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Article

Noncommutative Integration of Generalized Diffusion PDE

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Abstract: The article is devoted to the noncommutative integration of a diffusion partial differential equation (PDE). Its generalizations are also studied. This is motivated by the fact that many existing approaches for solutions of PDEs are based on evolutionary operators obtained as solutions of the corresponding stochastic PDEs. However, this is restricted to PDEs of an order not higher than 2 over the real or complex field. This article is aimed at extending such approaches to PDEs of an order higher than 2. For this purpose, measures and random functions having values in modules over complexified Cayley–Dickson algebras are investigated. Noncommutative integrals of hypercomplex random functions are investigated. By using them, the noncommutative integration of the generalized diffusion PDE is scrutinized. Possibilities are indicated for a subsequent solution of higher-order PDEs using their decompositions and noncommutative integration.

Keywords: diffusion; noncommutative integration; PDE; generalized; Cayley–Dickson algebras

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1. Introduction

For the studies and analysis of dynamical systems and inverse problems, random functions are frequently used. They play a very important role in the integration of partial differential equations (PDEs), diffusion-type PDEs (for example, [1–4]). For these purposes, matrix or operator measures are studied and used [1–3,5,6]. In [1,6], real and complex measures, stochastic PDEs, and their applications to solutions of second-order PDEs were described over real and complex fields. In [2,5], random functions, stochastic processes, Markov processes, and stochastic PDEs are described, and their applications to solutions of PDEs using evolutionary operators, generators of their semigroups are given. In [3,4], these themes are also provided, but the emphasis is on Feynman-type integrals, and their convergence in suitable domains of function spaces.

However, there are restrictions for these approaches because they work for partial differential operators (PDOs) of an order not higher than 2. Indeed, they are based on complex modifications of Gaussian measures. Nevertheless, if a characteristic function $\phi(t)$ of a measure has the form $\phi(t) = \exp(Q(t))$ where $Q(t)$ is a polynomial, then its degree is not higher than 2 according to the Marcinkievich theorem (Chapter II, §12 in [7]).

On the other side, hypercomplex numbers open new opportunities in these areas. For example, Dirac used the complexified quaternion algebra $\mathbf{H}_C$ for the solution of the Klein–Gordon hyperbolic PDE of second order with constant coefficients [8]. This is important in spin quantum mechanics, because $U(2) \subset \mathbf{H}$. It was proved in [9] that, in many variants, it is possible to reduce a PDE problem of a higher order to a subsequent solution of PDEs of an order not higher than 2 with hypercomplex coefficients. In general, the complex field is insufficient for this purpose.

On the other hand, algebras of hypercomplex numbers, in particular, the Cayley–Dickson algebras $\mathcal{A}_r$ over the real field $\mathbf{R}$ are natural generalizations of the complex field, where $\mathcal{A}_2 = \mathbf{H}$ is the quaternion skew field, $\mathcal{A}_3 = \mathbf{O}$ denotes the octonion algebra, $\mathcal{A}_0 = \mathbf{R}$, $\mathcal{A}_1 = \mathbf{C}$ denotes the complex field. Then, each subsequent algebra $\mathcal{A}_{r+1}$ is obtained from the preceding algebra $\mathcal{A}_r$ with the doubling procedure using the doubling generator [10–12] (see also Appendix A).

They are widely applied in PDEs, noncommutative analysis, mathematical physics, quantum field theory, hydrodynamics, industrial and computational mathematics, and noncommutative geometry [8,13–21].

This article is motivated by the fact that many existing approaches for solutions of PDEs are based on evolutionary operators obtained as solutions of the corresponding stochastic PDEs. However, this is restricted to PDEs of an order not higher than 2 over a real or complex field. This article is aimed at extending such approaches to PDEs of an order higher than 2.

Previously, measures with values in the complexified Cayley–Dickson algebra $\mathcal{A}_{r,C}$ were studied in [22]. They appear naturally with a solution of a second-order hyperbolic PDE with Cayley–Dickson coefficients. In this work, the results and notation of [22] are used. They are recalled in Appendix B. Relations between different forms of the diffusion PDE (such as backward Kolmogorov, Fokker-Planck-Kolmogorov, and stochastic) are discussed.

