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Bright Soliton Behaviours of Fractal Fractional Nonlinear Good Boussinesq Equation with Nonsingular Kernels

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Abstract: In this manuscript, we investigate the nonlinear Boussinesq equation (BEQ) under fractal-fractional derivatives in the sense of the Caputo–Fabrizio and Atangana–Baleanu operators. We use the double modified Laplace transform (LT) method to determine the general series solution of the Boussinesq equation. We study the convergence, existence, uniqueness, boundedness, and stability of the solution of the considered good BEQ under the aforementioned derivatives. The obtained solutions are presented with numerical illustrations considering a particular example by two cases based on both derivatives with suitable initial conditions. The results are illustrated graphically where good agreements are obtained. Our results show that fractal-fractional derivatives are a very effective tool for studying nonlinear systems. Furthermore, when t increases, the solitary waves of the system oscillate. As the fractional order α or fractal dimension β increases, the soliton solutions become coherently close to the exact solution. For compactness, an error analysis is performed. The absolute error reveals an approximate linear evolution in the soliton solutions as time increases and that the system does not blow up nonlinearly.

Keywords: Boussinesq equation; double Laplace transform; fractal-fractional operators; decomposition technique

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1. Introduction

In 1872, Joseph Boussinesq introduced the Boussinesq equation to model shallow water waves on ascending narrow canals and shores as well as to explain the motions of long waves under the action of gravity [1]. Subsequently, mathematical physics revealing the motion of wave phenomena has extensively applied this model. The Boussinesq equation has been improved into two forms known as the "good" and "bad" Boussinesq equations. Here, we consider a modified "good" Boussinesq equation under fractal-fractional derivatives as follows [2]:

$$u_{tt} - u_{xx} - u_{xxxx} - u_{xx}^2 = 0, \tag{1}$$

with the following conditions:

$$u(x,0) = f_1(x), \qquad u_t(x,0) = f_2(x).$$

This model represents the gravitational and one-dimensional nonlinear dynamics of long waves in shallow water [3,4]. The devising of the BEQ is explained in [5]. Several relevant papers have been presented in current years, and different methods have been used to formulate and investigate BEQ for nonlinear evolution of soliton solutions. Khater et al. used extended Riccati expansion to study soliton solutions of ill-posed BEQs [6]. Several other methods have been used for computing soliton solutions of nonlinear PDEs [7,8].

Saqib et al. utilized the extended expansion method to study new exact soliton solutions of the BEQ [9]. Traveling wave solutions of the BEQ have been analyzed through the Sardar sub-equation method [10]. A new set of exact solutions of the BEQ were constructed by [11], where Hirota's bilinear representation was implemented to obtain the propagation of wave packets and their interactions. An unusual BEQ was developed by Kaptsov [11] in which x is periodic and its amplitude rapidly leans towards zero for $t \to \pm\infty$. This concludes as a wave arises from naught during a short interval of time and then quickly devolves. Multi-soliton solutions of the BEQ have been presented in the literature as well [11,12]. There are other techniques to study BEQs, such as the auxiliary equation method [13] and the work of [14] solving non-linear Partial Differential Equations (NLPDEs) and existence and nonexistence of the interaction of solitons and anti-solitons in the good BEQ through Fourier analysis.

At present, the operators of fractional orders have attracted mathematicians and physicists due to their many applications in modelling and analysis. They show the past history of a process, and have the capability to preserve memory. In Fractional Calculus (FC), several fractional operators have been expressed with diverse forms of kernels, including Riemann-Liouville (RL), Caputo, Caputo–Fabrizio(CF), and Atangana–Baleanu in Caputo sense (ABC). The RL and Caputo operators are power law convolutions, while the CF operator is the convolution of the exponential decay law extended to a Mittag–Leffler type kernel, which provides better result when studying a variety of physical systems. These operators have many applications in different fields of science. For instance, fractional operators have been used in analysis of biodegradation modeling [15], biomath [16], mathematical physics [17,18], and engineering [19,20].

Recently, Atangana further generalized the nonlocal operators by including fractal derivatives in the fractional operators. He defined new sorts of operators called fractal-fractional (FF) operators [21]. In this approach, the notions of fractal and fractional operators are fused to produce operators, presenting an opportunity for researchers to analyze a wide variety of complicated systems. Fractal-fractional derivatives are now extensively used by researchers in many disciplines to study ever more complicated problems. FF operators have been used for the analysis of models in various fields of science. Researchers have used FF operators to analyse of chaotic systems [22,23], epidemic models [24,25], and mathematical physics [26,27]. Inspired by the above works, in this paper we study the BEQ using FF operators with nonsingular kernels. The theoretical aspects of the BEQ for both CF and ABC operators are investigated through nonlinear analysis. We implement the double modified LT method to find the approximate solution of Equation (1) under nonsingular FF operators. Stability and error analyses are presented for the proposed BEQ in both cases.

In Section 2, a few basic definitions of the Caputo–Fabrizio (FC) and ABC operators are provided. In Section 3, our proposed method is presented. A theoretical investigation of Equation (1) using the proposed operator is provided in Section 4. In Section 5, we present the Picard's φ-stability of the model.

2. Preliminaries

In this section, we provide a few basic definitions which relate to our problem. For further study of FF operators, readers are referred to [21]. Let $u(t)$ be a continuous and fractal differentiable on (a,b) and let α, β be the fractional order and fractal dimension, respectively.

Definition 1. *The FF derivative of $u(t)$ in CF sense is expressed as follows:*

$$ {}^{FFE}_{a}D^{\alpha,\beta}_{t}u(t) \quad = \quad \frac{M(\alpha)}{1-\alpha}\int_0^t \frac{du(y)}{dy^{\beta}} exp\Big(\frac{\alpha}{1-\alpha}(t-y)\Big)dy $$

where $M(\alpha)$ represents the normalization function.

Definition 2. *The FF derivative of $u(t)$ in ABC sense is expressed as follows:*

$$ {}_{a}^{FFM}D_t^{\alpha,\beta}u(t) \quad = \quad \frac{AB(\alpha)}{1-\alpha}\int_0^t \frac{du(y)}{dy^\beta}E_a\Big(-\frac{\alpha}{1-\alpha}(t-y)\Big)dy, $$

where $AB(\alpha) = 1-\alpha+\frac{\alpha}{\Gamma(\alpha)}$.

Definition 3. *The FF integral of $u(t)$ with an exponential-decay kernel is expressed as follows:*

$$ {}_{0}^{FFE}I_t^{\alpha,\beta}u(t) = \frac{\alpha\beta}{M(\alpha)}\int_0^t y^{\alpha-1}u(y)dy + \frac{\beta(1-\alpha)t^{\beta-1}u(t)}{M(\alpha)}. $$

Definition 4. *The FF integral of $u(t)$ wit ah Mittag–Leffler kernel is expressed as follows:*

$$ {}_{0}^{FFM}I_t^{\alpha,\beta}u(t) = \frac{\alpha\beta}{AB(\alpha)}\int_0^t y^{\alpha-1}u(y)(t-y)^{\alpha-1}dy + \frac{\beta(1-\alpha)t^{\beta-1}u(t)}{AB(\alpha)}. $$

Definition 5. *The formula for the double LT of CF operators is*

$$ \mathcal{L}_x\mathcal{L}_t\Big\{{}^{CF}D_x^{\alpha+n}\,u(x,t)\Big\} = \frac{M(\alpha)}{p+(1-p)\alpha}\left[p^{n+1}\,\overline{u}(p,s) - \sum_{k=0}^{n} p^{n-k}\mathcal{L}_t\left\{\frac{\partial^k u(0,t)}{\partial x^k}\right\}\right], $$

and

$$ \mathcal{L}_x\mathcal{L}_t\Big\{{}^{CF}D_t^{\beta+m}\,u(x,t)\Big\} = \frac{M(\beta)}{s+(1-s)\beta}\left[s^{m+1}\,\overline{u}(p,s) - \sum_{k=0}^{m} s^{m-k}\mathcal{L}_x\left\{\frac{\partial^k u(x,0)}{\partial t^k}\right\}\right], $$

where $m \,\&\, n = 0,1,2,\dots$ *and* $\overline{u}(p,s) = \mathcal{L}_x\mathcal{L}_t\{u(x,t)\}$.

