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Solution of Integral Equation with Neutrosophic Rectangular Triple Controlled Metric Spaces

Gunaseelan Mani [1], Rajagopalan Ramaswamy [2,*], Arul Joseph Gnanaprakasam [3], Ola A Ashour Abdelnaby [1,4], Slobodan Radojevic [5] and Stojan Radenović [5]

1 Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 602105, India
2 Department of Mathematics, College of Science and Humanities in Alkharj, Prince Sattam Bin Abdulaziz University, ALkharj 11942, Saudi Arabia
3 Department of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur, Chennai 603203, India
4 Department of Mathematics, Cairo University, Cairo 12613, Egypt
5 Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Belgrad, Serbia
* Correspondence: r.gopalan@psau.edu.sa

Abstract: In this paper, we introduce the notion of neutrosophic rectangular triple-controlled metric space, relaxing the symmetry requirement of neutrosophic metric spaces, by replacing triangular inequalities with rectangular inequalities, and prove fixed point theorems. We have derived several interesting results for contraction mappings supplemented with non-trivial examples. The derived results have been applied to prove the existence of a unique analytical solution as well as a closed form of the unique solution to the integral equation.

Keywords: neutrosophic metric space; neutrosophic rectangular triple-controlled metric space; fixed point results; integral equation

MSC: 47H10; 54H25

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1. Introduction

Pursuant upon the reporting of the famous Contraction Mapping Theorem (CMT) by S.Banach [1] in 1922, the study of the existence and uniqueness of fixed points and common fixed points in metric and metric-like spaces and their applications has become a subject of great interest. In 1979, Itoh [2] presented the application of fixed point results to differential equations in Banach spaces. Many authors proved the Banach contraction principle in various generalized metric spaces. In the sequel, the notion of rectangular metric space was introduced by Branciari [3] in 2000. He replaced the right-hand side of the triangular inequality of the metric space with a three-term expression and established an analogous proof of the CMT. Since then, many fixed point theorems for various contractions on rectangular metric spaces have appeared in the literature [4–10].

In 1965, Zadeh [11] introduced the concept of fuzzy sets, which has varied applications in logical semantics. The concept of the continuous t-norm was introduced by Schweizer et al. [12]. Kramosil and Michlek [13] were the first to introduce the notion of fuzzy metric space, using continuous t-norms as an analog to metric spaces, and analyzed the notions with the probabilistic/statistical extension of metric spaces. The concept of fuzzy sets and fuzzy metric space has varied applications in applied sciences, such as fixed point theorems, signal and image processing including medical imaging, decision making, etc. Garbiec [14] reported the fuzzy extension of the Banach contraction mapping theorem. Since then, many fixed point results have been reported by researchers using different types of contractive conditions in the setting of fuzzy metric space, dislocated fuzzy metric space, intuitionistic fuzzy metric space, etc.; see [15–29]. More recently, Ali et al. [30] have presented some applications of the best proximity points of non-self maps in the setting of non-Archimedian fuzzy metric spaces.

In the recent past, many researchers have reported fixed point results in the setting of fuzzy metric spaces and the like. For instance, in 2018, Mlaiki [31] presented fixed point results by defining the concept of controlled metric spaces. Konwar [32], in 2020, defined intutionistic fuzzy b-metric space and established fixed point results under various contractive conditions. Saif Ur Rehman et al. [33] proved some $\alpha - \phi$ fuzzy cone contraction results with integral-type application.

In the sequel, the concept of neutrosophic metric spaces was introduced by Kirisci and Simsek and various fixed point results were established by them in the setting of these spaces [34–36]. Subsequently, Sowndrarajan et al. [37] reported several fixed point results in neutrosophic metric spaces. Sezen [38] presented the concept of controlled fuzzy metric spaces and derived various fixed point results. The concept of fuzzy double-controlled metric space was given by Saleem et al. [39] in 2021. More recently, Uddin et al. [40] established the fixed point theorem on neutrosophic double-controlled metric space and presented an application to the derived result thereon.

Inspired by the above, in the present work, we define the notion of neutrosophic rectangular triple-controlled metric space and establish fixed point theorems. Accordingly, we have organized the rest of the manuscript as follows. Some preliminaries and a monograph are presented in Section 2. In Section 3, we define the neutrosophic metric space and define the Cauchy sequence and its convergence and establish fixed point results. We support the derived results with non-trivial examples. In Section 4, we establish the existence of a unique analytical solution to the Fredholm integral equation. We have also supplemented the derived results by finding the closed form of the unique solution to the intergral equation.

2. Preliminaries

A quick review of the following definitions and monograph will be useful in the sequel.

Definition 1 ([19])**.** *A binary operation* $\star\colon [0,1] \times [0,1] \to [0,1]$ *is called a continuous triangle norm if:*

(1) $\wp \star \sigma = \sigma \star \wp, (\forall)\wp, \sigma \in [0,1]$;
(2) $\star$ *is continuous;*
(3) $\wp \star 1 = \wp, (\forall)\wp \in [0,1]$;
(4) $(\wp \star \sigma) \star \kappa = \wp \star (\sigma \star \kappa)$, *for all* $\wp, \sigma, \kappa \in [0,1]$;
(5) *If* $\wp \leq \kappa$ *and* $\sigma \leq \nu$, *with* $\wp, \sigma, \kappa, \nu \in [0,1]$, *then* $\wp \star \sigma \leq \kappa \star \nu$.

Definition 2 ([19])**.** *A binary operation* $\diamond\colon [0,1] \times [0,1] \to [0,1]$ *is called a continuous triangle co-norm if:*

(1) $\wp \diamond \sigma = \sigma \diamond \wp$, *for all* $\wp, \sigma \in [0,1]$;
(2) $\diamond$ *is continuous;*
(3) $\wp \diamond 0 = 0$, *for all* $\wp \in [0,1]$;
(4) $(\wp \diamond \sigma) \diamond \kappa = \wp \diamond (\sigma \diamond \kappa)$, *for all* $\wp, \sigma, \kappa \in [0,1]$;
(5) *If* $\wp \leq \kappa$ *and* $\sigma \leq \nu$, *with* $\wp, \sigma, \kappa, \nu \in [0,1]$, *then* $\wp \diamond \sigma \leq \kappa \diamond \nu$.

Definition 3 ([2])**.** *Given that* $\beta, \Gamma\colon \Im \times \Im \to [1, +\infty)$ *are non-comparable functions, if* $\partial\colon \Im \times \Im \to [0, +\infty)$ *satisfies the following conditions*

(a) $\partial(\vartheta, \mathcal{Q}) = 0$ *iff* $\vartheta = \mathcal{Q}$;
(b) $\partial(\vartheta, \mathcal{Q}) = \partial(\mathcal{Q}, \vartheta)$;
(c) $\partial(\vartheta, \mathcal{Q}) \leq \beta(\vartheta, \psi)\partial(\vartheta, \psi) + \Gamma(\psi, \mathcal{Q})\partial(\psi, \mathcal{Q})$,

for all $\vartheta, \mathcal{Q}, \psi \in \Im$. *Then,* $(\Im, \partial)$ *is said to be a double-controlled metric space.*

Definition 4 ([39]). *Suppose $\Im \neq \emptyset$ and $\beta, \Gamma\colon \Im \times \Im \to [1, +\infty)$ are two non-comparable functions, $\star$ is a continuous t-norm and $\mathcal{W}$ is a fuzzy set on $\Im \times \Im \times (0, +\infty)$. It is said to be a fuzzy double-controlled metric on $\Im$, for all $\vartheta, \mathcal{Q}, \psi \in \Im$ if*

(i) $\mathcal{W}(\vartheta, \mathcal{Q}, 0) = 0$;
(ii) $\mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) = 1$ *for all* $\varsigma > 0$, *if and only if* $\vartheta = \mathcal{Q}$;
(iii) $\mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) = \mathcal{W}(\mathcal{Q}, \vartheta, \varsigma)$;
(iv) $\mathcal{W}(\vartheta, \psi, \varsigma + \varsigma) \geq \mathcal{W}\left(\vartheta, \mathcal{Q}, \frac{\varsigma}{\beta(\vartheta, \mathcal{Q})}\right) \star \mathcal{W}\left(\mathcal{Q}, \psi, \frac{\varsigma}{\Gamma(\mathcal{Q}, \psi)}\right)$;
(v) $\mathcal{W}(\vartheta, \mathcal{Q}, \cdot)\colon (0, +\infty) \to [0, 1]$ *is left continuous.*

Then, $(\Im, \mathcal{W}, \mathcal{E}, \star)$ is said to be a fuzzy double-controlled metric space.

Definition 5 ([32]). *Take $\Im \neq \emptyset$. Let $\star$ be a continuous t-norm, $\diamond$ be a continuous t-co-norm, $\mathfrak{b} \geq 1$ and $\mathcal{W}, \mathcal{E}$ be fuzzy sets on $\Im \times \Im \times (0, +\infty)$. If $(\Im, \mathcal{W}, \mathcal{E}, \star, \diamond)$ fullfils all $\vartheta, \mathcal{Q} \in \Im$ and $\chi, \varsigma > 0$,*

(I) $\mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) + \mathcal{E}(\vartheta, \mathcal{Q}, \varsigma) \leq 1$;
(II) $\mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) > 0$;
(III) $\mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) = 1 \Leftrightarrow \vartheta = \mathcal{Q}$;
(IV) $\mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) = \mathcal{W}(\mathcal{Q}, \vartheta, \varsigma)$;
(V) $\mathcal{W}(\vartheta, \psi, \mathfrak{b}(\varsigma + \chi)) \geq \mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) \star \mathcal{W}(\mathcal{Q}, \psi, \chi)$;
(VI) $\mathcal{W}(\vartheta, \mathcal{Q}, \cdot)$ *is a non-decreasing function of* $\mathbb{R}^+$ *and* $\lim_{\varsigma \to +\infty} \mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) = 1$;
(VII) $\mathcal{E}(\vartheta, \mathcal{Q}, \varsigma) > 0$;
(VIII) $\mathcal{E}(\vartheta, \mathcal{Q}, \varsigma) = 0 \Leftrightarrow \vartheta = \mathcal{Q}$;
(IX) $\mathcal{E}(\vartheta, \mathcal{Q}, \varsigma) = \mathcal{E}(\mathcal{Q}, \vartheta, \varsigma)$;
(X) $\mathcal{E}(\vartheta, \psi, \mathfrak{b}(\varsigma + \chi)) \leq \mathcal{E}(\vartheta, F\mathcal{Q}, \varsigma) \diamond \mathcal{E}(\mathcal{Q}, \psi, \chi)$;
(XI) $\mathcal{E}(\vartheta, \mathcal{Q}, \cdot)$ *is a non-increasing function of* $\mathbb{R}^+$ *and* $\lim_{\varsigma \to +\infty} \mathcal{E}(\vartheta, \mathcal{Q}, \varsigma) = 0$.

Then, $(\Im, \mathcal{W}, \mathcal{E}, \star, \diamond)$ is an intuitionistic fuzzy $\mathfrak{b}$-metric space.

Definition 6 ([36]). *Let $\Im \neq \emptyset, \star$ be a continuous t-norm, $\diamond$ be a continuous t-co-norm, and $\mathcal{W}, \mathcal{E}, \Upsilon$ are neutrosophic sets on $\Im \times \Im \times (0, +\infty)$. It is said to be a neutosophic metric on $\Im$, if, for all $\vartheta, \mathcal{Q}, \psi \in \Im$, the following conditions are satisfied:*

(1) $\mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) + \mathcal{E}(\vartheta, \mathcal{Q}, \varsigma) + \Upsilon(\vartheta, \mathcal{Q}, \varsigma) \leq 3$;
(2) $\mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) > 0$;
(3) $\mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) = 1$ *for all* $\varsigma > 0$, *if and only if* $\vartheta = \mathcal{Q}$;
(4) $\mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) = \mathcal{W}(\mathcal{Q}, \vartheta, \varsigma)$;
(5) $\mathcal{W}(\vartheta, \psi, \varsigma + \chi) \geq \mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) \star \mathcal{W}(\mathcal{Q}, \psi, \chi)$;
(6) $\mathcal{W}(\vartheta, \mathcal{Q}, \cdot)\colon (0, +\infty) \to [0, 1]$ *is continuous and* $\lim_{\varsigma \to +\infty} \mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) = 1$;
(7) $\mathcal{E}(\vartheta, \mathcal{Q}, \varsigma) < 1$;
(8) $\mathcal{E}(\vartheta, \mathcal{Q}, \varsigma) = 0$ *for all* $\varsigma > 0$, *if and only if* $\vartheta = \mathcal{Q}$;
(9) $\mathcal{E}(\vartheta, \mathcal{Q}, \varsigma) = \mathcal{E}(\mathcal{Q}, \vartheta, \varsigma)$;
(10) $\mathcal{E}(\vartheta, \psi, \varsigma + \chi) \leq \mathcal{E}(\vartheta, \mathcal{Q}, \varsigma) \diamond \mathcal{E}(\mathcal{Q}, \psi, \chi)$;
(11) $\mathcal{E}(\vartheta, \mathcal{Q}, \cdot)\colon (0, +\infty) \to [0, 1]$ *is continuous and* $\lim_{\varsigma \to +\infty} \mathcal{E}(\vartheta, \mathcal{Q}, \varsigma) = 0$;
(12) $\Upsilon(\vartheta, \mathcal{Q}, \varsigma) < 1$;
(13) $\Upsilon(\vartheta, \mathcal{Q}, \varsigma) = 0$ *for all* $\varsigma > 0$, *if and only if* $\vartheta = \mathcal{Q}$;
(14) $\Upsilon(\vartheta, \mathcal{Q}, \varsigma) = \Upsilon(\mathcal{Q}, \vartheta, \varsigma)$;
(15) $\Upsilon(\vartheta, \psi, \varsigma + \chi) \leq \Upsilon(\vartheta, \mathcal{Q}, \varsigma) \diamond \Upsilon(\mathcal{Q}, \psi, \chi)$;
(16) $\Upsilon(\vartheta, \mathcal{Q}, \cdot)\colon (0, +\infty) \to [0, 1]$ *is continuous and* $\lim_{\varsigma \to +\infty} \Upsilon(\vartheta, \mathcal{Q}, \varsigma) = 0$;
(17) *If* $\varsigma \leq 0$, *then* $\mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) = 0, \mathcal{E}(\vartheta, \mathcal{Q}, \varsigma) = 0$.

Then, $(\Im, \mathcal{W}, \mathcal{E}, \Upsilon, \star, \diamond)$ is called a neutrosophic metric space.

Now, we present our main results.

3. Main Results

We commence this section by defining neutrosophic rectangular triple-controlled metric space.

Definition 7. *Let $\Im \neq \emptyset$ and $\beta, \Gamma, \eta \colon \Im \times \Im \to [1, +\infty)$ be given non-comparable functions, $\star$ be a continuous t-norm, and $\diamond$ be a continuous t-co-norm. The neutrosophic set $\mathcal{W}, \mathcal{E}, \mathcal{G}$ on $\Im \times \Im \times (0, +\infty)$ is said to be a neutrosophic rectangular triple-controlled metric on $\Im$, if, for any $\vartheta, \psi \in \Im$ and all distinct $\mathfrak{r}, \mathcal{Q} \in \Im \backslash \{\vartheta, \psi\}$, the following conditions are satisfied:*

(i) $\mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) + \mathcal{E}(\vartheta, \mathcal{Q}, \varsigma) + \mathcal{G}(\vartheta, \mathcal{Q}, \varsigma) \leq 3$;

(ii) $\mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) > 0$;

(iii) $\mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) = 1$ *for all* $\varsigma > 0$, *if and only if* $\vartheta = \mathcal{Q}$;

(iv) $\mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) = \mathcal{W}(\mathcal{Q}, \vartheta, \varsigma)$;

(v) $\mathcal{W}(\vartheta, \psi, \varsigma + \chi + \breve{w}) \geq \mathcal{W}\left(\vartheta, \mathcal{Q}, \frac{\varsigma}{\beta(\vartheta, \mathcal{Q})}\right) \star \mathcal{W}\left(\mathcal{Q}, \mathfrak{r}, \frac{\chi}{\Gamma(\mathcal{Q}, \mathfrak{r})}\right) \star \mathcal{W}\left(\mathfrak{r}, \psi, \frac{\breve{w}}{\eta(\mathfrak{r}, \psi)}\right)$;

(vi) $\mathcal{W}(\vartheta, \mathcal{Q}, \cdot) \colon (0, +\infty) \to [0, 1]$ *is continuous and* $\lim_{\varsigma \to +\infty} \mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) = 1$;

(vii) $\mathcal{E}(\vartheta, \mathcal{Q}, \varsigma) < 1$;

(viii) $\mathcal{E}(\vartheta, \mathcal{Q}, \varsigma) = 0$ *for all* $\varsigma > 0$, *if and only if* $\vartheta = \mathcal{Q}$;

(ix) $\mathcal{E}(\vartheta, \mathcal{Q}, \varsigma) = \mathcal{E}(\mathcal{Q}, \vartheta, \varsigma)$;

(x) $\mathcal{E}(\vartheta, \psi, \varsigma + \chi + \breve{w}) \leq \mathcal{E}\left(\vartheta, \mathcal{Q}, \frac{\varsigma}{\beta(\vartheta, \mathcal{Q})}\right) \diamond \mathcal{E}\left(\mathcal{Q}, \mathfrak{r}, \frac{\chi}{\Gamma(\mathcal{Q}, \mathfrak{r})}\right) \diamond \mathcal{E}\left(\mathfrak{r}, \psi, \frac{\breve{w}}{\eta(\mathfrak{r}, \psi)}\right)$;

(xi) $\mathcal{E}(\vartheta, \mathcal{Q}, \cdot) \colon (0, +\infty) \to [0, 1]$ *is continuous and* $\lim_{\varsigma \to +\infty} \mathcal{E}(\vartheta, \mathcal{Q}, \varsigma) = 0$;

(xii) $\mathcal{G}(\vartheta, \mathcal{Q}, \varsigma) < 1$;

(xiii) $\mathcal{G}(\vartheta, \mathcal{Q}, \varsigma) = 0$ *for all* $\varsigma > 0$, *if and only if* $\vartheta = \mathcal{Q}$;

(xiv) $\mathcal{G}(\vartheta, \mathcal{Q}, \varsigma) = \mathcal{G}(\mathcal{Q}, \vartheta, \varsigma)$;

(xv) $\mathcal{G}(\vartheta, \psi, \varsigma + \chi + \breve{w}) \leq \mathcal{G}\left(\vartheta, \mathcal{Q}, \frac{\varsigma}{\beta(\vartheta, \mathcal{Q})}\right) \diamond \mathcal{G}\left(\mathcal{Q}, \mathfrak{r}, \frac{\chi}{\Gamma(\mathcal{Q}, \mathfrak{r})}\right) \diamond \mathcal{G}\left(\mathfrak{r}, \psi, \frac{\chi}{\eta(\mathfrak{r}, \psi)}\right)$;

(xvi) $\mathcal{G}(\vartheta, \mathcal{Q}, \cdot) \colon (0, +\infty) \to [0, 1]$ *is continuous and* $\lim_{\varsigma \to +\infty} \mathcal{G}(\vartheta, \mathcal{Q}, \varsigma) = 0$;

(xvii) *If* $\varsigma \leq 0$, *then* $\mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) = 0, \mathcal{E}(\vartheta, \mathcal{Q}, \varsigma) = 1$ *and* $\Upsilon(\vartheta, \mathcal{Q}, \varsigma) = 1$.

Then, $(\Im, \mathcal{W}, \mathcal{E}, \mathcal{G}, \star, \diamond)$ is called a neutrosophic rectangular triple-controlled metric space.

Example 1. *Let $\Im = \{1, 2, 3, 4\}$ and $\beta, \Gamma, \eta \colon \Im \times \Im \to [1, +\infty)$ be three non-comparable functions given by $\beta(\vartheta, \mathcal{Q}) = \vartheta + \mathcal{Q} + 1$, $\Gamma(\vartheta, \mathcal{Q}) = \vartheta^2 + \mathcal{Q}^2 + 1$, and $\eta(\vartheta, \mathcal{Q}) = \vartheta^2 + \mathcal{Q}^2 - 1$. Define $\mathcal{W}, \mathcal{E}, \mathcal{G} \colon \Im \times \Im \times (0, +\infty) \to [0, 1]$ as*

$$\mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) = \begin{cases} 1, & \text{if} \quad \vartheta = \mathcal{Q} \\ \frac{\varsigma}{\varsigma + \max\{\vartheta, \mathcal{Q}\}}, & \text{if} \quad \text{otherwise}, \end{cases}$$

$$\mathcal{E}(\vartheta, \mathcal{Q}, \varsigma) = \begin{cases} 0, & \text{if} \quad \vartheta = \mathcal{Q} \\ \frac{\max\{\vartheta, \mathcal{Q}\}}{\varsigma + \max\{\vartheta, \mathcal{Q}\}}, & \text{if} \quad \text{otherwise}, \end{cases}$$

and

$$\mathcal{G}(\vartheta, \mathcal{Q}, \varsigma) = \begin{cases} 0, & \text{if} \quad \vartheta = \mathcal{Q} \\ \frac{\max\{\vartheta, \mathcal{Q}\}}{\varsigma}, & \text{if} \quad \text{otherwise}. \end{cases}$$

Then, $(\Im, \mathcal{W}, \mathcal{E}, \mathcal{G}, \star, \diamond)$ is a neutrosophic rectangular triple-controlled metric space with continuous t-norm $\wp \star \sigma = \wp\sigma$ and continuous t-co-norm, $\wp \diamond \bar{a} = \max\{\wp, \bar{a}\}$.

Proof. We need to prove only (v), (x) and (xv).

