$, then Equations (27) and (28) become

$$u(x,y,t) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}\csc(\zeta_1 x + \zeta_2 y - (2\zeta_1\rho_1 + 2\zeta_2\rho_2)t)e^{(i\rho - \sigma\beta(t) - \sigma^2 t)}, \tag{31}$$

$$v(x,y,t) = \frac{2\zeta_1^2\kappa\ell_1}{(\zeta_1^2 - \delta^2\zeta_2^2)\gamma_1}\csc^2(\zeta_1 x + \zeta_2 y - (2\zeta_1\rho_1 + 2\zeta_2\rho_2)t)e^{(-2\sigma\beta(t) - 2\sigma^2 t)}. \tag{32}$$

Fourth case: If $\ell_1 = \frac{\mathbf{m}^2}{2}$, $\ell_2 = \frac{(\mathbf{m}^2-2)}{2}$ and $\ell_3 = \frac{1}{4}$ (or $\frac{\mathbf{m}^2}{4}$), then $Z(\zeta) = \frac{sn(\zeta)}{1 \pm dn(\zeta)}$ and the solutions of the wave Equation (13) are

$$\varphi(\zeta) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}\frac{sn(\zeta)}{1 \pm dn(\zeta)}.$$

Therefore, the solution of SDSEs (1) and (2) has the form

$$u(x,y,t) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}\frac{sn(\zeta)}{1 \pm dn(\zeta)}e^{(i\rho-\sigma\beta(t)-\sigma^2 t)}, \quad (33)$$

$$v(x,y,t) = \frac{2\zeta_1^2\kappa\ell_1}{(\zeta_1^2 - \delta^2\zeta_2^2)\gamma_1}[\frac{sn(\zeta)}{1 \pm dn(\zeta)}]^2 e^{(-2\sigma\beta(t)-2\sigma^2 t)}. \quad (34)$$

If $m \to 1$, then Equations (33) and (34) become

$$u(x,y,t) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}\frac{\tanh(\zeta)}{1 \pm \operatorname{sech}(\zeta)}e^{(i\rho-\sigma\beta(t)-\sigma^2 t)}, \quad (35)$$

$$v(x,y,t) = \frac{2\zeta_1^2\kappa\ell_1}{(\zeta_1^2 - \delta^2\zeta_2^2)\gamma_1}[\frac{\tanh(\zeta)}{1 \pm \operatorname{sech}(\zeta)}]^2 e^{(-2\sigma\beta(t)-2\sigma^2 t)}, \quad (36)$$

where $\zeta = \zeta_1 x + \zeta_2 y - (2\zeta_1\rho_1 + 2\zeta_2\rho_2)t$ and $\rho = \rho_1 x + \rho_2 y + \rho_3 t$.

Fifth case: If $\ell_1 = \frac{1-\mathbf{m}^2}{2}$, $\ell_2 = \frac{(1-\mathbf{m}^2)}{2}$ and $\ell_3 = \frac{(1-\mathbf{m}^2)}{4}$, then $Z(\zeta) = \frac{cn(\zeta)}{1 \pm sn(\zeta)}$ and the solutions of the wave Equation (13) are

$$\varphi(\zeta) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}\frac{cn(\zeta)}{1 \pm sn(\zeta)}.$$

Therefore, the solution of SDSEs (1) and (2) has the form

$$u(x,y,t) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}\frac{cn(\zeta)}{1 \pm sn(\zeta)}e^{(i\rho-\sigma\beta(t)-\sigma^2 t)}, \quad (37)$$

$$v(x,y,t) = \frac{2\zeta_1^2\kappa\ell_1}{(\zeta_1^2 - \delta^2\zeta_2^2)\gamma_1}[\frac{cn(\zeta)}{1 \pm sn(\zeta)}]^2 e^{(-2\sigma\beta(t)-2\sigma^2 t)}. \quad (38)$$

If $m \to 0$, then Equations (37) and (38) become

$$u(x,y,t) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}\frac{\cos(\zeta)}{1 \pm \sin(\zeta)}e^{(i\rho-\sigma\beta(t)-\sigma^2 t)}, \quad (39)$$

$$v(x,y,t) = \frac{2\zeta_1^2\kappa\ell_1}{(\zeta_1^2 - \delta^2\zeta_2^2)\gamma_1}[\frac{\cos(\zeta)}{1 \pm \sin(\zeta)}]^2 e^{(-2\sigma\beta(t)-2\sigma^2 t)}. \quad (40)$$