Definition 6. *The formula for the double LT of ABC operators is provided by*

$$ \mathcal{L}_x\mathcal{L}_t\Big\{{}^{ABC}D_x^{\alpha}\,u(x,t)\Big\} = \frac{B(\alpha)}{(1-\alpha)(p^\alpha+\frac{\alpha}{(1-\alpha)})}\left[p^{\alpha}\,\overline{u}(p,s) - \sum_{k=0}^{n-1} p^{\alpha-1-k}\mathcal{L}_t\left\{\frac{\partial^k u(0,t)}{\partial x^k}\right\}\right], $$

and

$$ \mathcal{L}_x\mathcal{L}_t\Big\{{}^{ABC}D_t^{\beta}\,u(x,t)\Big\} = \frac{B(\beta)}{(1-\beta)(s^\beta+\frac{\beta}{(1-\beta)})}\left[s^{\beta}\,\overline{u}(p,s) - \sum_{k=0}^{m-1} s^{\beta-1-k}\mathcal{L}_x\left\{\frac{\partial^k u(x,0)}{\partial t^k}\right\}\right], $$

where $n = [\alpha]+1, m = [\beta]+1$.

3. Proposed Method

Here, we use the MDLDM of a fractal-fractional operator with CF and ABC to obtain the approximate solution to the proposed system.

3.1. Strategy of Solution for the Caputo–Fabrizio Case

Consider a BEQ in Caputo sense with a kernel of the exponential decay type:

$$ {}_{0}^{FFE}D_t^{\alpha,\beta}u(x,t) = u_{xx} + u_{xxxx} + u_{xx}^2, \qquad 1<\alpha\le 2, 0<\beta\le 1, \tag{2} $$

with conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. Equivalently, we can write

$$ {}^{CF}D_t^{\alpha}u(x,t) = \beta t^{\beta-1}\Big[u_{xx} + u_{xxxx} + u_{xx}^2\Big]. $$

Applying double LT and using the given conditions, we reach

$$L_xL_tu(x,t) = \frac{1}{s}L_xL_tf(x) + \frac{1}{s^2}L_xL_tg(x) + \frac{s+(1-s)\alpha}{s}L_xL_t\beta t^{\beta-1}[u_{xx} + u_{xxxx} + u_{xx}^2]. \quad (3)$$

Consider

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \quad (4)$$

now, u_{xx}^2 can be decomposed as

$$u_{xx}^2 = \sum_{n=0}^{\infty} A_n, \quad (5)$$

where A_n represent the Adomian polynomials of the function $u_0, u_1, u_2, \dots$ provided by the formula

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\Big[\sum_{k=0}^{n} \lambda^k\, u_{kxx}\Big]^2_{\lambda=0}. \quad (6)$$

Implementing inverse double LT in Equation (9), we obtain

$$\sum_{n=0}^{\infty} u(x,t) = f(x) + tg(x) + L_x^{-1}L_t^{-1}\Big[\frac{s+(1-s)\alpha}{s}L_xL_t\beta t^{\beta-1}\Big\{(\sum_{n=0}^{\infty} u_{nxx} + \sum_{n=0}^{\infty} u_{nxxxx} + \sum_{n=0}^{\infty} A_n)\Big\}\Big]. \quad (7)$$

By comparing the terms, we obtain the following series:

$$\begin{aligned}
u_0 &= f(x) + tg(x), \\
u_1 &= L_x^{-1}L_t^{-1}\left[\frac{s+\alpha-\alpha s}{s}L_xL_t\{\beta t^{\beta-1}\{u_{0xx} + u_{0xxxx} + A_0\}\}\right], \\
u_2 &= L_x^{-1}L_t^{-1}\left[\frac{s+\alpha-\alpha s}{s}L_xL_t\{\beta t^{\beta-1}\{u_{1xx} + u_{1xxxx} + A_1\}\}\right], \\
u_3 &= L_x^{-1}L_t^{-1}\left[\frac{s+\alpha-\alpha s}{s}L_xL_t\{\beta t^{\beta-1}\{u_{2xx} + u_{2xxxx} + A_2\}\}\right], \\
u_4 &= L_x^{-1}L_t^{-1}\left[\frac{s+\alpha-\alpha s}{s}L_xL_t\{\beta t^{\beta-1}\{u_{3xx} + u_{3xxxx} + A_3\}\}\right], \\
u_5 &= L_x^{-1}L_t^{-1}\left[\frac{s+\alpha-\alpha s}{s}L_xL_t\{\beta t^{\beta-1}\{u_{4xx} + u_{4xxxx} + A_4\}\}\right].
\end{aligned}$$

It should be mentioned that more terms can be determined in an equivalent fashion. The final result can be written as

$$u(x,t) = \sum_{n=0}^{\infty} u(x,t). \quad (8)$$

3.2. Strategy for the Solution of the Atangana–Baleanu Case

Consider a Boussinesq Equation in Caputo sense with a Mittag–Leffler type kernel, as follows:

$$^{FFM}_{0}D_t^{\alpha,\beta}u(x,t) = u_{xx} + u_{xxxx} + u_{xx}^2, \qquad 1 < \alpha \leq 2, 0 < \beta \leq 1, \quad (9)$$

with conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. We can rewrite the above equation as

$$^{ABC}D_t^{\alpha}u(x,t) = \beta t^{\beta-1}\Big[u_{xx} + u_{xxxx} + u_{xx}^2\Big].$$

Applying double LT as discussed in Section 2 using the initial conditions, we reach

$$L_xL_tu(x,t) = \frac{1}{s}L_xL_tf(x) + \frac{1}{s^2}L_xL_tg(x) + \frac{1}{AB(\alpha)}(1-\alpha+\frac{\alpha}{s^\alpha})L_xL_t[\beta t^{\beta-1}[u_{xx}+u_{xxxx}+u_{xx}^2]]. \quad (10)$$

Consider

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \quad (11)$$

the term u_{xx}^2 can be decomposed as

$$u_{xx}^2 = \sum_{n=0}^{\infty} A_n, \quad (12)$$

where A_n are the Adomian polynomials of the function $u_0, u_1, u_2, \ldots$, provided by the formula

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\Big[\sum_{k=0}^{n}\lambda^k\, u_{kxx}\Big]^2_{\lambda=0}. \quad (13)$$

Applying inverse double LT to Equation (5), we have

$$\begin{aligned}\sum_{n=0}^{\infty} u(x,t) &= f(x)+tg(x)+\frac{1}{AB(\alpha)}L_x^{-1}L_t^{-1}\Bigg[(1-\alpha+\frac{\alpha}{s^\alpha})L_xL_t\beta t^{\beta-1}\Bigg\{\sum_{n=0}^{\infty}u_{nxx} \quad (14)\\ &+\sum_{n=0}^{\infty}u_{nxxxx}+\sum_{n=0}^{\infty}A_n\Bigg\}\Bigg].\end{aligned}$$

By comparing the terms, we obtain the following series:

$$\begin{aligned}
u_0 &= f(x)+tg(x),\\
u_1 &= \frac{1}{AB(\alpha)}-1_xL_t^{-1}\Big[(1-\alpha+\frac{\alpha}{s^\alpha})L_xL_t\{\beta t^{\beta-1}\{u_{0xx}+u_{0xxxx}+A_0\}\}\Big],\\
u_2 &= \frac{1}{AB(\alpha)}L_x^{-1}L_t^{-1}\Big[(1-\alpha+\frac{\alpha}{s^\alpha})L_xL_t\{\beta t^{\beta-1}\{u_{1xx}+u_{1xxxx}+A_1\}\}\Big],\\
u_3 &= \frac{1}{AB(\alpha)}L_x^{-1}L_t^{-1}\Big[(1-\alpha+\frac{\alpha}{s^\alpha})L_xL_t\{\beta t^{\beta-1}\{u_{2xx}+u_{2xxxx}+A_2\}\}\Big],\\
u_4 &= \frac{1}{AB(\alpha)}L_x^{-1}L_t^{-1}\Big[(1-\alpha+\frac{\alpha}{s^\alpha})L_xL_t\{\beta t^{\beta-1}\{u_{3xx}+u_{3xxxx}+A_3\}\}\Big],\\
u_5 &= \frac{1}{AB(\alpha)}L_x^{-1}L_t^{-1}\Big[(1-\alpha+\frac{\alpha}{s^\alpha})L_xL_t\{\beta t^{\beta-1}\{u_{4xx}+u_{4xxxx}+A_4\}\}\Big].
\end{aligned}$$

It should be mentioned that more terms can be determined in an equivalent fashion. The final result is

$$u(x,t) = \sum_{n=0}^{\infty} u(x,t). \quad (15)$$

4. Theoretical Investigation

In this section, a theoretical investigation, including existence, uniqueness, and stability analysis, is carried out for both the CF and ABC operators.

