

Article

Study of Transmission Dynamics of *Streptococcus suis* Infection Mathematical Model between Pig and Human under ABC Fractional Order Derivative

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Abstract: In this paper, a mathematical model for *Streptococcus suis* infection is improved by using the fractional order derivative. The modified model also investigates the transmission between pigs and humans. The proposed model can classify the pig population density into four classes, which are pig susceptible class, pig infectious class, pig quarantine class, and pig recovery class. Moreover, the human population density has been separated into three classes, these are human susceptible class, human infectious class, and human recovery class. The spread of the infection is analyzed by considering the contact between humans and pigs. The basic reproduction number (R_0), the infectious indicator, is carried out using the next generation matrix. The disease-free equilibrium is locally asymptotically stable if $R_0 < 1$, and the endemic equilibrium is locally asymptotically stable if $R_0 > 1$. The theoretical analyses of the fractional order derivative model, existence and uniqueness, have been proposed. The numerical examples were illustrated to support the proposed stability theorems. The results show that the fractional order derivative model provides the various possible solution trajectories with different fractional orders for the same parameters. In addition, transmission between pigs and humans resulted in the spread of *Streptococcus suis* infection.

Keywords: *Streptococcus suis*; endemic disease model; stability analysis; Atagana–Baleanu–Caputo fractional order derivative; numerical simulation

MSC: 34A34; 34D20; 37N25; 92-10

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1. Introduction

One of the biggest problems is the zoonotic pathogen. Recently, the world has faced SARS, influenza, Yellow fever, Avian influenza, MERS, and COVID-19 [1–3]. These diseases affect the world's economy. If we focus only on the swine industry, *Streptococcus suis* (*S. suis*) is one of the major problems [4–6]. *S. suis* is a Gram-positive bacterium that is an infectious agent found widely in piglets and pigs around the world [7,8]. It can be found in the respiratory tract, genitals, and alimentary tracts of pigs [9,10]. *S. suis* transmits rapidly across the farm from pig to pig via direct contact with sick pigs [5,11]. This pathogen has an at least 20% mortality rate in pigs around the world [12]. This disease was first found in 1954 [13,14]. There are at least 35 serotypes of *S. suis* [15,16]. Moreover, this pathogen can be transmitted to humans via contact with pork [4,17,18]. After that, the number of humans who are infected by this pathogen has increased over the last two decades. Especially, the most commonly found human cases are in Thailand, Vietnam, the Netherlands, and China [9,19–21].

Many people die due to the disease. Mathematical models can help to prevent the spread of the disease. They are efficient tools to predict the dynamics of the disease [22–25]. Fatmawati et al. [26] proposed a new model for dengue fever by using fractional calculus. They also investigated the relationship between mosquitoes and humans. Huang et al. [27]

used a price function to express the relation between price and pig stock. Then, they proposed the time delay dynamical model to analyze the dynamics of African swine fever (ASF) disease transmission. Carvalho et al. [28] proposed the mathematical modeling of a dengue epidemic by considering the relation between vectors and humans. Kumar et al. [29] studied the time-delay Caputo fractional order model for plant disease. Peter et al. [30] proposed the fractional order of COVID-19 in Nigeria using the Atangana–Baleanu operator. However, fewer research studies have been conducted on the mathematical models for *S. suis* infection. Shen et al. [31] introduced the SIQRW model to study the dynamic of *S. Suis* by using ordinary differential equations. Giang et al. [16] introduced the mathematical model to predict the behavior of the disease, and estimated the parameters in the model with collected data. To our knowledge, no research study has reported on a mathematical model for *S. suis* using fractional calculus. Fractional calculus is a suitable tool for demonstrating some real-world situations, because it is more generalized than integer-order differential equation systems. In addition, the model can reduce financial losses in the swine industry because the model can forecast disease transmission. Then, the strategies to control the disease are obtained from the model parameters.

In this paper, we improved the mathematical model to predict the disease transmission of *S. suis* between humans and pigs by using fractional calculus. Firstly, we formulated the model in the form of the system of integer-order differential equations. We also found the basic reproduction number by using the next-generation matrix. The symmetric and asymmetric properties are investigated in the next-generation matrix. Then, we apply the Atagana–Baleanu–Caputo fractional order derivative to the model. The analysis of the proposed model is investigated. We propose the theorem to confirm that the solution of the proposed model exists. We also present numerical examples to confirm the result from the analysis part which are the stability conditions. This means that if the parameters satisfies the stability condition, the solution trajectories will go to the equilibrium. In addition, the basic reproduction number of the model, which is used to describe how fast the disease transmission is, is also studied in this paper.

According to the argument mentioned above, to study endemic *S. Suis*, a mathematical model is developed by considering the disease transmission between pigs and humans. Basic knowledge about fractional calculus is presented in Section 2. Then, the SIQR-SIR (Susceptible-Infectious-Quarantine-Recovery-Susceptible-Infectious-Recovery) model is introduced in Section 3. The analysis of the mathematical model is shown in Section 4. The fractional order model is presented in Section 5. Then, Section 6 discusses the numerical examples of the proposed model. Finally, conclusions are presented in Section 7.

2. Fractional Calculus

In this section, the related definitions of the fractional operators that are used in this paper are shown as follows.

Definition 1 ([32]). *The Riemann–Liouville integral operator of the fractional order α for $0 < \alpha < 1$ for a function $f : (0, \infty) \rightarrow \mathbb{R}$ is given by:*

$$J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad (1)$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2 ([32]). *The Caputo fractional derivative for order $\rho > 0$ is defined as:*

$${}^c D_t^\rho f(t) = \frac{1}{\Gamma(n-\rho)} \int_0^t (t-s)^{n-\rho-1} f^{(n)}(s) ds, \quad (2)$$

where $n-1 < \rho \leq n$, $n \in \mathbb{N}$ and $f \in C^{n-1}[0, t]$.

Definition 3 ([33]). The Atangana–Baleanu fractional derivative for a given function for order ζ in the Caputo sense (ABC) is defined as:

$${}_a^{ABC}D_t^\zeta f(t) = \frac{B(\zeta)}{1-\zeta} \int_a^t \frac{df(s)}{ds} E_k \left[\frac{-\zeta(1-s)^\zeta}{1-\zeta} \right] ds, \quad (3)$$

where $B(\zeta) = (1-\zeta) + \frac{\zeta}{\Gamma(\zeta)}$ is a normalization function and $E_\alpha(\cdot)$ is the Mittag-Leffler function.

Definition 4 ([33]). The Atangana–Baleanu fractional integral order ζ is defined as:

$${}_0^{ABC}I_t^\zeta f(t) = \frac{1-\zeta}{B(\zeta)} f(t) + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_a^t f(s)(t-s)^{\zeta-1} ds, \quad (4)$$

where $B(\zeta) = (1-\zeta) + \frac{\zeta}{\Gamma(\zeta)}$ is a normalization function.

3. Formulation of the Mathematical Model

In this section, we express the model formulation of the disease transmitted in humans and pigs. We classified the pig population into four classes, which are susceptible class (S_p), infectious class (I_p), quarantine class (Q_p), and recovery class (R_p). Since the transmission of *S. Suis* can be transferred between pigs and humans, therefore, the susceptible human class (S_h), infectious class (I_h), and recovery class (R_h) are taken into the model. The flowchart of the SIQR-SIR model of pigs and humans described by the system (5) is shown in Figure 1.

$$\begin{aligned} \frac{dS_p}{dt} &= N_p - M\beta S_p I_p - nS_p, \\ \frac{dI_p}{dt} &= M\beta S_p I_p - \delta I_p - mI_p - nI_p, \\ \frac{dQ_p}{dt} &= \delta I_p - mQ_p - nQ_p - \varepsilon Q_p, \\ \frac{dR_p}{dt} &= \varepsilon Q_p - nR_p, \\ \frac{dS_h}{dt} &= N_h - \mu S_h - \gamma S_h I_p, \\ \frac{dI_h}{dt} &= \gamma S_h I_p - \alpha I_h - \mu I_h - \beta_2 I_h, \\ \frac{dR_h}{dt} &= \beta_2 I_h - \mu R_h. \end{aligned} \quad (5)$$

The model parameter defined as β is the coefficient transmission per unit of time per pig in the pig susceptible class contact with infectious pig class, M is the relative humidity, m is the pig death rate induced by the disease, n is the pig natural death rate, δ is the rate from infectious class to quarantine class in pigs, ε is the pig recovered rate, μ is the human natural death rate, γ is the transmission rate from infected pig to human, α is the disease death rate, and β_2 is the human recovered rate.

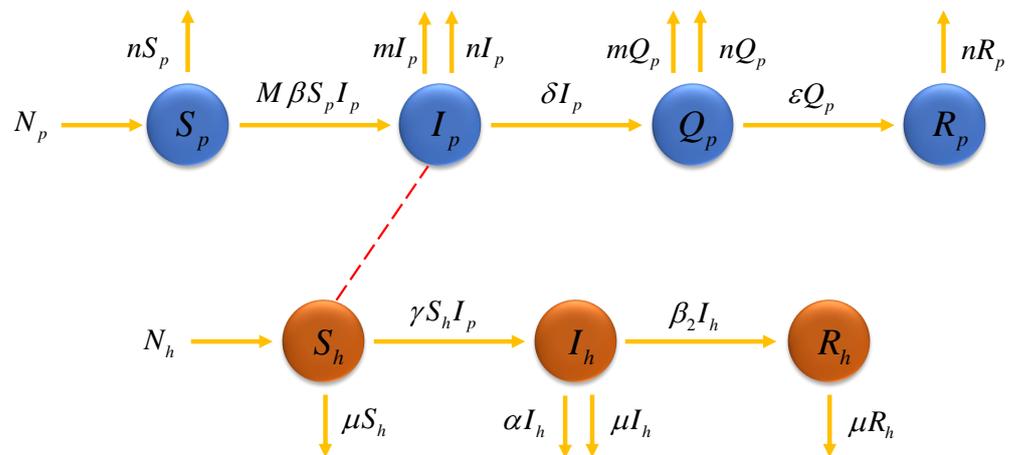


Figure 1. The compartmental diagram for disease transmission dynamics of *S. Suis* infection for pigs and humans.