Let $\vartheta = 1, \mathcal{Q} = 2, \mathfrak{r} = 3$ and $\psi = 4$. Then

$$\mathcal{W}(1, 4, \varsigma + \chi + \breve{w}) = \frac{\varsigma + \chi + \breve{w}}{\varsigma + \chi + \breve{w} + \max\{1, 4\}} = \frac{\varsigma + \chi + \breve{w}}{\varsigma + \chi + \breve{w} + 4},$$

whereas

$$\mathcal{W}\left(1,2,\frac{\varsigma}{\beta(1,2)}\right)=\frac{\frac{\varsigma}{\beta(1,2)}}{\frac{\varsigma}{\beta(1,2)}+\max\{1,2\}}=\frac{\frac{\varsigma}{4}}{\frac{\varsigma}{4}+2}=\frac{\varsigma}{\varsigma+8},$$

$$\mathcal{W}\left(2,3,\frac{\chi}{\Gamma(2,3)}\right)=\frac{\frac{\chi}{\Gamma(2,3)}}{\frac{\chi}{\Gamma(2,3)}+\max\{2,3\}}=\frac{\frac{\chi}{12}}{\frac{\chi}{12}+3}=\frac{\chi}{\chi+36},$$

and

$$\mathcal{W}\left(3,4,\frac{\breve{w}}{\eta(3,4)}\right)=\frac{\frac{\breve{w}}{\eta(3,4)}}{\frac{\breve{w}}{\eta(3,4)}+\max\{3,4\}}=\frac{\frac{\breve{w}}{24}}{\frac{\breve{w}}{24}+4}=\frac{\breve{w}}{\breve{w}+96}.$$

That is,

$$\frac{\varsigma+\chi+\breve{w}}{\varsigma+\chi+\breve{w}+3}\geq\frac{\varsigma}{\varsigma+8}\cdot\frac{\chi}{\chi+36}.\frac{\breve{w}}{\breve{w}+96}.$$

Then, this satisfies all $\varsigma,\chi,\breve{w}>0$. Hence,

$$\mathcal{W}(\vartheta,\psi,\varsigma+\chi+\breve{w})\geq\mathcal{W}\left(\vartheta,\mathcal{Q},\frac{\varsigma}{\beta(\vartheta,\mathcal{Q})}\right)\star\mathcal{W}\left(\mathcal{Q},\mathfrak{r},\frac{\chi}{\Gamma(\mathcal{Q},\mathfrak{r})}\right)\star\mathcal{W}\left(\mathfrak{r},\psi,\frac{\breve{w}}{\eta(\mathfrak{r},\psi)}\right).$$

Now,

$$\mathcal{E}(1,4,\varsigma+\chi+\breve{w})=\frac{\max\{1,4\}}{\varsigma+\chi+\breve{w}+\max\{1,4\}}=\frac{4}{\varsigma+\chi+\breve{w}+4},$$

whereas

$$\mathcal{E}\left(1,2,\frac{\varsigma}{\beta(1,2)}\right)=\frac{\max\{1,2\}}{\frac{\varsigma}{\beta(1,2)}+\max\{1,2\}}=\frac{2}{\frac{\varsigma}{4}+2}=\frac{8}{\varsigma+8},$$

$$\mathcal{E}\left(2,3,\frac{\chi}{\Gamma(2,3)}\right)=\frac{\max\{2,3\}}{\frac{\chi}{\Gamma(2,3)}+\max\{2,3\}}=\frac{3}{\frac{\chi}{12}+3}=\frac{36}{\chi+36},$$

and

$$\mathcal{E}\left(3,4,\frac{\breve{w}}{\eta(3,4)}\right)=\frac{\max\{3,4\}}{\frac{\breve{w}}{\eta(3,4)}+\max\{3,4\}}=\frac{4}{\frac{\breve{w}}{24}+4}=\frac{96}{\breve{w}+96}.$$

That is,

$$\frac{4}{\varsigma+\chi+\breve{w}+4}\leq\max\left\{\frac{8}{\varsigma+8},\frac{36}{\chi+36},\frac{96}{\breve{w}+96}\right\}.$$

The above expression is true for all $\varsigma,\chi,\breve{w}>0$.
Hence,

$$\mathcal{E}(\vartheta,\psi,\varsigma+\chi+\breve{w})\leq\mathcal{E}\left(\vartheta,\mathcal{Q},\frac{\varsigma}{\beta(\vartheta,\mathcal{Q})}\right)\diamond\mathcal{E}\left(\mathfrak{r},\psi,\frac{\chi}{\Gamma(\mathfrak{r},\psi)}\right)\diamond\mathcal{E}\left(\mathfrak{r},\psi,\frac{\breve{w}}{\eta(\mathfrak{r},\psi)}\right).$$

Now,

$$\mathcal{G}(1,3,\varsigma+\chi+\breve{w})=\frac{\max\{1,3\}}{\varsigma+\chi+\breve{w}}=\frac{3}{\varsigma+\chi+\breve{w}},$$

whereas

$$\mathcal{G}\left(1,2,\frac{\varsigma}{\Gamma(1,2)}\right)=\frac{\max\{1,2\}}{\frac{\varsigma}{\Gamma(1,2)}}=\frac{2}{\frac{\varsigma}{4}}=\frac{8}{\varsigma},$$

$$\mathcal{G}\left(2,3,\frac{\chi}{\Gamma(2,3)}\right)=\frac{\max\{2,3\}}{\frac{\chi}{\Gamma(2,3)}}=\frac{3}{\frac{\chi}{12}}=\frac{36}{\chi},$$

and

$$\mathcal{G}\left(3,4,\frac{\check{w}}{\eta(3,4)}\right)=\frac{\max\{3,4\}}{\frac{\check{w}}{\eta(3,4)}}=\frac{4}{\frac{\check{w}}{24}}=\frac{96}{\check{w}}.$$

That is,

$$\frac{3}{\varsigma+\chi+\check{w}}\leq\max\left\{\frac{8}{\varsigma},\frac{36}{\chi},\frac{96}{\check{w}}\right\}.$$

Then, it satisfies all $\varsigma,\chi>0$. Hence,

$$\mathcal{G}(\vartheta,\psi,\varsigma+\chi+\check{w})\leq\mathcal{G}\left(\vartheta,\mathcal{Q},\frac{\varsigma}{\beta(\vartheta,\mathcal{Q})}\right)\diamond\mathcal{G}\left(\mathcal{Q},\mathfrak{r},\frac{\chi}{\Gamma(\mathcal{Q},\mathfrak{r})}\right)\diamond\mathcal{G}\left(\mathfrak{r},\psi,\frac{\chi}{\eta(\mathfrak{r},\psi)}\right).$$

Hence, $(\Im,\mathcal{W},\mathcal{E},\mathcal{G},\star,\diamond)$ is a neutrosophic rectangular triple-controlled metric space. □

Remark 1. *It can be seen that the preceding example satisfies both the continuous t-norm* $\wp\star\bar{a}=\min\{\wp,\bar{a}\}$ *and continuous t-co-norm* $\wp\diamond\bar{a}=\max\{\wp,\bar{a}\}$.

Example 2. *Let* $\Im=\mathcal{B}\cup\mathcal{U}$, *where* $\mathcal{B}=\{0,\frac{1}{2},\frac{1}{3},\frac{1}{4}\}$ *and* $\mathcal{U}=[1,2]$ *and* $\beta,\Gamma,\eta\colon\Im\times\Im\to[1,+\infty)$ *be given by* $\beta(\vartheta,\mathcal{Q})=1$, $\Gamma(\vartheta,\mathcal{Q})=1$ *and* $\eta(\vartheta,\mathcal{Q})=1$. *Define* $\nu:\Im\times\Im\to[0,+\infty)$ *as follows:*

$$\begin{cases}\nu(\vartheta,\mathcal{Q})=\nu(\mathcal{Q},\vartheta)\text{ for all }\vartheta,\mathcal{Q}\in\Im\\ \nu(\vartheta,\mathcal{Q})=0\iff\vartheta=\mathcal{Q},\end{cases}$$

and

$$\begin{cases}\nu(0,\frac{1}{2})=\nu(\frac{1}{2},\frac{1}{3})=0.2\\ \nu(0,\frac{1}{3})=\nu(\frac{1}{3},\frac{1}{4})=0.02\\ \nu(0,\frac{1}{4})=\nu(\frac{1}{2},\frac{1}{4})=0.5\\ \nu(\vartheta,\mathcal{Q})=|\vartheta-\mathcal{Q}|,\ \text{otherwise}.\end{cases}$$

Define $\mathcal{W},\mathcal{E},\mathcal{G}\colon\Im\times\Im\times(0,+\infty)\to[0,1]$ *as*

$$\mathcal{W}(\vartheta,\mathcal{Q},\varsigma)=\frac{\varsigma}{\varsigma+\nu(\vartheta,\mathcal{Q})},$$
$$\mathcal{E}(\vartheta,\mathcal{Q},\varsigma)=\frac{\nu(\vartheta,\mathcal{Q})}{\varsigma+\nu(\vartheta,\mathcal{Q})},\quad\mathcal{G}(\vartheta,\mathcal{Q},\varsigma)=\frac{\nu(\vartheta,\mathcal{Q})}{\varsigma}.$$

Then, we have

$$\mathcal{W}(\vartheta,\psi,\varsigma+\chi+\check{w})\geq\mathcal{W}\left(\vartheta,\mathcal{Q},\frac{\varsigma}{\beta(\vartheta,\mathcal{Q})}\right)\star\mathcal{W}\left(\mathcal{Q},\mathfrak{r},\frac{\chi}{\Gamma(\mathcal{Q},\mathfrak{r})}\right)\star\mathcal{W}\left(\mathfrak{r},\psi,\frac{\check{w}}{\eta(\mathfrak{r},\psi)}\right),$$

$$\mathcal{E}(\vartheta,\psi,\varsigma+\chi+\check{w})\leq\mathcal{E}\left(\vartheta,\mathcal{Q},\frac{\varsigma}{\beta(\vartheta,\mathcal{Q})}\right)\diamond\mathcal{E}\left(\mathfrak{r},\psi,\frac{\chi}{\Gamma(\mathfrak{r},\psi)}\right)\diamond\mathcal{E}\left(\mathfrak{r},\psi,\frac{\check{w}}{\eta(\mathfrak{r},\psi)}\right),$$

$$\mathcal{G}(\vartheta, \psi, \varsigma + \chi + \breve{w}) \leq \mathcal{G}\left(\vartheta, \mathcal{Q}, \frac{\varsigma}{\beta(\vartheta, \mathcal{Q})}\right) \diamond \mathcal{G}\left(\mathcal{Q}, \mathfrak{r}, \frac{\chi}{\Gamma(\mathcal{Q}, \mathfrak{r})}\right) \diamond \mathcal{G}\left(\mathfrak{r}, \psi, \frac{\chi}{\eta(\mathfrak{r}, \psi)}\right).$$

Then, $(\Im, \mathcal{W}, \mathcal{E}, \mathcal{G}, \star, \diamond)$ is a neutrosophic rectangular triple-controlled metric space with continuous t-norm $\wp \star \bar{a} = \wp\bar{a}$ and continuous t-co-norm $\wp \diamond \bar{a} = \max\{\wp, \bar{a}\}$.

Definition 8. *Let $(\Im, \mathcal{W}, \mathcal{E}, \mathcal{G}, \star, \diamond)$ be a neutrosophic rectangular triple-controlled metric space, an open ball with center ϑ, radius $\mathfrak{r}, 0 < \mathfrak{r} < 1$ and $\varsigma > 0$ on $\mathcal{G}(\vartheta, \mathfrak{r}, \varsigma)$ is defined as below:*

$$\mathcal{G}(\vartheta, \mathfrak{r}, \varsigma) = \{\mathcal{Q} \in \Im \colon \mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) > 1 - \mathfrak{r}, \mathcal{E}(\vartheta, \mathcal{Q}, \varsigma) < \mathfrak{r}, \mathcal{G}(\vartheta, \mathcal{Q}, \varsigma) < \mathfrak{r}\}.$$

Definition 9. *Let $(\Im, \mathcal{W}, \mathcal{E}, \mathcal{G}, \star, \diamond)$ be a neutrosophic rectangular triple-controlled metric space and $\{\vartheta_{\mathrm{j}}\}$ be a sequence in $\Im$. Then, $\{\vartheta_{\mathrm{j}}\}$ is said to be*

(a) *convergent if there exists $\vartheta \in \Im$ such that*

$$\lim_{\mathrm{j}\to+\infty} \mathcal{W}(\vartheta_{\mathrm{j}}, \vartheta, \varsigma) = 1, \lim_{\mathrm{j}\to+\infty} \mathcal{E}(\vartheta_{\mathrm{j}}, \vartheta, \varsigma) = 0, \lim_{\mathrm{j}\to+\infty} \mathcal{G}(\vartheta_{\mathrm{j}}, \vartheta, \varsigma) = 0 \quad \textit{for all} \quad \varsigma > 0,$$

(b) *Cauchy, if and only if, for each $\bar{a} > 0, \varsigma > 0$, there exists $\mathrm{j}_0 \in \mathbb{N}$ such that*

$$\mathcal{W}(\vartheta_{\mathrm{j}}, \vartheta_{\mathfrak{m}}, \varsigma) \geq 1 - \bar{a}, \mathcal{E}(\vartheta_{\mathrm{j}}, \vartheta_{\mathfrak{m}}, \varsigma) \leq \bar{a}, \mathcal{E}(\vartheta_{\mathrm{j}}, \vartheta_{\mathfrak{m}}, \varsigma) \leq \bar{a} \quad \textit{for all} \quad \mathrm{j}, \mathfrak{m} \geq \mathrm{j}_0,$$

$(\Im, \mathcal{W}, \mathcal{E}, \mathcal{G}, \star, \diamond)$ is called a complete neutrosophic rectangular triple-controlled metric space if every Cauchy seqeunce is convergent in $\Im$.

Lemma 1. *Let $(\Im, \mathcal{W}, \mathcal{E}, \mathcal{G}, \star, \diamond)$ be a neutrosophic rectangular triple-controlled metric space. If, for some $0 < \rho < 1$ and for any $\vartheta, \mathcal{Q} \in \Im, \varsigma > 0$,*

$$\mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) \geq \mathcal{W}\left(\vartheta, \mathcal{Q}, \frac{\varsigma}{\rho}\right), \mathcal{E}(\vartheta, \mathcal{Q}, \varsigma) \leq \mathcal{E}\left(\vartheta, \mathcal{Q}, \frac{\varsigma}{\rho}\right), \mathcal{G}(\vartheta, \mathcal{Q}, \varsigma) \leq \mathcal{G}\left(\vartheta, \mathcal{Q}, \frac{\varsigma}{\rho}\right), \quad (1)$$

then, $\vartheta = \mathcal{Q}$.

Proof. (1) implies that

$$\mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) \geq \mathcal{W}\left(\vartheta, \mathcal{Q}, \frac{\varsigma}{\rho^{\mathrm{j}}}\right), \mathcal{E}(\vartheta, \mathcal{Q}, \varsigma) \leq \mathcal{E}\left(\vartheta, \mathcal{Q}, \frac{\varsigma}{\rho^{\mathrm{j}}}\right), \mathcal{G}(\vartheta, \mathcal{Q}, \varsigma) \leq \mathcal{G}\left(\vartheta, \mathcal{Q}, \frac{\varsigma}{\rho^{\mathrm{j}}}\right), \mathrm{j} \in \mathbb{N}, \varsigma > 0.$$

Now,

$$\mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) \geq \lim_{\mathrm{j}\to+\infty} \mathcal{W}\left(\vartheta, \mathcal{Q}, \frac{\varsigma}{\rho^{\mathrm{j}}}\right) = 1,$$
$$\mathcal{E}(\vartheta, \mathcal{Q}, \varsigma) \leq \lim_{\mathrm{j}\to+\infty} \mathcal{E}\left(\vartheta, \mathcal{Q}, \frac{\varsigma}{\rho^{\mathrm{j}}}\right) = 0,$$
$$\mathcal{G}(\vartheta, \mathcal{Q}, \varsigma) \leq \lim_{\mathrm{j}\to+\infty} \mathcal{G}\left(\vartheta, \mathcal{Q}, \frac{\varsigma}{\rho^{\mathrm{j}}}\right) = 0, \varsigma > 0.$$

Moreover, from (iii), (viii), (xiii), we have $\vartheta = \mathcal{Q}$. □

Theorem 1. *Suppose $(\Im, \mathcal{W}, \mathcal{E}, \mathcal{G}, \star, \diamond)$ is a complete neutrosophic rectangular triple-controlled metric space in the company of $\beta, \Gamma, \eta \colon \Im \times \Im \to [1, +\infty)$ with $0 < \rho < 1$. Let $\mathfrak{p} \colon \Im \to \Im$ be a mapping satisfying*

$$\mathcal{W}(\mathfrak{p}\vartheta, \mathfrak{p}\mathcal{Q}, \rho\varsigma) \geq \mathcal{W}(\vartheta, \mathcal{Q}, \varsigma),$$
$$\mathcal{E}(\mathfrak{p}\vartheta, \mathfrak{p}\mathcal{Q}, \rho\varsigma) \leq \mathcal{E}(\vartheta, \mathcal{Q}, \varsigma) \quad \textit{and} \quad \mathcal{G}(\mathfrak{p}\vartheta, \mathfrak{p}\mathcal{Q}, \rho\varsigma) \leq \mathcal{G}(\vartheta, \mathcal{Q}, \varsigma), \quad (2)$$

for all $\vartheta, \mathcal{Q} \in \Im$ and $\varsigma > 0$. Then, $\mathfrak{p}$ has a unique fixed point.

Proof. Let $\vartheta_0 \in \Im$. Define the sequence ϑ_j by $\vartheta_j = \mathfrak{p}^j \vartheta_0 = \mathfrak{p}\vartheta_{j-1}, j \in \mathbb{N}$.

For all $\varsigma > 0$, we have

$$\begin{aligned}\mathcal{W}(\vartheta_j, \vartheta_{j+1}, \rho\varsigma) &= \mathcal{W}(\mathfrak{p}\vartheta_{j-1}, \mathfrak{p}\vartheta_j, \rho\varsigma) \geq \mathcal{W}(\vartheta_{j-1}, \vartheta_j, \varsigma) \geq \mathcal{W}\left(\vartheta_{j-2}, \vartheta_{j-1}, \frac{\varsigma}{\rho}\right) \\ &\geq \mathcal{W}\left(\vartheta_{j-3}, \vartheta_{j-2}, \frac{\varsigma}{\rho^2}\right) \geq \cdots \geq \mathcal{W}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{\rho^{j-1}}\right), \\ \mathcal{E}(\vartheta_j, \vartheta_{j+1}, \rho\varsigma) &= \mathcal{E}(\mathfrak{p}\vartheta_{j-1}, \mathfrak{p}\vartheta_j, \rho\varsigma) \leq \mathcal{E}(\vartheta_{j-1}, \vartheta_j, \varsigma) \leq \mathcal{E}\left(\vartheta_{j-2}, \vartheta_{j-1}, \frac{\varsigma}{\rho}\right) \\ &\leq \mathcal{E}\left(\vartheta_{j-3}, \vartheta_{j-2}, \frac{\varsigma}{\rho^2}\right) \leq \cdots \leq \mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{\rho^{j-1}}\right),\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}(\vartheta_j, \vartheta_{j+1}, \rho\varsigma) &= \mathcal{G}(\mathfrak{p}\vartheta_{j-1}, \mathfrak{p}\vartheta_j, \varsigma) \leq \mathcal{G}(\vartheta_{j-1}, \vartheta_j, \varsigma) \leq \mathcal{G}\left(\vartheta_{j-2}, \vartheta_{j-1}, \frac{\varsigma}{\rho}\right) \\ &\leq \mathcal{G}\left(\vartheta_{j-3}, \vartheta_{j-2}, \frac{\varsigma}{\rho^2}\right) \leq \cdots \leq \mathcal{G}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{\rho^{j-1}}\right).\end{aligned}$$

We obtain

$$\mathcal{W}(\vartheta_j, \vartheta_{j+1}, \rho\varsigma) \geq \mathcal{W}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{\rho^{j-1}}\right),$$
$$\mathcal{E}(\vartheta_j, \vartheta_{j+1}, \rho\varsigma) \leq \mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{\rho^{j-1}}\right) \quad \text{and} \quad \mathcal{G}(\vartheta_j, \vartheta_{j+1}, \rho\varsigma) \leq \mathcal{G}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{\rho^{j-1}}\right). \tag{3}$$

Using (v), (x) and (xv), we have the following cases:

Case 1. When $i = 2m + 1$, i.e., i is odd, then

$$\begin{aligned}&\mathcal{W}(\vartheta_j, \vartheta_{j+2m+1}, \varsigma) \\ &\quad\geq \mathcal{W}\left(\vartheta_j, \vartheta_{j+1}, \frac{\varsigma}{3(\beta(\vartheta_j, \vartheta_{j+1}))}\right) \star \mathcal{W}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\ &\quad\star \mathcal{W}\left(\vartheta_{j+2}, \vartheta_{j+2m+1}, \frac{\varsigma}{3(\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\ &\quad\geq \mathcal{W}\left(\vartheta_j, \vartheta_{j+1}, \frac{\varsigma}{3(\beta(\vartheta_j, \vartheta_{j+1}))}\right) \star \mathcal{W}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\ &\quad\star \mathcal{W}\left(\vartheta_{j+2}, \vartheta_{j+3}, \frac{\varsigma}{3^2(\beta(\vartheta_{j+2}, \vartheta_{j+3})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\ &\quad\star \mathcal{W}\left(\vartheta_{j+3}, \vartheta_{j+4}, \frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3}, \vartheta_{j+4})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\ &\quad\star \mathcal{W}\left(\vartheta_{j+4}, \vartheta_{j+2m+1}, \frac{\varsigma}{3^2(\eta(\vartheta_{j+4}, \vartheta_{j+2m+1})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right)\end{aligned}$$