This article is devoted to dynamical systems such as hypercomplex generalized diffusion PDEs. For this purpose, measures and random functions having values in modules over the complexified Cayley–Dickson algebras are investigated. An integration of generalized diffusion processes is investigated. For their study, hypercomplex transition measures are used. Noncommutative integrals of hypercomplex random functions are studied. The existence of novel random functions and Markov processes over hypercomplex numbers is studied in Theorem 1, Corollary 1. Integrals of hypercomplex random functions and operators acting on them are investigated in Theorems 2–5. Properties of hypercomplex stochastic integrals are described in Propositions 1–3. In Theorem 6, their stochastic continuity is investigated. Necessary specific novel definitions are given. Notation is described in detail. Lemmas 1–5 are given in order to prove the theorems and propositions. These lemmas concern estimates of hypercomplex stochastic integrals, which was not performed before. In Theorems 7 and 8, and Corollary 2, solutions of generalized diffusion PDEs with hypercomplex random functions and operators are scrutinized. Ordered products of appearing operators are studied. Generators of semigroups of evolutionary operators are also studied for the generalized diffusion PDE in its stochastic form over the complexified Cayley–Dickson algebra. The stochastic Cauchy problem related with the generalized diffusion PDE is investigated for modules over complexified Cayley–Dickson algebras. Basics of hypercomplex numbers and measures are recalled in Appendices A and B (Formulas (A1)–(A40)). This opens new possibilities for a subsequent solution of higher-order PDEs using their decompositions and noncommutative integration, which is also discussed in the conclusion.

The main results of this work were obtained for the first time. The noncommutative integration developed in this paper permits to subsequently analyze and integrate PDEs of orders higher than 2 of different types, including parabolic, elliptic, and hyperbolic. The obtained results open new opportunities for subsequent studies of PDEs and their solutions regarding inverse problems.

2. Generalized Diffusion PDEs

Definition 1. *Suppose that $\mathbf{\Lambda}$ is an additive group contained in $\mathbf{R}$. Suppose also that T is a subset in $\mathbf{\Lambda}$ containing a point t_0. Let $X_t = X$ be a locally $\mathbf{R}$-convex space that is also a two-sided $\mathcal{A}_{r,C}$-module for each $t \in T$, where $2 \le r < \infty$. Then,*

$$(\tilde{X}_T, \tilde{\mathsf{U}}) := \prod_{t\in T}(X_t, \mathsf{U}_t)$$

for the product of measurable spaces, where U_t is the Borel σ-algebra of X_t, $\tilde{\mathsf{U}}$ is an algebra of cylindrical subsets of $\tilde{X}_T$ generated by projections $\tilde{\pi}_q : \tilde{X}_T \to X^q$, where $X^q := {}_l\prod_{t\in q} X_t$ is a left-ordered direct product, $q \subset T$ is a finite subset of T, $X^{\{t\}} = X_t$, $X^{t_1,\ldots,t_{n+1}} = X_{t_{n+1}} \times (X^{t_1,\ldots,t_n})$ for each $t_1 < \ldots < t_{n+1}$ in T.

Function $P(t_1, x_1, t_2, A)$ with values in the complexified Cayley–Dickson algebra $\mathcal{A}_{r,C}$ for each $t_1 < t_2 \in T$, $x_1 \in X_{t_1}$, $A \in \mathsf{U}_{t_2}$ is called a transitional measure if it satisfies the following conditions:

$$\text{the set function } \nu_{x_1,t_1,t_2}(A) := P(t_1, x_1, t_2, A) \text{ is a measure on } (X_{t_2}, \mathsf{U}_{t_2}); \tag{1}$$

the function $\alpha_{t_1,t_2,A}(x_1) := P(t_1, x_1, t_2, A)$ of the variable x_1 is U_{t_1}-measurable, that is,