4.1. Existence and Uniqueness Theorems

The existence and uniqueness of a solution of the Boussinesq equation under exponential-decay kernel and Mittag–Leffler kernel operators are determined through the following series of theorems.

4.1.1. Exponential Decay Kernel Operator

Here, we elucidate the boundedness, existence, and uniqueness of the solution of the considered BEQ under our CF approach. For this, we consider Equation (1) in the operator sense as follows:

$$ {}_0^{FFE}D_t^{\alpha,\beta} u(x,t) = \psi(x,t;u), \tag{16} $$

where $u_{xx} + u_{xxxx} + u_{xx}^2$. Applying the fractal-fractional integral on both sides of Equation (1), we have

$$ u(x,t) - C_0 u(x,0) - C_1 u_t(x,0) = \frac{\alpha\beta}{M(\alpha)} \int_0^t y^{\alpha-1} u(y) dy + \frac{\beta(1-\alpha)t^{\beta-1}u(t)}{M(\alpha)}. \tag{17} $$

To indicate that kernel $\psi(x,t;u)$ has the Lipschitz condition, we first consider a bounded function $||u(x,t)|| \leq \gamma_1$ and $||v(x,t)|| \leq \gamma_2$. Employing the triangle property of norms, we have

$$ \begin{aligned} \Big\|\psi(x,t;u) - \psi(x,t;v)\Big\| &= \Big\|u_{xx} + u_{xxxx} + u_{xx}^2 - v_{xx} - v_{xxxx} - v_{xx}^2\Big\| \\ &= \Big\|(u_{xx} - v_{xx}) + (u_{xxxx} - v_{xxxx}) + (u_{xx}^2 - v_{xx}^2)\Big\| \\ &= \Big\|\frac{\partial^2}{\partial x^2}(u-v) + \frac{\partial^4}{\partial x^4}(u-v) + \frac{\partial^2}{\partial x^2}(u^2 - v^2)\Big\| \\ &\leq B\Big\|u-v\Big\| + C\Big\|u-v\Big\| - D\Big\|u^2 - v^2\Big\| \\ &\leq \Big(B + C + D(\gamma_1 + \gamma_2)\Big)\Big\|u-v\Big\|, \end{aligned} $$

therefore,

$$ \Big\|\psi(x,t;u) - \psi(x,t;v)\Big\| \quad \leq \lambda \quad \|u-v\|, $$

where

$$ \lambda = \Big(A + B + C + D(\gamma_1 + \gamma_2)\Big) \leq 0. $$

This satisfies the Lipschitz condition for the kernel $\psi(x,t;u)$. Now, we consider a recursive scheme as follows:

$$ u_{n+1}(x,t) = \frac{\alpha\beta}{M(\alpha)} \int_0^t y^{\alpha-1} u(y) dy + \frac{\beta(1-\alpha)t^{\beta-1}u(t)}{M(\alpha)}, $$

where

$$ u_0(x,t) = C_0 u(x,0) + C_1 u_t(x,t). $$

It is apparent that

$$ \begin{aligned} e_n(x,t) &= u_n(x,t) - u_{n-1}(x,t), \\ &= \frac{\alpha\beta}{M(\alpha)} \int_0^t \Big(\psi(x,y;u_{n-1}) - \psi(x,y;u_{n-2})\Big) dy + \frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)} \Big(\psi(x,t;u_{n-1}) - \psi(x,t;u_{n-2})\Big), \end{aligned} $$

and

$$ u_n(x,t) = \sum_{i=0}^{n} e_i(x,t). \tag{18} $$

Now, we prove the existence of a solution of the BEQ equation in the following theorem.

Theorem 1. *Let the function $u(x,t)$ be bounded; then,*

$$\left\|e_n(x,t)\right\| \leq \left(\frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)}\lambda + \frac{\alpha\beta}{M\alpha}\lambda t\right)^n \left\|u_0(x,t)\right\|$$

Proof. Suppose we have $n=1$; then, we can rewrite it as follows:

$$\begin{aligned}
\left\|e_1(x,t)\right\| &= \left\|u_1(x,t)-u_0(x,t)\right\| \\
&\leq \frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)}\left\|\psi(x,t;u_0)-\psi(x,t;u_{-1})\right\| + \frac{\alpha\beta}{M(\alpha)}\int_0^t \left\|\psi(x,y;u_0)-\psi(x,y;u_{-1})\right\|dy, \\
&= \frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)}\lambda\left\|u_0-u_{-1}\right\| + \frac{\alpha\beta}{M(\alpha)}\int_0^t \lambda\left\|u_0-u_{-1}\right\|dy, \\
&\leq \frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)}\lambda\left\|u_0(x,t)\right\| + \frac{\alpha\beta}{M(\alpha)}\lambda\left\|u_0(x,t)\right\|\int_0^t dy, \\
&\leq \frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)}\lambda\left\|u_0(x,t)\right\| + \frac{\alpha\beta}{M(\alpha)}\lambda\left\|u_0(x,t)\right\|t, \\
&\leq \left(\frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)}\lambda + \frac{\alpha\beta}{M(\alpha)}\lambda t\right)\left\|u_0(x,t)\right\|,
\end{aligned}$$

Now, if the relation holds for $n=k$

$$\left\|e_k(x,t)\right\| \leq \left(\frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\lambda + \frac{\alpha\beta}{AB(\alpha)}\lambda t\right)^k \left\|u_0(x,t)\right\|,$$

then it will be proved for $n=k+1$

$$\left\|e_{k+1}(x,t)\right\| \leq \left(\frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)}\lambda + \frac{\alpha\beta}{M(\alpha)}\lambda t\right)^{k+1} \left\|u_0(x,t)\right\|.$$

To prove this, we continue as follows:

$$\begin{aligned}
\left\|e_{K+1}(x,t)\right\| &= \left\|u_{K+1}(x,t)-u_K(x,t)\right\|, \\
&\leq \frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)}\left\|\psi(x,t;u_k)-\psi(x,t;u_{k-1})\right\| + \frac{\alpha\beta}{M(\alpha)}\int_0^t \left\|\psi(x,y;u_k)-\psi(x,y;u_{k-1})\right\|dy, \\
&\leq \frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)}\lambda\left\|u_k-u_{k-1}\right\| + \frac{\alpha\beta}{M(\alpha)}\int_0^t \lambda\left\|u_k-u_{k-1}\right\|dy, \\
&= \frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)}\lambda\left\|e_K(x,y)\right\| + \frac{\alpha\beta}{M(\alpha)}\int_0^t \lambda\left\|e_K(x,y)\right\|dy, \\
&= \left(\frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)}\lambda + \frac{\alpha\beta}{M(\alpha)}\lambda\int_0^t dy\right)\left\|e_K(x,y)\right\|, \\
&= \left(\frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)}\lambda + \frac{\alpha\beta}{M(\alpha)}\lambda t\right)\left(\frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)}\lambda + \frac{\alpha\beta}{M(\alpha)}\lambda t\right)^k\left\|u_0(x,t)\right\|, \\
&= \left(\frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)}\lambda + \frac{\alpha\beta}{M(\alpha)}\lambda t\right)^{k+1}\left\|u_0(x,t)\right\|.
\end{aligned}$$

The proof is finished. □

To show that Equation (1) has at least one solution, the following theorem is applied.