$$
\begin{aligned}
&\geq \mathcal{W}\left(\vartheta_{j}, \vartheta_{j+1}, \frac{\varsigma}{3(\beta(\vartheta_{j}, \vartheta_{j+1}))}\right) \star \mathcal{W}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+2}, \vartheta_{j+3}, \frac{\varsigma}{3^2(\beta(\vartheta_{j+2}, \vartheta_{j+3})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+3}, \vartheta_{j+4}, \frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3}, \vartheta_{j+4})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+4}, \vartheta_{j+5}, \frac{\varsigma}{3^3(\beta(\vartheta_{j+4}, \vartheta_{j+5})\eta(\vartheta_{j+4}, \vartheta_{j+2m+1})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+5}, \vartheta_{j+6}, \frac{\varsigma}{3^3(\Gamma(\vartheta_{j+5}, \vartheta_{j+6})\eta(\vartheta_{j+4}, \vartheta_{j+2m+1})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+6}, \vartheta_{j+2m+1}, \frac{\varsigma}{3^3(\eta(\vartheta_{j+6}, \vartheta_{j+2m+1})\eta(\vartheta_{j+4}, \vartheta_{j+2m+1})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{W}(\vartheta_{j}, \vartheta_{j+2m+1}, \varsigma) \\
&\geq \mathcal{W}\left(\vartheta_{j}, \vartheta_{j+1}, \frac{\varsigma}{3(\beta(\vartheta_{j}, \vartheta_{j+1}))}\right) \star \mathcal{W}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+2}, \vartheta_{j+3}, \frac{\varsigma}{3^2(\beta(\vartheta_{j+2}, \vartheta_{j+3})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+3}, \vartheta_{j+4}, \frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3}, \vartheta_{j+4})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+4}, \vartheta_{j+5}, \frac{\varsigma}{3^3(\beta(\vartheta_{j+4}, \vartheta_{j+5})\eta(\vartheta_{j+4}, \vartheta_{j+2m+1})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+5}, \vartheta_{j+6}, \frac{\varsigma}{3^3(\Gamma(\vartheta_{j+5}, \vartheta_{j+6})\eta(\vartheta_{j+4}, \vartheta_{j+2m+1})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+6}, \vartheta_{j+2m+1}, \frac{\varsigma}{3^3(\eta(\vartheta_{j+6}, \vartheta_{j+2m+1})\eta(\vartheta_{j+4}, \vartheta_{j+2m+1})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\star \cdots \star \\
&\mathcal{W}\left(\vartheta_{j+2m-2}, \vartheta_{j+2m-1}, \frac{\varsigma}{3^m(\beta(\vartheta_{j+2m-2}, \vartheta_{j+2m-1})\eta(\vartheta_{j+2m-2}, \vartheta_{j+2m+1})\cdots\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+2m-1}, \vartheta_{j+2m}, \frac{\varsigma}{3^m(\Gamma(\vartheta_{j+2m-1}, \vartheta_{j+2m})\eta(\vartheta_{j+2m-2}, \vartheta_{j+2m+1})\cdots\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+2m}, \vartheta_{j+2m+1}, \frac{\varsigma}{3^m(\eta(\vartheta_{j+2m}, \vartheta_{j+2m+1})\eta(\vartheta_{j+2m-2}, \vartheta_{j+2m+1})\cdots\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}(\vartheta_{j}, \vartheta_{j+2m+1}, \varsigma) &\leq \mathcal{E}\left(\vartheta_{j}, \vartheta_{j+1}, \frac{\varsigma}{3(\beta(\vartheta_{j}, \vartheta_{j+1}))}\right) \diamond \mathcal{E}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+2}, \vartheta_{j+2m+1}, \frac{\varsigma}{3(\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\leq \mathcal{E}\left(\vartheta_{j}, \vartheta_{j+1}, \frac{\varsigma}{3(\beta(\vartheta_{j}, \vartheta_{j+1}))}\right) \diamond \mathcal{E}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\diamond \mathcal{E}\left(\vartheta_{j+2}, \vartheta_{j+3}, \frac{\varsigma}{3^2(\beta(\vartheta_{j+2}, \vartheta_{j+3})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+3}, \vartheta_{j+4}, \frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3}, \vartheta_{j+4})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+4}, \vartheta_{j+2m+1}, \frac{\varsigma}{3^2(\eta(\vartheta_{j+4}, \vartheta_{j+2m+1})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
\leq{}& \mathcal{E}\left(\vartheta_{j}, \vartheta_{j+1}, \frac{\varsigma}{3(\beta(\vartheta_{j}, \vartheta_{j+1}))}\right) \diamond \mathcal{E}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+2}, \vartheta_{j+3}, \frac{\varsigma}{3^2(\beta(\vartheta_{j+2}, \vartheta_{j+3})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+3}, \vartheta_{j+4}, \frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3}, \vartheta_{j+4})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+4}, \vartheta_{j+5}, \frac{\varsigma}{3^3(\beta(\vartheta_{j+4}, \vartheta_{j+5})\eta(\vartheta_{j+4}, \vartheta_{j+2m+1})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+5}, \vartheta_{j+6}, \frac{\varsigma}{3^3(\Gamma(\vartheta_{j+5}, \vartheta_{j+6})\eta(\vartheta_{j+4}, \vartheta_{j+2m+1})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+6}, \vartheta_{j+2m+1}, \frac{\varsigma}{3^3(\eta(\vartheta_{j+6}, \vartheta_{j+2m+1})\eta(\vartheta_{j+4}, \vartheta_{j+2m+1})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{E}(\vartheta_{j}, \vartheta_{j+2m+1}, \varsigma) \\
&\leq \mathcal{E}\left(\vartheta_{j}, \vartheta_{j+1}, \frac{\varsigma}{3(\beta(\vartheta_{j}, \vartheta_{j+1}))}\right) \diamond \mathcal{E}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+2}, \vartheta_{j+3}, \frac{\varsigma}{3^2(\beta(\vartheta_{j+2}, \vartheta_{j+3})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+3}, \vartheta_{j+4}, \frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3}, \vartheta_{j+4})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+4}, \vartheta_{j+5}, \frac{\varsigma}{3^3(\beta(\vartheta_{j+4}, \vartheta_{j+5})\eta(\vartheta_{j+4}, \vartheta_{j+2m+1})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+5}, \vartheta_{j+6}, \frac{\varsigma}{3^3(\Gamma(\vartheta_{j+5}, \vartheta_{j+6})\eta(\vartheta_{j+4}, \vartheta_{j+2m+1})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+6}, \vartheta_{j+2m+1}, \frac{\varsigma}{3^3(\eta(\vartheta_{j+6}, \vartheta_{j+2m+1})\eta(\vartheta_{j+4}, \vartheta_{j+2m+1})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\diamond \cdots \diamond \\
&\mathcal{E}\left(\vartheta_{j+2m-2}, \vartheta_{j+2m-1}, \frac{\varsigma}{3^m(\beta(\vartheta_{j+2m-2}, \vartheta_{j+2m-1})\eta(\vartheta_{j+2m-2}, \vartheta_{j+2m+1})\cdots\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+2m-1}, \vartheta_{j+2m}, \frac{\varsigma}{3^m(\Gamma(\vartheta_{j+2m-1}, \vartheta_{j+2m})\eta(\vartheta_{j+2m-2}, \vartheta_{j+2m+1})\cdots\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+2m}, \vartheta_{j+2m+1}, \frac{\varsigma}{3^m(\eta(\vartheta_{j+2m}, \vartheta_{j+2m+1})\eta(\vartheta_{j+2m-2}, \vartheta_{j+2m+1})\cdots\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right),
\end{aligned}
$$

and

$$\begin{aligned}
\mathcal{G}(\vartheta_{j},\vartheta_{j+2m+1},\varsigma) \leq\ & \mathcal{G}\left(\vartheta_{j},\vartheta_{j+1},\frac{\varsigma}{3(\beta(\vartheta_{j},\vartheta_{j+1}))}\right)\diamond\mathcal{G}\left(\vartheta_{j+1},\vartheta_{j+2},\frac{\varsigma}{3(\Gamma(\vartheta_{j+1},\vartheta_{j+2}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+2},\vartheta_{j+2m+1},\frac{\varsigma}{3(\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
\leq\ & \mathcal{G}\left(\vartheta_{j},\vartheta_{j+1},\frac{\varsigma}{3(\beta(\vartheta_{j},\vartheta_{j+1}))}\right)\diamond\mathcal{G}\left(\vartheta_{j+1},\vartheta_{j+2},\frac{\varsigma}{3(\Gamma(\vartheta_{j+1},\vartheta_{j+2}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+2},\vartheta_{j+3},\frac{\varsigma}{3^{2}(\beta(\vartheta_{j+2},\vartheta_{j+3})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+3},\vartheta_{j+4},\frac{\varsigma}{3^{2}(\Gamma(\vartheta_{j+3},\vartheta_{j+4})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+4},\vartheta_{j+2m+1},\frac{\varsigma}{3^{2}(\eta(\vartheta_{j+4},\vartheta_{j+2m+1})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
\leq\ & \mathcal{G}\left(\vartheta_{j},\vartheta_{j+1},\frac{\varsigma}{3(\beta(\vartheta_{j},\vartheta_{j+1}))}\right)\diamond\mathcal{G}\left(\vartheta_{j+1},\vartheta_{j+2},\frac{\varsigma}{3(\Gamma(\vartheta_{j+1},\vartheta_{j+2}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+2},\vartheta_{j+3},\frac{\varsigma}{3^{2}(\beta(\vartheta_{j+2},\vartheta_{j+3})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+3},\vartheta_{j+4},\frac{\varsigma}{3^{2}(\Gamma(\vartheta_{j+3},\vartheta_{j+4})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+4},\vartheta_{j+5},\frac{\varsigma}{3^{3}(\beta(\vartheta_{j+4},\vartheta_{j+5})\eta(\vartheta_{j+4},\vartheta_{j+2m+1})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+5},\vartheta_{j+6},\frac{\varsigma}{3^{3}(\Gamma(\vartheta_{j+5},\vartheta_{j+6})\eta(\vartheta_{j+4},\vartheta_{j+2m+1})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+6},\vartheta_{j+2m+1},\frac{\varsigma}{3^{3}(\eta(\vartheta_{j+6},\vartheta_{j+2m+1})\eta(\vartheta_{j+4},\vartheta_{j+2m+1})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)
\end{aligned}$$

$$\begin{aligned}
&\mathcal{G}(\vartheta_{j},\vartheta_{j+2m+1},\varsigma)\\
&\leq \mathcal{G}\left(\vartheta_{j},\vartheta_{j+1},\frac{\varsigma}{3(\beta(\vartheta_{j},\vartheta_{j+1}))}\right)\diamond\mathcal{G}\left(\vartheta_{j+1},\vartheta_{j+2},\frac{\varsigma}{3(\Gamma(\vartheta_{j+1},\vartheta_{j+2}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+2},\vartheta_{j+3},\frac{\varsigma}{3^{2}(\beta(\vartheta_{j+2},\vartheta_{j+3})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+3},\vartheta_{j+4},\frac{\varsigma}{3^{2}(\Gamma(\vartheta_{j+3},\vartheta_{j+4})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+4},\vartheta_{j+5},\frac{\varsigma}{3^{3}(\beta(\vartheta_{j+4},\vartheta_{j+5})\eta(\vartheta_{j+4},\vartheta_{j+2m+1})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+5},\vartheta_{j+6},\frac{\varsigma}{3^{3}(\Gamma(\vartheta_{j+5},\vartheta_{j+6})\eta(\vartheta_{j+4},\vartheta_{j+2m+1})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+6},\vartheta_{j+2m+1},\frac{\varsigma}{3^{3}(\eta(\vartheta_{j+6},\vartheta_{j+2m+1})\eta(\vartheta_{j+4},\vartheta_{j+2m+1})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\diamond\cdots\diamond
\end{aligned}$$

$$\begin{aligned}
&\mathcal{G}\left(\vartheta_{j+2m-2},\vartheta_{j+2m-1},\frac{\varsigma}{3^{m}(\beta(\vartheta_{j+2m-2},\vartheta_{j+2m-1})\eta(\vartheta_{j+2m-2},\vartheta_{j+2m+1})\cdots\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+2m-1},\vartheta_{j+2m},\frac{\varsigma}{3^{m}(\Gamma(\vartheta_{j+2m-1},\vartheta_{j+2m})\eta(\vartheta_{j+2m-2},\vartheta_{j+2m+1})\cdots\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+2m},\vartheta_{j+2m+1},\frac{\varsigma}{3^{m}(\eta(\vartheta_{j+2m},\vartheta_{j+2m+1})\eta(\vartheta_{j+2m-2},\vartheta_{j+2m+1})\cdots\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right).
\end{aligned}$$

Using (3) in the above inequalities, we have

$$\begin{aligned}
\mathcal{W}(\vartheta_{\mathrm{j}}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}, \varsigma) \geq{}& \mathcal{W}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3\rho^{\mathrm{j}-1}(\beta(\vartheta_{\mathrm{j}}, \vartheta_{\mathrm{j}+1}))}\right) \star \mathcal{W}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3\rho^{\mathrm{j}}(\Gamma(\vartheta_{\mathrm{j}+1}, \vartheta_{\mathrm{j}+2}))}\right) \\
&\star \mathcal{W}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^2\rho^{\mathrm{j}+1}(\beta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+3})\eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \\
&\star \mathcal{W}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^2\rho^{\mathrm{j}+2}(\Gamma(\vartheta_{\mathrm{j}+3}, \vartheta_{\mathrm{j}+4})\eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \\
&\star \mathcal{W}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^3\rho^{\mathrm{j}+3}(\beta(\vartheta_{\mathrm{j}+4}, \vartheta_{\mathrm{j}+5})\eta(\vartheta_{\mathrm{j}+4}, \vartheta_{\mathrm{j}+2\mathrm{m}+1})\eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \\
&\star \mathcal{W}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^3\rho^{\mathrm{j}+4}(\Gamma(\vartheta_{\mathrm{j}+5}, \vartheta_{\mathrm{j}+6})\eta(\vartheta_{\mathrm{j}+4}, \vartheta_{\mathrm{j}+2\mathrm{m}+1})\eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \\
&\star \mathcal{W}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^3\rho^{\mathrm{j}+5}(\eta(\vartheta_{\mathrm{j}+6}, \vartheta_{\mathrm{j}+2\mathrm{m}+1})\eta(\vartheta_{\mathrm{j}+4}, \vartheta_{\mathrm{j}+2\mathrm{m}+1})\eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \star \cdots \star \\
&\mathcal{W}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^{\mathrm{m}}\rho^{\mathrm{j}+2\mathrm{m}-3}(\beta(\vartheta_{\mathrm{j}+2\mathrm{m}-2}, \vartheta_{\mathrm{j}+2\mathrm{m}-1})\eta(\vartheta_{\mathrm{j}+2\mathrm{m}-2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}) \cdots \eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \\
&\star \mathcal{W}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^{\mathrm{m}}\rho^{\mathrm{j}+2\mathrm{m}-2}(\Gamma(\vartheta_{\mathrm{j}+2\mathrm{m}-1}, \vartheta_{\mathrm{j}+2\mathrm{m}})\eta(\vartheta_{\mathrm{j}+2\mathrm{m}-2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}) \cdots \eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \\
&\star \mathcal{W}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^{\mathrm{m}}\rho^{\mathrm{j}+2\mathrm{m}-1}(\eta(\vartheta_{\mathrm{j}+2\mathrm{m}}, \vartheta_{\mathrm{j}+2\mathrm{m}+1})\eta(\vartheta_{\mathrm{j}+2\mathrm{m}-2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}) \cdots \eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right),
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}(\vartheta_{\mathrm{j}}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}, \varsigma) \leq{}& \mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3\rho^{\mathrm{j}-1}(\beta(\vartheta_{\mathrm{j}}, \vartheta_{\mathrm{j}+1}))}\right) \diamond \mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3\rho^{\mathrm{j}}(\Gamma(\vartheta_{\mathrm{j}+1}, \vartheta_{\mathrm{j}+2}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^2\rho^{\mathrm{j}+1}(\beta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+3})\eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^2\rho^{\mathrm{j}+2}(\Gamma(\vartheta_{\mathrm{j}+3}, \vartheta_{\mathrm{j}+4})\eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^3\rho^{\mathrm{j}+3}(\beta(\vartheta_{\mathrm{j}+4}, \vartheta_{\mathrm{j}+5})\eta(\vartheta_{\mathrm{j}+4}, \vartheta_{\mathrm{j}+2\mathrm{m}+1})\eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^3\rho^{\mathrm{j}+4}(\Gamma(\vartheta_{\mathrm{j}+5}, \vartheta_{\mathrm{j}+6})\eta(\vartheta_{\mathrm{j}+4}, \vartheta_{\mathrm{j}+2\mathrm{m}+1})\eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^3\rho^{\mathrm{j}+5}(\eta(\vartheta_{\mathrm{j}+6}, \vartheta_{\mathrm{j}+2\mathrm{m}+1})\eta(\vartheta_{\mathrm{j}+4}, \vartheta_{\mathrm{j}+2\mathrm{m}+1})\eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \diamond \cdots \diamond \\
&\mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^{\mathrm{m}}\rho^{\mathrm{j}+2\mathrm{m}-3}(\beta(\vartheta_{\mathrm{j}+2\mathrm{m}-2}, \vartheta_{\mathrm{j}+2\mathrm{m}-1})\eta(\vartheta_{\mathrm{j}+2\mathrm{m}-2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}) \cdots \eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^{\mathrm{m}}\rho^{\mathrm{j}+2\mathrm{m}-2}(\Gamma(\vartheta_{\mathrm{j}+2\mathrm{m}-1}, \vartheta_{\mathrm{j}+2\mathrm{m}})\eta(\vartheta_{\mathrm{j}+2\mathrm{m}-2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}) \cdots \eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^{\mathrm{m}}\rho^{\mathrm{j}+2\mathrm{m}-1}(\eta(\vartheta_{\mathrm{j}+2\mathrm{m}}, \vartheta_{\mathrm{j}+2\mathrm{m}+1})\eta(\vartheta_{\mathrm{j}+2\mathrm{m}-2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}) \cdots \eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right),
\end{aligned}$$

$$\begin{aligned}
&\mathcal{G}(\vartheta_{j},\vartheta_{j+2m+1},\varsigma)\\
&\le \mathcal{G}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3\rho^{j-1}(\beta(\vartheta_j,\vartheta_{j+1}))}\right)\diamond\mathcal{G}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3\rho^{j}(\Gamma(\vartheta_{j+1},\vartheta_{j+2}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^2\rho^{j+1}(\beta(\vartheta_{j+2},\vartheta_{j+3})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^2\rho^{j+2}(\Gamma(\vartheta_{j+3},\vartheta_{j+4})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^3\rho^{j+3}(\beta(\vartheta_{j+4},\vartheta_{j+5})\eta(\vartheta_{j+4},\vartheta_{j+2m+1})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^3\rho^{j+4}(\Gamma(\vartheta_{j+5},\vartheta_{j+6})\eta(\vartheta_{j+4},\vartheta_{j+2m+1})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^3\rho^{j+5}(\eta(\vartheta_{j+6},\vartheta_{j+2m+1})\eta(\vartheta_{j+4},\vartheta_{j+2m+1})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\diamond\cdots\diamond\\
&\mathcal{G}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^m\rho^{j+2m-3}(\beta(\vartheta_{j+2m-2},\vartheta_{j+2m-1})\eta(\vartheta_{j+2m-2},\vartheta_{j+2m+1})\cdots\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^m\rho^{j+2m-2}(\Gamma(\vartheta_{j+2m-1},\vartheta_{j+2m})\eta(\vartheta_{j+2m-2},\vartheta_{j+2m+1})\cdots\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^m\rho^{j+2m-1}(\eta(\vartheta_{j+2m},\vartheta_{j+2m+1})\eta(\vartheta_{j+2m-2},\vartheta_{j+2m+1})\cdots\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right).
\end{aligned}$$

Case 2. When $i = 2m$, i.e., i is even, then

$$\begin{aligned}
&\mathcal{W}(\vartheta_j,\vartheta_{j+2m},\varsigma)\\
&\ge \mathcal{W}\left(\vartheta_j,\vartheta_{j+1},\frac{\varsigma}{3(\beta(\vartheta_j,\vartheta_{j+1}))}\right)\star\mathcal{W}\left(\vartheta_{j+1},\vartheta_{j+2},\frac{\varsigma}{3(\Gamma(\vartheta_{j+1},\vartheta_{j+2}))}\right)\\
&\star\mathcal{W}\left(\vartheta_{j+2},\vartheta_{j+2m},\frac{\varsigma}{3(\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right)\\
&\ge \mathcal{W}\left(\vartheta_j,\vartheta_{j+1},\frac{\varsigma}{3(\beta(\vartheta_j,\vartheta_{j+1}))}\right)\star\mathcal{W}\left(\vartheta_{j+1},\vartheta_{j+2},\frac{\varsigma}{3(\Gamma(\vartheta_{j+1},\vartheta_{j+2}))}\right)\\
&\star\mathcal{W}\left(\vartheta_{j+2},\vartheta_{j+3},\frac{\varsigma}{3^2(\beta(\vartheta_{j+2},\vartheta_{j+3})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right)\\
&\star\mathcal{W}\left(\vartheta_{j+3},\vartheta_{j+4},\frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3},\vartheta_{j+4})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right)\\
&\star\mathcal{W}\left(\vartheta_{j+4},\vartheta_{j+2m},\frac{\varsigma}{3^2(\eta(\vartheta_{j+4},\vartheta_{j+2m})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right)\\
&\ge \mathcal{W}\left(\vartheta_j,\vartheta_{j+1},\frac{\varsigma}{3(\beta(\vartheta_j,\vartheta_{j+1}))}\right)\star\mathcal{W}\left(\vartheta_{j+1},\vartheta_{j+2},\frac{\varsigma}{3(\Gamma(\vartheta_{j+1},\vartheta_{j+2}))}\right)\\
&\star\mathcal{W}\left(\vartheta_{j+2},\vartheta_{j+3},\frac{\varsigma}{3^2(\beta(\vartheta_{j+2},\vartheta_{j+3})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right)\\
&\star\mathcal{W}\left(\vartheta_{j+3},\vartheta_{j+4},\frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3},\vartheta_{j+4})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right)\\
&\star\mathcal{W}\left(\vartheta_{j+4},\vartheta_{j+5},\frac{\varsigma}{3^3(\beta(\vartheta_{j+4},\vartheta_{j+5})\eta(\vartheta_{j+4},\vartheta_{j+2m})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right)\\
&\star\mathcal{W}\left(\vartheta_{j+5},\vartheta_{j+6},\frac{\varsigma}{3^3(\Gamma(\vartheta_{j+5},\vartheta_{j+6})\eta(\vartheta_{j+4},\vartheta_{j+2m})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right)\\
&\star\mathcal{W}\left(\vartheta_{j+6},\vartheta_{j+2m},\frac{\varsigma}{3^3(\eta(\vartheta_{j+6},\vartheta_{j+2m})\eta(\vartheta_{j+4},\vartheta_{j+2m})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right)
\end{aligned}$$