Sixth case: If $\ell_1 = \frac{(1-\mathbf{m}^2)^2}{2}$, $\ell_2 = \frac{(1-\mathbf{m}^2)^2}{2}$ and $\ell_3 = \frac{1}{4}$, then $Z(\zeta) = \frac{sn(\zeta)}{dn \pm cn(\zeta)}$ and the solutions of the wave Equation (13) are

$$\varphi(\zeta) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}\frac{sn(\zeta)}{dn \pm cn(\zeta)}.$$

Therefore, the solution of SDSEs (1) and (2) has the form

$$u(x,y,t) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}\frac{sn(\zeta)}{dn \pm cn(\zeta)}e^{(i\rho-\sigma\beta(t)-\sigma^2 t)}, \tag{41}$$

$$v(x,y,t) = \frac{2\zeta_1^2\kappa\ell_1}{(\zeta_1^2-\delta^2\zeta_2^2)\gamma_1}[\frac{sn(\zeta)}{dn \pm cn(\zeta)}]^2 e^{(-2\sigma\beta(t)-2\sigma^2 t)}. \tag{42}$$

If $m \to 0$, then Equations (41) and (42) become

$$u(x,y,t) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}\frac{\sin(\zeta)}{1 \pm \cos(\zeta)}e^{(i\rho-\sigma\beta(t)-\sigma^2 t)}, \tag{43}$$

$$v(x,y,t) = \frac{2\zeta_1^2\kappa\ell_1}{(\zeta_1^2-\delta^2\zeta_2^2)\gamma_1}[\frac{\sin(\zeta)}{1 \pm \cos(\zeta)}]^2 e^{(-2\sigma\beta(t)-2\sigma^2 t)}, \tag{44}$$

where $\zeta = \zeta_1 x + \zeta_2 y - (2\zeta_1\rho_1 + 2\zeta_2\rho_2)t$ and $\rho = \rho_1 x + \rho_2 y + \rho_3 t$.

Seventh case: If $\ell_1 = 2,\ \ell_2 = 0$ and $\ell_3 = 0$, then $Z(\zeta) = \frac{c}{\zeta}$ and the solutions of the wave Equation (13) are

$$\varphi(\zeta) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}\frac{c}{\zeta}.$$

Therefore, the solution of SDSEs (1) and (2) has the form

$$u(x,y,t) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}\frac{c}{[\zeta_1 x + \zeta_2 y - (2\zeta_1\rho_1 + 2\zeta_2\rho_2)t]}e^{(i\rho-\sigma\beta(t)-\sigma^2 t)}, \tag{45}$$

$$v(x,y,t) = \frac{2\zeta_1^2\kappa\ell_1}{(\zeta_1^2-\delta^2\zeta_2^2)\gamma_1}[\frac{c}{\zeta_1 x + \zeta_2 y - (2\zeta_1\rho_1 + 2\zeta_2\rho_2)t}]^2 e^{(-2\sigma\beta(t)-2\sigma^2 t)}. \tag{46}$$

Second set: If $\ell_1 < 0$ and $\gamma_1 < 0$, then there are many cases:

First case: If $\ell_1 = -2m^2,\ \ell_2 = 2m^2 - 1$ and $\ell_3 = (1-m^2)$, then $Z(\zeta) = cn(\zeta)$ and the solutions of the wave Equation (13) are

$$\varphi(\zeta) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}cn(\zeta).$$

Therefore, the solution of SDSEs (1) and (2) takes the form

$$u(x,y,t) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}cn(\zeta_1 x + \zeta_2 y - (2\zeta_1\rho_1 + 2\zeta_2\rho_2)t)e^{(i\rho-\sigma\beta(t)-\sigma^2 t)}, \tag{47}$$

$$v(x,y,t) = \frac{2\zeta_1^2\kappa\ell_1}{(\zeta_1^2-\delta^2\zeta_2^2)\gamma_1}cn^2(\zeta_1 x + \zeta_2 y - (2\zeta_1\rho_1 + 2\zeta_2\rho_2)t)e^{(-2\sigma\beta(t)-2\sigma^2 t)}. \tag{48}$$