Theorem 2. *For $t = t_0$, we have*

$$0 \leq \frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)}\lambda + \frac{\alpha\beta}{M(\alpha)}\lambda t < 1,$$

then, the solution of the improved Boussinesq equation involving a fractal-fractional CF derivative exists.

Proof. Based on Equation (18), we can write

$$\begin{aligned}\left\|u_n(x,t)\right\| &\leq \sum_{i=0}^{n}\left\|e_i\left(x,t\right)\right\|,\\ &\leq \sum_{i=0}^{n}\left(\frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)}\lambda + \frac{\alpha\beta}{M(\alpha)}\lambda t\right)^i\left\|u_0(x,t)\right\|\end{aligned}$$

for $t = t_0$, we obtain

$$\leq \sum_{i=0}^{n}\left(\frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)}\lambda + \frac{\alpha\beta}{M(\alpha)}\lambda t_0\right)^i\left\|u_0(x,t)\right\|,$$

consequently,

$$\lim_{n\to\infty}\left\|u_n(x,t)\right\| \leq \sum_{i=0}^{\infty}\left(\frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)}\lambda + \frac{\alpha\beta}{M(\alpha)}\lambda t_0\right)^i\left\|u_0(x,t)\right\|,$$

because

$$0 \leq \frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)}\lambda + \frac{\alpha\beta}{M(\alpha)}\lambda t_0 < 1.$$

Thus, the above series is convergent, and therefore, $u_n(x,t)$ exists and is bounded for any n. □

4.1.2. Uniqueness

The uniqueness of the BEQ equation is proved in the following theorem.

Theorem 3. *At $t = t_0$, we have*

$$0 \leq \frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)}\lambda + \frac{\alpha\beta}{M(\alpha)}\lambda t < 1,$$

then, the solution of improved Boussinesq equation involving a fractal-fractional CF derivative is unique.

Proof. To start, let us consider two solutions $u(x,t)$ and $v(x,t)$ for the model. We can write

$$u(x,t) - v(x,t) = \frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)}\psi(x,t;u) - \psi(x,t;v) + \frac{\alpha\beta}{M(\alpha)}\int_0^t\left\|\psi(x,y;u) - \psi(x,y;v)\right\|dy,$$

consequently,

$$\begin{aligned}\Big\|u(x,t)-v(x,t)\Big\| &\leq \frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)}\Big\|\psi(x,t;u)-\psi(x,t;v)\Big\|+\frac{\alpha\beta}{M(\alpha)}\int_0^t\Big\|\psi(x,y;u)-\psi(x,y;v)\Big\|dy,\\ &\leq \frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)}\lambda\Big\|u-v\Big\|+\frac{\alpha\beta}{M(\alpha)}\int_0^t\lambda\Big\|u-v\Big\|dy,\\ &\leq \Big(\frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)}\lambda+\frac{\alpha\beta}{M(\alpha)}\int_0^t\lambda dy\Big)\Big\|u-v\Big\|,\\ &\leq \Big(\frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)}\lambda+\frac{\alpha\beta}{M(\alpha)}\lambda t\Big)\Big\|u-v\Big\|,\end{aligned}$$

however,

$$0 \leq \frac{\beta(1-\alpha)t^{\beta-1}}{M(\alpha)}\lambda+\frac{\alpha\beta}{M(\alpha)}\lambda t<1,$$

then,

$$\Big\|u(x,t)-v(x,t)\Big\| = 0$$

Thus, the solution of the BEQ is unique. □

4.2. Existence Theory for ABC Operator

Here, we provide the existence and uniqueness of the solution of the BEQ under a fractal-fractional ABC operator. For this, we consider Equation (1) in the operator sense as follows:

$${}_0^{FFM}D_t^{\alpha,\beta}u(x,t)=u_{xx}+u_{xxxx}+u_{xx}^2,\quad 1<\alpha\leq 2,\ \ 0<\beta\leq 1,$$

where

$${}_0^{FFM}D_t^{\alpha,\beta}u(x,t)=\psi(x,t;u), \tag{19}$$

where $\psi(x,t;u)=u_{xx}+u_{xxxx}+u_{xx}^2$. Applying the ABC fractal-fractional integral on both sides of Equation (1),

$$u(x,t)-C_0u(x,0)-C_1u_t(x,0) = \frac{\alpha\beta}{AB(\alpha)}\int_0^t y^{\alpha-1}u(y)(t-y)^{\alpha-1}dy+\frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}u(t) \tag{20}$$

The following is based on Equation (20) and the fixed point theorem; a recursive scheme is provided as follows:

$$u_{n+1}(x,t) = \frac{\alpha\beta}{AB(\alpha)}\int_0^t y^{\alpha-1}u(y)(t-y)^{\alpha-1}dy+\frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}u(t),$$

now, by

$$u_0(x,t)=C_0u(x,0)+C_1u_t(x,0)$$

it is apparent that

$$\begin{aligned}e_n(x,t) &= u_n(x,t)-u_{n-1}(x,t),\\ &= \frac{\alpha\beta}{AB(\alpha)}\int_0^t\Big(\psi(x,y;u_{n-1})-\psi(x,y;u_{n-2})\Big)dy+\frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\Big(\psi(x,t;u_{n-1})-\psi(x,t;u_{n-2})\Big),\end{aligned}$$

and

$$u_n(x,t) = \sum_{i=0}^{n} e_i(x,t). \tag{21}$$

Theorem 4. *Let the function $u(x,t)$ be bounded; then,*

$$\Big\|e_n(x,t)\Big\| \leq \Big(\frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\lambda + \frac{\alpha\beta}{AB\alpha}\lambda t\Big)^n \Big\|u_0(x,t)\Big\|$$

Proof. Suppose we have $n=1$; then, we can write it as

$$\begin{aligned}
\Big\|e_1(x,t)\Big\| &= \Big\|u_1(x,t) - u_0(x,t)\Big\| \\
&\leq \frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\Big\|\psi(x,t;u_0) - \psi(x,t;u_{-1})\Big\| + \frac{\alpha\beta}{AB(\alpha)}\int_0^t \Big\|\psi(x,y;u_0) - \psi(x,y;u_{-1})\Big\|dy, \\
&= \frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\lambda\Big\|u_0 - u_{-1}\Big\| + \frac{\alpha\beta}{AB(\alpha)}\int_0^t \lambda\Big\|u_0 - u_{-1}\Big\|, dy \\
&\leq \frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\lambda\Big\|u_0(x,t)\Big\| + \frac{\alpha\beta}{AB(\alpha)}\lambda\Big\|u_0(x,t)\Big\|\int_0^t dy, \\
&\leq \frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\lambda\Big\|u_0(x,t)\Big\| + \frac{\alpha\beta}{AB(\alpha)}\lambda\Big\|u_0(x,t)\Big\|t, \\
&\leq \Big(\frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\lambda + \frac{\alpha\beta}{AB(\alpha)}\lambda t\Big)\Big\|u_0(x,t)\Big\|,
\end{aligned}$$

Now, if the relation holds for $n=k$

$$\Big\|e_k(x,t)\Big\| \leq \Big(\frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\lambda + \frac{\alpha\beta}{AB(\alpha)}\lambda t\Big)^k \Big\|u_0(x,t)\Big\|,$$

it is proven for $n=k+1$ as well

$$\Big\|e_{k+1}(x,t)\Big\| \leq \Big(\frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\lambda + \frac{\alpha\beta}{AB(\alpha)}\lambda t\Big)^{k+1} \Big\|u_0(x,t)\Big\|.$$