$$
\begin{aligned}
&\mathcal{W}(\vartheta_j, \vartheta_{j+2m}, \varsigma) \\
&\geq \mathcal{W}\left(\vartheta_j, \vartheta_{j+1}, \frac{\varsigma}{3(\beta(\vartheta_j, \vartheta_{j+1}))}\right) \star \mathcal{W}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+2}, \vartheta_{j+3}, \frac{\varsigma}{3^2(\beta(\vartheta_{j+2}, \vartheta_{j+3})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+3}, \vartheta_{j+4}, \frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3}, \vartheta_{j+4})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+4}, \vartheta_{j+5}, \frac{\varsigma}{3^3(\beta(\vartheta_{j+4}, \vartheta_{j+5})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+5}, \vartheta_{j+6}, \frac{\varsigma}{3^3(\Gamma(\vartheta_{j+5}, \vartheta_{j+6})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+6}, \vartheta_{j+2m}, \frac{\varsigma}{3^3(\eta(\vartheta_{j+6}, \vartheta_{j+2m})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \star \cdots \star \\
&\mathcal{W}\left(\vartheta_{j+2m-4}, \vartheta_{j+2m-3}, \frac{\varsigma}{3^{m-1}(\beta(\vartheta_{j+2m-4}, \vartheta_{j+2m-3})\eta(\vartheta_{j+2m-4}, \vartheta_{j+2m})\cdots\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+2m-3}, \vartheta_{j+2m-2}, \frac{\varsigma}{3^{m-1}(\Gamma(\vartheta_{j+2m-3}, \vartheta_{j+2m-2})\eta(\vartheta_{j+2m-4}, \vartheta_{j+2m})\cdots\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+2m-2}, \vartheta_{j+2m}, \frac{\varsigma}{3^{m-1}(\eta(\vartheta_{j+2m-2}, \vartheta_{j+2m})\eta(\vartheta_{j+2m-4}, \vartheta_{j+2m})\cdots\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{E}(\vartheta_j, \vartheta_{j+2m}, \varsigma) \\
&\leq \mathcal{E}\left(\vartheta_j, \vartheta_{j+1}, \frac{\varsigma}{3(\beta(\vartheta_j, \vartheta_{j+1}))}\right) \diamond \mathcal{E}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+2}, \vartheta_{j+2m}, \frac{\varsigma}{3(\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\leq \mathcal{E}\left(\vartheta_j, \vartheta_{j+1}, \frac{\varsigma}{3(\beta(\vartheta_j, \vartheta_{j+1}))}\right) \diamond \mathcal{E}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+2}, \vartheta_{j+3}, \frac{\varsigma}{3^2(\beta(\vartheta_{j+2}, \vartheta_{j+3})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+3}, \vartheta_{j+4}, \frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3}, \vartheta_{j+4})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+4}, \vartheta_{j+2m}, \frac{\varsigma}{3^2(\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\leq \mathcal{E}\left(\vartheta_j, \vartheta_{j+1}, \frac{\varsigma}{3(\beta(\vartheta_j, \vartheta_{j+1}))}\right) \diamond \mathcal{E}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+2}, \vartheta_{j+3}, \frac{\varsigma}{3^2(\beta(\vartheta_{j+2}, \vartheta_{j+3})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+3}, \vartheta_{j+4}, \frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3}, \vartheta_{j+4})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+4}, \vartheta_{j+5}, \frac{\varsigma}{3^3(\beta(\vartheta_{j+4}, \vartheta_{j+5})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+5}, \vartheta_{j+6}, \frac{\varsigma}{3^3(\Gamma(\vartheta_{j+5}, \vartheta_{j+6})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+6}, \vartheta_{j+2m}, \frac{\varsigma}{3^3(\eta(\vartheta_{j+6}, \vartheta_{j+2m})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{E}(\vartheta_{j}, \vartheta_{j+2m}, \varsigma) \\
&\leq \mathcal{E}\left(\vartheta_{j}, \vartheta_{j+1}, \frac{\varsigma}{3(\beta(\vartheta_{j}, \vartheta_{j+1}))}\right) \diamond \mathcal{E}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\
&\quad \diamond \mathcal{E}\left(\vartheta_{j+2}, \vartheta_{j+3}, \frac{\varsigma}{3^2(\beta(\vartheta_{j+2}, \vartheta_{j+3})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\quad \diamond \mathcal{E}\left(\vartheta_{j+3}, \vartheta_{j+4}, \frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3}, \vartheta_{j+4})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\quad \diamond \mathcal{E}\left(\vartheta_{j+4}, \vartheta_{j+5}, \frac{\varsigma}{3^3(\beta(\vartheta_{j+4}, \vartheta_{j+5})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\quad \diamond \mathcal{E}\left(\vartheta_{j+5}, \vartheta_{j+6}, \frac{\varsigma}{3^3(\Gamma(\vartheta_{j+5}, \vartheta_{j+6})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\quad \diamond \mathcal{E}\left(\vartheta_{j+6}, \vartheta_{j+2m}, \frac{\varsigma}{3^3(\eta(\vartheta_{j+6}, \vartheta_{j+2m})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \diamond \cdots \diamond \\
&\mathcal{E}\left(\vartheta_{j+2m-4}, \vartheta_{j+2m-3}, \frac{\varsigma}{3^{m-1}(\beta(\vartheta_{j+2m-4}, \vartheta_{j+2m-3})\eta(\vartheta_{j+2m-4}, \vartheta_{j+2m}) \cdots \eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\quad \diamond \mathcal{E}\left(\vartheta_{j+2m-3}, \vartheta_{j+2m-2}, \frac{\varsigma}{3^{m-1}(\Gamma(\vartheta_{j+2m-3}, \vartheta_{j+2m-2})\eta(\vartheta_{j+2m-4}, \vartheta_{j+2m}) \cdots \eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\quad \diamond \mathcal{E}\left(\vartheta_{j+2m-2}, \vartheta_{j+2m}, \frac{\varsigma}{3^{m-1}(\eta(\vartheta_{j+2m-2}, \vartheta_{j+2m})\eta(\vartheta_{j+2m-4}, \vartheta_{j+2m}) \cdots \eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{G}(\vartheta_{j}, \vartheta_{j+2m}, \varsigma) \\
&\leq \mathcal{G}\left(\vartheta_{j}, \vartheta_{j+1}, \frac{\varsigma}{3(\beta(\vartheta_{j}, \vartheta_{j+1}))}\right) \diamond \mathcal{G}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\
&\quad \diamond \mathcal{G}\left(\vartheta_{j+2}, \vartheta_{j+2m}, \frac{\varsigma}{3(\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\leq \mathcal{G}\left(\vartheta_{j}, \vartheta_{j+1}, \frac{\varsigma}{3(\beta(\vartheta_{j}, \vartheta_{j+1}))}\right) \diamond \mathcal{G}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\
&\quad \diamond \mathcal{G}\left(\vartheta_{j+2}, \vartheta_{j+3}, \frac{\varsigma}{3^2(\beta(\vartheta_{j+2}, \vartheta_{j+3})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\quad \diamond \mathcal{G}\left(\vartheta_{j+3}, \vartheta_{j+4}, \frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3}, \vartheta_{j+4})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\quad \diamond \mathcal{G}\left(\vartheta_{j+4}, \vartheta_{j+2m}, \frac{\varsigma}{3^2(\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\leq \mathcal{G}\left(\vartheta_{j}, \vartheta_{j+1}, \frac{\varsigma}{3(\beta(\vartheta_{j}, \vartheta_{j+1}))}\right) \diamond \mathcal{G}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\
&\quad \diamond \mathcal{G}\left(\vartheta_{j+2}, \vartheta_{j+3}, \frac{\varsigma}{3^2(\beta(\vartheta_{j+2}, \vartheta_{j+3})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\quad \diamond \mathcal{G}\left(\vartheta_{j+3}, \vartheta_{j+4}, \frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3}, \vartheta_{j+4})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\quad \diamond \mathcal{G}\left(\vartheta_{j+4}, \vartheta_{j+5}, \frac{\varsigma}{3^3(\beta(\vartheta_{j+4}, \vartheta_{j+5})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\quad \diamond \mathcal{G}\left(\vartheta_{j+5}, \vartheta_{j+6}, \frac{\varsigma}{3^3(\Gamma(\vartheta_{j+5}, \vartheta_{j+6})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\quad \diamond \mathcal{G}\left(\vartheta_{j+6}, \vartheta_{j+2m}, \frac{\varsigma}{3^3(\eta(\vartheta_{j+6}, \vartheta_{j+2m})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{G}(\vartheta_{j}, \vartheta_{j+2m}, \varsigma) \\
&\leq \mathcal{G}\left(\vartheta_{j}, \vartheta_{j+1}, \frac{\varsigma}{3(\beta(\vartheta_{j}, \vartheta_{j+1}))}\right) \diamond \mathcal{G}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\
&\diamond \mathcal{G}\left(\vartheta_{j+2}, \vartheta_{j+3}, \frac{\varsigma}{3^{2}(\beta(\vartheta_{j+2}, \vartheta_{j+3})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{G}\left(\vartheta_{j+3}, \vartheta_{j+4}, \frac{\varsigma}{3^{2}(\Gamma(\vartheta_{j+3}, \vartheta_{j+4})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{G}\left(\vartheta_{j+4}, \vartheta_{j+5}, \frac{\varsigma}{3^{3}(\beta(\vartheta_{j+4}, \vartheta_{j+5})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{G}\left(\vartheta_{j+5}, \vartheta_{j+6}, \frac{\varsigma}{3^{3}(\Gamma(\vartheta_{j+5}, \vartheta_{j+6})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{G}\left(\vartheta_{j+6}, \vartheta_{j+2m}, \frac{\varsigma}{3^{3}(\eta(\vartheta_{j+6}, \vartheta_{j+2m})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \diamond \cdots \diamond \\
&\mathcal{G}\left(\vartheta_{j+2m-4}, \vartheta_{j+2m-3}, \frac{\varsigma}{3^{m-1}(\beta(\vartheta_{j+2m-4}, \vartheta_{j+2m-3})\eta(\vartheta_{j+2m-4}, \vartheta_{j+2m}) \cdots \eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{G}\left(\vartheta_{j+2m-3}, \vartheta_{j+2m-2}, \frac{\varsigma}{3^{m-1}(\Gamma(\vartheta_{j+2m-3}, \vartheta_{j+2m-2})\eta(\vartheta_{j+2m-4}, \vartheta_{j+2m}) \cdots \eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{G}\left(\vartheta_{j+2m-2}, \vartheta_{j+2m}, \frac{\varsigma}{3^{m-1}(\eta(\vartheta_{j+2m-2}, \vartheta_{j+2m})\eta(\vartheta_{j+2m-4}, \vartheta_{j+2m}) \cdots \eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right).
\end{aligned}
$$

Using (3) in the above inequalities, we have

$$
\begin{aligned}
&\mathcal{W}(\vartheta_{j}, \vartheta_{j+2m}, \varsigma) \\
&\geq \mathcal{W}\left(\vartheta_{0}, \vartheta_{1}, \frac{\varsigma}{3\rho^{j-1}\beta(\vartheta_{j}, \vartheta_{j+1})}\right) \star \mathcal{W}\left(\vartheta_{0}, \vartheta_{1}, \frac{\varsigma}{3\rho^{j}(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{0}, \vartheta_{1}, \frac{\varsigma}{3^{2}\rho^{j+1}(\beta(\vartheta_{j+2}, \vartheta_{j+3})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{0}, \vartheta_{1}, \frac{\varsigma}{3^{2}\rho^{j+2}(\Gamma(\vartheta_{j+3}, \vartheta_{j+4})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{0}, \vartheta_{1}, \frac{\varsigma}{3^{3}\rho^{j+3}(\beta(\vartheta_{j+4}, \vartheta_{j+5})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{0}, \vartheta_{1}, \frac{\varsigma}{3^{3}\rho^{j+4}(\Gamma(\vartheta_{j+5}, \vartheta_{j+6})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{0}, \vartheta_{1}, \frac{\varsigma}{3^{3}\rho^{j+5}(\eta(\vartheta_{j+6}, \vartheta_{j+2m})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \star \cdots \star \\
&\mathcal{W}\left(\vartheta_{0}, \vartheta_{1}, \frac{\varsigma}{3^{m-1}\rho^{j+2m-5}(\beta(\vartheta_{j+2m-4}, \vartheta_{j+2m-3})\eta(\vartheta_{j+2m-4}, \vartheta_{j+2m}) \cdots \eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{0}, \vartheta_{1}, \frac{\varsigma}{3^{m-1}\rho^{j+2m-4}(\Gamma(\vartheta_{j+2m-3}, \vartheta_{j+2m-2})\eta(\vartheta_{j+2m-4}, \vartheta_{j+2m}) \cdots \eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{0}, \vartheta_{1}, \frac{\varsigma}{3^{m-1}\rho^{j+2m-3}(\eta(\vartheta_{j+2m-2}, \vartheta_{j+2m})\eta(\vartheta_{j+2m-4}, \vartheta_{j+2m}) \cdots \eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}(\vartheta_{j}, \vartheta_{j+2m}, \varsigma) \leq\, &\mathcal{E}\left(\vartheta_{0}, \vartheta_{1}, \frac{\varsigma}{3\rho^{j-1}(\beta(\vartheta_{j}, \vartheta_{j+1}))}\right) \diamond \mathcal{E}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3\rho^{j}(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{0}, \vartheta_{1}, \frac{\varsigma}{3^{2}\rho^{j+1}(\beta(\vartheta_{j+2}, \vartheta_{j+3})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{0}, \vartheta_{1}, \frac{\varsigma}{3^{2}\rho^{j+2}(\Gamma(\vartheta_{j+3}, \vartheta_{j+4})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\diamond \mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^3\rho^{j+3}(\beta(\vartheta_{j+4}, \vartheta_{j+5})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^3\rho^{j+4}(\Gamma(\vartheta_{j+5}, \vartheta_{j+6})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^3\rho^{j+5}(\eta(\vartheta_{j+6}, \vartheta_{j+2m})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \diamond \cdots \diamond \\
&\mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^{m-1}\rho^{j+2m-5}(\beta(\vartheta_{j+2m-4}, \vartheta_{j+2m-3})\eta(\vartheta_{j+2m-4}, \vartheta_{j+2m}) \cdots \eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^{m-1}\rho^{j+2m-4}(\Gamma(\vartheta_{j+2m-3}, \vartheta_{j+2m-2})\eta(\vartheta_{j+2m-4}, \vartheta_{j+2m}) \cdots \eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^{m-1}\rho^{j+2m-3}(\eta(\vartheta_{j+2m-2}, \vartheta_{j+2m})\eta(\vartheta_{j+2m-4}, \vartheta_{j+2m}) \cdots \eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right),
\end{aligned}
$$

and,

$$
\begin{aligned}
&\mathcal{G}(\vartheta_j, \vartheta_{j+2m}, \varsigma) \leq \mathcal{G}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3\rho^{j-1}(\beta(\vartheta_j, \vartheta_{j+1}))}\right) \diamond \mathcal{G}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3\rho^{j}(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\
&\diamond \mathcal{G}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^2\rho^{j+1}(\beta(\vartheta_{j+2}, \vartheta_{j+3})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{G}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^2\rho^{j+2}(\Gamma(\vartheta_{j+3}, \vartheta_{j+4})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{G}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^3\rho^{j+3}(\beta(\vartheta_{j+4}, \vartheta_{j+5})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{G}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^3\rho^{j+4}(\Gamma(\vartheta_{j+5}, \vartheta_{j+6})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{G}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^3\rho^{j+5}(\eta(\vartheta_{j+6}, \vartheta_{j+2m})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \diamond \cdots \diamond \\
&\mathcal{G}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^{m-1}\rho^{j+2m-5}(\beta(\vartheta_{j+2m-4}, \vartheta_{j+2m-3})\eta(\vartheta_{j+2m-4}, \vartheta_{j+2m}) \cdots \eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{G}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^{m-1}\rho^{j+2m-4}(\Gamma(\vartheta_{j+2m-3}, \vartheta_{j+2m-2})\eta(\vartheta_{j+2m-4}, \vartheta_{j+2m}) \cdots \eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{G}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^{m-1}\rho^{j+2m-3}(\eta(\vartheta_{j+2m-2}, \vartheta_{j+2m})\eta(\vartheta_{j+2m-4}, \vartheta_{j+2m}) \cdots \eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right).
\end{aligned}
$$

As $j \to +\infty$, we have

$$
\begin{aligned}
\lim_{j \to +\infty} \mathcal{W}(\vartheta_j, \vartheta_{j+i}, \varsigma) &= 1 \star 1 \star \cdots \star 1 = 1, \\
\lim_{j \to +\infty} \mathcal{E}(\vartheta_j, \vartheta_{j+i}, \varsigma) &= 0 \diamond 0 \diamond \cdots \diamond 0 = 0,
\end{aligned}
$$

and

$$
\lim_{j \to +\infty} \mathcal{G}(\vartheta_j, \vartheta_{j+i}, \varsigma) = 0 \diamond 0 \diamond \cdots \diamond 0 = 0.
$$

Therefore, $\{\vartheta_j\}$ is a Cauchy sequence. Since $(\Im, \mathcal{W}, \mathcal{E}, \mathcal{G}, \star, \diamond)$ is a complete neutrosophic rectangular triple-controlled metric space, we have

$$
\lim_{j \to +\infty} \vartheta_j = \vartheta.
$$

Now, due to the fact that ϑ is a fixed point of $\mathfrak{p}$, utilizing (v), (x), and (xv), we obtain

$$\begin{aligned}
\mathcal{W}(\vartheta, \mathfrak{p}\vartheta, \varsigma) &\geq \mathcal{W}\left(\vartheta, \vartheta_{\mathrm{j}}, \frac{\varsigma}{3(\beta(\vartheta, \vartheta_{\mathrm{j}}))}\right) \star \mathcal{W}\left(\vartheta_{\mathrm{j}}, \vartheta_{\mathrm{j}+1}, \frac{\varsigma}{3(\Gamma(\vartheta_{\mathrm{j}}, \vartheta_{\mathrm{j}+1}))}\right) \\
&\quad \star \mathcal{W}\left(\vartheta_{\mathrm{j}+1}, \mathfrak{p}\vartheta, \frac{\varsigma}{3(\eta(\vartheta_{\mathrm{j}+1}, \mathfrak{p}\vartheta))}\right) \\
&= \mathcal{W}\left(\vartheta, \vartheta_{\mathrm{j}}, \frac{\varsigma}{3(\beta(\vartheta, \vartheta_{\mathrm{j}}))}\right) \star \mathcal{W}\left(\mathfrak{p}\vartheta_{\mathrm{j}-1}, \mathfrak{p}\vartheta_{\mathrm{j}}, \frac{\varsigma}{3(\Gamma(\vartheta_{\mathrm{j}}, \vartheta_{\mathrm{j}+1}))}\right) \\
&\quad \star \mathcal{W}\left(\mathfrak{p}\vartheta_{\mathrm{j}}, \mathfrak{p}\vartheta, \frac{\varsigma}{3(\eta(\vartheta_{\mathrm{j}+1}, \mathfrak{p}\vartheta))}\right) \\
&\geq \mathcal{W}\left(\vartheta, \vartheta_{\mathrm{j}}, \frac{\varsigma}{3(\beta(\vartheta, \vartheta_{\mathrm{j}}))}\right) \star \mathcal{W}\left(\vartheta_{\mathrm{j}-1}, \vartheta_{\mathrm{j}}, \frac{\varsigma}{3(\Gamma(\vartheta_{\mathrm{j}}, \vartheta_{\mathrm{j}+1}))}\right) \\
&\quad \star \mathcal{W}\left(\vartheta_{\mathrm{j}}, \vartheta, \frac{\varsigma}{3(\eta(\vartheta_{\mathrm{j}+1}, \mathfrak{p}\vartheta))}\right) \\
&\to 1 \star 1 \star 1 = 1 \quad \text{as} \quad \mathrm{j} \to +\infty,
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}(\vartheta, \mathfrak{p}\vartheta, \varsigma) &\leq \mathcal{E}\left(\vartheta, \vartheta_{\mathrm{j}}, \frac{\varsigma}{3(\beta(\vartheta, \vartheta_{\mathrm{j}}))}\right) \diamond \mathcal{E}\left(\vartheta_{\mathrm{j}}, \vartheta_{\mathrm{j}+1}, \frac{\varsigma}{3(\Gamma(\vartheta_{\mathrm{j}}, \vartheta_{\mathrm{j}+1}))}\right) \diamond \mathcal{E}\left(\vartheta_{\mathrm{j}+1}, \mathfrak{p}\vartheta, \frac{\varsigma}{3(\eta(\vartheta_{\mathrm{j}+1}, \mathfrak{p}\vartheta))}\right) \\
&= \mathcal{E}\left(\vartheta, \vartheta_{\mathrm{j}}, \frac{\varsigma}{3(\beta(\vartheta, \vartheta_{\mathrm{j}}))}\right) \diamond \mathcal{E}\left(\mathfrak{p}\vartheta_{\mathrm{j}-1}, \mathfrak{p}\vartheta_{\mathrm{j}}, \frac{\varsigma}{3(\Gamma(\vartheta_{\mathrm{j}}, \vartheta_{\mathrm{j}+1}))}\right) \diamond \mathcal{E}\left(\mathfrak{p}\vartheta_{\mathrm{j}}, \mathfrak{p}\vartheta, \frac{\varsigma}{3(\eta(\vartheta_{\mathrm{j}+1}, \mathfrak{p}\vartheta))}\right) \\
&\leq \mathcal{E}\left(\vartheta, \vartheta_{\mathrm{j}}, \frac{\varsigma}{3(\beta(\vartheta, \vartheta_{\mathrm{j}}))}\right) \diamond \mathcal{E}\left(\vartheta_{\mathrm{j}-1}, \vartheta_{\mathrm{j}}, \frac{\varsigma}{3(\Gamma(\vartheta_{\mathrm{j}}, \vartheta_{\mathrm{j}+1}))}\right) \diamond \mathcal{E}\left(\vartheta_{\mathrm{j}}, \vartheta, \frac{\varsigma}{3(\eta(\vartheta_{\mathrm{j}+1}, \mathfrak{p}\vartheta))}\right) \\
&\to 0 \diamond 0 \diamond 0 = 0 \quad \text{as} \quad \mathrm{j} \to +\infty,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{G}(\vartheta, \mathfrak{p}\vartheta, \varsigma) &\leq \mathcal{G}\left(\vartheta, \vartheta_{\mathrm{j}}, \frac{\varsigma}{3(\beta(\vartheta, \vartheta_{\mathrm{j}}))}\right) \diamond \mathcal{G}\left(\vartheta_{\mathrm{j}}, \vartheta_{\mathrm{j}+1}, \frac{\varsigma}{3(\Gamma(\vartheta_{\mathrm{j}}, \vartheta_{\mathrm{j}+1}))}\right) \diamond \mathcal{G}\left(\vartheta_{\mathrm{j}+1}, \mathfrak{p}\vartheta, \frac{\varsigma}{3(\eta(\vartheta_{\mathrm{j}+1}, \mathfrak{p}\vartheta))}\right) \\
&= \mathcal{G}\left(\vartheta, \vartheta_{\mathrm{j}}, \frac{\varsigma}{3(\beta(\vartheta, \vartheta_{\mathrm{j}}))}\right) \diamond \mathcal{G}\left(\mathfrak{p}\vartheta_{\mathrm{j}-1}, \mathfrak{p}\vartheta_{\mathrm{j}}, \frac{\varsigma}{3(\Gamma(\vartheta_{\mathrm{j}}, \vartheta_{\mathrm{j}+1}))}\right) \diamond \mathcal{G}\left(\mathfrak{p}\vartheta_{\mathrm{j}}, \mathfrak{p}\vartheta, \frac{\varsigma}{3(\eta(\vartheta_{\mathrm{j}+1}, \mathfrak{p}\vartheta))}\right) \\
&\leq \mathcal{G}\left(\vartheta, \vartheta_{\mathrm{j}}, \frac{\varsigma}{3(\beta(\vartheta, \vartheta_{\mathrm{j}}))}\right) \diamond \mathcal{G}\left(\vartheta_{\mathrm{j}-1}, \vartheta_{\mathrm{j}}, \frac{\varsigma}{3(\Gamma(\vartheta_{\mathrm{j}}, \vartheta_{\mathrm{j}+1}))}\right) \diamond \mathcal{G}\left(\vartheta_{\mathrm{j}}, \vartheta, \frac{\varsigma}{3(\eta(\vartheta_{\mathrm{j}+1}, \mathfrak{p}\vartheta))}\right) \\
&\to 0 \diamond 0 \diamond 0 = 0 \quad \text{as} \quad \mathrm{j} \to +\infty.
\end{aligned}$$

Hence, $\mathfrak{p}\vartheta = \vartheta$.