$$\alpha^{-1}_{t_1,t_2,A}(\mathcal{B}(\mathcal{A}_{r,C})) \subset \mathsf{U}_{t_1}; \tag{2}$$

$$P(t_1, x_1, t_2, A) = \int_{X_z} P(t, y, t_2, A) P(t_1, x_1, t, dy) \text{ for each } t_1 < t < t_2 \in T \tag{3}$$

so that $P(t, y, t_2, A)$ as the function by y is in $L^1((X_t, \mathsf{U}_t), \nu_{x_1,t_1,t}, \mathcal{A}_{r,C})$. A transition measure $P(t_1, x_1, t_2, A)$ is called unital if

$$P(t_1, x_1, t_2, X_{t_2}) = 1 \text{ for each } t_1 < t_2 \in T. \tag{4}$$

Then, for each finite set $q = (t_0, t_1, \ldots, t_{n+1})$ of points in T, such that $t_0 < t_1 < \ldots < t_{n+1}$; there is defined a measure in X^g

$$\mu^q_{x_0}(D) = \int_D {}_l\prod_{k=1}^{n+1} P(t_{k-1}, x_{k-1}, t_k, dx_k),\ D \in \mathsf{U}^g := {}_l\prod_{t \in g} \mathsf{U}_t, \tag{5}$$

where $g = q \setminus \{t_0\}$, variables $x_1, \ldots, x_{n+1}$ are such that $(x_1, \ldots, x_{n+1}) \in D$, $x_0 \in X_{t_0}$ is fixed.

Let the transitional measure $P(t, x_1, t_2, dx_2)$ be unital. Then, for the product $D = D_2 \times (X_{t_j} \times D_1)$, where $D_1 \in {}_l\prod_{i=1}^{j-1} \mathsf{U}_{t_i}$, $D_2 \in {}_l\prod_{i=j+1}^{n+1} \mathsf{U}_{t_i}$, the equality

$$\mu^q_{x_0}(D) = \int_{D_2 \times D_1} \left[{}_l\prod_{k=j+1}^{n+1} P(t_{k-1}, x_{k-1}, t_k, dx_k) \right]$$

$$\times \left[\left[\int_{X_{t_j}} P(t_{j-1}, x_{j-1}, t_j, dx_j) \right] \left[{}_l\prod_{k=1}^{j-1} P(t_{k-1}, x_{k-1}, t_k, dx_k) \right] \right] = \mu^r_{x_0}(D_2 \times D_1) \tag{6}$$

is fulfilled, where $r = q \setminus \{t_j\}$. Equation (6) implies that

$$[\mu^q_{x_0}]^{\pi^q_v} = \mu^v_{x_0} \tag{7}$$

for each $v < q$, where finite sets are ordered by inclusion: $v < q$ if and only if $v \subset q$, where $\pi^q_w : X^g \to X^w$ is the natural projection, $g = q \setminus \{t_0\}$, $w = v \setminus \{t_0\}$.

Υ_T denotes the family of all finite linearly ordered subsets q in T, such that $t_0 \in q \subset T$, $v \le q \in \Upsilon_T$, $\pi_q : \tilde{X}_T \to X^g$ is the natural projection, $g = q \setminus \{t_0\}$. Hence, Conditions (4), (5), (7) imply that: $\{\mu^q_{x_0}; \pi^q_v; \Upsilon_T\}$ is the consistent family of measures that induces a cylindrical distribution $\tilde{\mu}_{x_0}$ on the measurable space $(\tilde{X}_T, \tilde{\mathsf{U}})$ such that

$$\tilde{\mu}_{x_0}(\pi_q^{-1}(D)) = \mu^q_{x_0}(D) \tag{8}$$

for each $D \in \mathsf{U}^g$.

The cylindrical distribution given by Formulas (1)–(5), (7) and (8) is called the $\mathcal{A}_{r,C}$-valued Markov distribution with time t in T.