4. Model Analysis

In this section, we investigate the invariant region of the solution of the model, which is biologically meaningful because the proposed model involves the population number. All variables and parameters in the model are assumed to be non-negative. Then, the equilibria of the model and the basic reproduction number are obtained. Moreover, we also study the local stability of all equilibria.

4.1. The Invariant Region

The total population of humans and pigs can be written as

$$N(t) = S_p + I_p + Q_p + R_p + S_h + I_h + R_h.$$

Differentiating with respect to t of the total population equation, we have

$$\begin{aligned} \frac{dN(t)}{dt} &= \frac{dS_p}{dt} + \frac{dI_p}{dt} + \frac{dQ_p}{dt} + \frac{dR_p}{dt} + \frac{dS_h}{dt} + \frac{dI_h}{dt} + \frac{dR_h}{dt} \\ &= N_p + N_h - n(S_p + R_p) - (m + n)(I_p + Q_p) - \mu(S_h + R_h) - (\alpha + \mu)I_p \\ &\leq N_p + N_h - AN \end{aligned}$$

where A is a constant.

Then, we have

$$N(t) \leq \frac{N_p + N_h}{A} + \left(N(0) - \frac{N_p + N_h}{A} \right) e^{-At}.$$

As $t \rightarrow \infty$, the above inequality leads to

$$N(t) \leq \frac{N_p + N_h}{A}.$$

The feasible solution set of population in the system (5) remains in the region

$$\Omega = \left\{ (S_p, I_p, Q_p, R_p, S_h, I_h, R_h) \in \mathbb{R}_+^7 : N \leq \frac{N_p + N_h}{A} \right\}.$$

4.2. Equilibria

Setting all equations in the system (5) to be zeros, we obtain

$$\begin{aligned}
 N_p - M\beta S_p I_p - nS_p &= 0, \\
 M\beta S_p I_p - \delta I_p - mI_p - nI_p &= 0, \\
 \delta I_p - mQ_p - nQ_p - \varepsilon Q_p &= 0, \\
 \varepsilon Q_p - nR_p &= 0, \\
 N_h - \mu S_h - \gamma S_h I_p &= 0, \\
 \gamma S_h I_p - \alpha I_h - \mu I_h - \beta_2 I_h &= 0, \\
 \beta_2 I_h - \mu R_h &= 0.
 \end{aligned} \tag{6}$$

The system (5) has two types of equilibrium points, which are disease-free equilibrium (DFE), the point at which no disease is, and endemic equilibrium, where the infection is constantly maintained at some level [34,35].

The disease-free equilibrium (DFE) of the system (5) can be written as

$$E_1 = \left(\frac{N_p}{n}, 0, 0, 0, \frac{N_h}{\mu}, 0, 0 \right).$$

The endemic equilibrium of the system (5) is

$$E_2 = \left(\frac{\delta + m + n}{M\beta}, \frac{n(R_0 - 1)}{M\beta}, \frac{\delta I_p}{m + n + \varepsilon}, \frac{\varepsilon Q_p}{n}, \frac{N_h M\beta}{\mu M\beta + n\gamma(R_0 - 1)}, \frac{\gamma S_h I_p}{\alpha + \mu + \beta_2}, \frac{\beta_2 I_h}{\mu} \right).$$

4.3. Basic Reproduction Number (R_0)

The next generation matrix method [36] is used to compute the basic reproduction number (R_0). In addition, the local stability of the equilibrium of the system (5) can be considered by using the basic reproduction number. By following the process as [37], we have

$$f = \begin{bmatrix} M\beta S_p I_p \\ \delta I_p \\ \gamma S_h I_p \end{bmatrix}, v = \begin{bmatrix} \delta I_p + mI_p + nI_p \\ mQ_p + nQ_p + \varepsilon Q_p \\ \alpha I_h + \mu I_h + \beta_2 I_h \end{bmatrix}.$$

The corresponding asymmetry Jacobian matrix of f and v are

$$F = \begin{bmatrix} M\beta S_p & 0 & 0 \\ \delta & 0 & 0 \\ \gamma S_h & 0 & 0 \end{bmatrix}, V = \begin{bmatrix} \delta + m + n & 0 & 0 \\ 0 & m + n + \varepsilon & 0 \\ \alpha & 0 & \mu + \beta_2 \end{bmatrix}.$$

Then, we have

$$V^{-1} = \begin{bmatrix} \frac{1}{\delta + m + n} & 0 & 0 \\ 0 & \frac{1}{m + n + \varepsilon} & 0 \\ \frac{-\alpha}{(\delta + m + n)(\mu + \beta_2)} & 0 & \frac{1}{\mu + \beta_2} \end{bmatrix}.$$

Therefore,

$$FV^{-1} = \begin{bmatrix} \frac{M\beta S_p}{\delta + m + n} & 0 & 0 \\ \frac{\delta}{\delta + m + n} & 0 & 0 \\ \frac{\gamma S_h}{\delta + m + n} & 0 & 0 \end{bmatrix}.$$

The eigenvalues of FV^{-1} are

$$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = \frac{M\beta S_p}{\delta + m + n}.$$

We thus have the basic reproduction number, which is the spectral radius

$$R_0 = \frac{M\beta S_p}{\delta + m + n}.$$

4.4. The Stability of Disease-Free Equilibrium

In this subsection, the disease-free equilibrium E_1 is shown, where E_1 is locally asymptotically stable when $R_0 < 1$. The method is to investigate the eigenvalues of the Jacobian matrix of the linearized system of the model (5) at the equilibrium.

Theorem 1. *The disease-free equilibrium point (E_1) is locally asymptotically stable if $R_0 < 1$.*

Proof. The Jacobian matrix at the disease-free equilibrium (E_1) of the system (5) is

$$J(E_1) = \begin{bmatrix} -n & -M\beta \frac{N_p}{n} & 0 & 0 & 0 & 0 & 0 \\ 0 & M\beta \frac{N_p}{n} - \delta - \omega & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta & -\omega - \varepsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & -n & 0 & 0 & 0 \\ 0 & -\gamma \frac{N_h}{\mu} & 0 & 0 & -\mu & 0 & 0 \\ 0 & \gamma \frac{N_h}{\mu} & 0 & 0 & 0 & -\alpha - \mu - \beta_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_2 & -\mu \end{bmatrix}.$$

The eigenvalues of the Jacobian matrix $J(E_1)$ are

$$\begin{aligned} \lambda_1 &= -\alpha - \mu - \beta_2, \\ \lambda_2 &= -\omega - \varepsilon, \\ \lambda_3 &= \lambda_4 = -\mu, \\ \lambda_5 &= \lambda_6 = -n, \\ \lambda_7 &= \frac{M\beta N_p - n(\delta - \omega)}{n} = (\delta + \omega)(R_0 - 1). \end{aligned}$$

We have that $\lambda_7 < 0$ when $R_0 < 1$. Therefore, the disease-free equilibrium (E_1) of the system (5) is locally asymptotically stable when $R_0 < 1$. \square

4.5. The Stability of Endemic Equilibrium

In this subsection, the endemic equilibrium E_2 is shown, where E_2 is locally asymptotically stable when $R_0 > 1$. The method is to investigate the eigenvalues of the Jacobian matrix $J(E_2)$ of the linearized system of the model (5) at the equilibrium. The Jacobian matrix method has been used in [38].

Theorem 2. *The endemic equilibrium (E_2) is locally asymptotically stable if $R_0 > 1$.*

Proof. The endemic equilibrium

$$E_2 = \left(\frac{\delta + \omega}{M\beta}, \frac{n(R_0 - 1)}{M\beta}, \frac{\delta I_p}{\omega + \varepsilon}, \frac{\varepsilon Q_p}{n}, \frac{N_h M\beta}{\mu M\beta + n\gamma(R_0 - 1)}, \frac{\gamma S_h I_p}{\alpha + \mu + \beta_2}, \frac{\beta_2 I_h}{\mu} \right)$$

is positive and epidemiologically meaningful if $R_0 > 1$.

The Jacobian matrix at the endemic equilibrium (E_2) of the system (5) is

$$J(E_2) = \begin{bmatrix} -n(R_0 - 1) - n & -\delta - \omega & 0 & 0 & 0 & 0 & 0 & 0 \\ n(R_0 - 1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta & -\omega - \varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & -n & 0 & 0 & 0 & 0 \\ 0 & -B & 0 & 0 & -\mu - \gamma C & 0 & 0 & 0 \\ 0 & B & 0 & 0 & \gamma C & -\alpha - \mu - \beta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_2 & -\mu & 0 \end{bmatrix}$$

where

$$B = \frac{\gamma N_h}{\frac{n(R_0 - 1)\gamma}{M\beta} + \mu}, C = \frac{n(R_0 - 1)}{M\beta}.$$

The eigenvalues of the Jacobian matrix $J(E_2)$ are

$$\lambda_1 = -n,$$

$$\lambda_2 = -\mu,$$

$$\lambda_3 = -\omega - \varepsilon,$$

$$\lambda_4 = -\alpha - \mu\beta_2,$$

$$\lambda_5 = -\frac{M\beta\mu + n\gamma(R_0 - 1)}{M\beta},$$

$$\begin{aligned} \lambda_6 &= -\frac{1}{2}nR_0 + \frac{1}{2}\sqrt{n^2R_0^2 - 4\delta nR_0 - 4mnR_0 - 4n^2R_0 + 4\delta n + 4mn + 4n^2} \\ &= -\frac{1}{2}nR_0 + \frac{1}{2}\sqrt{n^2R_0^2 - 4(\delta n + mn + n^2)(R_0 - 1)}, \end{aligned}$$

$$\begin{aligned} \lambda_7 &= -\frac{1}{2}nR_0 - \frac{1}{2}\sqrt{n^2R_0^2 - 4\delta nR_0 - 4mnR_0 - 4n^2R_0 + 4\delta n + 4mn + 4n^2} \\ &= -\frac{1}{2}nR_0 - \frac{1}{2}\sqrt{n^2R_0^2 - 4(\delta n + mn + n^2)(R_0 - 1)}. \end{aligned}$$

We have that $\lambda_5 < 0$ when $R_0 > 1$.