Now, we examine the uniqueness. Let $\mathfrak{p}\kappa = \kappa$ for some $\kappa \in \Im$, then

$$\begin{aligned}
1 \geq \mathcal{W}(\kappa, \vartheta, \varsigma) &= \mathcal{W}(\mathfrak{p}\kappa, \mathfrak{p}\vartheta, \varsigma) \geq \mathcal{W}\left(\kappa, \vartheta, \frac{\varsigma}{\rho}\right) = \mathcal{W}\left(\mathfrak{p}\kappa, \mathfrak{p}\vartheta, \frac{\varsigma}{\rho}\right) \\
&\geq \mathcal{W}\left(\kappa, \vartheta, \frac{\varsigma}{\rho^2}\right) \geq \cdots \geq \mathcal{W}\left(\kappa, \vartheta, \frac{\varsigma}{\rho^{\mathrm{j}}}\right) \to 1 \quad \text{as} \quad \mathrm{j} \to +\infty, \\
0 \leq \mathcal{E}(\kappa, \vartheta, \varsigma) &= \mathcal{E}(\mathfrak{p}\kappa, \mathfrak{p}\vartheta, \varsigma) \leq \mathcal{E}\left(\kappa, \vartheta, \frac{\varsigma}{\rho}\right) = \mathcal{E}\left(\mathfrak{p}\kappa, \mathfrak{p}\vartheta, \frac{\varsigma}{\rho}\right) \\
&\leq \mathcal{E}\left(\kappa, \vartheta, \frac{\varsigma}{\rho^2}\right) \leq \cdots \leq \mathcal{E}\left(\kappa, \vartheta, \frac{\varsigma}{\rho^{\mathrm{j}}}\right) \to 0 \quad \text{as} \quad \mathrm{j} \to +\infty,
\end{aligned}$$

and

$$\begin{aligned}
0 \leq \mathcal{G}(\kappa, \vartheta, \varsigma) &= \mathcal{G}(\mathfrak{p}\kappa, \mathfrak{p}\vartheta, \varsigma) \leq \mathcal{G}\left(\kappa, \vartheta, \frac{\varsigma}{\rho}\right) = \mathcal{G}\left(\mathfrak{p}\kappa, \mathfrak{p}\vartheta, \frac{\varsigma}{\rho}\right) \\
&\leq \mathcal{G}\left(\kappa, \vartheta, \frac{\varsigma}{\rho^2}\right) \leq \cdots \leq \mathcal{G}\left(\kappa, \vartheta, \frac{\varsigma}{\rho^{\mathrm{j}}}\right) \to 0 \quad \text{as} \quad \mathrm{j} \to +\infty,
\end{aligned}$$

by using (iii), (viii) and (xiii), $\vartheta = \kappa$. □

Definition 10. *Let* $(\Im, \mathcal{W}, \mathcal{E}, \mathcal{G}, \star, \diamond)$ *be a neutrosophic rectangular triple-controlled metric space. A map* $\mathfrak{p}\colon \Im \to \Im$ *is an NRT(neutrosophic rectangular triple)-controlled contraction if there exists* $0 < \rho < 1$*, such that*

$$\frac{1}{\mathcal{W}(\mathfrak{p}\vartheta, \mathfrak{p}\mathcal{Q}, \varsigma)} - 1 \le \rho\left[\frac{1}{\mathcal{W}(\vartheta, \mathcal{Q}, \varsigma)} - 1\right] \tag{4}$$

$$\mathcal{E}(\mathfrak{p}\vartheta, \mathfrak{p}\mathcal{Q}, \varsigma) \le \rho\mathcal{E}(\vartheta, \mathcal{Q}, \varsigma), \tag{5}$$

and

$$\mathcal{G}(\mathfrak{p}\vartheta, \mathfrak{p}\mathcal{Q}, \varsigma) \le \rho\mathcal{G}(\vartheta, \mathcal{Q}, \varsigma), \tag{6}$$

for all $\vartheta, \mathcal{Q} \in \Im$ *and* $\varsigma > 0$.

We now present the fixed point result for an NRT (neutrosophic rectangular triple)-controlled contraction.

Theorem 2. *Let* $(\Im, \mathcal{W}, \mathcal{E}, \mathcal{G}, \star, \diamond)$ *be a complete neutrosophic rectangular triple-controlled metric space with* $\beta, \Gamma, \eta\colon \Im \times \Im \to [1, +\infty)$*. Let* $\mathfrak{p}\colon \Im \to \Im$ *be an ND-controlled contraction. Further, suppose that for an arbitrary* $\vartheta_0 \in \Im$*, and* $\mathrm{j}, \mathfrak{q} \in \mathbb{N}$*, where* $\vartheta_{\mathrm{j}} = \mathfrak{p}^{\mathrm{j}}\vartheta_0 = \mathfrak{p}\vartheta_{\mathrm{j}-1}$*. Then,* $\mathfrak{p}$ *has a unique fixed point.*

Proof. Let $\vartheta_0 \in \Im$. Define ϑ_{j} by $\vartheta_{\mathrm{j}} = \mathfrak{p}^{\mathrm{j}}\vartheta_0 = \mathfrak{p}\vartheta_{\mathrm{j}-1}, \mathrm{j} \in \mathbb{N}$. From (4)–(6), for all $\varsigma > 0, \mathrm{j} > \mathfrak{q}$, we deduce

$$\begin{aligned}
\frac{1}{\mathcal{W}(\vartheta_{\mathrm{j}}, \vartheta_{\mathrm{j}+1}, \varsigma)} - 1 &= \frac{1}{\mathcal{W}(\mathfrak{p}\vartheta_{\mathrm{j}-1}, \mathfrak{p}\vartheta_{\mathrm{j}}, \varsigma)} - 1 \le \rho\left[\frac{1}{\mathcal{W}(\vartheta_{\mathrm{j}-1}, \vartheta_{\mathrm{j}}, \varsigma)}\right] = \frac{\rho}{\mathcal{W}(\vartheta_{\mathrm{j}-1}, \vartheta_{\mathrm{j}}, \varsigma)} - \rho \\
\Rightarrow \frac{1}{\mathcal{W}(\vartheta_{\mathrm{j}}, \vartheta_{\mathrm{j}+1}, \varsigma)} &\le \frac{\rho}{\mathcal{W}(\vartheta_{\mathrm{j}-1}, \vartheta_{\mathrm{j}}, \varsigma)} + (1-\rho) \le \frac{\rho^2}{\mathcal{W}(\vartheta_{\mathrm{j}-2}, \vartheta_{\mathrm{j}-1}, \varsigma)} + \rho(1-\rho) + (1-\rho).
\end{aligned}$$

Proceeding in this way, we have

$$\begin{aligned}
\frac{1}{\mathcal{W}(\vartheta_{\mathrm{j}}, \vartheta_{\mathrm{j}+1}, \varsigma)} &\le \frac{\rho^{\mathrm{j}}}{\mathcal{W}(\vartheta_0, \vartheta_1, \varsigma)} + \rho^{\mathrm{j}-1}(1-\rho) + \rho^{\mathrm{j}-2}(1-\rho) + \cdots + \rho(1-\rho) + (1-\rho) \\
&\le \frac{\rho^{\mathrm{j}}}{\mathcal{W}(\vartheta_0, \vartheta_1, \varsigma)} + (\rho^{\mathrm{j}-1} + \rho^{\mathrm{j}-2} + \cdots + 1)(1-\rho) \le \frac{\rho^{\mathrm{j}}}{\mathcal{W}(\vartheta_0, \vartheta_1, \varsigma)} + (1-\rho^{\mathrm{j}}).
\end{aligned}$$

We obtain

$$\frac{1}{\frac{\rho^{\mathrm{j}}}{\mathcal{W}(\vartheta_0, \vartheta_1, \varsigma)} + (1-\rho^{\mathrm{j}})} \le \mathcal{W}(\vartheta_{\mathrm{j}}, \vartheta_{\mathrm{j}+1}, \varsigma) \tag{7}$$

$$\begin{aligned}
\mathcal{E}(\vartheta_{\mathrm{j}}, \vartheta_{\mathrm{j}+1}, \varsigma) &= \mathcal{E}(\mathfrak{p}\vartheta_{\mathrm{j}-1}, \mathfrak{p}\vartheta_{\mathrm{j}}, \varsigma) \le \rho\mathcal{E}(\vartheta_{\mathrm{j}-1}, \vartheta_{\mathrm{j}}, \varsigma) = \mathcal{E}(\mathfrak{p}\vartheta_{\mathrm{j}-2}, \mathfrak{p}\vartheta_{\mathrm{j}-1}, \varsigma) \\
&\le \rho^2\mathcal{E}(\vartheta_{\mathrm{j}-2}, \vartheta_{\mathrm{j}-1}, \varsigma) \le \cdots \le \rho^{\mathrm{j}}\mathcal{E}(\vartheta_0, \vartheta_1, \varsigma),
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
\mathcal{G}(\vartheta_{\mathrm{j}}, \vartheta_{\mathrm{j}+1}, \varsigma) &= \mathcal{G}(\mathfrak{p}\vartheta_{\mathrm{j}-1}, \mathfrak{p}\vartheta_{\mathrm{j}}, \varsigma) \le \rho\mathcal{G}(\vartheta_{\mathrm{j}-1}, \vartheta_{\mathrm{j}}, \varsigma) = \mathcal{G}(\mathfrak{p}\vartheta_{\mathrm{j}-2}, \mathfrak{p}\vartheta_{\mathrm{j}-1}, \varsigma) \\
&\le \rho^2\mathcal{G}(\vartheta_{\mathrm{j}-2}, \vartheta_{\mathrm{j}-1}, \varsigma) \le \cdots \le \rho^{\mathrm{j}}\mathcal{G}(\vartheta_0, \vartheta_1, \varsigma).
\end{aligned} \tag{9}$$

Using (v), (x) and (xv), we have the following cases:
Case 1. When $\mathrm{i} = 2\mathfrak{m} + 1$, i.e., i is odd, then

$$\begin{aligned}
\mathcal{W}(\vartheta_{j},\vartheta_{j+2m+1},\varsigma) &\geq \mathcal{W}\left(\vartheta_{j},\vartheta_{j+1},\frac{\varsigma}{3(\beta(\vartheta_{j},\vartheta_{j+1}))}\right)\star\mathcal{W}\left(\vartheta_{j+1},\vartheta_{j+2},\frac{\varsigma}{3(\Gamma(\vartheta_{j+1},\vartheta_{j+2}))}\right)\\
&\star\mathcal{W}\left(\vartheta_{j+2},\vartheta_{j+2m+1},\frac{\varsigma}{3(\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\geq \mathcal{W}\left(\vartheta_{j},\vartheta_{j+1},\frac{\varsigma}{3(\beta(\vartheta_{j},\vartheta_{j+1}))}\right)\star\mathcal{W}\left(\vartheta_{j+1},\vartheta_{j+2},\frac{\varsigma}{3(\Gamma(\vartheta_{j+1},\vartheta_{j+2}))}\right)\\
&\star\mathcal{W}\left(\vartheta_{j+2},\vartheta_{j+3},\frac{\varsigma}{3^2(\beta(\vartheta_{j+2},\vartheta_{j+3})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\star\mathcal{W}\left(\vartheta_{j+3},\vartheta_{j+4},\frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3},\vartheta_{j+4})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\star\mathcal{W}\left(\vartheta_{j+4},\vartheta_{j+2m+1},\frac{\varsigma}{3^2(\eta(\vartheta_{j+4},\vartheta_{j+2m+1})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\geq \mathcal{W}\left(\vartheta_{j},\vartheta_{j+1},\frac{\varsigma}{3(\beta(\vartheta_{j},\vartheta_{j+1}))}\right)\star\mathcal{W}\left(\vartheta_{j+1},\vartheta_{j+2},\frac{\varsigma}{3(\Gamma(\vartheta_{j+1},\vartheta_{j+2}))}\right)\\
&\star\mathcal{W}\left(\vartheta_{j+2},\vartheta_{j+3},\frac{\varsigma}{3^2(\beta(\vartheta_{j+2},\vartheta_{j+3})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\star\mathcal{W}\left(\vartheta_{j+3},\vartheta_{j+4},\frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3},\vartheta_{j+4})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)
\end{aligned}$$

$$\begin{aligned}
&\star\mathcal{W}\left(\vartheta_{j+4},\vartheta_{j+5},\frac{\varsigma}{3^3(\beta(\vartheta_{j+4},\vartheta_{j+5})\eta(\vartheta_{j+4},\vartheta_{j+2m+1})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\star\mathcal{W}\left(\vartheta_{j+5},\vartheta_{j+6},\frac{\varsigma}{3^3(\Gamma(\vartheta_{j+5},\vartheta_{j+6})\eta(\vartheta_{j+4},\vartheta_{j+2m+1})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\star\mathcal{W}\left(\vartheta_{j+6},\vartheta_{j+2m+1},\frac{\varsigma}{3^3(\eta(\vartheta_{j+6},\vartheta_{j+2m+1})\eta(\vartheta_{j+4},\vartheta_{j+2m+1})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)
\end{aligned}$$

$$\begin{aligned}
&\mathcal{W}(\vartheta_{j},\vartheta_{j+2m+1},\varsigma)\\
&\geq \mathcal{W}\left(\vartheta_{j},\vartheta_{j+1},\frac{\varsigma}{3(\beta(\vartheta_{j},\vartheta_{j+1}))}\right)\star\mathcal{W}\left(\vartheta_{j+1},\vartheta_{j+2},\frac{\varsigma}{3(\Gamma(\vartheta_{j+1},\vartheta_{j+2}))}\right)\\
&\star\mathcal{W}\left(\vartheta_{j+2},\vartheta_{j+3},\frac{\varsigma}{3^2(\beta(\vartheta_{j+2},\vartheta_{j+3})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\star\mathcal{W}\left(\vartheta_{j+3},\vartheta_{j+4},\frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3},\vartheta_{j+4})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\star\mathcal{W}\left(\vartheta_{j+4},\vartheta_{j+5},\frac{\varsigma}{3^3(\beta(\vartheta_{j+4},\vartheta_{j+5})\eta(\vartheta_{j+4},\vartheta_{j+2m+1})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\star\mathcal{W}\left(\vartheta_{j+5},\vartheta_{j+6},\frac{\varsigma}{3^3(\Gamma(\vartheta_{j+5},\vartheta_{j+6})\eta(\vartheta_{j+4},\vartheta_{j+2m+1})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\star\mathcal{W}\left(\vartheta_{j+6},\vartheta_{j+2m+1},\frac{\varsigma}{3^3(\eta(\vartheta_{j+6},\vartheta_{j+2m+1})\eta(\vartheta_{j+4},\vartheta_{j+2m+1})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\star\cdots\star\\
&\mathcal{W}\left(\vartheta_{j+2m-2},\vartheta_{j+2m-1},\frac{\varsigma}{3^m(\beta(\vartheta_{j+2m-2},\vartheta_{j+2m-1})\eta(\vartheta_{j+2m-2},\vartheta_{j+2m+1})\cdots\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\star\mathcal{W}\left(\vartheta_{j+2m-1},\vartheta_{j+2m},\frac{\varsigma}{3^m(\Gamma(\vartheta_{j+2m-1},\vartheta_{j+2m})\eta(\vartheta_{j+2m-2},\vartheta_{j+2m+1})\cdots\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\star\mathcal{W}\left(\vartheta_{j+2m},\vartheta_{j+2m+1},\frac{\varsigma}{3^m(\eta(\vartheta_{j+2m},\vartheta_{j+2m+1})\eta(\vartheta_{j+2m-2},\vartheta_{j+2m+1})\cdots\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right),
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}(\vartheta_j, \vartheta_{j+2m+1}, \varsigma) \leq{}& \mathcal{E}\left(\vartheta_j, \vartheta_{j+1}, \frac{\varsigma}{3(\beta(\vartheta_j, \vartheta_{j+1}))}\right) \diamond \mathcal{E}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+2}, \vartheta_{j+2m+1}, \frac{\varsigma}{3(\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
\leq{}& \mathcal{E}\left(\vartheta_j, \vartheta_{j+1}, \frac{\varsigma}{3(\beta(\vartheta_j, \vartheta_{j+1}))}\right) \diamond \mathcal{E}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+2}, \vartheta_{j+3}, \frac{\varsigma}{3^2(\beta(\vartheta_{j+2}, \vartheta_{j+3})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+3}, \vartheta_{j+4}, \frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3}, \vartheta_{j+4})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+4}, \vartheta_{j+2m+1}, \frac{\varsigma}{3^2(\eta(\vartheta_{j+4}, \vartheta_{j+2m+1})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
\leq{}& \mathcal{E}\left(\vartheta_j, \vartheta_{j+1}, \frac{\varsigma}{3(\beta(\vartheta_j, \vartheta_{j+1}))}\right) \diamond \mathcal{E}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+2}, \vartheta_{j+3}, \frac{\varsigma}{3^2(\beta(\vartheta_{j+2}, \vartheta_{j+3})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+3}, \vartheta_{j+4}, \frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3}, \vartheta_{j+4})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right)
\end{aligned}$$

$$\begin{aligned}
&\diamond \mathcal{E}\left(\vartheta_{j+4}, \vartheta_{j+5}, \frac{\varsigma}{3^3(\beta(\vartheta_{j+4}, \vartheta_{j+5})\eta(\vartheta_{j+4}, \vartheta_{j+2m+1})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+5}, \vartheta_{j+6}, \frac{\varsigma}{3^3(\Gamma(\vartheta_{j+5}, \vartheta_{j+6})\eta(\vartheta_{j+4}, \vartheta_{j+2m+1})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+6}, \vartheta_{j+2m+1}, \frac{\varsigma}{3^3(\eta(\vartheta_{j+6}, \vartheta_{j+2m+1})\eta(\vartheta_{j+4}, \vartheta_{j+2m+1})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right)
\end{aligned}$$

$$\begin{aligned}
&\mathcal{E}(\vartheta_j, \vartheta_{j+2m+1}, \varsigma) \\
&\leq \mathcal{E}\left(\vartheta_j, \vartheta_{j+1}, \frac{\varsigma}{3(\beta(\vartheta_j, \vartheta_{j+1}))}\right) \diamond \mathcal{E}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+2}, \vartheta_{j+3}, \frac{\varsigma}{3^2(\beta(\vartheta_{j+2}, \vartheta_{j+3})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+3}, \vartheta_{j+4}, \frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3}, \vartheta_{j+4})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+4}, \vartheta_{j+5}, \frac{\varsigma}{3^3(\beta(\vartheta_{j+4}, \vartheta_{j+5})\eta(\vartheta_{j+4}, \vartheta_{j+2m+1})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+5}, \vartheta_{j+6}, \frac{\varsigma}{3^3(\Gamma(\vartheta_{j+5}, \vartheta_{j+6})\eta(\vartheta_{j+4}, \vartheta_{j+2m+1})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+6}, \vartheta_{j+2m+1}, \frac{\varsigma}{3^3(\eta(\vartheta_{j+6}, \vartheta_{j+2m+1})\eta(\vartheta_{j+4}, \vartheta_{j+2m+1})\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \diamond \cdots \diamond \\
&\mathcal{E}\left(\vartheta_{j+2m-2}, \vartheta_{j+2m-1}, \frac{\varsigma}{3^m(\beta(\vartheta_{j+2m-2}, \vartheta_{j+2m-1})\eta(\vartheta_{j+2m-2}, \vartheta_{j+2m+1})\cdots\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+2m-1}, \vartheta_{j+2m}, \frac{\varsigma}{3^m(\Gamma(\vartheta_{j+2m-1}, \vartheta_{j+2m})\eta(\vartheta_{j+2m-2}, \vartheta_{j+2m+1})\cdots\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+2m}, \vartheta_{j+2m+1}, \frac{\varsigma}{3^m(\eta(\vartheta_{j+2m}, \vartheta_{j+2m+1})\eta(\vartheta_{j+2m-2}, \vartheta_{j+2m+1})\cdots\eta(\vartheta_{j+2}, \vartheta_{j+2m+1}))}\right),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{G}(\vartheta_{j},\vartheta_{j+2m+1},\varsigma) \leq{}& \mathcal{G}\left(\vartheta_{j},\vartheta_{j+1},\frac{\varsigma}{3(\beta(\vartheta_{j},\vartheta_{j+1}))}\right)\diamond\mathcal{G}\left(\vartheta_{j+1},\vartheta_{j+2},\frac{\varsigma}{3(\Gamma(\vartheta_{j+1},\vartheta_{j+2}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+2},\vartheta_{j+2m+1},\frac{\varsigma}{3(\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
\leq{}& \mathcal{G}\left(\vartheta_{j},\vartheta_{j+1},\frac{\varsigma}{3(\beta(\vartheta_{j},\vartheta_{j+1}))}\right)\diamond\mathcal{G}\left(\vartheta_{j+1},\vartheta_{j+2},\frac{\varsigma}{3(\Gamma(\vartheta_{j+1},\vartheta_{j+2}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+2},\vartheta_{j+3},\frac{\varsigma}{3^2(\beta(\vartheta_{j+2},\vartheta_{j+3})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+3},\vartheta_{j+4},\frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3},\vartheta_{j+4})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+4},\vartheta_{j+2m+1},\frac{\varsigma}{3^2(\eta(\vartheta_{j+4},\vartheta_{j+2m+1})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
\leq{}& \mathcal{G}\left(\vartheta_{j},\vartheta_{j+1},\frac{\varsigma}{3(\beta(\vartheta_{j},\vartheta_{j+1}))}\right)\diamond\mathcal{G}\left(\vartheta_{j+1},\vartheta_{j+2},\frac{\varsigma}{3(\Gamma(\vartheta_{j+1},\vartheta_{j+2}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+2},\vartheta_{j+3},\frac{\varsigma}{3^2(\beta(\vartheta_{j+2},\vartheta_{j+3})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+3},\vartheta_{j+4},\frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3},\vartheta_{j+4})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)
\end{aligned}$$

$$\begin{aligned}
&\diamond\mathcal{G}\left(\vartheta_{j+4},\vartheta_{j+5},\frac{\varsigma}{3^3(\beta(\vartheta_{j+4},\vartheta_{j+5})\eta(\vartheta_{j+4},\vartheta_{j+2m+1})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+5},\vartheta_{j+6},\frac{\varsigma}{3^3(\Gamma(\vartheta_{j+5},\vartheta_{j+6})\eta(\vartheta_{j+4},\vartheta_{j+2m+1})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+6},\vartheta_{j+2m+1},\frac{\varsigma}{3^3(\eta(\vartheta_{j+6},\vartheta_{j+2m+1})\eta(\vartheta_{j+4},\vartheta_{j+2m+1})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}(\vartheta_{j},\vartheta_{j+2m+1},\varsigma) \leq{}& \mathcal{G}\left(\vartheta_{j},\vartheta_{j+1},\frac{\varsigma}{3(\beta(\vartheta_{j},\vartheta_{j+1}))}\right)\diamond\mathcal{G}\left(\vartheta_{j+1},\vartheta_{j+2},\frac{\varsigma}{3(\Gamma(\vartheta_{j+1},\vartheta_{j+2}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+2},\vartheta_{j+3},\frac{\varsigma}{3^2(\beta(\vartheta_{j+2},\vartheta_{j+3})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+3},\vartheta_{j+4},\frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3},\vartheta_{j+4})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+4},\vartheta_{j+5},\frac{\varsigma}{3^3(\beta(\vartheta_{j+4},\vartheta_{j+5})\eta(\vartheta_{j+4},\vartheta_{j+2m+1})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+5},\vartheta_{j+6},\frac{\varsigma}{3^3(\Gamma(\vartheta_{j+5},\vartheta_{j+6})\eta(\vartheta_{j+4},\vartheta_{j+2m+1})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+6},\vartheta_{j+2m+1},\frac{\varsigma}{3^3(\eta(\vartheta_{j+6},\vartheta_{j+2m+1})\eta(\vartheta_{j+4},\vartheta_{j+2m+1})\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\diamond\cdots\diamond\\
&\mathcal{G}\left(\vartheta_{j+2m-2},\vartheta_{j+2m-1},\frac{\varsigma}{3^m(\beta(\vartheta_{j+2m-2},\vartheta_{j+2m-1})\eta(\vartheta_{j+2m-2},\vartheta_{j+2m+1})\cdots\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+2m-1},\vartheta_{j+2m},\frac{\varsigma}{3^m(\Gamma(\vartheta_{j+2m-1},\vartheta_{j+2m})\eta(\vartheta_{j+2m-2},\vartheta_{j+2m+1})\cdots\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{j+2m},\vartheta_{j+2m+1},\frac{\varsigma}{3^m(\eta(\vartheta_{j+2m},\vartheta_{j+2m+1})\eta(\vartheta_{j+2m-2},\vartheta_{j+2m+1})\cdots\eta(\vartheta_{j+2},\vartheta_{j+2m+1}))}\right).
\end{aligned}$$