Remark 1. *Let $X_t = X$ for each $t \in T$, $\tilde{X}_{t_0,x_0} := \{x \in \tilde{X}_T : x(t_0) = x_0\}$. Put $\bar{\pi}_q : x \mapsto x_q$ for each $x = x(t)$ in $\tilde{X}_T$, where x_q is defined on $q = (t_0, \ldots, t_{n+1}) \in \Upsilon_T$ such that $x_q(t) = x(t)$ for* *each $t \in q$. To an arbitrary function $F : \tilde{X}_T \to \mathcal{A}^l_{r,C}$ a function can be posed $(S_qF)(x) := F(x_q) = F_q(y_0, \dots, y_n)$, where $y_j = x(t_j)$, $F_q : X^q \to \mathcal{A}^l_{r,C}$, $l \in \mathbf{N}$. Put*

$$\mathsf{F} := \{F | F : \tilde{X}_T \to \mathcal{A}^l_{r,C}, S_qF \text{ is } \mathsf{U}^q - \text{measurable for each } q \in \Upsilon_T\}.$$

If $F \in \mathsf{F}$, $\tau = t_0 \in q$, $t_0 < t_1 < \dots < t_{n+1}$, then the integral

$$J_q(F) = \int_{X^q} (S_qF)(x_0, \dots, x_n)\ {}_l\prod_{k=1}^{n+1} P(t_{k-1}, x_{k-1}, t_k, dx_k) \tag{9}$$

can be defined whenever it converges.

Definition 2. *A function F is called integrable relative to a Markov cylindrical distribution μ_{x_0} if the limit*

$$\lim_{q \in \Upsilon_T} J_q(F) =: J(F) \tag{10}$$

along the generalized net by finite subsets $q = (t_0, \dots, t_{n+1}) \in \Upsilon_T$ of T exists (see (9)). This limit is called a functional integral relative to the Markov cylindrical distribution:

$$J(F) = \int_{\tilde{X}_{t_0,x_0}} F(x)\mu_{x_0}(dx). \tag{11}$$

Remark 2. Spatially homogeneous transition measure. *Suppose that $P(t, A)$ is an $\mathcal{A}_{r,C}$-valued measure on (X, U) for each $t \in T$ such that $A - x \in \mathsf{U}$ for each $A \in \mathsf{U}$ and $x \in X$, where $A \in \mathsf{U}$, X is a locally $\mathbf{R}$-convex space which is also a two-sided $\mathcal{A}_{r,C}$-module, U is an algebra of subsets of X. Suppose also that P is a spatially homogeneous transition measure:*

$$P(t_1, x_1, t_2, A) = P(t_2 - t_1, A - x_1) \tag{12}$$

for each $A \in \mathsf{U}$, $t_1 < t_2 \in T$ and $t_2 - t_1 \in T$ and every $x_1 \in X$, where $P(t, A)$ also satisfies the following condition:

$$P(t_1 + t_2, A) = \int_X P(t_2, A - y)P(t_1, dy) \tag{13}$$

for each $t_1 < t_2$ and $t_1 + t_2$ in T.

Then,

$$\phi(t_1, x_1, t_2, y) := \int_X P(t_1, x_1, t_2, dx) \exp(\mathbf{i}y(x)) \tag{14}$$

is the characteristic functional of transitional measure $P(t_1, x_1, t_2, dx)$ for each $t_1 < t_2 \in T$ and each $x_1 \in X$, where $X^*_{\mathbf{R}}$ notates the topologically dual space of all continuous $\mathbf{R}$-linear real-valued functionals y on X, $y \in X^*_{\mathbf{R}}$. Particularly for P satisfying Conditions (12) and (13) with $t_0 = 0$ its characteristic functional ϕ satisfies the equalities:

$$\phi(t_1, x_1, t_2, y) = \psi(t_2 - t_1, y) \exp(\mathbf{i}y(x_1)), \tag{15}$$

where

$$\psi(t, y) := \int_X P(t, dx) \exp(\mathbf{i}y(x)) \quad \text{and} \tag{16}$$

$$\psi(t_1 + t_2, y) = \psi(t_2, y)\psi(t_1, y) \tag{17}$$

for each $t_1 < t_2 \in T$ and $t_2 - t_1 \in T$ and $t_1 + t_2 \in T$ respectively and $y \in X^*_{\mathbf{R}}$, $x_1 \in X$, since $Z(\mathcal{A}_{r,C}) = \mathbf{C}$.