We next consider λ_6 and λ_7 are negative if $(\delta n + mn + n^2)(R_0 - 1) < 0$. This means that $R_0 > 1$.

Therefore, the endemic equilibrium (E_2) of the system (5) is locally asymptotically stable when $R_0 > 1$. \square

5. The Fractional Order Model with the Atangana–Baleanu–Caputo Operator

The improved model by using Atangana–Baleanu–Caputo (ABC) fractional derivative to the proposed model (5) given by a system of ordinary differential equation can be written as the following system.

$$\begin{aligned} {}^{ABC}_0 D^\zeta S_p(t) &= \Psi_1(t, S_p(t)) = N_p - M\beta S_p I_p - nS_p \\ {}^{ABC}_0 D^\zeta I_p(t) &= \Psi_2(t, I(t)) = M\beta S_p I_p - \delta I_p - mI_p - nI_p \\ {}^{ABC}_0 D^\zeta Q_p(t) &= \Psi_3(t, Q(t)) = \delta I_p - mQ_p - nQ_p - \varepsilon Q_p \\ {}^{ABC}_0 D^\zeta R_p(t) &= \Psi_4(t, R_p(t)) = \varepsilon Q_p - nR_p \\ {}^{ABC}_0 D^\zeta S_h(t) &= \Psi_5(t, S_h(t)) = N_h - \mu S_h - \gamma S_h I_p \\ {}^{ABC}_0 D^\zeta I_h(t) &= \Psi_6(t, I_h(t)) = \gamma S_h I_p - \alpha I_h - \mu I_h \\ {}^{ABC}_0 D^\zeta R_h(t) &= \Psi_7(t, R_h(t)) = \beta_2 I_h - \mu R_h - \beta_2 I_h \end{aligned} \quad (7)$$

The notation ${}^{ABC}_0D^\zeta$ is the time fractional derivative in Atangana–Baleanu with Caputo sense and the parameter $0 < \zeta \leq 1$ is the order of the ABC operator with the initial conditions

$$\begin{aligned} S_{p_0}(t) &= S_p(0), I_{p_0}(t) = I_p(0), Q_{p_0}(t) = Q_p(0), R_{p_0}(t) = R_p(0), \\ S_{h_0}(t) &= S_h(0), I_{h_0}(t) = I_h(0), R_{h_0}(t) = R_h(0). \end{aligned} \tag{8}$$

The system (7) can be written as the Volterra integral equation by using the ABC integral operator as follows:

$$\begin{aligned} S_{p_0}(t) - S_p(0) &= \frac{1-\zeta}{B(\zeta)} \left(N_p - M\beta S_p(t)I_p(t) - nS_p(t) \right) \\ &\quad + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(N_p - M\beta S_p(z)I_p(z) - nS_p(z) \right) (t-z)^{\zeta-1} dz \\ I_{p_0}(t) - I_p(0) &= \frac{1-\zeta}{B(\zeta)} \left(M\beta S_p(t)I_p(t) - (\delta + m + n)I_p(t) \right) \\ &\quad + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(M\beta S_p(z)I_p(z) - (\delta + m + n)I_p(z) \right) (t-z)^{\zeta-1} dz \\ Q_{p_0}(t) - Q_p(0) &= \frac{1-\zeta}{B(\zeta)} \left(\delta I_p(t) - (m + n + \varepsilon)Q_p(t) \right) \\ &\quad + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(\delta I_p(z) - (m + n + \varepsilon)Q_p(z) \right) (t-z)^{\zeta-1} dz \\ R_{p_0}(t) - R_p(0) &= \frac{1-\zeta}{B(\zeta)} \left(\varepsilon Q_p(t) - nR_p(t) \right) \\ &\quad + \frac{k}{B(K)\Gamma(\zeta)} \int_0^t \left(\varepsilon Q_p(z) - nR_p(z) \right) (t-z)^{\zeta-1} dz \\ S_{h_0}(t) - S_h(0) &= \frac{1-\zeta}{B(\zeta)} \left(N_h - \mu S_h(t) - \gamma S_h(t)I_p(t) \right) \\ &\quad + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(N_h - \mu S_h(z) - \gamma S_h(z)I_p(z) \right) (t-z)^{\zeta-1} dz \\ I_{h_0}(t) - I_h(0) &= \frac{1-\zeta}{B(\zeta)} \left(\gamma S_h I_p - (\alpha + \mu + \beta - 2)I_h(t) \right) \\ &\quad + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(\gamma S_h I_p - (\alpha + \mu + \beta - 2)I_h(z) \right) (t-z)^{\zeta-1} dz \\ R_{h_0}(t) - R_h(0) &= \frac{1-\zeta}{B(\zeta)} \left(\beta_2 I_h - \mu R_h \right) \\ &\quad + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(\beta_2 I_h - \mu R_h \right) (t-z)^{\zeta-1} dz. \end{aligned} \tag{9}$$

To confirm the existence of the solution of modification model [39,40], we propose the following theorems.

Theorem 3. *The kernels $\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5, \Psi_6, \Psi_7$ given by the system (7) satisfy the Lipschitz condition and contraction if the following inequalities hold: $0 \leq \zeta_i < 1$ for all $i = 1, 2, \dots, 7$.*

Proof. Let the kernel $\Psi_1(t, S(t)) = N_p - M\beta S_p I_p - nS_p$.

Let S_{p_1} and S_{p_2} be two functions, then we obtain the following

$$\begin{aligned} \|\Psi_1(t, S_{p_1}(t)) - \Psi_1(t, S_{p_2}(t))\| &= \|(N_p - M\beta S_{p_1}(t)I_p(t) - nS_{p_1}(t)) \\ &\quad - (N_p - M\beta S_{p_2}(t)I_p(t) - nS_{p_2}(t))\| \\ &= \|-M\beta I_p(t)(S_{p_1} - S_{p_2})(t) - n(S_{p_1}(t) - S_{p_2}(t))\| \\ &\leq (M\beta \|I_p(t)\| + n) \|S_{p_1}(t) - S_{p_2}(t)\| \\ &\leq (M\beta L_1 + n) \|S_{p_1}(t) - S_{p_2}(t)\| \\ &\leq \zeta_1 \|S_{p_1}(t) - S_{p_2}(t)\|, \end{aligned}$$

where $\zeta_1 = M\beta L_1 + n$ and $L_1 = \max_{t \in J} \|I_p(t)\|$.

In the same manner, we obtain

$$\begin{aligned} \|\Psi_2(t, S_{p_1}(t)) - \Psi_2(t, S_{p_2}(t))\| &\leq \zeta_2 \|I_{p1}(t) - I_{p2}(t)\|, \\ \|\Psi_3(t, S_{p_1}(t)) - \Psi_3(t, S_{p_2}(t))\| &\leq \zeta_3 \|Q_{p1}(t) - Q_{p2}(t)\|, \\ \|\Psi_4(t, S_{p_1}(t)) - \Psi_4(t, S_{p_2}(t))\| &\leq \zeta_4 \|R_{p1}(t) - R_{p2}(t)\|, \\ \|\Psi_5(t, S_{p_1}(t)) - \Psi_5(t, S_{p_2}(t))\| &\leq \zeta_5 \|S_{h1}(t) - S_{h2}(t)\|, \\ \|\Psi_6(t, S_{p_1}(t)) - \Psi_6(t, S_{p_2}(t))\| &\leq \zeta_6 \|I_{h1}(t) - I_{h2}(t)\|, \\ \|\Psi_7(t, S_{p_1}(t)) - \Psi_7(t, S_{p_2}(t))\| &\leq \zeta_7 \|R_{h1}(t) - R_{h2}(t)\|, \end{aligned} \tag{10}$$

where

$$\begin{aligned} \zeta_2 &= M\beta L_2 - (\delta + m + n), \\ \zeta_3 &= m + n + \delta, \\ \zeta_4 &= n, \\ \zeta_5 &= \mu + \gamma L_3, \\ \zeta_6 &= \alpha + \mu + \beta_2, \\ \zeta_7 &= \mu, \end{aligned}$$

with $L_2 = \max_{t \in J} \|S_p(t)\|$, $L_3 = \max_{t \in J} \|I_p(t)\|$.