Using (3) in the above inequalities, we deduce

$$\mathcal{W}(\vartheta_{\mathrm{j}},\vartheta_{\mathrm{j}+2\mathrm{m}+1},\varsigma) \geq \frac{1}{\frac{\rho^{\mathrm{j}}}{\mathcal{W}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3(\beta(\vartheta_{\mathrm{j}},\vartheta_{\mathrm{j}+1}))}\right)}+(1-\rho^{\mathrm{j}})} \star \frac{1}{\frac{\rho^{\mathrm{j}+1}}{\mathcal{W}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3(\Gamma(\vartheta_{\mathrm{j}+1},\vartheta_{\mathrm{j}+2}))}\right)}+(1-\rho^{\mathrm{j}+1})}$$

$$\star \frac{1}{\frac{\rho^{\mathrm{j}+2}}{\mathcal{W}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^2(\beta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+3})\eta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right)}+(1-\rho^{\mathrm{j}+2})}$$

$$\star \frac{1}{\frac{\rho^{\mathrm{j}+3}}{\mathcal{W}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^2(\Gamma(\vartheta_{\mathrm{j}+3},\vartheta_{\mathrm{j}+4})\eta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right)}+(1-\rho^{\mathrm{j}+3})}$$

$$\star \frac{1}{\frac{\rho^{\mathrm{j}+4}}{\mathcal{W}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^3(\beta(\vartheta_{\mathrm{j}+4},\vartheta_{\mathrm{j}+5})\eta(\vartheta_{\mathrm{j}+4},\vartheta_{\mathrm{j}+2\mathrm{m}+1})\eta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right)}+(1-\rho^{\mathrm{j}+4})}$$

$$\star \frac{1}{\frac{\rho^{\mathrm{j}+5}}{\mathcal{W}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^3(\Gamma(\vartheta_{\mathrm{j}+5},\vartheta_{\mathrm{j}+6})\eta(\vartheta_{\mathrm{j}+4},\vartheta_{\mathrm{j}+2\mathrm{m}+1})\eta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right)}+(1-\rho^{\mathrm{j}+5})}$$

$$\star \frac{1}{\frac{\rho^{\mathrm{j}+6}}{\mathcal{W}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^3(\eta(\vartheta_{\mathrm{j}+6},\vartheta_{\mathrm{j}+2\mathrm{m}+1})\eta(\vartheta_{\mathrm{j}+4},\vartheta_{\mathrm{j}+2\mathrm{m}+1})\eta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right)}+(1-\rho^{\mathrm{j}+6})} \star \cdots \star$$

$$\star \frac{1}{\frac{\rho^{\mathrm{j}+2\mathrm{m}-2}}{\mathcal{W}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^{\mathrm{m}}(\beta(\vartheta_{\mathrm{j}+2\mathrm{m}-2},\vartheta_{\mathrm{j}+2\mathrm{m}-1})\eta(\vartheta_{\mathrm{j}+2\mathrm{m}-2},\vartheta_{\mathrm{j}+2\mathrm{m}+1})\cdots\eta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right)}+(1-\rho^{\mathrm{j}+2\mathrm{m}-2})}$$

$$\star \frac{1}{\frac{\rho^{\mathrm{j}+2\mathrm{m}-1}}{\mathcal{W}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^{\mathrm{m}}(\Gamma(\vartheta_{\mathrm{j}+2\mathrm{m}-1},\vartheta_{\mathrm{j}+2\mathrm{m}})\eta(\vartheta_{\mathrm{j}+2\mathrm{m}-2},\vartheta_{\mathrm{j}+2\mathrm{m}+1})\cdots\eta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right)}+(1-\rho^{\mathrm{j}+2\mathrm{m}-1})}$$

$$\star \frac{1}{\frac{\rho^{\mathrm{j}+2\mathrm{m}}}{\mathcal{W}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^{\mathrm{m}}(\eta(\vartheta_{\mathrm{j}+2\mathrm{m}},\vartheta_{\mathrm{j}+2\mathrm{m}+1})\eta(\vartheta_{\mathrm{j}+2\mathrm{m}-2},\vartheta_{\mathrm{j}+2\mathrm{m}+1})\cdots\eta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right)}+(1-\rho^{\mathrm{j}+2\mathrm{m}})},$$

$$
\begin{aligned}
&\mathcal{E}(\vartheta_{\mathrm{j}}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}, \varsigma) \\
&\quad \leq \rho^{\mathrm{j}} \mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3(\beta(\vartheta_{\mathrm{j}}, \vartheta_{\mathrm{j}+1}))}\right) \diamond \rho^{\mathrm{j}+1} \mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3(\Gamma(\vartheta_{\mathrm{j}+1}, \vartheta_{\mathrm{j}+2}))}\right) \\
&\quad \diamond \rho^{\mathrm{j}+2} \mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^2(\beta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+3})\eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \\
&\quad \diamond \rho^{\mathrm{j}+3} \mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^2(\Gamma(\vartheta_{\mathrm{j}+3}, \vartheta_{\mathrm{j}+4})\eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \\
&\quad \diamond \rho^{\mathrm{j}+4} \mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^3(\beta(\vartheta_{\mathrm{j}+4}, \vartheta_{\mathrm{j}+5})\eta(\vartheta_{\mathrm{j}+4}, \vartheta_{\mathrm{j}+2\mathrm{m}+1})\eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \\
&\quad \diamond \rho^{\mathrm{j}+5} \mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^3(\Gamma(\vartheta_{\mathrm{j}+5}, \vartheta_{\mathrm{j}+6})\eta(\vartheta_{\mathrm{j}+4}, \vartheta_{\mathrm{j}+2\mathrm{m}+1})\eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \\
&\quad \diamond \rho^{\mathrm{j}+6} \mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^3(\eta(\vartheta_{\mathrm{j}+6}, \vartheta_{\mathrm{j}+2\mathrm{m}+1})\eta(\vartheta_{\mathrm{j}+4}, \vartheta_{\mathrm{j}+2\mathrm{m}+1})\eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \diamond \cdots \diamond \\
&\quad \rho^{\mathrm{j}+2\mathrm{m}-2} \mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^{\mathrm{m}}(\beta(\vartheta_{\mathrm{j}+2\mathrm{m}-2}, \vartheta_{\mathrm{j}+2\mathrm{m}-1})\eta(\vartheta_{\mathrm{j}+2\mathrm{m}-2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1})\cdots\eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \\
&\quad \diamond \rho^{\mathrm{j}+2\mathrm{m}-1} \mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^{\mathrm{m}}(\Gamma(\vartheta_{\mathrm{j}+2\mathrm{m}-1}, \vartheta_{\mathrm{j}+2\mathrm{m}})\eta(\vartheta_{\mathrm{j}+2\mathrm{m}-2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1})\cdots\eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \\
&\quad \diamond \rho^{\mathrm{j}+2\mathrm{m}} \mathcal{E}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^{\mathrm{m}}(\eta(\vartheta_{\mathrm{j}+2\mathrm{m}}, \vartheta_{\mathrm{j}+2\mathrm{m}+1})\eta(\vartheta_{\mathrm{j}+2\mathrm{m}-2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1})\cdots\eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{G}(\vartheta_{\mathrm{j}}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}, \varsigma) &\leq \rho^{\mathrm{j}} \mathcal{G}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3(\beta(\vartheta_{\mathrm{j}}, \vartheta_{\mathrm{j}+1}))}\right) \diamond \rho^{\mathrm{j}+1} \mathcal{G}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3(\Gamma(\vartheta_{\mathrm{j}+1}, \vartheta_{\mathrm{j}+2}))}\right) \\
&\diamond \rho^{\mathrm{j}+2} \mathcal{G}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^2(\beta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+3})\eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \\
&\diamond \rho^{\mathrm{j}+3} \mathcal{G}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^2(\Gamma(\vartheta_{\mathrm{j}+3}, \vartheta_{\mathrm{j}+4})\eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \\
&\diamond \rho^{\mathrm{j}+4} \mathcal{G}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^3(\beta(\vartheta_{\mathrm{j}+4}, \vartheta_{\mathrm{j}+5})\eta(\vartheta_{\mathrm{j}+4}, \vartheta_{\mathrm{j}+2\mathrm{m}+1})\eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \\
&\diamond \rho^{\mathrm{j}+5} \mathcal{G}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^3(\Gamma(\vartheta_{\mathrm{j}+5}, \vartheta_{\mathrm{j}+6})\eta(\vartheta_{\mathrm{j}+4}, \vartheta_{\mathrm{j}+2\mathrm{m}+1})\eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\diamond \rho^{\mathrm{j}+6} \mathcal{G}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^3(\eta(\vartheta_{\mathrm{j}+6}, \vartheta_{\mathrm{j}+2\mathrm{m}+1})\eta(\vartheta_{\mathrm{j}+4}, \vartheta_{\mathrm{j}+2\mathrm{m}+1})\eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \diamond \cdots \diamond \\
&\rho^{\mathrm{j}+2\mathrm{m}-2} \mathcal{G}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^{\mathrm{m}}(\beta(\vartheta_{\mathrm{j}+2\mathrm{m}-2}, \vartheta_{\mathrm{j}+2\mathrm{m}-1})\eta(\vartheta_{\mathrm{j}+2\mathrm{m}-2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1})\cdots\eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \\
&\diamond \rho^{\mathrm{j}+2\mathrm{m}-1} \mathcal{G}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^{\mathrm{m}}(\Gamma(\vartheta_{\mathrm{j}+2\mathrm{m}-1}, \vartheta_{\mathrm{j}+2\mathrm{m}})\eta(\vartheta_{\mathrm{j}+2\mathrm{m}-2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1})\cdots\eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right) \\
&\diamond \rho^{\mathrm{j}+2\mathrm{m}} \mathcal{G}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^{\mathrm{m}}(\eta(\vartheta_{\mathrm{j}+2\mathrm{m}}, \vartheta_{\mathrm{j}+2\mathrm{m}+1})\eta(\vartheta_{\mathrm{j}+2\mathrm{m}-2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1})\cdots\eta(\vartheta_{\mathrm{j}+2}, \vartheta_{\mathrm{j}+2\mathrm{m}+1}))}\right).
\end{aligned}
$$

Case 2. When $\mathfrak{i} = 2\mathfrak{m}$, i.e., $\mathfrak{i}$ is even, then

$$\begin{aligned}
\mathcal{W}(\vartheta_{j}, \vartheta_{j+2\mathfrak{m}}, \varsigma) \geq{}& \mathcal{W}\left(\vartheta_{j}, \vartheta_{j+1}, \frac{\varsigma}{3(\beta(\vartheta_{j}, \vartheta_{j+1}))}\right) \star \mathcal{W}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+2}, \vartheta_{j+2\mathfrak{m}}, \frac{\varsigma}{3(\eta(\vartheta_{j+2}, \vartheta_{j+2\mathfrak{m}}))}\right) \\
\geq{}& \mathcal{W}\left(\vartheta_{j}, \vartheta_{j+1}, \frac{\varsigma}{3(\beta(\vartheta_{j}, \vartheta_{j+1}))}\right) \star \mathcal{W}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+2}, \vartheta_{j+3}, \frac{\varsigma}{3^2(\beta(\vartheta_{j+2}, \vartheta_{j+3})\eta(\vartheta_{j+2}, \vartheta_{j+2\mathfrak{m}}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+3}, \vartheta_{j+4}, \frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3}, \vartheta_{j+4})\eta(\vartheta_{j+2}, \vartheta_{j+2\mathfrak{m}}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+4}, \vartheta_{j+2\mathfrak{m}}, \frac{\varsigma}{3^2(\eta(\vartheta_{j+4}, \vartheta_{j+2\mathfrak{m}})\eta(\vartheta_{j+2}, \vartheta_{j+2\mathfrak{m}}))}\right) \\
\geq{}& \mathcal{W}\left(\vartheta_{j}, \vartheta_{j+1}, \frac{\varsigma}{3(\beta(\vartheta_{j}, \vartheta_{j+1}))}\right) \star \mathcal{W}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+2}, \vartheta_{j+3}, \frac{\varsigma}{3^2(\beta(\vartheta_{j+2}, \vartheta_{j+3})\eta(\vartheta_{j+2}, \vartheta_{j+2\mathfrak{m}}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+3}, \vartheta_{j+4}, \frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3}, \vartheta_{j+4})\eta(\vartheta_{j+2}, \vartheta_{j+2\mathfrak{m}}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+4}, \vartheta_{j+5}, \frac{\varsigma}{3^3(\beta(\vartheta_{j+4}, \vartheta_{j+5})\eta(\vartheta_{j+4}, \vartheta_{j+2\mathfrak{m}})\eta(\vartheta_{j+2}, \vartheta_{j+2\mathfrak{m}}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+5}, \vartheta_{j+6}, \frac{\varsigma}{3^3(\Gamma(\vartheta_{j+5}, \vartheta_{j+6})\eta(\vartheta_{j+4}, \vartheta_{j+2\mathfrak{m}})\eta(\vartheta_{j+2}, \vartheta_{j+2\mathfrak{m}}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+6}, \vartheta_{j+2\mathfrak{m}}, \frac{\varsigma}{3^3(\eta(\vartheta_{j+6}, \vartheta_{j+2\mathfrak{m}})\eta(\vartheta_{j+4}, \vartheta_{j+2\mathfrak{m}})\eta(\vartheta_{j+2}, \vartheta_{j+2\mathfrak{m}}))}\right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{W}(\vartheta_{j}, \vartheta_{j+2\mathfrak{m}}, \varsigma) \geq{}& \mathcal{W}\left(\vartheta_{j}, \vartheta_{j+1}, \frac{\varsigma}{3(\beta(\vartheta_{j}, \vartheta_{j+1}))}\right) \star \mathcal{W}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+2}, \vartheta_{j+3}, \frac{\varsigma}{3^2(\beta(\vartheta_{j+2}, \vartheta_{j+3})\eta(\vartheta_{j+2}, \vartheta_{j+2\mathfrak{m}}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+3}, \vartheta_{j+4}, \frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3}, \vartheta_{j+4})\eta(\vartheta_{j+2}, \vartheta_{j+2\mathfrak{m}}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+4}, \vartheta_{j+5}, \frac{\varsigma}{3^3(\beta(\vartheta_{j+4}, \vartheta_{j+5})\eta(\vartheta_{j+4}, \vartheta_{j+2\mathfrak{m}})\eta(\vartheta_{j+2}, \vartheta_{j+2\mathfrak{m}}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+5}, \vartheta_{j+6}, \frac{\varsigma}{3^3(\Gamma(\vartheta_{j+5}, \vartheta_{j+6})\eta(\vartheta_{j+4}, \vartheta_{j+2\mathfrak{m}})\eta(\vartheta_{j+2}, \vartheta_{j+2\mathfrak{m}}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+6}, \vartheta_{j+2\mathfrak{m}}, \frac{\varsigma}{3^3(\eta(\vartheta_{j+6}, \vartheta_{j+2\mathfrak{m}})\eta(\vartheta_{j+4}, \vartheta_{j+2\mathfrak{m}})\eta(\vartheta_{j+2}, \vartheta_{j+2\mathfrak{m}}))}\right) \star \cdots \star
\end{aligned}$$

$$\begin{aligned}
&\mathcal{W}\left(\vartheta_{j+2\mathfrak{m}-4}, \vartheta_{j+2\mathfrak{m}-3}, \frac{\varsigma}{3^{\mathfrak{m}-1}(\beta(\vartheta_{j+2\mathfrak{m}-4}, \vartheta_{j+2\mathfrak{m}-3})\eta(\vartheta_{j+2\mathfrak{m}-4}, \vartheta_{j+2\mathfrak{m}}) \cdots \eta(\vartheta_{j+2}, \vartheta_{j+2\mathfrak{m}}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+2\mathfrak{m}-3}, \vartheta_{j+2\mathfrak{m}-2}, \frac{\varsigma}{3^{\mathfrak{m}-1}(\Gamma(\vartheta_{j+2\mathfrak{m}-3}, \vartheta_{j+2\mathfrak{m}-2})\eta(\vartheta_{j+2\mathfrak{m}-4}, \vartheta_{j+2\mathfrak{m}}) \cdots \eta(\vartheta_{j+2}, \vartheta_{j+2\mathfrak{m}}))}\right) \\
&\star \mathcal{W}\left(\vartheta_{j+2\mathfrak{m}-2}, \vartheta_{j+2\mathfrak{m}}, \frac{\varsigma}{3^{\mathfrak{m}-1}(\eta(\vartheta_{j+2\mathfrak{m}-2}, \vartheta_{j+2\mathfrak{m}})\eta(\vartheta_{j+2\mathfrak{m}-4}, \vartheta_{j+2\mathfrak{m}+1}) \cdots \eta(\vartheta_{j+2}, \vartheta_{j+2\mathfrak{m}}))}\right),
\end{aligned}$$

$$
\begin{aligned}
\mathcal{E}(\vartheta_{j},\vartheta_{j+2m},\varsigma) \leq{}& \mathcal{E}\left(\vartheta_{j},\vartheta_{j+1},\frac{\varsigma}{3(\beta(\vartheta_{j},\vartheta_{j+1}))}\right) \diamond \mathcal{E}\left(\vartheta_{j+1},\vartheta_{j+2},\frac{\varsigma}{3(\Gamma(\vartheta_{j+1},\vartheta_{j+2}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+2},\vartheta_{j+2m},\frac{\varsigma}{3(\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right) \\
\leq{}& \mathcal{E}\left(\vartheta_{j},\vartheta_{j+1},\frac{\varsigma}{3(\beta(\vartheta_{j},\vartheta_{j+1}))}\right) \diamond \mathcal{E}\left(\vartheta_{j+1},\vartheta_{j+2},\frac{\varsigma}{3(\Gamma(\vartheta_{j+1},\vartheta_{j+2}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+2},\vartheta_{j+3},\frac{\varsigma}{3^2(\beta(\vartheta_{j+2},\vartheta_{j+3})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+3},\vartheta_{j+4},\frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3},\vartheta_{j+4})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+4},\vartheta_{j+2m},\frac{\varsigma}{3^2(\eta(\vartheta_{j+4},\vartheta_{j+2m})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right) \\
\leq{}& \mathcal{E}\left(\vartheta_{j},\vartheta_{j+1},\frac{\varsigma}{3(\beta(\vartheta_{j},\vartheta_{j+1}))}\right) \diamond \mathcal{E}\left(\vartheta_{j+1},\vartheta_{j+2},\frac{\varsigma}{3(\Gamma(\vartheta_{j+1},\vartheta_{j+2}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+2},\vartheta_{j+3},\frac{\varsigma}{3^2(\beta(\vartheta_{j+2},\vartheta_{j+3})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+3},\vartheta_{j+4},\frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3},\vartheta_{j+4})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+4},\vartheta_{j+5},\frac{\varsigma}{3^3(\beta(\vartheta_{j+4},\vartheta_{j+5})\eta(\vartheta_{j+4},\vartheta_{j+2m})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+5},\vartheta_{j+6},\frac{\varsigma}{3^3(\Gamma(\vartheta_{j+5},\vartheta_{j+6})\eta(\vartheta_{j+4},\vartheta_{j+2m})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+6},\vartheta_{j+2m},\frac{\varsigma}{3^3(\eta(\vartheta_{j+6},\vartheta_{j+2m})\eta(\vartheta_{j+4},\vartheta_{j+2m})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}(\vartheta_{j},\vartheta_{j+2m},\varsigma) \leq{}& \mathcal{E}\left(\vartheta_{j},\vartheta_{j+1},\frac{\varsigma}{3(\beta(\vartheta_{j},\vartheta_{j+1}))}\right) \diamond \mathcal{E}\left(\vartheta_{j+1},\vartheta_{j+2},\frac{\varsigma}{3(\Gamma(\vartheta_{j+1},\vartheta_{j+2}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+2},\vartheta_{j+3},\frac{\varsigma}{3^2(\beta(\vartheta_{j+2},\vartheta_{j+3})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+3},\vartheta_{j+4},\frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3},\vartheta_{j+4})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+4},\vartheta_{j+5},\frac{\varsigma}{3^3(\beta(\vartheta_{j+4},\vartheta_{j+5})\eta(\vartheta_{j+4},\vartheta_{j+2m})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+5},\vartheta_{j+6},\frac{\varsigma}{3^3(\Gamma(\vartheta_{j+5},\vartheta_{j+6})\eta(\vartheta_{j+4},\vartheta_{j+2m})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+6},\vartheta_{j+2m},\frac{\varsigma}{3^3(\eta(\vartheta_{j+6},\vartheta_{j+2m})\eta(\vartheta_{j+4},\vartheta_{j+2m})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right) \diamond \cdots \diamond \\
&\mathcal{E}\left(\vartheta_{j+2m-4},\vartheta_{j+2m-3},\frac{\varsigma}{3^{m-1}(\beta(\vartheta_{j+2m-4},\vartheta_{j+2m-3})\eta(\vartheta_{j+2m-4},\vartheta_{j+2m+1})\cdots\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\diamond \mathcal{E}\left(\vartheta_{j+2m-3},\vartheta_{j+2m-2},\frac{\varsigma}{3^{m-1}(\Gamma(\vartheta_{j+2m-3},\vartheta_{j+2m-2})\eta(\vartheta_{j+2m-4},\vartheta_{j+2m})\cdots\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right) \\
&\diamond \mathcal{E}\left(\vartheta_{j+2m-2},\vartheta_{j+2m},\frac{\varsigma}{3^{m-1}(\eta(\vartheta_{j+2m-2},\vartheta_{j+2m})\eta(\vartheta_{j+2m-4},\vartheta_{j+2m+1})\cdots\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right),
\end{aligned}
$$