To prove this, we continue as follows:

$$\begin{aligned}
\Big\|e_{K+1}(x,t)\Big\| &= \Big\|u_{K+1}(x,t) - u_K(x,t)\Big\|, \\
&\leq \frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\Big\|\psi(x,t;u_k) - \psi(x,t;u_{k-1})\Big\| + \frac{\alpha\beta}{AB(\alpha)}\int_0^t \Big\|\psi(x,y;u_k) - \psi(x,y;u_{k-1})\Big\|dy, \\
&\leq \frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\lambda\Big\|u_k - u_{k-1}\Big\| + \frac{\alpha\beta}{AB(\alpha)}\int_0^t \lambda\Big\|u_k - u_{k-1}\Big\|dy, \\
&= \frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\lambda\Big\|e_K(x,t)\Big\| + \frac{\alpha\beta}{AB(\alpha)}\int_0^t \lambda\Big\|e_K(x,t)\Big\|dy, \\
&= \Big(\frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\lambda + \frac{\alpha\beta}{AB(\alpha)}\lambda\int_0^t dy\Big)\Big\|e_K(x,t)\Big\|, \\
&= \Big(\frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\lambda + \frac{\alpha\beta}{AB(\alpha)}\lambda t\Big)\Big(\frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\lambda + \frac{\alpha\beta}{AB(\alpha)}\lambda t\Big)^k\Big\|u_0(x,t)\Big\|, \\
&= \Big(\frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\lambda + \frac{\alpha\beta}{AB(\alpha)}\lambda t\Big)^{k+1}\Big\|u_0(x,t)\Big\|.
\end{aligned}$$

Thus, the result is proved. □

4.2.1. Existence

Theorem 5. *At* $t = t_0$*, we have*

$$0 \leq \frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\lambda + \frac{\alpha\beta}{AB(\alpha)}\lambda t < 1,$$

then, the solution of the improved Boussinesq equation involving the fractal-fractional ABC derivative exists.

Proof. Based on Equation (22), we can write

$$\begin{aligned} \Big\| u_n(x,t) \Big\| &\leq \sum_{i=0}^{n} \Big\| ei\Big(x,y,t\Big) \Big\|, \\ &\leq \sum_{i=0}^{n} \Big(\frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\lambda + \frac{\alpha\beta}{AB(\alpha)}\lambda t \Big)^{i} \Big\| u_0(x,t) \Big\| \end{aligned}$$

for $t = t_0$, we obtain

$$\leq \sum_{i=0}^{n} \Big(\frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\lambda + \frac{\alpha\beta}{AB(\alpha)}\lambda t_0 \Big)^{i} \Big\| u_0(x,t) \Big\|,$$

consequently,

$$\lim_{n\to\infty} \Big\| u_n(x,t) \Big\| \leq \sum_{i=0}^{\infty} \Big(\frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\lambda + \frac{\alpha\beta}{AB(\alpha)}\lambda t_0 \Big)^{i} \Big\| u_0(x,t) \Big\|,$$

because

$$0 \leq \frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\lambda + \frac{\alpha\beta}{AB(\alpha)}\lambda t_0 < 1.$$

Thus, the above series is convergent and therefore, $u_n(x,t)$ exists and is bounded for any n. □

4.2.2. Uniqueness

Theorem 6. *At* $t = t_0$*, we have*

$$0 \leq \frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\lambda + \frac{\alpha\beta}{AB(\alpha)}\lambda t < 1,$$

then, the solution of the improved Boussinesq equation involving the fractal-fractional ABC derivative is unique.

Proof. To start, let us consider two solutions $u(x,t)$ and $v(x,t)$ for the model; we can write

$$u(x,t) - v(x,t) = \frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\psi(x,t;u) - \psi(x,t;v) + \frac{\alpha\beta}{AB(\alpha)}\int_0^t \Big\| \psi(x,t;u) - \psi(x,t;v) \Big\|,$$

consequently,

$$\begin{aligned}\left\|u(x,t)-v(x,t)\right\| &\leq \frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\left\|\psi(x,t;u)-\psi(x,t;v)\right\|+\frac{\alpha\beta}{AB(\alpha)}\int_0^t\left\|\psi(x,t;u)-\psi(x,t;v)\right\|dy,\\ &\leq \frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\lambda\left\|u-v\right\|+\frac{\alpha\beta}{AB(\alpha)}\int_0^t\lambda\left\|u-v\right\|dy,\\ &\leq \Big(\frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\lambda+\frac{\alpha\beta}{AB(\alpha)}\int_0^t\lambda dy\Big)\left\|u-v\right\|,\\ &\leq \Big(\frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\lambda+\frac{\alpha\beta}{AB(\alpha)}\lambda t\Big)\left\|u-v\right\|,\end{aligned}$$

however,

$$0 \leq \frac{\beta(1-\alpha)t^{\beta-1}}{AB(\alpha)}\lambda+\frac{\alpha\beta}{AB(\alpha)}\lambda t<1,$$

then,

$$\left\|u(x,t)-v(x,t)\right\| = 0$$

Thus, the solution of the BEQ is unique. □

5. Stability Analysis

This portion is devoted to the Picard's φ-stability of the model using an Exponential Decay Kernel and Mittag–Leffler operators in the following theorems.

Theorem 7. *Let φ be a self-mapping which is defined as follows:*

$$\varphi u_n(x,t) = u_n(x,t)+\frac{1}{M(\alpha)}L_x^{-1}L_t^{-1}\left[\frac{s+(1-s)\alpha}{s^\alpha}L_xL_t\{\beta t^{\beta-1}\{u_{nxx}+u_{nxxxx}+u_{nxx}^2\}\}\right].$$

Then, the iteration in the CF case is φ-stable in the $L^1_{(a,b)}$ if the condition $(B+C+D(\gamma_1+\gamma_2))G<1$ is satisfied.

Proof. With the help of Banach contraction theorem, we first show that the mapping φ posseses a unique fixed point. For this, we assume that the bounded iteration for $(n,m)\ \epsilon\ N*N$.

Consider

$$\begin{aligned}(\varphi u_m-\varphi u_n) &= u_m(x,t)-u_n(x,t)+\frac{1}{M(\alpha)}L_x^{-1}L_t^{-1}\left[\frac{s+(1-s)\alpha}{s^\alpha}L_xL_t\Big\{\beta t^{\beta-1}(u_{mxx}+u_{mxxxx}+u_{mxx}^2)\Big\}\right]\\ &\quad-\frac{1}{M(\alpha)}L_x^{-1}L_t^{-1}\left[\frac{s+(1-s)\alpha}{s^\alpha}L_xL_t\Big\{\beta t^{\beta-1}(u_{nxx}+u_{nxxxx}+u_{nxx}^2)\Big\}\right],\\ &= u_m(x,t)-u_n(x,t)+\frac{1}{M(\alpha)}L_x^{-1}L_t^{-1}\left[\frac{s+(1-s)\alpha}{s^\alpha}L_xL_t\Big\{\beta t^{\beta-1}(u_{mxx}+u_{mxxxx}\right.\\ &\quad\left.+u_{mxx}^2-u_{nxx}-u_{nxxxx}-u_{nxx}^2\Big)\Big\}\right],\\ &= u_m(x,t)-u_n(x,t)+\frac{1}{M(\alpha)}L_x^{-1}L_t^{-1}\left[\frac{s+(1-s)\alpha}{s^\alpha}L_xL_t\Big\{\beta t^{\beta-1}(B(u_m-u_n)\right.\\ &\quad\left.+C(u_m-u_n)+(u_{mxx}^2-u_{nxx}^2)\Big)\Big\}\right],\end{aligned}$$

now, using triangular inequality, we obtain

$$\|\varphi(u_m - \varphi u_n)\| \leq \|u_m(x,t) - u_n(x,t)\| + \Big\| \frac{1}{M(\alpha)} L_x^{-1} L_t^{-1} \Big[\frac{s + (1-s)\alpha}{s^\alpha} L_x L_t \Big\{ \beta t^{\beta-1} \Big(B(u_m - u_n) + C(u_m - u_n) + D(u_{mxx}^2 - u_{nxx}^2) \Big) \Big\} \Big] \Big\|$$