Considering the kernels of the model, we can write the Equation (9) as

$$\begin{aligned} S_p(t) &= S_p(0) + \frac{1-\zeta}{B(\zeta)} \left(\Psi_1(t, S_p(t)) \right) + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(\Psi_1(z, S_p(z)) \right) (t-z)^{\zeta-1} dz, \\ I_p(t) &= I_p(0) + \frac{1-\zeta}{B(\zeta)} \left(\Psi_2(t, I_p(t)) \right) + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(\Psi_2(z, I_p(z)) \right) (t-z)^{\zeta-1} dz, \\ Q_p(t) &= Q_p(0) + \frac{1-\zeta}{B(\zeta)} \left(\Psi_3(t, Q_p(t)) \right) + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(\Psi_3(z, Q_p(z)) \right) (t-z)^{\zeta-1} dz, \\ R_p(t) &= R_p(0) + \frac{1-\zeta}{B(\zeta)} \left(\Psi_4(t, R_p(t)) \right) + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(\Psi_4(z, R_p(z)) \right) (t-z)^{\zeta-1} dz, \\ S_h(t) &= S_h(0) + \frac{1-\zeta}{B(\zeta)} \left(\Psi_5(t, S_h(t)) \right) + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(\Psi_5(z, S_h(z)) \right) (t-z)^{\zeta-1} dz, \\ I_h(t) &= I_h(0) + \frac{1-\zeta}{B(\zeta)} \left(\Psi_6(t, I_h(t)) \right) + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(\Psi_6(z, I_h(z)) \right) (t-z)^{\zeta-1} dz, \\ R_h(t) &= R_h(0) + \frac{1-\zeta}{B(\zeta)} \left(\Psi_7(t, R_h(t)) \right) + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(\Psi_7(z, R_h(z)) \right) (t-z)^{\zeta-1} dz. \end{aligned} \tag{11}$$

Therefore, we obtain the following recursive formula.

$$\begin{aligned}
 S_{p_n}(t) &= S_p(0) + \frac{1-\zeta}{B(\zeta)} \left(\Psi_1(t, S_{p_{n-1}}(t)) \right) + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(\Psi_1(z, S_{p_{n-1}}(z)) \right) (t-z)^{\zeta-1} dz, \\
 I_{p_n}(t) &= I_p(0) + \frac{1-\zeta}{B(\zeta)} \left(\Psi_2(t, I_{p_{n-1}}(t)) \right) + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(\Psi_2(z, I_{p_{n-1}}(z)) \right) (t-z)^{\zeta-1} dz, \\
 Q_{p_n}(t) &= Q_p(0) + \frac{1-\zeta}{B(\zeta)} \left(\Psi_3(t, Q_{p_{n-1}}(t)) \right) + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(\Psi_3(z, Q_{p_{n-1}}(z)) \right) (t-z)^{\zeta-1} dz, \\
 R_{p_n}(t) &= R_p(0) + \frac{1-\zeta}{B(\zeta)} \left(\Psi_4(t, R_{p_{n-1}}(t)) \right) + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(\Psi_4(z, R_{p_{n-1}}(z)) \right) (t-z)^{\zeta-1} dz, \\
 S_{h_n}(t) &= S_h(0) + \frac{1-\zeta}{B(\zeta)} \left(\Psi_5(t, S_{h_{n-1}}(t)) \right) + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(\Psi_5(z, S_{h_{n-1}}(z)) \right) (t-z)^{\zeta-1} dz, \\
 I_{h_n}(t) &= I_h(0) + \frac{1-\zeta}{B(\zeta)} \left(\Psi_6(t, I_{h_{n-1}}(t)) \right) + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(\Psi_6(z, I_{h_{n-1}}(z)) \right) (t-z)^{\zeta-1} dz, \\
 R_{h_n}(t) &= R_h(0) + \frac{1-\zeta}{B(\zeta)} \left(\Psi_7(t, R_{h_{n-1}}(t)) \right) + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(\Psi_7(z, R_{h_{n-1}}(z)) \right) (t-z)^{\zeta-1} dz.
 \end{aligned}
 \tag{12}$$

We then obtain the difference between the iterative terms in the expression

$$\begin{aligned}
 Y_{1_n} &= S_{p_n}(t) - S_{p_{n-1}}(t) = \frac{1-\zeta}{B(\zeta)} \left(\Psi_1(t, S_{p_{n-1}}(t)) - \Psi_1(t, S_{p_{n-2}}(t)) \right) \\
 &\quad + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(\Psi_1(\xi, S_{p_{n-1}}(t)) - \Psi_1(\xi, S_{p_{n-2}}(\xi)) \right) (t-\xi)^{\zeta-1} d\xi, \\
 Y_{2_n} &= I_{p_n}(t) - I_{p_{n-1}}(t) = \frac{1-\zeta}{B(\zeta)} \left(\Psi_2(t, I_{p_{n-1}}(t)) - \Psi_2(t, I_{p_{n-2}}(t)) \right) \\
 &\quad + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(\Psi_2(\xi, S_{p_{n-1}}(t)) - \Psi_2(\xi, I_{p_{n-2}}(\xi)) \right) (t-\xi)^{\zeta-1} d\xi, \\
 Y_{3_n} &= Q_{p_n}(t) - Q_{p_{n-1}}(t) = \frac{1-\zeta}{B(\zeta)} \left(\Psi_3(t, Q_{p_{n-1}}(t)) - \Psi_3(t, Q_{p_{n-2}}(t)) \right) \\
 &\quad + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(\Psi_3(\xi, Q_{p_{n-1}}(t)) - \Psi_3(\xi, Q_{p_{n-2}}(\xi)) \right) (t-\xi)^{\zeta-1} d\xi, \\
 Y_{4_n} &= R_{p_n}(t) - R_{p_{n-1}}(t) = \frac{1-\zeta}{B(\zeta)} \left(\Psi_4(t, R_{p_{n-1}}(t)) - \Psi_4(t, R_{p_{n-2}}(t)) \right) \\
 &\quad + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(\Psi_4(\xi, R_{p_{n-1}}(t)) - \Psi_4(\xi, R_{p_{n-2}}(\xi)) \right) (t-\xi)^{\zeta-1} d\xi, \\
 Y_{5_n} &= S_{h_n}(t) - S_{h_{n-1}}(t) = \frac{1-\zeta}{B(\zeta)} \left(\Psi_5(t, S_{h_{n-1}}(t)) - \Psi_5(t, S_{h_{n-2}}(t)) \right) \\
 &\quad + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(\Psi_5(\xi, S_{h_{n-1}}(t)) - \Psi_5(\xi, S_{h_{n-2}}(\xi)) \right) (t-\xi)^{\zeta-1} d\xi, \\
 Y_{6_n} &= I_{h_n}(t) - I_{h_{n-1}}(t) = \frac{1-\zeta}{B(\zeta)} \left(\Psi_6(t, I_{h_{n-1}}(t)) - \Psi_6(t, I_{h_{n-2}}(t)) \right) \\
 &\quad + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(\Psi_6(\xi, I_{h_{n-1}}(t)) - \Psi_6(\xi, I_{h_{n-2}}(\xi)) \right) (t-\xi)^{\zeta-1} d\xi, \\
 Y_{7_n} &= R_{h_n}(t) - R_{h_{n-1}}(t) = \frac{1-\zeta}{B(\zeta)} \left(\Psi_7(t, R_{h_{n-1}}(t)) - \Psi_7(t, R_{h_{n-2}}(t)) \right) \\
 &\quad + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t \left(\Psi_7(\xi, R_{h_{n-1}}(t)) - \Psi_7(\xi, R_{h_{n-2}}(\xi)) \right) (t-\xi)^{\zeta-1} d\xi,
 \end{aligned}
 \tag{13}$$

where

$$S_{p_n} = \sum_{m=1}^n Y_{1_m}, I_{p_n} = \sum_{m=1}^n Y_{2_m}, Q_{p_n} = \sum_{m=1}^n Y_{3_m}, R_{p_n} = \sum_{m=1}^n Y_{4_m},$$

$$S_{h_n} = \sum_{m=1}^n Y_{5_m}, I_{h_n} = \sum_{m=1}^n Y_{6_m}, R_{h_n} = \sum_{m=1}^n Y_{7_m}.$$

Applying the norm of both sides and considering the triangular inequality, Equation (13) becomes

$$\begin{aligned} \|Y_{1_n}(t)\| &= \|S_{p_n}(t) - S_{p_{n-1}}(t)\| \\ &\leq \frac{1-\zeta}{B(\zeta)} \|\Psi_1(t, S_{p_{n-1}}(t)) - \Psi_1(t, S_{p_{n-2}}(t))\| \\ &\quad + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \left\| \int_0^t \{\Psi_1(\Psi, S_{p_{n-1}}(\Psi)) - \Psi_1(\Psi, S_{p_{n-2}}(\Psi))\} (t-\Psi)^{\zeta-1} \right\|, \\ \|Y_{2_n}(t)\| &= \|I_{p_n}(t) - I_{p_{n-1}}(t)\| \\ &\leq \frac{1-\zeta}{B(\zeta)} \|\Psi_2(t, I_{p_{n-1}}(t)) - \Psi_2(t, I_{p_{n-2}}(t))\| \\ &\quad + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \left\| \int_0^t \{\Psi_2(\zeta, I_{p_{n-1}}(\zeta)) - \Psi_2(\zeta, I_{p_{n-2}}(\zeta))\} (t-\zeta)^{\zeta-1} \right\|, \\ \|Y_{3_n}(t)\| &= \|Q_{p_n}(t) - Q_{p_{n-1}}(t)\| \\ &\leq \frac{1-\zeta}{B(\zeta)} \|\Psi_3(t, Q_{p_{n-1}}(t)) - \Psi_3(t, Q_{p_{n-2}}(t))\| \\ &\quad + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \left\| \int_0^t \{\Psi_3(\zeta, Q_{p_{n-1}}(\zeta)) - \Psi_3(\zeta, Q_{p_{n-2}}(\zeta))\} (t-\zeta)^{\zeta-1} \right\|, \\ \|Y_{4_n}(t)\| &= \|R_{p_n}(t) - R_{p_{n-1}}(t)\| \\ &\leq \frac{1-\zeta}{B(\zeta)} \|\Psi_4(t, R_{p_{n-1}}(t)) - \Psi_4(t, R_{p_{n-2}}(t))\| \\ &\quad + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \left\| \int_0^t \{\Psi_4(\zeta, R_{p_{n-1}}(\zeta)) - \Psi_4(\zeta, R_{p_{n-2}}(\zeta))\} (t-\zeta)^{\zeta-1} \right\|, \\ \|Y_{5_n}(t)\| &= \|S_{h_n}(t) - S_{h_{n-1}}(t)\| \\ &\leq \frac{1-\zeta}{B(\zeta)} \|\Psi_5(t, S_{h_{n-1}}(t)) - \Psi_5(t, S_{h_{n-2}}(t))\| \\ &\quad + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \left\| \int_0^t \{\Psi_5(\zeta, S_{h_{n-1}}(\zeta)) - \Psi_5(\zeta, S_{h_{n-2}}(\zeta))\} (t-\zeta)^{\zeta-1} \right\|, \\ \|Y_{6_n}(t)\| &= \|I_{h_n}(t) - I_{h_{n-1}}(t)\| \\ &\leq \frac{1-\zeta}{B(\zeta)} \|\Psi_6(t, I_{h_{n-1}}(t)) - \Psi_6(t, I_{h_{n-2}}(t))\| \\ &\quad + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \left\| \int_0^t \{\Psi_6(\zeta, I_{h_{n-1}}(\zeta)) - \Psi_6(\zeta, I_{h_{n-2}}(\zeta))\} (t-\zeta)^{\zeta-1} \right\|, \\ \|Y_{7_n}(t)\| &= \|R_{h_n}(t) - R_{h_{n-1}}(t)\| \\ &\leq \frac{1-\zeta}{B(\zeta)} \|\Psi_7(t, R_{h_{n-1}}(t)) - \Psi_7(t, R_{h_{n-2}}(t))\| \\ &\quad + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \left\| \int_0^t \{\Psi_7(\zeta, R_{h_{n-1}}(\zeta)) - \Psi_7(\zeta, R_{h_{n-2}}(\zeta))\} (t-\zeta)^{\zeta-1} \right\|. \end{aligned} \tag{14}$$