Remark 3. *If T is a $T_1 \cap T_{3.5}$ topological space, we denote by $C^0_b(T, H)$ the Banach space of all continuous bounded functions $f : T \to H$ supplied with the norm:*

$$\|f\|_{C^0} := \sup_{t \in T} \|f(t)\|_H < \infty, \tag{18}$$

where H is a Banach space over $\mathbf{R}$ that may also be a two-sided $\mathcal{A}_{r,C}$-module. If T is compact, then $C^0_b(T,H)$ is isomorphic with the space $C^0(T,H)$ of all continuous functions $f : T \to H$.

For a set T and a complete locally $\mathbf{R}$-convex space H that may also be a two-sided $\mathcal{A}_{r,C}$-module, consider product $\mathbf{R}$-convex space $H^T := \prod_{t\in T} H_t$ in the product topology, where $H_t := H$ for each $t \in T$.

Suppose that B is a separating algebra on the space either $X := X(T,H) = L^q(T,\mathcal{B}(T),\lambda,H)$ or $X := X(T,H) = C^0_b(T,H)$ or on $X = X(T,H) = H^T$, where $\lambda : \mathcal{B}(T) \to [0,\infty)$ is a σ-additive measure on the Borel σ-algebra $\mathcal{B}(T)$ on T, $1 \le q \le \infty$. Consider a random variable $\xi : \omega \mapsto \xi(t,\omega)$ with values in (X,B), where $t \in T$, $\omega \in \Omega$, $(\Omega,\mathcal{R},P)$ is a measure space with an $\mathcal{A}_{r,C}$-valued measure P, $P : \mathcal{R} \to \mathcal{A}_{r,C}$.

Events $S_1,\dots,S_n$ are independent in total if $P({}_l\prod_{k=1}^n S_k) = {}_l\prod_{k=1}^n P(S_k)$. Subalgebras $\mathcal{R}^k \subset \mathcal{R}$ are independent if all collections of events $S_k \in \mathcal{R}^k$ are independent in total, where $k = 1,\dots,n$, $n \in \mathbf{N}$. To each collection of random variables ξ_γ on $(\Omega,\mathcal{R})$ with $\gamma \in \Upsilon$ is related the minimal algebra $\mathcal{R}_\Upsilon \subset \mathcal{R}$ for which all ξ_γ are measurable, where Υ is a set. Collections $\{\xi_\gamma : \gamma \in \Upsilon^l\}$ are independent if $\mathcal{R}_{\Upsilon^l}$, where $\Upsilon^l \subset \Upsilon$ for each $l = 1,\dots,n$, $n \in \mathbf{N}$.

For $X = C^0_b(T,H)$ or $X = H^T$ define $X(T,H;(t_1,\dots,t_n);(z_1,\dots,z_n))$ as a closed submanifold in X of all $f : T \to H$, $f \in X$ such that $f(t_1) = z_1,\dots,f(t_n) = z_n$, where $t_1,\dots,t_n$ are pairwise distinct points in T and $z_1,\dots,z_n$ are points in H. For $n = 1$ and $t_0 \in T$ and $z_1 = 0$, we denote $X_0 := X_0(T,H) := X(T,H;t_0;0)$.