Using the boundedness of u_m and u_n, we obtain

$$\|\varphi(u_m - \varphi u_n)\| \leq (B + C + D(\gamma_1 + \gamma_2)) N \|(u_m - u_n)\|,$$

where N is a function obtained from $\frac{1}{M(\alpha)} L_x^{-1} L_t^{-1} \Big[\frac{s+(1-s)\alpha}{s^\alpha} L_x L_t \beta t^{\beta-1}(*) \Big]$ by assumption and the mapping φ fulfills the contraction condition. Hence, from the Banach fixed point result, φ has a unique fixed point. In addition, the mapping fulfills all the condition of Picard stability, with $z_1 = 0$ and $z_2 = (B + C + D(\gamma_1 + \gamma_2))N$. Thus, the solution is Picard φ-stable. □

Theorem 8. *Let φ be a self-mapping defined as*

$$\varphi u_{n+1}(x,t) = u_n(x,t) + \frac{1}{AB(\alpha)} L_x^{-1} L_t^{-1} \Big[(1 - \alpha + \frac{\alpha}{s^\alpha}) L_x L_t \{ \beta t^{\beta-1} \{ u_{nxx} + u_{nxxxxx} + A_n \} \} \Big].$$

Then, the iteration in the ABC case is φ-stable in the $L^1_{(a,b)}$ if the condition $(B + C + D(\gamma_1 + \gamma_2))G < 1$ is satisfied.

Proof. With the help of Banach contraction theorem, we first show that the mapping φ posseses a unique fixed point. For this, we assume that the bounded iteration for $(n,m) \ \epsilon \ N * N$.

Consider

$$\begin{aligned}
(\varphi u_m - \varphi u_n) &= u_m(x,t) - u_n(x,t) + \frac{1}{M(\alpha)} L_x^{-1} L_t^{-1} \Big[(1 - \alpha + \frac{\alpha}{s^\alpha}) L_x L_t \Big\{ \beta t^{\beta-1} (u_{mxx} + u_{mxxxxx} + u_{mxx}^2) \Big\} \Big] \\
&\quad - L_x^{-1} L_t^{-1} \Big[(1 - \alpha + \frac{\alpha}{s^\alpha}) L_x L_t \Big\{ \beta t^{\beta-1} (u_{nxx} + u_{nxxxxx} + u_{nxx}^2) \Big\} \Big], \\
&= u_m(x,t) - u_n(x,t) + \frac{1}{M(\alpha)} L_x^{-1} L_t^{-1} \Big[(1 - \alpha + \frac{\alpha}{s^\alpha}) L_x L_t \Big\{ \beta t^{\beta-1} (u_{mxx} + u_{mxxxxx} \\
&\quad + u_{mxx}^2 - u_{nxx} - u_{nxxxxx} - u_{nxx}^2 \Big) \Big\} \Big], \\
&= u_m(x,t) - u_n(x,t) + \frac{1}{M(\alpha)} L_x^{-1} L_t^{-1} \Big[(1 - \alpha + \frac{\alpha}{s^\alpha}) L_x L_t \Big\{ \beta t^{\beta-1} (\mathbf{B}(u_m - u_n) \\
&\quad + \mathbf{C}(u_m - u_n) + (u_{mxx}^2 - u_{nxx}^2) \Big) \Big\} \Big],
\end{aligned}$$

now, using triangular inequality, we obtain

$$\|\varphi(u_m - \varphi u_n)\| \leq \|u_m(x,t) - u_n(x,t)\| + \Big\| \frac{1}{M(\alpha)} L_x^{-1} L_t^{-1} \Big[(1 - \alpha + \frac{\alpha}{s^\alpha}) L_x L_t \Big\{ \beta t^{\beta-1} \Big(B(u_m - u_n) + C(u_m - u_n) + D(u_m^2 - u_m^2) \Big) \Big\} \Big] \Big\|$$

Using the boundedness of u_m and u_n, we obtain

$$\|\varphi(u_m - \varphi u_n)\| \leq (B + C + D(\gamma_1 + \gamma_2)) N \|(u_m - u_n)\|,$$

where N is a function obtained from $\frac{1}{M(\alpha)} L_x^{-1} L_t^{-1} \Big[\frac{s+(1-s)\alpha}{s^\alpha} L_x L_t \beta t^{\beta-1}(*) \Big]$ using assumption and the mapping φ fulfills the contraction condition. Hence, from the Banach fixed point result, φ has a unique fixed point. In addition, the mapping fulfills all the condition of Picard stability, with $z_1 = 0$ and $z_2 = (B + C + D(\gamma_1 + \gamma_2))N$. Thus, the solution of the ABC model is Picard φ-stable. □

6. Numerical Examples

In this section, we illustrate our results with the help of examples.

Case 1. Consider the FF Beq equation with an exponential decay kernel

$$ {}_{0}^{FFE}D_t^{\alpha,\beta}u(x,t) \quad = \quad u_{xx} + u_{xxxx} + u_{xx}^2 \qquad 1 < \alpha \leq 2, 0 < \beta \leq 1, \tag{22} $$

with subsidiary conditions

$$ u(x,0) = 2\,\frac{ak^2e^{kx}}{\left(1+ae^{kx}\right)^2}, \qquad u_t(x,0) = -2\,\frac{\sqrt{k^2+1}ak^3e^{kx}\left(-1+ae^{kx}x\right)}{\left(1+ae^{kx}\right)^3}. \tag{23} $$

The exact solution of Equation (22) with integer order is [2]

$$ u(x,t) = 2\,\frac{ak^2e\left(kx+k\sqrt{1+k^2t}\right)}{1+ae\left(kx+k\sqrt{1+k^2t}\right)^2}. \tag{24} $$

Using the established method, we obtain the following series of solutions of Equation (22):

$$ \begin{aligned} u0 &= 2\frac{ak^2e^{kx}}{\left(1+ae^{kx}\right)^2} - 2\frac{\sqrt{k^2+1}ak^3e^{kx}\left(-1+ae^{kx}x\right)}{\left(1+ae^{kx}\right)^3}t \\ u1 &= \left[(\alpha t + \beta - \alpha\beta)t^{\beta-1}\right][4\,\frac{ak^4e^{kx}\left(2\,k^2+1\right)}{\left(1+ae^{kx}\right)^2} \\ &- 24\,\frac{a^2k^4e^{2\,kx}\left(10\,k^2+1\right)}{\left(1+ae^{kx}\right)^3} + 24\,\frac{a^3k^4e^{3\,kx}\left(50\,k^2+1\right)}{\left(1+ae^{kx}\right)^4} \\ &+ 1920\,\frac{a^4k^6e^{4\,kx}}{\left(1+ae^{kx}\right)^5} + 960\,\frac{a^5k^6e^{5\,kx}}{\left(1+ae^{kx}\right)^6} + 16\,\frac{a^2k^8e^{2\,kx}\left(e^{2\,kx}a^2 - 4\,ae^{kx}+1\right)^2}{\left(1+ae^{kx}\right)^8}] \end{aligned} $$

Case 2. Consider the FF Beq equation with a Mittage–Leffler kernel

$$ {}_{a}^{FFM}D_t^{\alpha,\beta}u(x,t) = u_{xx} + u_{xxxx} + u_{xx}^2 \qquad 1 < \alpha \leq 2, \quad 0 < \beta \leq 1, \tag{25} $$

with subsidiary conditions

$$ u(x,0) = 2\,\frac{ak^2e^{kx}}{\left(1+ae^{kx}\right)^2}, \qquad u_t(x,0) = -2\,\frac{\sqrt{k^2+1}ak^3e^{kx}\left(-1+ae^{kx}x\right)}{\left(1+ae^{kx}\right)^3}. \tag{26} $$

The exact solution of Equation (25) with integer order is [2]

$$ u(x,t) = 2\,\frac{ak^2e\left(kx+k\sqrt{1+k^2t}\right)}{1+ae\left(kx+k\sqrt{1+k^2t}\right)^2}. \tag{27} $$