Since the kernels satisfy the Lipschitz condition, we obtain the following equations.

$$\begin{aligned}
 \|Y_{1_n}(t)\| &\leq \frac{1-\zeta}{B(\zeta)} \zeta_1 \|Y_{1_{(n-1)}}(t)\| + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \zeta_1 \int_0^t \|Y_{1_{(n-1)}}(x)\| (t-\zeta)^{\zeta-1} d\zeta, \\
 \|Y_{2_n}(t)\| &\leq \frac{1-\zeta}{B(\zeta)} \zeta_2 \|Y_{2_{(n-1)}}(t)\| + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \zeta_2 \int_0^t \|Y_{2_{(n-1)}}(x)\| (t-\zeta)^{\zeta-1} d\zeta, \\
 \|Y_{3_n}(t)\| &\leq \frac{1-\zeta}{B(\zeta)} \zeta_3 \|Y_{3_{(n-1)}}(t)\| + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \zeta_3 \int_0^t \|Y_{3_{(n-1)}}(x)\| (t-\zeta)^{\zeta-1} d\zeta, \\
 \|Y_{4_n}(t)\| &\leq \frac{1-\zeta}{B(\zeta)} \zeta_4 \|Y_{4_{(n-1)}}(t)\| + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \zeta_4 \int_0^t \|Y_{4_{(n-1)}}(x)\| (t-\zeta)^{\zeta-1} d\zeta, \\
 \|Y_{5_n}(t)\| &\leq \frac{1-\zeta}{B(\zeta)} \zeta_5 \|Y_{5_{(n-1)}}(t)\| + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \zeta_5 \int_0^t \|Y_{5_{(n-1)}}(x)\| (t-\zeta)^{\zeta-1} d\zeta, \\
 \|Y_{6_n}(t)\| &\leq \frac{1-\zeta}{B(\zeta)} \zeta_6 \|Y_{6_{(n-1)}}(t)\| + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \zeta_6 \int_0^t \|Y_{6_{(n-1)}}(x)\| (t-\zeta)^{\zeta-1} d\zeta, \\
 \|Y_{7_n}(t)\| &\leq \frac{1-\zeta}{B(\zeta)} \zeta_7 \|Y_{7_{(n-1)}}(t)\| + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \zeta_7 \int_0^t \|Y_{7_{(n-1)}}(x)\| (t-\zeta)^{\zeta-1} d\zeta.
 \end{aligned}
 \tag{15}$$

This completes the proof of the theorem. □

Theorem 4. The system (7) has a solution under the conditions that we can find t_{\max} satisfying

$$\frac{\pi_i}{B(\zeta)} \left(1 - \zeta + \frac{t_{\max}^\zeta}{\Gamma(\zeta)} \right) < 1,$$

for $j = 1, 2, \dots, 7$.

Proof. Suppose that the functions $S_p(t), I_p(t), Q_p(t), R_p(t), S_h(t), I_h(t), R_h(t)$ are bounded. From Equation (15), we obtain the following inequalities.

$$\begin{aligned}
 \|Y_{1_n}(t)\| &\leq \|S_p(0)\| \left[\frac{\zeta_1}{B(\zeta)} \left(1 - \zeta + \frac{t_{\max}^\zeta}{\Gamma(\zeta)} \right) \right]^n, \\
 \|Y_{2_n}(t)\| &\leq \|I_p(0)\| \left[\frac{\zeta_2}{B(\zeta)} \left(1 - \zeta + \frac{t_{\max}^\zeta}{\Gamma(\zeta)} \right) \right]^n, \\
 \|Y_{3_n}(t)\| &\leq \|Q_p(0)\| \left[\frac{\zeta_3}{B(\zeta)} \left(1 - \zeta + \frac{t_{\max}^\zeta}{\Gamma(\zeta)} \right) \right]^n, \\
 \|Y_{4_n}(t)\| &\leq \|R_p(0)\| \left[\frac{\zeta_4}{B(\zeta)} \left(1 - \zeta + \frac{t_{\max}^\zeta}{\Gamma(\zeta)} \right) \right]^n, \\
 \|Y_{5_n}(t)\| &\leq \|S_h(0)\| \left[\frac{\zeta_5}{B(\zeta)} \left(1 - \zeta + \frac{t_{\max}^\zeta}{\Gamma(\zeta)} \right) \right]^n, \\
 \|Y_{6_n}(t)\| &\leq \|I_h(0)\| \left[\frac{\zeta_6}{B(\zeta)} \left(1 - \zeta + \frac{t_{\max}^\zeta}{\Gamma(\zeta)} \right) \right]^n, \\
 \|Y_{7_n}(t)\| &\leq \|R_h(0)\| \left[\frac{\zeta_7}{B(\zeta)} \left(1 - \zeta + \frac{t_{\max}^\zeta}{\Gamma(\zeta)} \right) \right]^n.
 \end{aligned}
 \tag{16}$$

Thus, the functions $Y_{1_n}(t), Y_{2_n}(t), Y_{3_n}(t), Y_{4_n}(t), Y_{5_n}(t), Y_{6_n}(t)$, and $Y_{7_n}(t)$, which are given in (16), exist and are smooth. In addition, to show that (16) is the solution of the system (7), we assume that

$$\begin{aligned}
 S_p(t) - S_p(0) &= S_{p_n} - \Phi_{1_n}(t), \\
 I_p(t) - I_p(0) &= S_{p_n} - \Phi_{2_n}(t), \\
 Q_p(t) - Q_p(0) &= Q_{p_n} - \Phi_{3_n}(t), \\
 R_p(t) - R_p(0) &= R_{p_n} - \Phi_{4_n}(t), \\
 S_h(t) - S_h(0) &= S_{h_n} - \Phi_{5_n}(t), \\
 I_h(t) - I_h(0) &= I_{h_n} - \Phi_{6_n}(t), \\
 R_h(t) - R_h(0) &= R_{h_n} - \Phi_{7_n}(t).
 \end{aligned}
 \tag{17}$$

where $\Phi_{1_n}(t), \Phi_{2_n}(t), \Phi_{3_n}(t), \Phi_{4_n}(t), \Phi_{5_n}(t), \Phi_{6_n}(t)$, and $\Phi_{7_n}(t)$ are the remainder terms of the series solution.

Then, we must show that these terms approach to zero at infinity; that is, $\|\Phi_{1(\infty)}(t)\| \rightarrow 0, \|\Phi_{2(\infty)}(t)\| \rightarrow 0, \|\Phi_{3(\infty)}(t)\| \rightarrow 0, \|\Phi_{4(\infty)}(t)\| \rightarrow 0, \|\Phi_{5(\infty)}(t)\| \rightarrow 0, \|\Phi_{6(\infty)}(t)\| \rightarrow 0$, and $\|\Phi_{7(\infty)}(t)\| \rightarrow 0$. Thus, we consider

$$\begin{aligned}
 \|\Phi_{1_n}(t)\| &\leq \frac{1-\zeta}{B(\zeta)} \|\Psi_1(t, S_p(t)) - \Psi_1(t, S_{p_{n-1}}(t))\| \\
 &\quad + \frac{\zeta}{B(\zeta)\Gamma(\zeta)} \int_0^t (t-\psi)^{\zeta-1} \|\Psi_1(t, S_p(t)) - \Psi_1(t, S_{p_{n-1}}(t))\| d\psi \\
 &\leq \frac{\zeta_1}{B(\zeta)} \left[1 - \zeta + \frac{t_{max}^\zeta}{\Gamma(\zeta)} \right] \|S_p(t) - S_{p_{n-1}}\|.
 \end{aligned}$$