Second case: If $\ell_1 = -2,\ \ell_2 = 2 - m^2$ and $\ell_3 = (m^2 - 1)$, then $Z(\zeta) = dn(\zeta)$ and the solutions of the wave Equation (13) are

$$\varphi(\zeta) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}dn(\zeta).$$

Therefore, the solution of SDSEs (1) and (2) takes the form

$$u(x,y,t) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}dn(\zeta_1 x + \zeta_2 y - (2\zeta_1\rho_1 + 2\zeta_2\rho_2)t)e^{(i\rho-\sigma\beta(t)-\sigma^2 t)}, \tag{49}$$

$$v(x,y,t) = \frac{2\zeta_1^2\kappa\ell_1}{(\zeta_1^2-\delta^2\zeta_2^2)\gamma_1}dn^2(\zeta_1 x+\zeta_2 y-(2\zeta_1\rho_1+2\zeta_2\rho_2)t)e^{(-2\sigma\beta(t)-2\sigma^2 t)}. \quad (50)$$

Third case: If $\ell_1 = \frac{-1}{2}$, $\ell_2 = \frac{(\mathbf{m}^2+1)}{2}$ and $\ell_3 = \frac{-(1-\mathbf{m}^2)^2}{4}$, then $Z(\zeta) = mcn(\zeta) \pm dn(\zeta)$ and the solutions of the wave Equation (13) are

$$\varphi(\zeta) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}[\mathbf{m}cn(\zeta)\pm dn(\zeta)].$$

Therefore, the solution of SDSEs (1) and (2) takes the form

$$u(x,y,t) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}[\mathbf{m}cn(\zeta)\pm dn(\zeta)]e^{(i\rho-\sigma\beta(t)-\sigma^2 t)}, \quad (51)$$

$$v(x,y,t) = \frac{2\zeta_1^2\kappa\ell_1}{(\zeta_1^2-\delta^2\zeta_2^2)\gamma_1}[\mathbf{m}cn(\zeta)\pm dn(\zeta)]^2 e^{(-2\sigma\beta(t)-2\sigma^2 t)}. \quad (52)$$

Fourth case: If $\ell_1 = \frac{\mathbf{m}^2-1}{2}$, $\ell_2 = \frac{(\mathbf{m}^2+1)}{2}$ and $\ell_3 = \frac{(\mathbf{m}^2-1)}{4}$, then $Z(\zeta) = \frac{dn(\zeta)}{1\pm sn(\zeta)}$ and the solutions of the wave Equation (13) are

$$\varphi(\zeta) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}\frac{dn(\zeta)}{1\pm sn(\zeta)}.$$

Therefore, the solution of SDSEs (1) and (2) takes the form

$$u(x,y,t) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}\frac{dn(\zeta)}{1\pm sn(\zeta)}e^{(i\rho-\sigma\beta(t)-\sigma^2 t)}, \quad (53)$$

$$v(x,y,t) = \frac{2\zeta_1^2\kappa\ell_1}{(\zeta_1^2-\delta^2\zeta_2^2)\gamma_1}[\frac{dn(\zeta)}{1\pm sn(\zeta)}]^2 e^{(-2\sigma\beta(t)-2\sigma^2 t)}. \quad (54)$$

If $m \to 1$ in Equations (47)–(50), then these equations transform into

$$u(x,y,t) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}sech(\zeta_1 x+\zeta_2 y-(2\zeta_1\rho_1+2\zeta_2\rho_2)t)e^{(i\rho-\sigma\beta(t)-\sigma^2 t)}, \quad (55)$$

$$v(x,y,t) = \frac{2\zeta_1^2\kappa\ell_1}{(\zeta_1^2-\delta^2\zeta_2^2)\gamma_1}sech^2(\zeta_1 x+\zeta_2 y-(2\zeta_1\rho_1+2\zeta_2\rho_2)t)e^{(-2\sigma\beta(t)-2\sigma^2 t)}. \quad (56)$$