Using the developed strategy, we obtain the following series of solutions of Equation (25)

$$
\begin{aligned}
u0 &= 2\frac{ak^2e^{kx}}{\left(1+ae^{kx}\right)^2} - 2\frac{\sqrt{k^2+1}ak^3e^{kx}\left(-1+ae^{kx}x\right)}{\left(1+ae^{kx}\right)^3}t \\
u1 &= \left[\frac{1}{AB(\alpha)}\left((1-\alpha)\beta + \frac{\alpha\Gamma(1+\beta)t^{\alpha}}{\Gamma(\alpha+\beta)}\right)t^{\beta-1}\right][4\,\frac{ak^4e^{kx}\left(2\,k^2+1\right)}{\left(1+ae^{kx}\right)^2} \\
&- 24\,\frac{a^2k^4e^{2\,kx}\left(10\,k^2+1\right)}{\left(1+ae^{kx}\right)^3} + 24\,\frac{a^3k^4e^{3\,kx}\left(50\,k^2+1\right)}{\left(1+ae^{kx}\right)^4} \\
&+ 1920\,\frac{a^4k^6e^{4\,kx}}{\left(1+ae^{kx}\right)^5} + 960\,\frac{a^5k^6e^{5\,kx}}{\left(1+ae^{kx}\right)^6} + 16\,\frac{a^2k^8e^{2\,kx}\left(e^{2\,kx}a^2-4\,ae^{kx}+1\right)^2}{\left(1+ae^{kx}\right)^8}]
\end{aligned}
$$

Discussion and Error Analysis

For the numerical demonstration of the single soliton solution we have considered the parameter $\gamma = 1$. The left panel of Figure 1 displays the absolute of the approximate solution provided in Equation (25) versus the exact solution (24). It can be noticed that the approximate solution identically matches the exact solution per A.M. Wazwaz [2], where a and k are arbitrary constants. The single soliton is in full agreement with the result of [2]. In the right panel, the approximate solution is plotted for a few distinct values of t. It can be observed that for longer times the solitary wave solution blows up. For better observation of the impact of spatial variable x on the wave solution of (25), the results are displayed in Figure 2 in the right panel, keeping $\beta = 1$ and varying α by 1, 1.5, and 2.0. It can be seen that for different values of α, the solitary wave solution changes with the change in the fractional derivative. Similarly, the solutions of the system are very close to each other for different values of β when 1, 1.5, and 2.0 are considered for time $t = 1$. Moreover, the dynamics of the obtained solution for Case 1 are displayed in Figure 3 for different values of α and β. Similarly, for the ABC case, we simulate the series solution for a few values of α and β. The simulation of the obtained results is provided in Figures 4–6. The left panel of Figure 4 shows the evolution of the exact and acquired solutions, while the right panel shows the behaviour of the solution for different values of t. We observe that the solution in the ABC case is highly sensitive to time. The last picture, Figure 5, represents the dynamics of the acquired solution for two sets of α and β. Figure 6 displays the 3D behaviour of the obtained solution for Case 2. We observe that the solution's behaviour changes when varying α or β. Further, to show the accuracy and validity of the proposed approach, we provide error analysis for the considered examples in Tables 1 and 2.

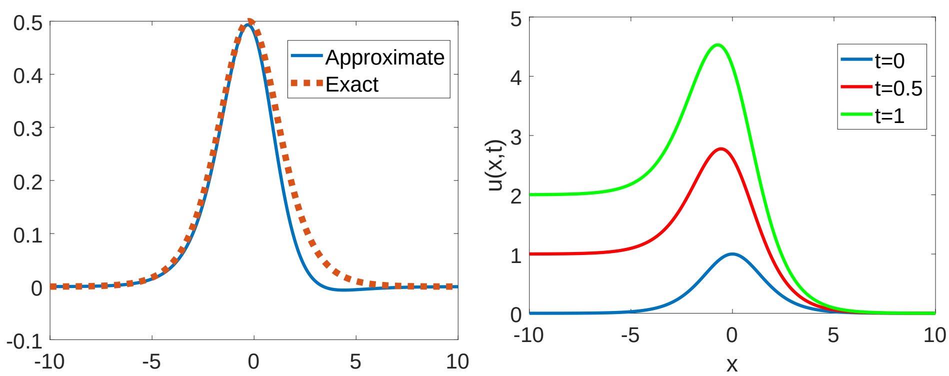

Figure 1. Comparison between Equation (24) and solution of Equation (22) for $\alpha = 2$, $\beta = 1$. The right panel shows the approximate solutions for different values of time (t).

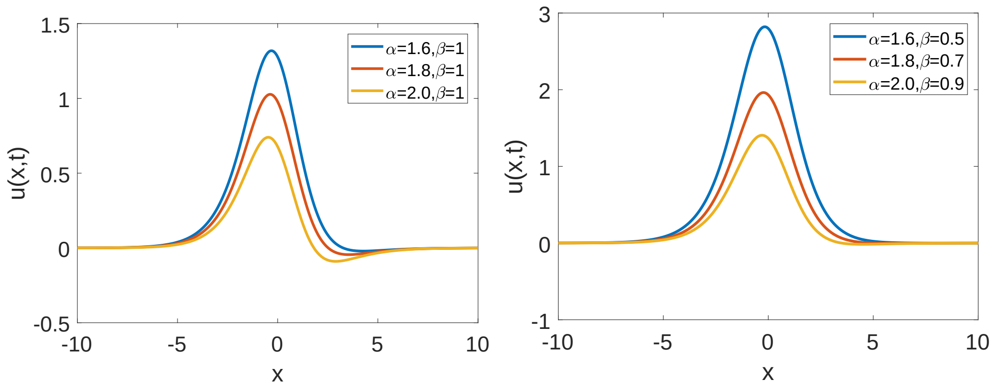

Figure 2. Solution profiles for Equation (22) $u(x,t)$ versus x for $\beta = 1$ in the left panel and with both varied in the right panel.

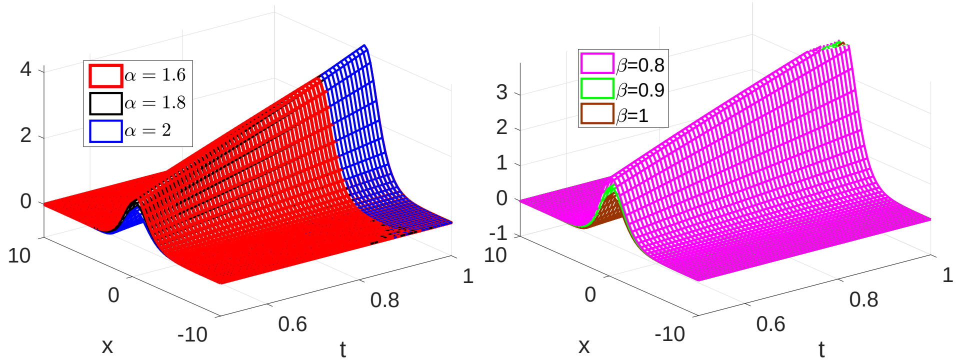

Figure 3. Solution profiles of Equation (22) in 3D space.

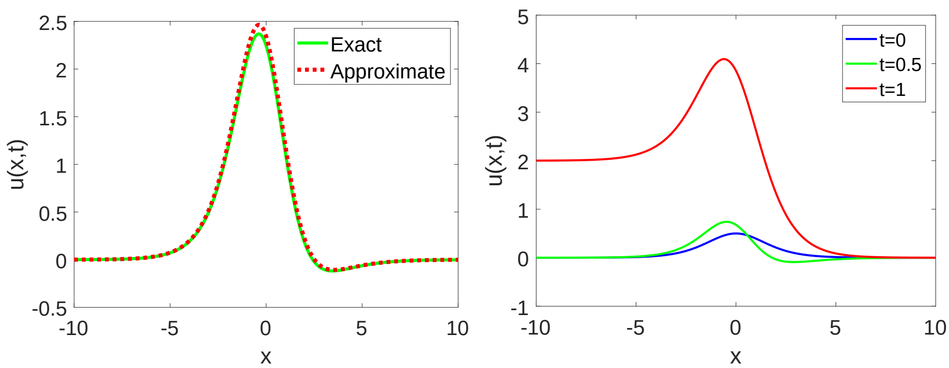

Figure 4. Comparison between Equation (27) and solution of Equation (25) for $\alpha = 2,\ \beta = 1$. The right panel shows the approximate solutions for different values of time (t).