Continuing in this way, we obtain

$$\|\Phi_{1_n}(t)\| \leq B\zeta_1^n \left[\frac{\zeta}{B(\zeta)} \left(1 - \zeta + \frac{t_{max}^\zeta}{\Gamma(\zeta)} \right) \right]^{n+1}
 \tag{18}$$

where $B = \|S_p(t) - S_{p_{n-1}}(t)\|$. After we take the limit of both sides, n tends to infinity, and we have

$$\|\Phi_{j(\infty)}(t)\| \rightarrow 0, \quad j = 1, 2, \dots, 7.$$

□

6. Numerical Examples and Discussion

6.1. Approximation Technique

Consider the Atagana–Baleanu–Caputo fractional order model (7), along with initial conditions (8). The terms $S_p I_p$ and $S_h I_p$ in the proposed model are nonlinear. Applying the Laplace transform on both sides of the system (7), we obtain,

$$\begin{aligned}
 \frac{B(\zeta)}{1-\zeta} \frac{s^\zeta \mathcal{L}\{S_p(t)\} - s^{\zeta-1} S_p(0)}{s^\zeta + \frac{\zeta}{1-\zeta}} &= \mathcal{L}\{N_p - M\beta S_p I_p - n S_p\}, \\
 \frac{B(\zeta)}{1-\zeta} \frac{s^\zeta \mathcal{L}\{I_p(t)\} - s^{\zeta-1} I_p(0)}{s^\zeta + \frac{\zeta}{1-\zeta}} &= \mathcal{L}\{M\beta S_p I_p - \delta I_p - m I_p - n I_p\}, \\
 \frac{B(\zeta)}{1-\zeta} \frac{s^\zeta \mathcal{L}\{Q_p(t)\} - s^{\zeta-1} Q_p(0)}{s^\zeta + \frac{\zeta}{1-\zeta}} &= \mathcal{L}\{\delta I_p - m Q_p - n Q_p - \epsilon Q_p\}, \\
 \frac{B(\zeta)}{1-\zeta} \frac{s^\zeta \mathcal{L}\{R_p(t)\} - s^{\zeta-1} R_p(0)}{s^\zeta + \frac{\zeta}{1-\zeta}} &= \mathcal{L}\{\epsilon Q_p - n R_p\}, \\
 \frac{B(\zeta)}{1-\zeta} \frac{s^\zeta \mathcal{L}\{S_h(t)\} - s^{\zeta-1} S_h(0)}{s^\zeta + \frac{\zeta}{1-\zeta}} &= \mathcal{L}\{N_h - \mu S_h - \gamma S_h I_p\}, \\
 \frac{B(\zeta)}{1-\zeta} \frac{s^\zeta \mathcal{L}\{I_h(t)\} - s^{\zeta-1} I_h(0)}{s^\zeta + \frac{\zeta}{1-\zeta}} &= \mathcal{L}\{\gamma S_h I_p - \alpha I_h - \mu I_h\}, \\
 \frac{B(\zeta)}{1-\zeta} \frac{s^\zeta \mathcal{L}\{R_h(t)\} - s^{\zeta-1} R_h(0)}{s^\zeta + \frac{\zeta}{1-\zeta}} &= \mathcal{L}\{\beta_2 I_h - \mu R_h - \beta_2 I_h\}.
 \end{aligned}
 \tag{19}$$

Rearranging, we obtain:

$$\begin{aligned}
 \mathcal{L}\{S_p(t)\} &= \frac{S_p(0)}{s} \frac{s^\zeta(1-\zeta) + \zeta}{s^\zeta(B(\zeta))} \mathcal{L}\{N_p - M\beta S_p I_p - nS_p\}, \\
 \mathcal{L}\{I_p(t)\} &= \frac{I_p(0)}{s} \frac{s^\zeta(1-\zeta) + \zeta}{s^\zeta(B(\zeta))} \mathcal{L}\{M\beta S_p I_p - \delta I_p - mI_p - nI_p\}, \\
 \mathcal{L}\{Q_p(t)\} &= \frac{Q_p(0)}{s} \frac{s^\zeta(1-\zeta) + \zeta}{s^\zeta(B(\zeta))} \mathcal{L}\{\delta I_p - mQ_p - nQ_p - \varepsilon Q_p\}, \\
 \mathcal{L}\{R_p(t)\} &= \frac{R_p(0)}{s} \frac{s^\zeta(1-\zeta) + \zeta}{s^\zeta(B(\zeta))} \mathcal{L}\{\varepsilon Q_p - nR_p\}, \\
 \mathcal{L}\{S_h(t)\} &= \frac{S_h(0)}{s} \frac{s^\zeta(1-\zeta) + \zeta}{s^\zeta(B(\zeta))} \mathcal{L}\{N_h - \mu S_h - \gamma S_h I_p\}, \\
 \mathcal{L}\{I_h(t)\} &= \frac{I_h(0)}{s} \frac{s^\zeta(1-\zeta) + \zeta}{s^\zeta(B(\zeta))} \mathcal{L}\{\gamma S_h I_p - \alpha I_h - \mu I_h\}, \\
 \mathcal{L}\{R_h(t)\} &= \frac{R_h(0)}{s} \frac{s^\zeta(1-\zeta) + \zeta}{s^\zeta(B(\zeta))} \mathcal{L}\{\beta_2 I_h - \mu R_h - \beta_2 I_h\}.
 \end{aligned} \tag{20}$$

Further, the inverse Laplace transform on (20) yields

$$\begin{aligned}
 S_p(t) &= S_p(0) + \mathcal{L}^{-1} \left[\left(\frac{s^\zeta(1-\zeta) + \zeta}{s^\zeta(B(\zeta))} \right) \mathcal{L}\{N_p - M\beta S_p I_p - nS_p\} \right], \\
 I_p(t) &= I_p(0) + \mathcal{L}^{-1} \left[\left(\frac{s^\zeta(1-\zeta) + \zeta}{s^\zeta(B(\zeta))} \right) \mathcal{L}\{M\beta S_p I_p - \delta I_p - mI_p - nI_p\} \right], \\
 Q_p(t) &= Q_p(0) + \mathcal{L}^{-1} \left[\left(\frac{s^\zeta(1-\zeta) + \zeta}{s^\zeta(B(\zeta))} \right) \mathcal{L}\{\delta I_p - mQ_p - nQ_p - \varepsilon Q_p\} \right], \\
 R_p(t) &= R_p(0) + \mathcal{L}^{-1} \left[\left(\frac{s^\zeta(1-\zeta) + \zeta}{s^\zeta(B(\zeta))} \right) \mathcal{L}\{\varepsilon Q_p - nR_p\} \right], \\
 S_h(t) &= S_h(0) + \mathcal{L}^{-1} \left[\left(\frac{s^\zeta(1-\zeta) + \zeta}{s^\zeta(B(\zeta))} \right) \mathcal{L}\{N_h - \mu S_h - \gamma S_h I_p\} \right], \\
 I_h(t) &= I_h(0) + \mathcal{L}^{-1} \left[\left(\frac{s^\zeta(1-\zeta) + \zeta}{s^\zeta(B(\zeta))} \right) \mathcal{L}\{\gamma S_h I_p - \alpha I_h - \mu I_h\} \right], \\
 R_h(t) &= R_h(0) + \mathcal{L}^{-1} \left[\left(\frac{s^\zeta(1-\zeta) + \zeta}{s^\zeta(B(\zeta))} \right) \mathcal{L}\{\beta_2 I_h - \mu R_h - \beta_2 I_h\} \right].
 \end{aligned} \tag{21}$$

The nonlinearity property of $S_p I_p = \sum_{n=0}^{\infty} H_n$, whereas H_n is further decomposed as

$$H_n = \sum_{j=0}^n S_{p_j} \sum_{j=0}^n I_{p_j} - \sum_{j=0}^{n-1} S_{p_j} \sum_{j=0}^{n-1} I_{p_j}.$$

Applying the initial conditions, the recursive formula are given by

$$\begin{aligned}
 S_{p_{n+1}}(t) &= S_{p_n}(0) + \mathcal{L}^{-1} \left[\left(\frac{s^\zeta(1-\zeta) + \zeta}{s^\zeta(B(\zeta))} \right) \mathcal{L}\{N_p - M\beta S_p I_p - nS_p\} \right], \\
 I_{p_{n-1}}(t) &= I_{p_n}(0) + \mathcal{L}^{-1} \left[\left(\frac{s^\zeta(1-\zeta) + \zeta}{s^\zeta(B(\zeta))} \right) \mathcal{L}\{M\beta S_p I_p - \delta I_p - mI_p - nI_p\} \right], \\
 Q_{p_{n-1}}(t) &= Q_{p_n}(0) + \mathcal{L}^{-1} \left[\left(\frac{s^\zeta(1-\zeta) + \zeta}{s^\zeta(B(\zeta))} \right) \mathcal{L}\{\delta I_p - mQ_p - nQ_p - \varepsilon Q_p\} \right], \\
 R_{p_{n-1}}(t) &= R_{p_n}(0) + \mathcal{L}^{-1} \left[\left(\frac{s^\zeta(1-\zeta) + \zeta}{s^\zeta(B(\zeta))} \right) \mathcal{L}\{\varepsilon Q_p - nR_p\} \right], \\
 S_{h_{n-1}}(t) &= S_{h_n}(0) + \mathcal{L}^{-1} \left[\left(\frac{s^\zeta(1-\zeta) + \zeta}{s^\zeta(B(\zeta))} \right) \mathcal{L}\{N_h - \mu S_h - \gamma S_h I_p\} \right], \\
 I_{h_{n-1}}(t) &= I_{h_n}(0) + \mathcal{L}^{-1} \left[\left(\frac{s^\zeta(1-\zeta) + \zeta}{s^\zeta(B(\zeta))} \right) \mathcal{L}\{\gamma S_h I_p - \alpha I_h - \mu I_h\} \right], \\
 R_{h_{n-1}}(t) &= R_{h_n}(0) + \mathcal{L}^{-1} \left[\left(\frac{s^\zeta(1-\zeta) + \zeta}{s^\zeta(B(\zeta))} \right) \mathcal{L}\{\beta_2 I_h - \mu R_h - \beta_2 I_h\} \right],
 \end{aligned}
 \tag{22}$$

where

$$S_{p_0}(t) = S_p(0), I_{p_0}(t) = I_p(0), Q_{p_0}(t) = Q_p(0), R_{p_0}(t) = R_p(0), S_{h_0}(t) = S_h(0), I_{h_0}(t) = I_h(0), R_{h_0}(t) = R_h(0).$$