and

$$\begin{aligned}
\mathcal{G}(\vartheta_{\mathrm{j}},\vartheta_{\mathrm{j}+2\mathrm{m}},\varsigma) &\leq \mathcal{G}\left(\vartheta_{\mathrm{j}},\vartheta_{\mathrm{j}+1},\frac{\varsigma}{3(\beta(\vartheta_{\mathrm{j}},\vartheta_{\mathrm{j}+1}))}\right)\diamond\mathcal{G}\left(\vartheta_{\mathrm{j}+1},\vartheta_{\mathrm{j}+2},\frac{\varsigma}{3(\Gamma(\vartheta_{\mathrm{j}+1},\vartheta_{\mathrm{j}+2}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+2\mathrm{m}},\frac{\varsigma}{3(\eta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+2\mathrm{m}}))}\right)\\
&\leq \mathcal{G}\left(\vartheta_{\mathrm{j}},\vartheta_{\mathrm{j}+1},\frac{\varsigma}{3(\beta(\vartheta_{\mathrm{j}},\vartheta_{\mathrm{j}+1}))}\right)\diamond\mathcal{G}\left(\vartheta_{\mathrm{j}+1},\vartheta_{\mathrm{j}+2},\frac{\varsigma}{3(\Gamma(\vartheta_{\mathrm{j}+1},\vartheta_{\mathrm{j}+2}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+3},\frac{\varsigma}{3^2(\beta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+3})\eta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+2\mathrm{m}}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{\mathrm{j}+3},\vartheta_{\mathrm{j}+4},\frac{\varsigma}{3^2(\Gamma(\vartheta_{\mathrm{j}+3},\vartheta_{\mathrm{j}+4})\eta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+2\mathrm{m}}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{\mathrm{j}+4},\vartheta_{\mathrm{j}+2\mathrm{m}},\frac{\varsigma}{3^2(\eta(\vartheta_{\mathrm{j}+4},\vartheta_{\mathrm{j}+2\mathrm{m}})\eta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+2\mathrm{m}}))}\right)\\
&\leq \mathcal{G}\left(\vartheta_{\mathrm{j}},\vartheta_{\mathrm{j}+1},\frac{\varsigma}{3(\beta(\vartheta_{\mathrm{j}},\vartheta_{\mathrm{j}+1}))}\right)\diamond\mathcal{G}\left(\vartheta_{\mathrm{j}+1},\vartheta_{\mathrm{j}+2},\frac{\varsigma}{3(\Gamma(\vartheta_{\mathrm{j}+1},\vartheta_{\mathrm{j}+2}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+3},\frac{\varsigma}{3^2(\beta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+3})\eta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+2\mathrm{m}}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{\mathrm{j}+3},\vartheta_{\mathrm{j}+4},\frac{\varsigma}{3^2(\Gamma(\vartheta_{\mathrm{j}+3},\vartheta_{\mathrm{j}+4})\eta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+2\mathrm{m}}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{\mathrm{j}+4},\vartheta_{\mathrm{j}+5},\frac{\varsigma}{3^3(\beta(\vartheta_{\mathrm{j}+4},\vartheta_{\mathrm{j}+5})\eta(\vartheta_{\mathrm{j}+4},\vartheta_{\mathrm{j}+2\mathrm{m}})\eta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+2\mathrm{m}}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{\mathrm{j}+5},\vartheta_{\mathrm{j}+6},\frac{\varsigma}{3^3(\Gamma(\vartheta_{\mathrm{j}+5},\vartheta_{\mathrm{j}+6})\eta(\vartheta_{\mathrm{j}+4},\vartheta_{\mathrm{j}+2\mathrm{m}})\eta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+2\mathrm{m}}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{\mathrm{j}+6},\vartheta_{\mathrm{j}+2\mathrm{m}},\frac{\varsigma}{3^3(\eta(\vartheta_{\mathrm{j}+6},\vartheta_{\mathrm{j}+2\mathrm{m}})\eta(\vartheta_{\mathrm{j}+4},\vartheta_{\mathrm{j}+2\mathrm{m}})\eta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+2\mathrm{m}}))}\right)
\end{aligned}$$

$$\begin{aligned}
&\mathcal{G}(\vartheta_{\mathrm{j}},\vartheta_{\mathrm{j}+2\mathrm{m}},\varsigma)\\
&\leq \mathcal{G}\left(\vartheta_{\mathrm{j}},\vartheta_{\mathrm{j}+1},\frac{\varsigma}{3(\beta(\vartheta_{\mathrm{j}},\vartheta_{\mathrm{j}+1}))}\right)\diamond\mathcal{G}\left(\vartheta_{\mathrm{j}+1},\vartheta_{\mathrm{j}+2},\frac{\varsigma}{3(\Gamma(\vartheta_{\mathrm{j}+1},\vartheta_{\mathrm{j}+2}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+3},\frac{\varsigma}{3^2(\beta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+3})\eta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+2\mathrm{m}}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{\mathrm{j}+3},\vartheta_{\mathrm{j}+4},\frac{\varsigma}{3^2(\Gamma(\vartheta_{\mathrm{j}+3},\vartheta_{\mathrm{j}+4})\eta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+2\mathrm{m}}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{\mathrm{j}+4},\vartheta_{\mathrm{j}+5},\frac{\varsigma}{3^3(\beta(\vartheta_{\mathrm{j}+4},\vartheta_{\mathrm{j}+5})\eta(\vartheta_{\mathrm{j}+4},\vartheta_{\mathrm{j}+2\mathrm{m}})\eta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+2\mathrm{m}}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{\mathrm{j}+5},\vartheta_{\mathrm{j}+6},\frac{\varsigma}{3^3(\Gamma(\vartheta_{\mathrm{j}+5},\vartheta_{\mathrm{j}+6})\eta(\vartheta_{\mathrm{j}+4},\vartheta_{\mathrm{j}+2\mathrm{m}})\eta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+2\mathrm{m}}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{\mathrm{j}+6},\vartheta_{\mathrm{j}+2\mathrm{m}},\frac{\varsigma}{3^3(\eta(\vartheta_{\mathrm{j}+6},\vartheta_{\mathrm{j}+2\mathrm{m}})\eta(\vartheta_{\mathrm{j}+4},\vartheta_{\mathrm{j}+2\mathrm{m}})\eta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+2\mathrm{m}}))}\right)\diamond\cdots\diamond\\
&\mathcal{G}\left(\vartheta_{\mathrm{j}+2\mathrm{m}-4},\vartheta_{\mathrm{j}+2\mathrm{m}-3},\frac{\varsigma}{3^{\mathrm{m}-1}(\beta(\vartheta_{\mathrm{j}+2\mathrm{m}-4},\vartheta_{\mathrm{j}+2\mathrm{m}-3})\eta(\vartheta_{\mathrm{j}+2\mathrm{m}-4},\vartheta_{\mathrm{j}+2\mathrm{m}})\cdots\eta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+2\mathrm{m}}))}\right)
\end{aligned}$$

$$\begin{aligned}
&\diamond\mathcal{G}\left(\vartheta_{\mathrm{j}+2\mathrm{m}-3},\vartheta_{\mathrm{j}+2\mathrm{m}-2},\frac{\varsigma}{3^{\mathrm{m}-1}(\Gamma(\vartheta_{\mathrm{j}+2\mathrm{m}-3},\vartheta_{\mathrm{j}+2\mathrm{m}-2})\eta(\vartheta_{\mathrm{j}+2\mathrm{m}-4},\vartheta_{\mathrm{j}+2\mathrm{m}})\cdots\eta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+2\mathrm{m}}))}\right)\\
&\diamond\mathcal{G}\left(\vartheta_{\mathrm{j}+2\mathrm{m}-2},\vartheta_{\mathrm{j}+2\mathrm{m}},\frac{\varsigma}{3^{\mathrm{m}-1}(\eta(\vartheta_{\mathrm{j}+2\mathrm{m}-2},\vartheta_{\mathrm{j}+2\mathrm{m}})\eta(\vartheta_{\mathrm{j}+2\mathrm{m}-4},\vartheta_{\mathrm{j}+2\mathrm{m}+1})\cdots\eta(\vartheta_{\mathrm{j}+2},\vartheta_{\mathrm{j}+2\mathrm{m}}))}\right).
\end{aligned}$$

Using (3) in the above inequalities, we deduce

$$
\begin{aligned}
\mathcal{W}(\vartheta_{j}, \vartheta_{j+2m}, \varsigma) \geq\ & \frac{1}{\frac{\rho^{j}}{\mathcal{W}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3(\beta(\vartheta_j, \vartheta_{j+1}))}\right)} + (1-\rho^{j})} \star \frac{1}{\frac{\rho^{j+1}}{\mathcal{W}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right)} + (1-\rho^{j+1})} \\
& \star \frac{1}{\frac{\rho^{j+2}}{\mathcal{W}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^2(\beta(\vartheta_{j+2}, \vartheta_{j+3})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right)} + (1-\rho^{j+2})} \\
& \star \frac{1}{\frac{\rho^{j+3}}{\mathcal{W}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3}, \vartheta_{j+4})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right)} + (1-\rho^{j+3})} \\
& \star \frac{1}{\frac{\rho^{j+4}}{\mathcal{W}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^3(\beta(\vartheta_{j+4}, \vartheta_{j+5})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right)} + (1-\rho^{j+4})} \\
& \star \frac{1}{\frac{\rho^{j+5}}{\mathcal{W}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^3(\Gamma(\vartheta_{j+5}, \vartheta_{j+6})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right)} + (1-\rho^{j+5})} \\
& \star \frac{1}{\frac{\rho^{j+6}}{\mathcal{W}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^3(\eta(\vartheta_{j+6}, \vartheta_{j+2m})\eta(\vartheta_{j+4}, \vartheta_{j+2m})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right)} + (1-\rho^{j+6})} \star \cdots \star \\
& \star \frac{1}{\frac{\rho^{j+2m-4}}{\mathcal{W}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^{m-1}(\beta(\vartheta_{j+2m-4}, \vartheta_{j+2m-3})\eta(\vartheta_{j+2m-4}, \vartheta_{j+2m})\cdots\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right)} + (1-\rho^{j+2m-4})} \\
& \star \frac{1}{\frac{\rho^{j+2m-3}}{\mathcal{W}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^{m-1}(\Gamma(\vartheta_{j+2m-3}, \vartheta_{j+2m-2})\eta(\vartheta_{j+2m-4}, \vartheta_{j+2m})\cdots\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right)} + (1-\rho^{j+2m-3})} \\
& \star \frac{1}{\frac{\rho^{j+2m-2}}{\mathcal{W}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^{m-1}(\eta(\vartheta_{j+2m-2}, \vartheta_{j+2m})\eta(\vartheta_{j+2m-4}, \vartheta_{j+2m+1})\cdots\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right)} + (1-\rho^{j+2m-2})}
\end{aligned}
$$

$$
\begin{aligned}
& \mathcal{G}(\vartheta_{j}, \vartheta_{j+2m}, \varsigma) \\
& \quad \leq \rho^{j}\mathcal{G}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3(\beta(\vartheta_j, \vartheta_{j+1}))}\right) \diamond \rho^{j+1}\mathcal{G}\left(\vartheta_{j+1}, \vartheta_{j+2}, \frac{\varsigma}{3(\Gamma(\vartheta_{j+1}, \vartheta_{j+2}))}\right) \\
& \quad \diamond \rho^{j+2}\mathcal{G}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^2(\beta(\vartheta_{j+2}, \vartheta_{j+3})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right) \\
& \quad \diamond \rho^{j+3}\mathcal{G}\left(\vartheta_0, \vartheta_1, \frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3}, \vartheta_{j+4})\eta(\vartheta_{j+2}, \vartheta_{j+2m}))}\right)
\end{aligned}
$$

$$\begin{aligned}
&\diamond \rho^{j+4}\mathcal{G}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^3(\beta(\vartheta_{j+4},\vartheta_{j+5})\eta(\vartheta_{j+4},\vartheta_{j+2m})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right)\\
&\diamond \rho^{j+5}\mathcal{G}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^3(\Gamma(\vartheta_{j+5},\vartheta_{j+6})\eta(\vartheta_{j+4},\vartheta_{j+2m})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right)\\
&\diamond \rho^{j+6}\mathcal{G}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^3(\eta(\vartheta_{j+6},\vartheta_{j+2m})\eta(\vartheta_{j+4},\vartheta_{j+2m})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right)\diamond\cdots\diamond\\
&\rho^{j+2m-4}\mathcal{G}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^{m-1}(\beta(\vartheta_{j+2m-4},\vartheta_{j+2m-3})\eta(\vartheta_{j+2m-4},\vartheta_{j+2m})\cdots\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right)\\
&\diamond \rho^{j+2m-3}\mathcal{G}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^{m-1}(\Gamma(\vartheta_{j+2m-3},\vartheta_{j+2m-2})\eta(\vartheta_{j+2m-4},\vartheta_{j+2m})\cdots\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right)\\
&\diamond \rho^{j+2m-2}\mathcal{G}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^{m-1}(\eta(\vartheta_{j+2m-2},\vartheta_{j+2m})\eta(\vartheta_{j+2m-4},\vartheta_{j+2m})\cdots\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right).
\end{aligned}$$

$$\begin{aligned}
&\mathcal{E}(\vartheta_j,\vartheta_{j+2m},\varsigma)\\
&\le \rho^{j}\mathcal{E}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3(\beta(\vartheta_j,\vartheta_{j+1}))}\right)\diamond\rho^{j+1}\mathcal{E}\left(\vartheta_{j+1},\vartheta_{j+2},\frac{\varsigma}{3(\Gamma(\vartheta_{j+1},\vartheta_{j+2}))}\right)\\
&\diamond \rho^{j+2}\mathcal{E}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^2(\beta(\vartheta_{j+2},\vartheta_{j+3})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right)\\
&\diamond \rho^{j+3}\mathcal{E}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^2(\Gamma(\vartheta_{j+3},\vartheta_{j+4})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right)\\
&\diamond \rho^{j+4}\mathcal{E}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^3(\beta(\vartheta_{j+4},\vartheta_{j+5})\eta(\vartheta_{j+4},\vartheta_{j+2m})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right)\\
&\diamond \rho^{j+5}\mathcal{E}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^3(\Gamma(\vartheta_{j+5},\vartheta_{j+6})\eta(\vartheta_{j+4},\vartheta_{j+2m})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right)\\
&\diamond \rho^{j+6}\mathcal{E}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^3(\eta(\vartheta_{j+6},\vartheta_{j+2m})\eta(\vartheta_{j+4},\vartheta_{j+2m})\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right)\diamond\cdots\diamond\\
&\rho^{j+2m-4}\mathcal{E}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^{m-1}(\beta(\vartheta_{j+2m-4},\vartheta_{j+2m-3})\eta(\vartheta_{j+2m-4},\vartheta_{j+2m})\cdots\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right)\\
&\diamond \rho^{j+2m-3}\mathcal{E}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^{m-1}(\Gamma(\vartheta_{j+2m-3},\vartheta_{j+2m-2})\eta(\vartheta_{j+2m-4},\vartheta_{j+2m})\cdots\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right)\\
&\diamond \rho^{j+2m-2}\mathcal{E}\left(\vartheta_0,\vartheta_1,\frac{\varsigma}{3^{m-1}(\eta(\vartheta_{j+2m-2},\vartheta_{j+2m})\eta(\vartheta_{j+2m-4},\vartheta_{j+2m})\cdots\eta(\vartheta_{j+2},\vartheta_{j+2m}))}\right).
\end{aligned}$$

As $j \to +\infty$, we deduce

$$\lim_{j\to+\infty}\mathcal{W}(\vartheta_j,\vartheta_{j+q},\varsigma)=1\star1\star\cdots\star=1,$$
$$\lim_{j\to+\infty}\mathcal{E}(\vartheta_j,\vartheta_{j+q},\varsigma)=0\diamond0\diamond\cdots\diamond0=0,$$

and

$$\lim_{j\to+\infty}\mathcal{G}(\vartheta_j,\vartheta_{j+q},\varsigma)=0\diamond0\diamond\cdots\diamond0=0.$$

Therefore, $\{\vartheta_j\}$ is a Cauchy sequence. Since $(\Im,\mathcal{W},\mathcal{E},\mathcal{G},\star,\diamond)$ is a complete neutrosophic rectangular triple-controlled metric space, we have

$$\lim_{j\to+\infty}\vartheta_j=\vartheta.$$

Now, we examine that ϑ is a fixed point of $\mathfrak{p}$. Utilizing (v), (x) and (xv), we obtain

$$\frac{1}{\mathcal{W}(\mathfrak{p}\vartheta_{\mathrm{j}},\mathfrak{p}\vartheta,\varsigma)}-1\leq\rho\left[\frac{1}{\mathcal{W}(\vartheta_{\mathrm{j}},\vartheta,\varsigma)}-1\right]=\frac{\rho}{\mathcal{W}(\vartheta_{\mathrm{j}},\vartheta,\varsigma)}-\rho$$
$$\Rightarrow\frac{1}{\frac{\rho}{\mathcal{W}(\vartheta_{\mathrm{j}},\vartheta,\varsigma)}+(1-\rho)}\leq\mathcal{W}(\mathfrak{p}\vartheta_{\mathrm{j}},\mathfrak{p}\vartheta,\varsigma).$$

Using the above inequality, we obtain

$$\begin{aligned}\mathcal{W}(\vartheta,\mathfrak{p}\vartheta,\varsigma)&\geq\mathcal{W}\left(\vartheta,\vartheta_{\mathrm{j}},\frac{\varsigma}{3\beta(\vartheta,\vartheta_{\mathrm{j}})}\right)\star\mathcal{W}\left(\vartheta_{\mathrm{j}},\vartheta_{\mathrm{j}+1},\frac{\varsigma}{3\Gamma(\vartheta_{\mathrm{j}},\vartheta_{\mathrm{j}+1})}\right)\star\mathcal{W}\left(\vartheta_{\mathrm{j}+1},\mathfrak{p}\vartheta,\frac{\varsigma}{3\eta(\vartheta_{\mathrm{j}+1},\mathfrak{p}\vartheta)}\right)\\&\geq\mathcal{W}\left(\vartheta,\vartheta_{\mathrm{j}},\frac{\varsigma}{3\beta(\vartheta,\vartheta_{\mathrm{j}})}\right)\star\mathcal{W}\left(\mathfrak{p}\vartheta_{\mathrm{j}-1},\mathfrak{p}\vartheta_{\mathrm{j}},\frac{\varsigma}{3\Gamma(\vartheta_{\mathrm{j}},\vartheta_{\mathrm{j}+1})}\right)\star\mathcal{W}\left(\mathfrak{p}\vartheta_{\mathrm{j}},\mathfrak{p}\vartheta,\frac{\varsigma}{3\eta(\vartheta_{\mathrm{j}+1},\mathfrak{p}\vartheta)}\right)\\&\geq\mathcal{W}\left(\vartheta,\vartheta_{\mathrm{j}},\frac{\varsigma}{(3\beta(\vartheta,\vartheta_{\mathrm{j}}))}\right)\star\frac{1}{\frac{\rho^{\mathrm{j}}}{\mathcal{W}(\vartheta_0,\vartheta_1,\frac{\varsigma}{3\Gamma(\vartheta_{\mathrm{j}},\vartheta_{\mathrm{j}+1})})+(1-\rho^{\mathrm{j}})}}\star\frac{1}{\frac{\rho}{\mathcal{W}\left(\vartheta_{\mathrm{j}},\vartheta,\frac{\varsigma}{3\eta(\vartheta_{\mathrm{j}+1},\mathfrak{p}\vartheta)}\right)+(1-\rho)}}\\&\to1\star1\star1=1\text{ as }\mathrm{j}\to+\infty,\end{aligned}$$

$$\begin{aligned}\mathcal{E}(\vartheta,\mathfrak{p}\vartheta,\varsigma)&\leq\mathcal{E}\left(\vartheta,\vartheta_{\mathrm{j}},\frac{\varsigma}{3\beta(\vartheta,\vartheta_{\mathrm{j}})}\right)\diamond\mathcal{E}\left(\vartheta_{\mathrm{j}},\vartheta_{\mathrm{j}+1},\frac{\varsigma}{3\Gamma(\vartheta_{\mathrm{j}},\vartheta_{\mathrm{j}+1})}\right)\diamond\mathcal{E}\left(\vartheta_{\mathrm{j}+1},\mathfrak{p}\vartheta,\frac{\varsigma}{3\eta(\vartheta_{\mathrm{j}+1},\mathfrak{p}\vartheta)}\right)\\&\leq\mathcal{E}\left(\vartheta,\vartheta_{\mathrm{j}},\frac{\varsigma}{3\beta(\vartheta,\vartheta_{\mathrm{j}})}\right)\diamond\mathcal{E}\left(\mathfrak{p}\vartheta_{\mathrm{j}-1},\mathfrak{p}\vartheta_{\mathrm{j}},\frac{\varsigma}{3\Gamma(\vartheta_{\mathrm{j}},\vartheta_{\mathrm{j}+1})}\right)\diamond\mathcal{E}\left(\mathfrak{p}\vartheta_{\mathrm{j}},\mathfrak{p}\vartheta,\frac{\varsigma}{3\eta(\vartheta_{\mathrm{j}+1},\mathfrak{p}\vartheta)}\right)\\&\leq\mathcal{E}\left(\vartheta,\vartheta_{\mathrm{j}},\frac{\varsigma}{3\beta(\vartheta,\vartheta_{\mathrm{j}})}\right)\diamond\rho^{\mathrm{j}-1}\mathcal{E}\left(\vartheta_{\mathrm{j}-1},\vartheta_{\mathrm{j}},\frac{\varsigma}{3\Gamma(\vartheta_{\mathrm{j}},\vartheta_{\mathrm{j}+1})}\right)\diamond\rho\mathcal{E}\left(\vartheta_{\mathrm{j}},\vartheta,\frac{\varsigma}{3\eta(\vartheta_{\mathrm{j}+1},\vartheta)}\right)\\&\to0\diamond0\diamond0=0\text{ as }\mathrm{j}\to+\infty,\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}(\vartheta,\mathfrak{p}\vartheta,\varsigma)&\leq\mathcal{G}\left(\vartheta,\vartheta_{\mathrm{j}},\frac{\varsigma}{3\beta(\vartheta,\vartheta_{\mathrm{j}})}\right)\diamond\mathcal{G}\left(\vartheta_{\mathrm{j}},\vartheta_{\mathrm{j}+1},\frac{\varsigma}{3\Gamma(\vartheta_{\mathrm{j}},\vartheta_{\mathrm{j}+1})}\right)\diamond\mathcal{G}\left(\vartheta_{\mathrm{j}+1},\mathfrak{p}\vartheta,\frac{\varsigma}{3\eta(\vartheta_{\mathrm{j}+1},\mathfrak{p}\vartheta)}\right)\\&\leq\mathcal{G}\left(\vartheta,\vartheta_{\mathrm{j}},\frac{\varsigma}{3\beta(\vartheta,\vartheta_{\mathrm{j}})}\right)\diamond\mathcal{G}\left(\mathfrak{p}\vartheta_{\mathrm{j}-1},\mathfrak{p}\vartheta_{\mathrm{j}},\frac{\varsigma}{3\Gamma(\vartheta_{\mathrm{j}},\vartheta_{\mathrm{j}+1})}\right)\diamond\mathcal{G}\left(\mathfrak{p}\vartheta_{\mathrm{j}},\mathfrak{p}\vartheta,\frac{\varsigma}{3\eta(\vartheta_{\mathrm{j}+1},\mathfrak{p}\vartheta)}\right)\\&\leq\mathcal{G}\left(\vartheta,\vartheta_{\mathrm{j}},\frac{\varsigma}{3\beta(\vartheta,\vartheta_{\mathrm{j}})}\right)\diamond\rho^{\mathrm{j}-1}\mathcal{G}\left(\vartheta_{\mathrm{j}-1},\vartheta_{\mathrm{j}},\frac{\varsigma}{3\Gamma(\vartheta_{\mathrm{j}},\vartheta_{\mathrm{j}+1})}\right)\diamond\rho\mathcal{G}\left(\vartheta_{\mathrm{j}},\vartheta,\frac{\varsigma}{3\eta(\vartheta_{\mathrm{j}+1},\vartheta)}\right)\\&\to0\diamond0\diamond0=0\text{ as }\mathrm{j}\to+\infty.\end{aligned}$$

Hence, $\mathfrak{p}\vartheta=\vartheta$. For uniqueness, consider that $\mathfrak{p}\kappa=\kappa$ for some $\kappa\in\Im$. Then,

$$\begin{aligned}\frac{1}{\mathcal{W}(\vartheta,\kappa,\varsigma)}-1&=\frac{1}{\mathcal{W}(\mathfrak{p}\vartheta,\mathfrak{p}\kappa,\varsigma)}-1\\&\leq\rho\left[\frac{1}{\mathcal{W}(\vartheta,\kappa,\varsigma)}-1\right]<\frac{1}{\mathcal{W}(\vartheta,\kappa,\varsigma)}-1,\end{aligned}$$

$$\mathcal{E}(\vartheta,\kappa,\varsigma)=\mathcal{E}(\mathfrak{p}\vartheta,\mathfrak{p}\kappa,\varsigma)\leq\rho\mathcal{E}(\vartheta,\kappa,\varsigma)<\mathcal{E}(\vartheta,\kappa,\varsigma),$$

and

$$\mathcal{G}(\vartheta,\kappa,\varsigma)=\mathcal{G}(\mathfrak{p}\vartheta,\mathfrak{p}\kappa,\varsigma)\leq\rho\mathcal{G}(\vartheta,\kappa,\varsigma)<\mathcal{G}(\vartheta,\kappa,\varsigma),$$

which are contradictions.