If $m \to 0$ in Equations (53) and (54), then these equations transform into

$$u(x,y,t) = \pm\sqrt{\frac{\ell_1}{\gamma_1}}[\frac{1}{1\pm\sin(\zeta)}]e^{(i\rho-\sigma\beta(t)-\sigma^2 t)}, \quad (57)$$

$$v(x,y,t) = \frac{2\zeta_1^2\kappa\ell_1}{(\zeta_1^2-\delta^2\zeta_2^2)\gamma_1}[\frac{1}{1\pm\sin(\zeta)}]^2 e^{(-2\sigma\beta(t)-2\sigma^2 t)}. \quad (58)$$

Remark 1. *If we set $\sigma = 0$ in Equations (19)–(32), then we attain the same results stated in [31].*

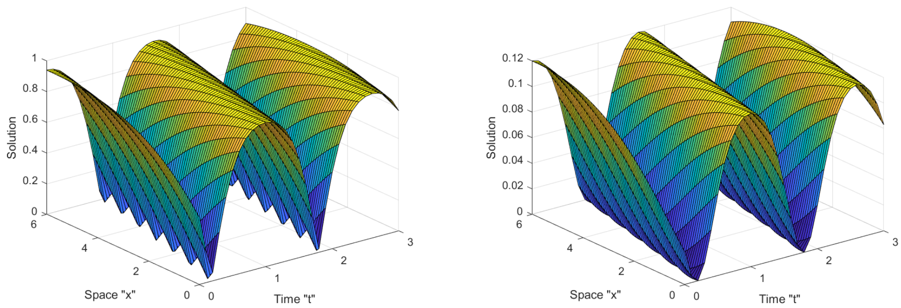
Remark 2. *If we set $\sigma = 0$ in Equations (25), (29), (31) and (55), then we attain the same results stated in [38].*

Remark 3. *If we set $\sigma = 0$ in Equations (55) and (56), then we attain Equations (49) and (50) with $n = 1$ as stated in [37].*

4. The Impact of Noise on the SDSE Solutions

In this article, the impact of noise on the acquired solutions of the SDSEs (1) and (2) is addressed. Depending on the research on the topic [40–44], the stabilizing and destabilizing influences caused by noisy terms in deterministic systems are currently well understood. It is now beyond question that these effects are important for understanding the long-term behavior of actual systems. For various noise strengths σ, we utilize the MATLAB tools (for more details, see, for example, [45]) to create some figures for some solutions such as (19) and (20). The following parameters are fixed: $\kappa = -1, \delta = i$, $\rho_1 = \zeta_1 = 0.3$, $\rho_2 = \zeta_2 = 1$, $\rho_3 = 0.2$ and $y = 0.5$. Then, $\zeta_3 = 2.18$, $\gamma_1 = \frac{2}{1.09}$ and $\gamma_2 = \frac{149}{91}$. In this case, $m = 0.8$, $\ell_1 = 1.28$ and $\zeta = 0.3x + 0.5 - 2.8t$.

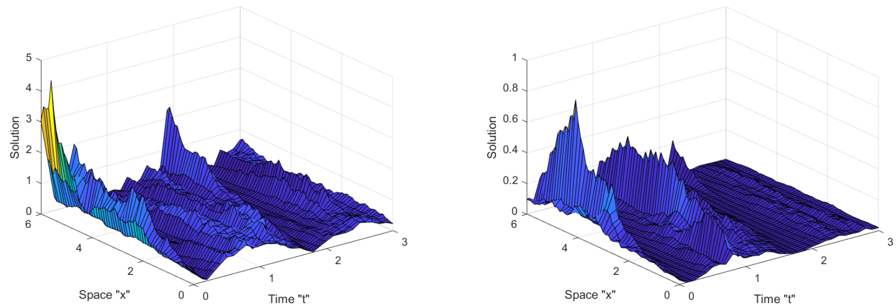
In Figure 1, when $\sigma = 0$, we notice that the surface fluctuates.