The approximate solution is assumed to obtain as a limit when n tends to infinity.

$$\begin{aligned}
 S_p(t) &= \lim_{n \rightarrow \infty} S_{p_n}(t), I_p(t) = \lim_{n \rightarrow \infty} I_{p_n}(t), Q_p(t) = \lim_{n \rightarrow \infty} Q_{p_n}(t), R_p(t) = \lim_{n \rightarrow \infty} R_{p_n}(t), \\
 S_h(t) &= \lim_{n \rightarrow \infty} S_{h_n}(t), I_h(t) = \lim_{n \rightarrow \infty} I_{h_n}(t), R_h(t) = \lim_{n \rightarrow \infty} R_{h_n}(t).
 \end{aligned}$$

6.2. Numerical Examples

The numerical examples of the Atagana–Baleanu–Caputo fractional derivative system (7) are computed for demonstration of the disease transmission with the given initial values

$$S_p(0) = 100, I_p(0) = 5, Q_p(0) = 5, R_p(0) = 0, S_h(0) = 100, I_h(0) = 5, R_h(0) = 0.$$

The parameter values of the numerical results of the integer-order model (9) are defined as $N_p = 0.175, N_h = 10, M = 0.9, m = 0.001, n = 0.001, \delta = 0.5, \varepsilon = 0.7, \alpha = 0.1, \mu = 0.1, \gamma = 0.02, \beta = 0.02,$ and $\beta_2 = 0.03,$ which gives $R_0 < 1.$ The trajectories of the solution approaches to disease-free equilibrium (E_1), as shown in Figure 2.

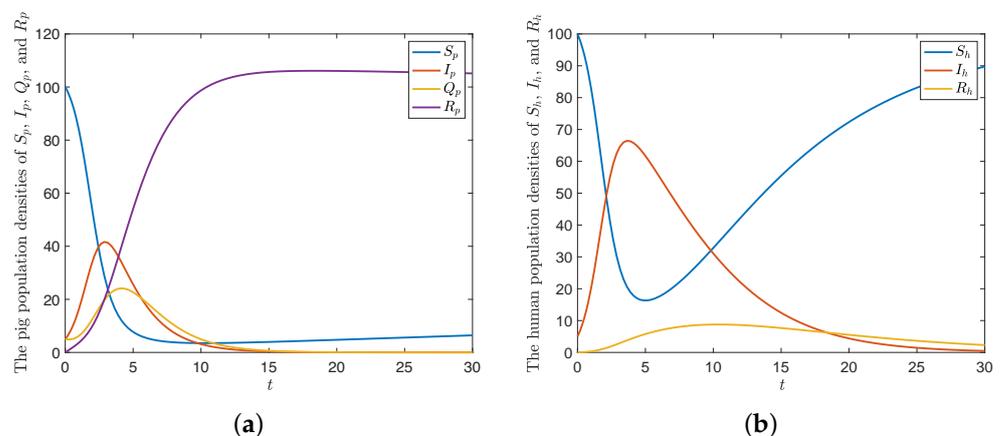


Figure 2. The trajectories of the solution of the model SIQR-SIR approaching to E_1 when $R_0 < 1$ for (a) pig population (b) human population.

The numerical simulation in Figure 2 shows that there are only pig susceptible, pig infected, human susceptible, and human infected populations at the beginning. After that, the disease spreads from the infected population to the susceptible population. The

densities of pig- and human infected increase. Then, the densities of human infected and pig infected decrease to a lower value. The density of the pig quarantine class increases after the density of pig infected class increases, and then it decreases after the density of pig infected decreases. This means that the pig quarantine reduces the transmission of the disease. In the long run, the numerical results approached to the equilibrium point in the analysis part.

The parameter values of the numerical results of the ABC fractional order model (7) with various orders are defined as $N_p = 0.175, N_h = 10, M = 0.9, m = 0.001, n = 0.001, \delta = 0.5, \varepsilon = 0.7, \alpha = 0.1, \mu = 0.1, \gamma = 0.02, \beta = 0.02,$ and $\beta_2 = 0.03,$ which give $R_0 < 1.$ The trajectories of the solution approaches to disease-free equilibrium (E_1), as shown in Figure 3.

Figure 3 shows the solutions of the fractional order model with the ABC operator. The various values of the fractional order $\zeta = 0.6, 0.7, 0.8,$ and 0.9 were investigated. The results indicated that the fractional order provided different trajectories. This is the major benefit of the fractional order model. From a mathematical point of view, we can change the order of fractional order to find the best fit of the curve. In the long run, the numerical results show that higher fractional order provides the lower densities of all classes except human recovered classes. Then, the solution trajectories of all fractional orders leads to the disease-free equilibrium.

The parameter values of the numerical results of the integer-order model (9) are defined as $N_p = 9, N_h = 5, M = 0.3, m = 0.3, n = 0.1, \delta = 0.6, \varepsilon = 0.3, \alpha = 0.1, \mu = 0.01, \gamma = 0.1, \beta = 0.03,$ and $\beta_2 = 0.1,$ which give $R_0 > 1.$ The trajectories of the solution approaches to endemic equilibrium (E_2), as shown in Figure 4.

The parameter values of the numerical results of the ABC fractional order model (7) with various orders are defined as $N_p = 9, N_h = 5, M = 0.3, m = 0.3, n = 0.1, \delta = 0.6, \varepsilon = 0.3, \alpha = 0.1, \mu = 0.01, \gamma = 0.1, \beta = 0.03,$ and $\beta_2 = 0.1,$ which give $R_0 < 1.$ The trajectories of the solution approaches to disease-free equilibrium (E_2), as shown in Figure 5.

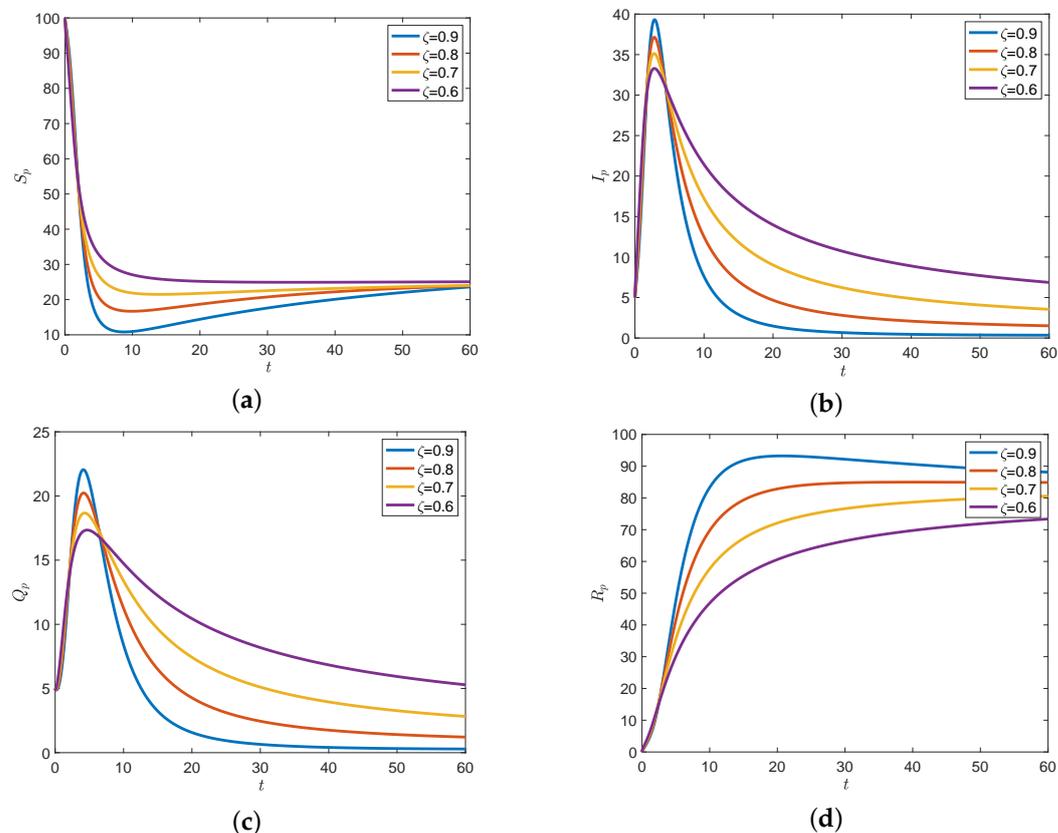


Figure 3. Cont.