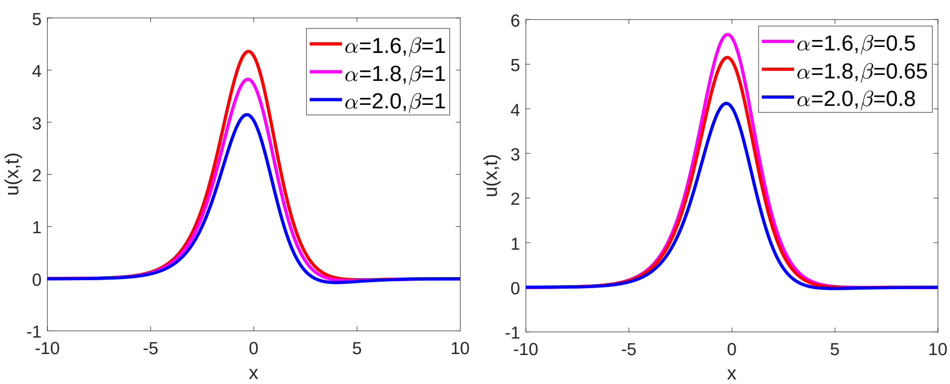

Figure 5. Solution profiles for Equation (25) $u(x,t)$ versus x for $\beta = 1$ in the left panel and with both varied in the right panel.

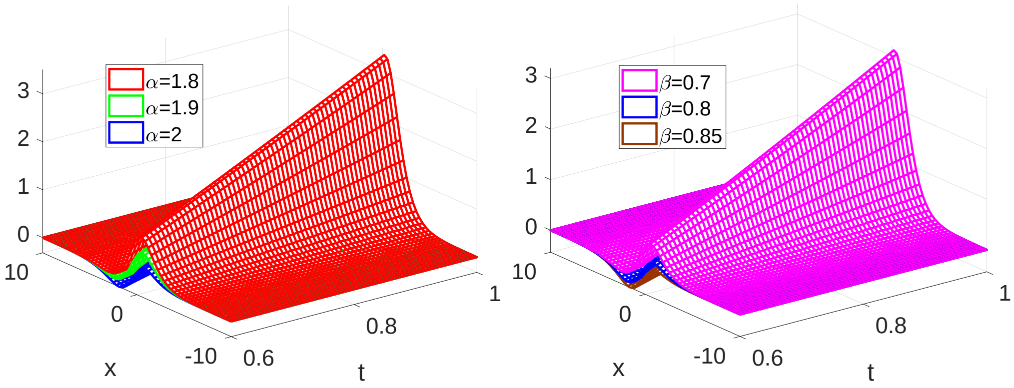

Figure 6. Solution profiles for Equation (25) in 3D space.

Case 1. Exponential Decay Kernel

Table 1. Absolute error between the approximate versus exact solution for $\alpha = 1$, $k = 1$, $a = 1$, $\beta = 1$, and $t = 0.05$.

(x,t)	Exact	u	| Exact$-u$|	(x,t)	Exact	u	| Exact$-u$|
(−5,0.05)	0.0143	0.1189	3.2593×10^{-5}	(−4.5,0.05)	0.1303	0.0232	8.2544×10^{-5}
(−4,0.05)	0.0378	0.1485	1.9948×10^{-4}	(−3.5,0.05)	0.0608	0.1766	4.4540×10^{-4}
(−3,0.05)	0.0963	0.2192	8.6756×10^{-4}	(−2.5,0.05)	0.1635	0.1635	3.5154×10^{-6}
(−2,0.05)	0.1756	0.1755	8.6007×10^{-6}	(−1.5,0.05)	0.3117	0.4684	1.9665×10^{-4}
(−1,0.05)	0.4049	0.5737	1.6855×10^{-5}	(−0.5,0.05)	0.4776	0.6508	7.427×10^{-4}
(0,0.05)	0.4993	0.6676	1.250×10^{-2}	(1,0.05)	0.3802	0.5047	9.9831×10^{-4}
(1.5,0.05)	0.2849	0.3796	1.9665×10^{-4}	(2,0.05)	0.1988	0.2662	1.2075×10^{-3}
(2.5,0.05)	0.1319	0.1776	1.3197×10^{-3}	(3,0.05)	0.0847	0.1145	8.6756×10^{-4}
(3.5,0.05)	0.0532	0.0722	4.4540×10^{-4}	(4,0.05)	0.0329	0.0449	1.9948×10^{-4}
(4.5,0.05)	0.0202	0.0277	8.2544×10^{-5}	(5,0.05)	0.01239	0.0170	3.2592×10^{-5}

We performed an error analysis for both Caputo–Fabrizio (CF) and Atangana–Baleanu (ABC) operators between the approximate and exact solutions for the values of $k = 1$, $a = 1$, $\alpha = 2$, and $\beta = 1$ for a very small value of $t = 0.18$. The absolute error between the acquired and exact solutions in the CF case is a little higher than in the ABC case. Thus, we can say that the fractal-fractional ABC BEQ produces better dynamics to the CF BEQ.
Case 2. Mittag–Leffler Kernel

Table 2. Absolute error between the approximate versus exact solution for $\alpha = 2$, $k = 1$, $a = 1$, $\beta = 1$, and $t = 0.18$.

(x,t)	Exact	u	| Exact$-u$|	(x,t)	Exact	u	| Exact$-u$|
(−5,0.05)	0.0142	0.1178	3.2490×10^{-5}	(−4.5,0.05)	0.1303	0.0132	8.2431×10^{-5}
(−4,0.05)	0.0378	0.1385	1.9846×10^{-4}	(−3.5,0.05)	0.0608	0.1661	4.4438×10^{-4}
(−3,0.05)	0.0963	0.2088	8.6216×10^{-4}	(−2.5,0.05)	0.1635	0.1521	3.5001×10^{-6}
(−2,0.05)	0.1756	0.1450	8.5013×10^{-6}	(−1.5,0.05)	0.3117	0.4480	1.9562×10^{-4}
(−1,0.05)	0.4049	0.5632	1.6550×10^{-5}	(−0.5,0.05)	0.4776	0.6428	7.411×10^{-4}
(0,0.05)	0.4993	0.6316	1.201×10^{-2}	(1,0.05)	0.3802	0.4947	9.9430×10^{-4}
(1.5,0.05)	0.2849	0.3694	1.9575×10^{-4}	(2,0.05)	0.1988	0.2460	1.1975×10^{-3}
(2.5,0.05)	0.1319	0.1673	1.3092×10^{-3}	(3,0.05)	0.0847	0.1041	8.6351×10^{-4}
(3.5,0.05)	0.0532	0.0712	4.4330×10^{-4}	(4,0.05)	0.0329	0.0348	1.9847×10^{-4}
(4.5,0.05)	0.0202	0.0176	8.2430×10^{-5}	(5,0.05)	0.01239	0.0169	3.2491×10^{-5}

7. Conclusions

In this article, we have investigated the BEQ using fractal-fractional operators with two different nonsingular kernels. The theoretical aspects such as the existence, boundedness, uniqueness, and stability of the solution of the considered BEQ for both operators have been discussed using the concept of fixed point theorems. A computational analysis was carried out via double LT and ADM. The results were verified and validated by simulating them via MATLAB-18. The main goal was to show the effect of the fractional order α and fractal dimension β on the behaviour of the waves produced by the BEQ. These behaviours are shown in Figures 2 and 4. The results show full agreement with [2] in terms of comparison between the exact solution as provided in [2] and the approximate solution obtained through the MDLDM technique. Furthermore, we have obtained results for different values of time t; the results show that when the time is increased by a large extent, the soliton wave blows up, while with increasing values for $1 < \alpha \leq 2$ and $0 < \beta \leq 1$ the soliton wave linearly increases and shows good results. In the future, this technique could be used for the periodic solution of the two-soliton Boussinesq equation.

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