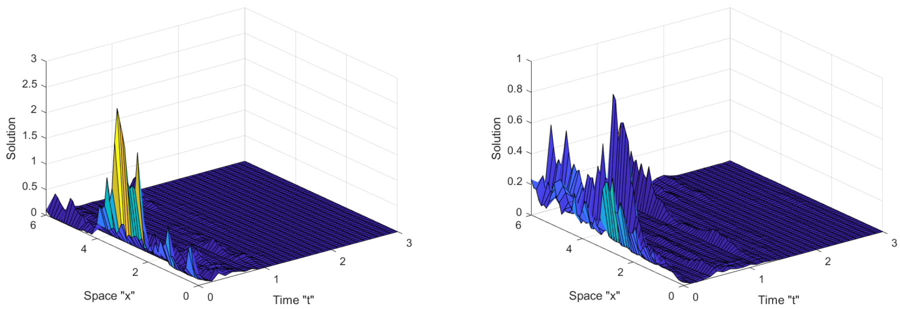
(**a**) Equation (19) with $\sigma = 0$ (**b**) Equation (20) with $\sigma = 0$

Figure 1. A 3D plot of Equations (19) and (20) with $\sigma = 0$.

Meanwhile, in Figure 2, if the intensity of the noise is raised, the surface becomes more planer after small transit behaviors, as follows:



(**a**) Equation (19) with $\sigma = 1$ (**b**) Equation (20) with $\sigma = 1$



(**c**) Equation (19) with $\sigma = 2$ (**d**) Equation (20) with $\sigma = 2$

Figure 2. A 3D plot of Equations (19) and (20) with $\sigma = 1, 2$.

We conclude from the previous figures that the noise term must be included in the Davey–Stewartson Equations (1) and (2) in order to produce accurate results and stable solutions that are near to zero.

5. Conclusions

This work took into account the stochastic (2+1)-dimensional Davey–Stewartson Equations (1) and (2) forced by multiplicative noise. Using the mapping method, we were able to generate stochastic trigonometric, elliptic, hyperbolic and rational solutions. For further study in fields such as hydrodynamics nonlinear optics, plasma physics, and others, the discovered solutions will be very beneficial. Due to the significance of the Davey–Stewartson equations in plasma physics, nonlinear optics, hydrodynamics and other fields, the obtained solutions are useful in explaining a number of intriguing physical phenomena. Additionally, we generalized previously obtained results, such as the one described in [31,37,38]. As a result of our results, we deduced that multiplicative Brownian motion stabilizes the solutions at zero. Finally, a demonstration of how multiplicative Brownian motion influences the exact solutions of the SDSEs is provided. We may take into account the additive noise in future work, as the multiplicative noise was covered in this paper.

Author Contributions: Data curation, F.M.A.-A.; Formal analysis, W.W.M., F.M.A.-A. and C.C.; Funding acquisition, F.M.A.-A.; Methodology, C.C. and F.M.A.-A.; Project administration, W.W.M.; Software, W.W.M.; Supervision, C.C.; Visualization, F.M.A.-A.; Writing—original draft, F.M.A.-A.; Writing—review & editing, W.W.M. and C.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: All the data are available in this paper.

Acknowledgments: Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2022R273), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Wazwaz, A.M. The tanh method: Exact solutions of the Sine–Gordon and Sinh–Gordon equations. *Appl. Math. Comput.* **2005**, *167*, 1196–1210. [CrossRef]
2. Mohammed, W.W.; Alshammari, M.; Cesarano, C.; El-Morshedy, M. Brownian Motion Effects on the Stabilization of Stochastic Solutions to Fractional Diffusion Equations with Polynomials. *Mathematics* **2022**, *10*, 1458. [CrossRef]
3. Al-Askar, E.M.; Mohammed, W.W.; Albalahi, A.M.; El-Morshedy, M. The Impact of the Wiener process on the analytical solutions of the stochastic (2+ 1)-dimensional breaking soliton equation by using tanh–coth method. *Mathematics* **2022**, *10*, 817. [CrossRef]
4. Zaman, U.H.M.; Arefin, M.A.; Akbar, M.A.; Uddin, M.H. Analytical behavior of soliton solutions to the couple type fractional-order nonlinear evolution equations utilizing a novel technique. *Alex. Eng. J.* **2022**, *61*, 11947–11958. [CrossRef]
5. Khatun, M.A.; Arefin, M.A.; Islam, M.Z.; Akbar, M.A.; Uddin, M.H. New dynamical soliton propagation of fractional type couple modified equal-width and Boussinesq equations. *Alex. Eng. J.* **2022**, *61*, 9949–9963. [CrossRef]
6. Wazwaz, A.M. An analytic study of compactons structures in a class of nonlinear dispersive equations. *Math. Comput. Simul.* **2003**, *63*, 35–44. [CrossRef]
7. Wen-Xiu, M.; Sumayah, B. A binary darboux transformation for multicomponent NLS equations and their reductions. *Anal. Math. Phys.* **2021**, *11*, 44.
8. Yan, Z.L. Abunbant families of Jacobi elliptic function solutions of the-dimensional integrable Davey-Stewartson-type equation via a new method. *Chaos Solitons Fract.* **2003**, *18*, 299–309. [CrossRef]
9. Fan, E.; Zhang, J. Applications of the Jacobi elliptic function method to special-type nonlinear equations. *Phys. Lett. A* **2002**, *305*, 383–392. [CrossRef]
10. Wazwaz, A.M. A sine-cosine method for handling nonlinear wave equations. *Math. Comput. Model.* **2004**, *40*, 499–508. [CrossRef]
11. Yan, C. A simple transformation for nonlinear waves. *Phys. Lett. A* **1996**, *224*, 77–84. [CrossRef]
12. Wang, M.L.; Li, X.Z.; Zhang, J.L. The (G'/G) -expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. *Phys. Lett. A* **2008**, *372*, 417–423. [CrossRef]