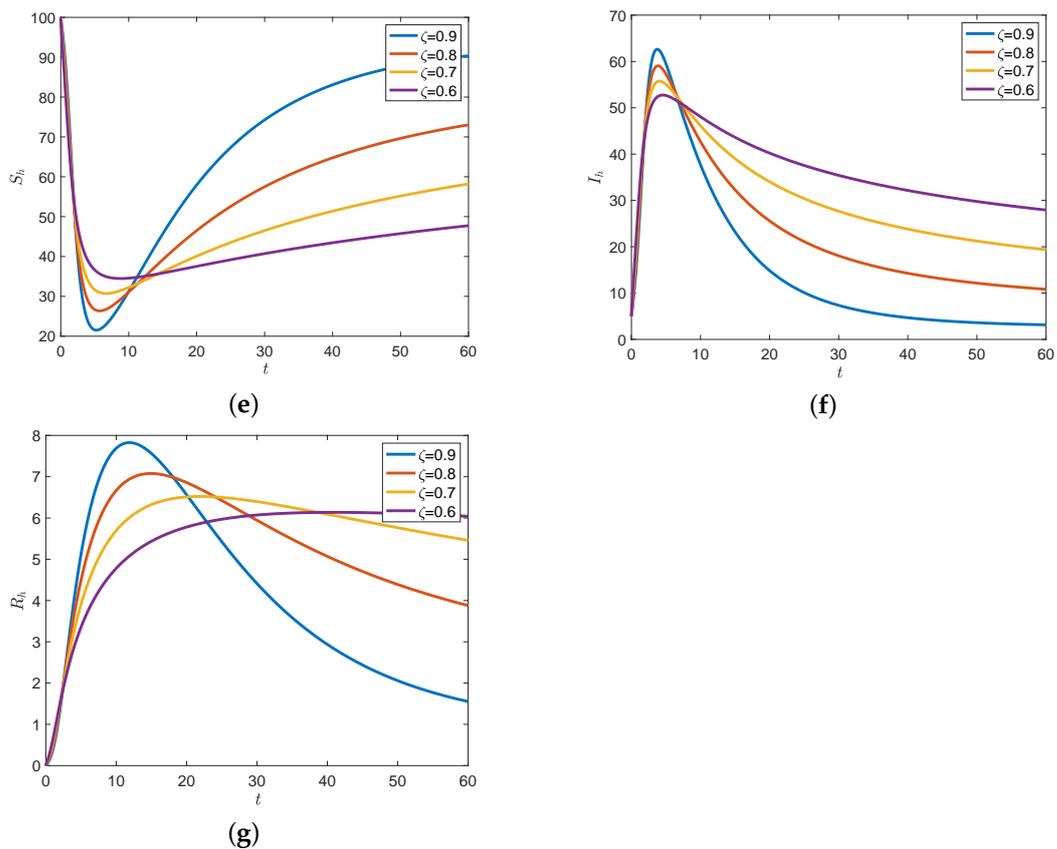


Figure 3. The solution trajectories of the system (7) when $R_0 < 1$ for (a) pig susceptible class, (b) pig infectious class, (c) pig quarantine class, (d) pig removable class, (e) human susceptible class, (f) human infectious class, and (g) human removable class with various time fractional orders $\zeta = 0.6, 0.7, 0.8,$ and 0.9 .

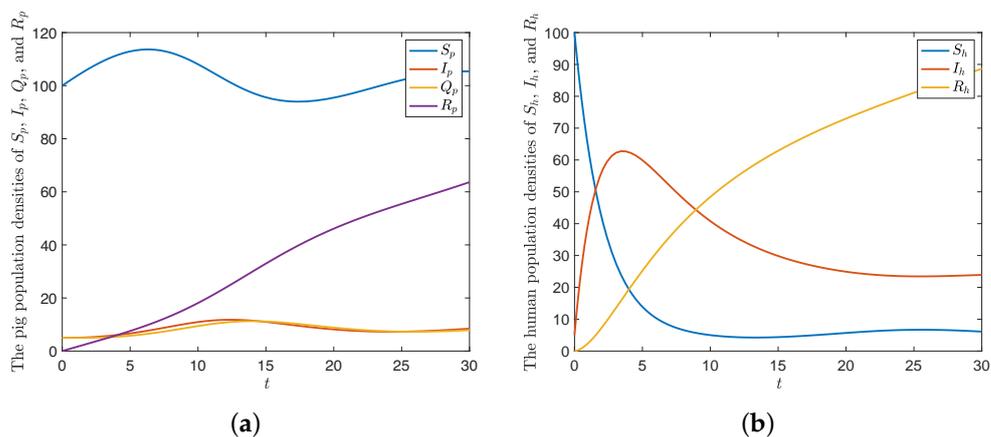


Figure 4. The trajectories of the solution of the model SIQR-SIR approaching to E_2 when $R_0 > 1$ for (a) pig population (b) human population.

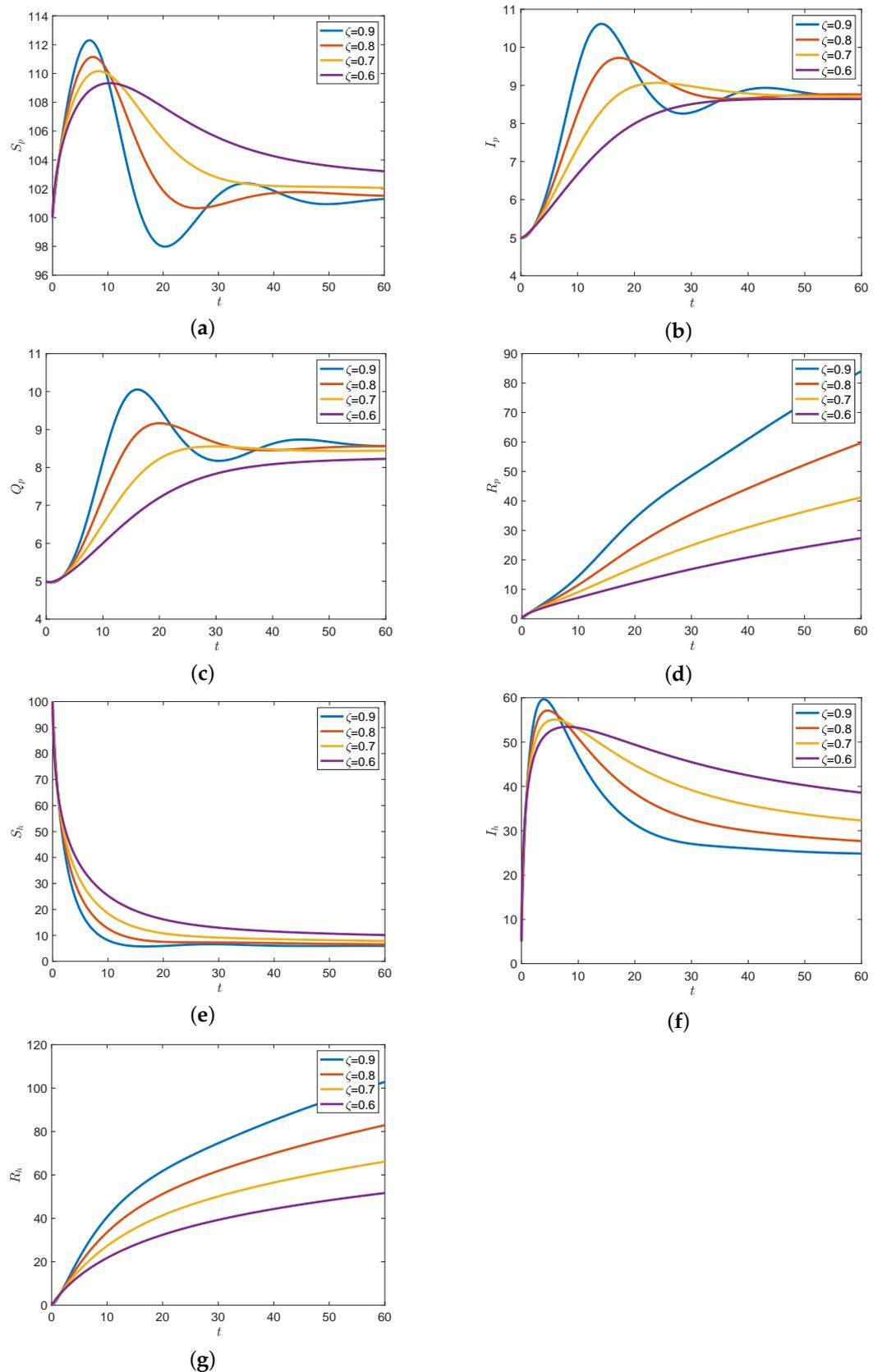


Figure 5. The solution trajectories of the system (7) when $R_0 > 1$ for (a) pig susceptible class, (b) pig infectious class, (c) pig quarantine class, (d) pig removable class, (e) human susceptible class, (f) human infectious class, and (g) human removable class with various time fractional orders $\zeta = 0.6, 0.7, 0.8,$ and 0.9 .

According to the numerical result in Figure 4, the solution trajectories lead to the endemic equilibrium when $R_0 > 1$. The endemic equilibrium is the state where the disease still transmits with constant densities in each class. Figure 5 shows that the solution trajectories of all classes have different values with different fractional-orders. Higher fractional orders provide lower densities for pig susceptible, pig infected, pig quarantine, and human susceptible. Otherwise, the higher fractional order provides higher densities for the pig removable, human infected, and human removable classes.

7. Conclusions

The dynamics of disease transmission for *Streptococcus suis* by considering the relation between humans and pigs is investigated. We have proposed the fractional order mathematical model based on the Atangana–Baleanu–Caputo fractional derivative. The proposed model is improved from the SIQR model, which can be expressed in the system of ordinary differential equations. The model is composed with seven compartments, these are the pig susceptible (S_p), infectious (I_p), quarantine (Q_p), and recovery (R_p) populations, and the human susceptible (S_h), infectious (I_h), and recovery (R_h) populations. The proposed model has investigated the contact between pigs and humans. Many properties of the model have been investigated: the invariant region, equilibria, stability analysis, and the existence and uniqueness of the solution. The basic reproduction number (R_0) of the model is also presented to explain the behavior of the disease. The conditions of equilibrium points are obtained. The disease-free equilibrium point is locally asymptotically stable if $R_0 < 1$. The endemic equilibrium is locally asymptotically stable if $R_0 > 1$. The numerical examples confirm the results from the proposed theorems, which lead the solution trajectories to the equilibria. Moreover, the fractional order derivative provides different solutions with the same parameters. Using the concept of the fractional order derivative, we can extend the mathematical models that are represented in the ordinary differential equation in this way.

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