Thus, we have $\mathcal{W}(\vartheta,\kappa,\varsigma)=1,\mathcal{E}(\vartheta,\kappa,\varsigma)=0$ and $\mathcal{G}(\vartheta,\kappa,\varsigma)=0$, and accordingly, $\vartheta=\kappa$. $\square$

Example 3. *Let $\Im = [0,1]$ and $\beta, \Gamma, \eta \colon \Im \times \Im \to [1, +\infty)$ be three non-comparable functions given by*

$$\beta(\vartheta, \mathcal{Q}) = \begin{cases} 1 & \text{if} \quad \vartheta = \mathcal{Q}, \\ \frac{1+\max\{\vartheta, \mathcal{Q}\}}{1+\min\{\vartheta, \mathcal{Q}\}} & \text{if} \quad \vartheta \neq \mathcal{Q}, \end{cases}$$

$$\Gamma(\vartheta, \mathcal{Q}) = \begin{cases} 1 & \text{if} \quad \vartheta = \mathcal{Q}, \\ \frac{1+\max\{\vartheta^2, \mathcal{Q}^2\}}{1+\min\{\vartheta^2, \mathcal{Q}^2\}} & \text{if} \quad \vartheta \neq \mathcal{Q}, \end{cases}$$

and

$$\eta(\vartheta, \mathcal{Q}) = \begin{cases} 1 & \text{if} \quad \vartheta = \mathcal{Q}, \\ \frac{1+\max\{\vartheta^2, \mathcal{Q}\}}{1+\min\{\vartheta^2, \mathcal{Q}\}} & \text{if} \quad \vartheta \neq \mathcal{Q}. \end{cases}$$

Define $\mathcal{W}, \mathcal{E}, \mathcal{G} \colon \Im \times \Im \times (0, +\infty) \to [0,1]$ as

$$\begin{aligned} \mathcal{W}(\vartheta, \mathcal{Q}, \varsigma) &= \frac{\varsigma}{\varsigma + |\vartheta - \mathcal{Q}|}, \\ \mathcal{E}(\vartheta, \mathcal{Q}, \varsigma) &= \frac{|\vartheta - \mathcal{Q}|}{\varsigma + |\vartheta - \mathcal{Q}|}, \\ \mathcal{G}(\vartheta, \mathcal{Q}, \varsigma) &= \frac{|\vartheta - \mathcal{Q}|}{\varsigma}. \end{aligned}$$

Then, $(\Im, \mathcal{W}, \mathcal{E}, \mathcal{G}, \star, \diamond)$ is a complete neutrosophic rectangular triple-controlled metric space with continuous t-norm $\wp \star \sigma = \wp\sigma$ and continuous t-co-norm $\wp \diamond \sigma = \max\{\wp, \sigma\}$.

Define $\mathfrak{p} \colon \Im \to \Im$ by $\mathfrak{p}(\vartheta) = \frac{1-3^{-\vartheta}}{5}$ and take $\rho \in [\frac{1}{2}, 1)$, then

$$\begin{aligned} \mathcal{W}(\mathfrak{p}\vartheta, \mathfrak{p}\mathcal{Q}, \rho\varsigma) &= \mathcal{W}\left(\frac{1-3^{-\vartheta}}{5}, \frac{1-3^{-\mathcal{Q}}}{5}, \rho\varsigma\right) \\ &= \frac{\rho\varsigma}{\rho\varsigma + \left|\frac{1-3^{-\vartheta}}{5} - \frac{1-3^{-\mathcal{Q}}}{5}\right|} = \frac{\rho\varsigma}{\rho\varsigma + \frac{|3^{-\vartheta}-3^{-\mathcal{Q}}|}{5}} \\ &\geq \frac{\rho\varsigma}{\rho\varsigma + \frac{|\vartheta-\mathcal{Q}|}{5}} = \frac{5\rho\varsigma}{5\rho\varsigma + |\vartheta - \mathcal{Q}|} \geq \frac{\varsigma}{\varsigma + |\vartheta - \mathcal{Q}|} = \mathcal{W}(\vartheta, \mathcal{Q}, \varsigma), \end{aligned}$$

$$\begin{aligned} \mathcal{E}(\mathfrak{p}\vartheta, \mathfrak{p}\mathcal{Q}, \rho\varsigma) &= \mathcal{E}\left(\frac{1-3^{-\vartheta}}{5}, \frac{1-3^{-\mathcal{Q}}}{5}, \rho\varsigma\right) \\ &= \frac{\left|\frac{1-3^{-\vartheta}}{5} - \frac{1-3^{-\mathcal{Q}}}{5}\right|}{\rho\varsigma + \left|\frac{1-3^{-\vartheta}}{5} - \frac{1-3^{-\mathcal{Q}}}{5}\right|} = \frac{\frac{|3^{-\vartheta}-3^{-\mathcal{Q}}|}{5}}{\rho\varsigma + \frac{|3^{-\vartheta}-3^{-\mathcal{Q}}|}{5}} \\ &= \frac{|3^{-\vartheta} - 3^{-\mathcal{Q}}|}{5\rho\varsigma + |3^{-\vartheta} - 3^{-\mathcal{Q}}|} \leq \frac{|\vartheta - \mathcal{Q}|}{5\rho\varsigma + |\vartheta - \mathcal{Q}|} \leq \frac{|\vartheta - \mathcal{Q}|}{\varsigma + |\vartheta - \mathcal{Q}|} = \mathcal{E}(\vartheta, \mathcal{Q}, \varsigma), \end{aligned}$$

and

$$\begin{aligned}\mathcal{G}(\mathfrak{p}\vartheta,\mathfrak{p}\mathcal{Q},\rho\varsigma) &= \mathcal{G}\left(\frac{1-3^{-\vartheta}}{5},\frac{1-3^{-\mathcal{Q}}}{5},\rho\varsigma\right)\\ &= \frac{\left|\frac{1-3^{-\vartheta}}{5}-\frac{1-3^{-\mathcal{Q}}}{5}\right|}{\rho\varsigma} = \frac{\frac{|3^{-\vartheta}-3^{-\mathcal{Q}}|}{5}}{\rho\varsigma}\\ &= \frac{|3^{-\vartheta}-3^{-\mathcal{Q}}|}{5\rho\varsigma} \leq \frac{|\vartheta-\mathcal{Q}|}{5\rho\varsigma} \leq \frac{|\vartheta-\mathcal{Q}|}{\varsigma} = \mathcal{G}(\vartheta,\mathcal{Q},\varsigma).\end{aligned}$$

Thus, all conditions of Theorem 1 are satisfied with 0 *as the unique fixed point for* $\mathfrak{p}$.

Application

Let $\Im = \mathcal{C}([\mathfrak{c},\mathfrak{a}],\mathbb{R})$, the set of real-valued continuous functions defined on $[\mathfrak{c},\mathfrak{a}]$. Consider the integral equation

$$\vartheta(\tau) = \wedge(\tau) + \delta\int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\tau,\mathfrak{v})\vartheta(\tau)\nu\mathfrak{v} \quad \text{for} \quad \tau,\mathfrak{v}\in[\mathfrak{c},\mathfrak{a}] \tag{10}$$

where $\delta > 0, \wedge(\mathfrak{v})$ is a fuzzy function of $\mathfrak{v}\colon \mathfrak{v}\in[\mathfrak{c},\mathfrak{a}]$ and $\mho\colon \mathcal{C}([\mathfrak{c},\mathfrak{a}]\times\mathbb{R})\to\mathbb{R}^+$. Define $\mathcal{W}$ and $\mathcal{E}$ by means of

$$\mathcal{W}(\vartheta(\tau),\mathcal{Q}(\tau),\varsigma) = \sup_{\tau\in[\mathfrak{c},\mathfrak{a}]} \frac{\varsigma}{\varsigma+|\vartheta(\tau)-\mathcal{Q}(\tau)|} \quad \text{for all} \quad \vartheta,\mathcal{Q}\in\Im \text{ and } \varsigma>0,$$
$$\mathcal{E}(\vartheta(\tau),\mathcal{Q}(\tau),\varsigma) = 1-\sup_{\tau\in[\mathfrak{c},\mathfrak{a}]} \frac{\varsigma}{\varsigma+|\vartheta(\tau)-\mathcal{Q}(\tau)|} \quad \text{for all} \quad \vartheta,\mathcal{Q}\in\Im \text{ and } \varsigma>0,$$

and

$$\mathcal{G}(\vartheta(\tau),\mathcal{Q}(\tau),\varsigma) = \sup_{\tau\in[\mathfrak{c},\mathfrak{a}]} \frac{|\vartheta(\tau)-\mathcal{Q}(\tau)|}{\varsigma} \quad \text{for all} \quad \vartheta,\mathcal{Q}\in\Im \text{ and } \varsigma>0,$$

with the continuous t-norm and continuous t-co-norm defined by $\wp\star\sigma = \wp\cdot\sigma$ and $\wp\diamond\sigma = \max\{\wp,\sigma\}$, respectively. Define $\beta,\Gamma,\eta\colon \Im\times\Im\to[1,+\infty)$ as

$$\beta(\vartheta,\mathcal{Q}) = \begin{cases} 1 & \text{if} \quad \vartheta=\mathcal{Q},\\ \frac{1+\max\{\vartheta,\mathcal{Q}\}}{1+\min\{\vartheta,\mathcal{Q}\}} & \text{if} \quad \vartheta\neq\mathcal{Q},\end{cases}$$

$$\Gamma(\vartheta,\mathcal{Q}) = \begin{cases} 1 & \text{if} \quad \vartheta=\mathcal{Q},\\ \frac{1+\max\{\vartheta^2,\mathcal{Q}^2\}}{1+\min\{\vartheta^2,\mathcal{Q}^2\}} & \text{if} \quad \vartheta\neq\mathcal{Q},\end{cases}$$

and

$$\eta(\vartheta,\mathcal{Q}) = \begin{cases} 1 & \text{if} \quad \vartheta=\mathcal{Q},\\ \frac{1+\max\{\vartheta^2,\mathcal{Q}\}}{1+\min\{\vartheta^2,\mathcal{Q}\}} & \text{if} \quad \vartheta\neq\mathcal{Q}.\end{cases}$$

Then, $(\Im,\mathcal{W},\mathcal{E},\mathcal{G},\star,\diamond)$ is a complete neutrosophic rectangular triple-controlled metric space. Let $|\mho(\tau,\mathfrak{v})\vartheta(\tau)-\mho(\tau,\mathfrak{v})\mathcal{Q}(\tau)| \leq |\vartheta(\tau)-\mathcal{Q}(\tau)|$ for $\vartheta,\mathcal{Q}\in\Im,\rho\in(0,1)$ and for all$\tau,\mathfrak{v}\in[\mathfrak{c},\mathfrak{a}]$. Moreover, let $\mho(\tau,\mathfrak{v})(\delta\int_{\mathfrak{c}}^{\mathfrak{a}}\nu\mathfrak{v}) \leq \rho < 1$. Then, the integral Equation (10) has a unique solution.

Proof. Define $\mathfrak{p}\colon \Im\to\Im$ by

$$\mathfrak{p}\vartheta(\tau) = \wedge(\tau) + \delta\int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\tau,\mathfrak{v})\vartheta(\tau)\nu\mathfrak{v} \quad \text{for all} \quad \tau,\mathfrak{v}\in[\mathfrak{c},\mathfrak{a}].$$

Now, for all $\vartheta, \mathcal{Q} \in \Im$, we deduce

$$\begin{aligned}
\mathcal{W}(\mathfrak{p}\vartheta(\tau),\mathfrak{p}\mathcal{Q}(\tau),\rho\varsigma) &= \sup_{\tau\in[\mathfrak{c},\mathfrak{a}]} \frac{\rho\varsigma}{\rho\varsigma + |\mathfrak{p}\vartheta(\tau) - \mathfrak{p}\mathcal{Q}(\tau)|} \\
&= \sup_{\tau\in[\mathfrak{c},\mathfrak{a}]} \frac{\rho\varsigma}{\rho\varsigma + |\wedge(\tau) + \delta\int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\tau,\mathfrak{v})\vartheta(\tau)\nu\mathfrak{v} - \wedge(\tau) - \delta\int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\tau,\mathfrak{v})\vartheta(\tau)\nu\mathfrak{v}|} \\
&= \sup_{\tau\in[\mathfrak{c},\mathfrak{a}]} \frac{\rho\varsigma}{\rho\varsigma + |\delta\int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\tau,\mathfrak{v})\vartheta(\tau)\nu\mathfrak{v} - \delta\int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\tau,\mathfrak{v})\vartheta(\tau)\nu\mathfrak{v}|} \\
&= \sup_{\tau\in[\mathfrak{c},\mathfrak{a}]} \frac{\rho\varsigma}{\rho\varsigma + |\mho(\tau,\mathfrak{v})\vartheta(\tau) - \mho(\tau,\mathfrak{v})\mathcal{Q}(\tau)|(\delta\int_{\mathfrak{c}}^{\mathfrak{a}}\nu\mathfrak{v})} \\
&\geq \sup_{\tau\in[\mathfrak{c},\mathfrak{a}]} \frac{\varsigma}{\varsigma + |\vartheta(\tau) - \mathcal{Q}(\tau)|} \\
&\geq \mathcal{W}(\vartheta(\tau),\mathcal{Q}(\tau),\varsigma),
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}(\mathfrak{p}\vartheta(\tau),\mathfrak{p}\mathcal{Q}(\tau),\rho\varsigma) &= 1 - \sup_{\tau\in[\mathfrak{c},\mathfrak{a}]} \frac{\rho\varsigma}{\rho\varsigma + |\mathfrak{p}\vartheta(\tau) - \mathfrak{p}\mathcal{Q}(\tau)|} \\
&= 1 - \sup_{\tau\in[\mathfrak{c},\mathfrak{a}]} \frac{\rho\varsigma}{\rho\varsigma + |\wedge(\tau) + \delta\int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\tau,\mathfrak{v})\vartheta(\tau)\nu\mathfrak{v} - \wedge(\tau) - \delta\int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\tau,\mathfrak{v})\vartheta(\tau)\nu\mathfrak{v}|} \\
&= 1 - \sup_{\tau\in[\mathfrak{c},\mathfrak{a}]} \frac{\rho\varsigma}{\rho\varsigma + |\delta\int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\tau,\mathfrak{v})\vartheta(\tau)\nu\mathfrak{v} - \delta\int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\tau,\mathfrak{v})\vartheta(\tau)\nu\mathfrak{v}|} \\
&= 1 - \sup_{\tau\in[\mathfrak{c},\mathfrak{a}]} \frac{\rho\varsigma}{\rho\varsigma + |\mho(\tau,\mathfrak{v})\vartheta(\tau) - \mho(\tau,\mathfrak{v})\mathcal{Q}(\tau)|(\delta\int_{\mathfrak{c}}^{\mathfrak{a}}\nu\mathfrak{v})} \\
&\leq 1 - \sup_{\tau\in[\mathfrak{c},\mathfrak{a}]} \frac{\varsigma}{\varsigma + |\vartheta(\tau) - \mathcal{Q}(\tau)|} \\
&\leq \mathcal{E}(\vartheta(\tau),\mathcal{Q}(\tau),\varsigma),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{G}(\mathfrak{p}\vartheta(\tau),\mathfrak{p}\mathcal{Q}(\tau),\rho\varsigma) &= \sup_{\tau\in[\mathfrak{c},\mathfrak{a}]} \frac{\rho\varsigma}{\rho\varsigma + |\mathfrak{p}\vartheta(\tau) - \mathfrak{p}\mathcal{Q}(\tau)|} \\
&= \sup_{\tau\in[\mathfrak{c},\mathfrak{a}]} \frac{\rho\varsigma}{\rho\varsigma + |\wedge(\tau) + \delta\int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\tau,\mathfrak{v})\vartheta(\tau)\nu\mathfrak{v} - \wedge(\tau) - \delta\int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\tau,\mathfrak{v})\vartheta(\tau)\nu\mathfrak{v}|} \\
&= \sup_{\tau\in[\mathfrak{c},\mathfrak{a}]} \frac{\rho\varsigma}{\rho\varsigma + |\delta\int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\tau,\mathfrak{v})\vartheta(\tau)\nu\mathfrak{v} - \delta\int_{\mathfrak{c}}^{\mathfrak{a}} \mho(\tau,\mathfrak{v})\vartheta(\tau)\nu\mathfrak{v}|} \\
&= \sup_{\tau\in[\mathfrak{c},\mathfrak{a}]} \frac{\rho\varsigma}{\rho\varsigma + |\mho(\tau,\mathfrak{v})\vartheta(\tau) - \mho(\tau,\mathfrak{v})\mathcal{Q}(\tau)|(\delta\int_{\mathfrak{c}}^{\mathfrak{a}}\nu\mathfrak{v})} \\
&\leq \sup_{\tau\in[\mathfrak{c},\mathfrak{a}]} \frac{\varsigma}{\varsigma + |\vartheta(\tau) - \mathcal{Q}(\tau)|} \\
&\leq \mathcal{W}(\vartheta(\tau),\mathcal{Q}(\tau),\varsigma),
\end{aligned}$$

Thus, all the conditions of Theorem (1) are satisfied and operator $\mathfrak{p}$ has a unique fixed point. This proves that (10) has a unique solution. □

Example 4. *Consider the integral equation*

$$\vartheta(\tau) = |\cos\tau| + \frac{1}{7}\int_0^1 \mathfrak{v}\vartheta(\mathfrak{v})\nu\mathfrak{v}, \quad \textit{for all} \quad \mathfrak{v} \in [0,1].$$

Then, it has a solution in $\Im$.

Proof. Let $\mathfrak{p}: \Im \to \Im$ be defined by

$$\mathfrak{p}\vartheta(\tau) = |\cos\tau| + \frac{1}{7}\int_0^1 \mathfrak{v}\vartheta(\mathfrak{v})\nu\mathfrak{v},$$

and set $\mho(\tau,\mathfrak{v})\vartheta(\tau) = \frac{1}{7}\mathfrak{v}\vartheta(\mathfrak{v})$ and $\mho(\tau,\mathfrak{v})\mathcal{Q}(\tau) = \frac{1}{7}\mathfrak{v}\mathcal{Q}(\mathfrak{v})$, where $\vartheta, \mathcal{Q} \in \Im$, and for all $\tau, \mathfrak{v} \in [0,1]$. Then, we have

$$\begin{aligned} |\mho(\tau,\mathfrak{v})\vartheta(\tau) - \mho(\tau,\mathfrak{v})\mathcal{Q}(\tau)| &= |\frac{1}{7}\mathfrak{v}\vartheta(\mathfrak{v}) - \frac{1}{7}\mathfrak{v}\mathcal{Q}(\mathfrak{v})| \\ &= \frac{\mathfrak{v}}{7}|\vartheta(\mathfrak{v}) - \mathcal{Q}(\mathfrak{v})| \leq |\vartheta(\mathfrak{v}) - \mathcal{Q}(\mathfrak{v})|. \end{aligned}$$

Furthermore, see that $\frac{1}{7}\int_0^1 \mathfrak{v}\nu\mathfrak{v} = \frac{1}{7}\left(\frac{(1)^2}{2} - \frac{(0)^2}{2}\right) = \rho < 1$, where $\delta = \frac{1}{7}$. It is easy to verify all other conditions of the preceding application and hence a solution exists in $\Im$. □

Indeed, the closed form of the unique solution for the integral equation of Example 4 using the software is found to be

$$\mathfrak{a}(\tau) = |\cos\tau| + 0.0574712,$$

and the graph of the solution is shown in Figure 1.

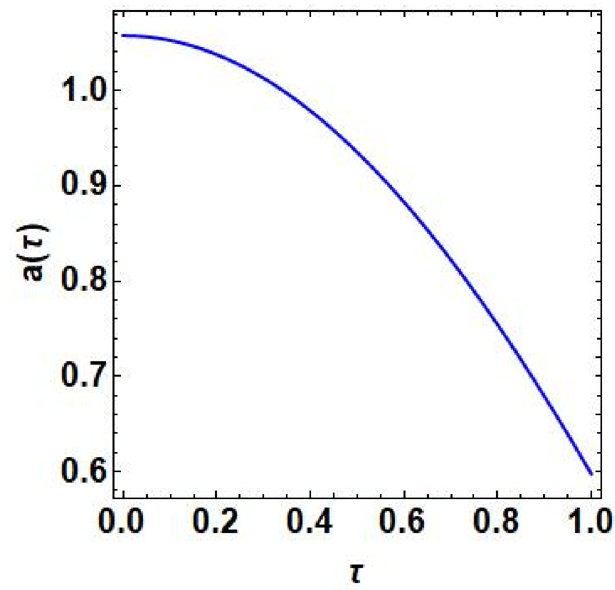

Figure 1. Solution of Example 4.

4. Conclusions

In the above work, we have defined neutrosophic rectangular triple-controlled metric spaces and defined some topological properties of such spaces. We have proven the existence of a unique fixed point under various contractive conditions in these spaces, supported with non-trivial examples. To substantiate the derived results, we have presented the existence of a unique analytical solution to the Fredholm integral equations, outperforming those present in the literature. We have also presented the closed form of the unique solution to Example 4 to supplement the derived results. It is an open problem to explore the possibility of extending our results thorugh various contractive conditions, such as Meir-Keeler contractions, Ciric-type contractions or by using more generalized verisons of the defined spaces.

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