13. Zhang, H. New application of the (G'/G) -expansion method. *Commun. Nonlinear Sci. Numer. Simul.* **2009**, *14*, 3220–3225. [CrossRef]
14. Hirota, R. Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons. *Phys. Rev. Lett.* **1971**, *27*, 1192–1194. [CrossRef]
15. Khan, K.; Akbar, M.A. The $exp(-\Phi(\varsigma))$ -expansion method for finding travelling wave solutions of Vakhnenko-Parkes equation. *Int. J. Dyn. Syst. Differ. Equ*. **2014**, *5*, 72–83.
16. Mohammed, W.W.; Blömker, D. Fast-Diffusion Limit with Large Noise for Systems of Stochastic Reaction-Diffusion Equations. *Stoch. Anal. Appl*. **2016**, *34*, 961–978. [CrossRef]
17. Mohammed, W.W.; Blömker, D. Fast-diffusion limit for reaction-diffusion equations with multiplicative noise. *J. Math. Anal. Appl.* **2021**, *496*, 124808. [CrossRef]
18. Elbrolosy, M.E.; Elmandouh, A.A. Dynamical behaviour of nondissipative double dispersive microstrain wave in the microstructured solids. *Eur. Phys. J.* **2021**, *136*, 955. [CrossRef]
19. Elmandouha, A.A.; Ibrahim, A.G. Bifurcation and travelling wave solutions for a (2+1)-dimensional KdV equation. *J. Taibah Univ. Sci.* **2020**, *14*, 139–147. [CrossRef]
20. Elmandouha, A.A. Bifurcation and new traveling wave solutions for the 2D Ginzburg–Landau equation. *Eur. Phys. Plus* **2020**, *135*, 1–13.
21. Elbrolosy, M.E.; Elmandouh, A.A. Construction of new traveling wave solutions for the (2+1) dimensional extended kadomtsev-petviashvili equation. *J. Appl. Anal. Comput.* **2022**, *12*, 533–550. [CrossRef]
22. Elmandouh, A.A.; Elbrolosy, M.E. New traveling wave solutions for Gilson–Pickering equation in plasma via bifurcation analysis and direct method. *Math. Methods Appl. Sci.* **2022**, *accepted.* [CrossRef]
23. Yang, X.F.; Deng, Z.C.; Wei, Y. A Riccati-Bernoulli sub-ODE method for nonlinear partial differential equations and its application. *Adv. Diff. Equa.* **2015**, *1*, 117–133. [CrossRef]
24. Filiz, A.; Ekici, M. Sonmezoglu A. F-expansion method and new exact solutions of the Schrödinger-KdV equation. *Sci. World J.* **2014**, *2014*, 14. [CrossRef]
25. Arnold, L. *Random Dynamical Systems*; Springer: Berlin/Heidelberg, Germany, 1998.
26. Weinan, E.; Li, X.; Vanden-Eijnden, E. Some recent progress in multiscale modeling. In *Multiscale Modeling and Simulation*; Lecture Notes in Computational Science and Engineering; Springer: Berlin/Heidelberg, Germany, 2004; Volume 39, pp. 3–21.
27. Mohammed, W.W. Amplitude equation with quintic nonlinearities for the generalized Swift-Hohenberg equation with additive degenerate noise. *Adv. Differ. Equ.* **2016**, *2016*, 84. [CrossRef]
28. Mohammed, W.W.; Iqbal, N.; Botmart, T. Additive noise effects on the stabilization of fractional-space diffusion equation solutions. *Mathematics* **2020**, *10*, 130. [CrossRef]
29. Kloeden, P.E.; Platen, E. *Numerical Solution of Stochastic Differential Equations*; Springer: New York, NY, USA, 1995.
30. Davey, A.; Stewartson, K. On three-dimensional packets of surface waves. *Proc. Royal. Soc. Lond. Ser. A* **1974**, *338*, 101–110.
31. Bhrawy, A.H.; Abdelkawy, M.A.; Biswas, A. Cnoidal and snoidal wave solutions to coupled nonlinear wave equations by the extended Jacobi's elliptic function method. *Commun. Nonlinear Sci. Numer. Simul.* **2013**, *18*, 915–925. [CrossRef]
32. Jafari, H.; Sooraki, A.; Talebi, Y.; Biswas, A. The first integral method and traveling wave solutions to Davey–Stewartson equation. *Nonlinear Anal. Model. Control* **2012**, *17*, 182–193. [CrossRef]
33. Mirzazadeh, M. Soliton solutions of Davey–Stewartson equation by trial equation method and ansatz approach. *Nonlinear Dyn.* **2015**, *82*, 1775–1780. [CrossRef]
34. El Achab, A. Constructing new wave solutions to the (2+1)-dimensional Davey–Stewartson equation (DSE) which arises in fluid dynamics. *JMST Adv*. **2019**, *1*, 227–232. [CrossRef]
35. Fu, H.M.; Dai, Z.D. Double exp-function method and application. *Int. J. Nonlinear Sci. Numer. Simul.* **2009**, *10*, 927–933. [CrossRef]
36. Abdelaziz, M.A.M.; Moussa, A.E.; Alrahal, D.M. Exact Solutions for the nonlinear (2+1)-dimensional Davey-Stewartson equation using the generalized (G'/G)-expansion method. *J. Math. Res.* **2014**, *6*, 91–99. [CrossRef]
37. Ebadi, G.; Biswas, A. The (G'/G) method and 1-soliton solution of the Davey–Stewartson equation. *Math. Comput. Model.* **2011**, *53*, 694–698. [CrossRef]
38. Zedan, H.A.; Monaquel, S.J. The sine-cosine method for the Davey-Stewartson equations. *Appl. Math. E-Notes* **2010**, *10*, 103–111.
39. Peng, Y.Z. Exact solutions for some nonlinear partial differential equations. *Phys. Lett. A* **2003**, *314*, 401–408. [CrossRef]
40. Caraballo, T.; Langa, J.A.; Valero, J. Stabilisation of differential inclusions and PDEs without uniqueness by noise. *Commun. Pure Appl. Anal.* **2008**, *7*, 1375–1392. [CrossRef]
41. Caraballo, T.; Robinson, J.C. Stabilisation of linear PDEs by Stratonovich noise. *Syst. Control. Lett.* **2004**, *53*, 41–50. [CrossRef]
42. Mackey, M.C.; Longtin, A.; Lasota, A. Noise-induced global asymptotic stability. *J. Stat. Phys.* **1990**, *60*, 735–751. [CrossRef]
43. Blömker, D.; Fu, H. The impact of multiplicative noise in SPDEs close to bifurcation via amplitude equations. *Nonlinearity* **2020**, *33*, 3905. [CrossRef]
44. Caraballo, T.; Kloeden, P.E. Stabilization of evolution equations by noise. *Interdiscip. Math. Sci.* **2009**, *8*, 43–66.
45. Higham, D.J. An algorithmic introduction to numerical simulation of stochastic differential equations. *SIAM Rev.* **2001**, *43*, 525–546. [CrossRef]