Definition 3. *Suppose that H is a real Banach space that may also be a two-sided $\mathcal{A}_{r,C}$-module. Consider a random function $w(t,\omega)$ with values in the space H as a random variable such that:*

$$\text{the random variable } \omega(t,\omega) - \omega(u,\omega) \text{ has a distribution } \mu^{F_{t,u}}, \tag{19}$$

where μ is an $\mathcal{A}_{r,C}$-valued measure on $(X(T,H),\mathsf{B})$, $\mu^g(A) := \mu(g^{-1}(A))$ for $g : X \to H$ such that $g^{-1}(\mathcal{R}_H) \subset \mathsf{B}$ and each $A \in \mathcal{R}_H$. Thereby, $F_{t,u}$ a $\mathbf{R}$-linear operator $F_{t,u} : X \to H$ is denoted, which is prescribed by the following formula:

$$F_{t,u}(w) := w(t,\omega) - w(u,\omega)$$

for each $u < t$ in T, where $\mathcal{R}_H$ is a separating algebra of H such that $F^{-1}_{t,u}(\mathcal{R}_H) \subset \mathsf{B}$ for each $u < t$ in T, where $T = [0,b]$ with $0 < b < \infty$ or $T = [0,\infty)$, $\Omega \ne \emptyset$;

$$\text{the vectors } w(t_m,\omega) - w(t_{m-1},\omega),\dots,w(t_1,\omega) - w(0,\omega) \text{ and } w(0,\omega) \tag{20}$$

are mutually independent for each chosen $0 < t_1 < \ldots < t_m$ in T and each $m \ge 2$, where $\omega \in \Omega$.

Then, $\{w(t) : t \in T\}$ is the random function with independent increments, where $w(t)$ is the shortened notation of $w(t,\omega)$.

In addition,

$$w(0,\omega) = 0. \tag{21}$$

Remark 4. *Random function $w(t,\omega)$ satisfying Conditions (19)–(21) in Definition 3 possesses a Markovian property with transitional measure*

$P(u,x,t,A) = \mu^{F_{t,u}}(A - x)$ (see also (10)-(17)).

As usual, it is put for the expectation

$$E_P f = \int_\Omega f(\omega)P(d\omega) = P^L(f)$$

of a random variable $f : \Omega \to \mathcal{A}^h_{r,C}$ whenever this integral exists, where $P = P_{[r]}$ is the $\mathcal{A}_{r,C}$-valued measure on a measure space $(\Omega_{[r]}, {}_{[r]}\mathcal{F})$ shortly denoted by $(\Omega,\mathcal{F})$, where f is $(\mathcal{F},\mathcal{B}(\mathcal{A}^h_{r,C}))$-measurable, $h \in \mathbf{N}$, $\mathcal{B}(\mathcal{A}^h_{r,C})$ denotes the Borel σ-algebra on $\mathcal{A}^h_{r,C}$. If P is specified, it may be shortly *written E instead of E_P. If $\mathcal{G}$ is a sub-σ-algebra in the σ-algebra $\mathcal{F}$ and if there exists a random variable $g : \Omega \to \mathcal{A}^h_{r,C}$ such that g is $(\mathcal{G}, \mathcal{B}(\mathcal{A}^h_{r,C}))$-measurable and*

$$\int_A f(\omega)P(d\omega) = \int_A g(\omega)P(d\omega)$$

for each $A \in \mathcal{G}$, then g is called the conditional expectation relative to $\mathcal{G}$ and denoted by $g = E(f|\mathcal{G})$.

An operator $J : \mathcal{A}^n_{r,C} \to \mathcal{A}^h_{r,C}$ is called right $\mathcal{A}_{r,C}$-linear in the weak sense if

$$J(xb + yc) = (Jx)b + (Jy)c \tag{22}$$

for each x and y in $\mathbf{R}^n$ and b and c in $\mathcal{A}_{r,C}$, where real field $\mathbf{R}$ is canonically embedded into the complexified Cayley–Dickson algebra $\mathcal{A}_{r,C}$ as $\mathbf{R}i_0$, $i_0 = 1$. Over the algebra $\mathbf{H_C} = \mathcal{A}_{2,C}$, this gives right linear operators $J(xb + yc) = (Jx)b + (Jy)c$ for each x and y in $\mathcal{A}^n_{2,C}$ and b and c in $\mathcal{A}_{2,C}$, since $\mathbf{H_C}$ is associative. For brevity, we omitted "in the weak sense". We notate such a set of operators with $L_r(\mathcal{A}^n_{r,C}, \mathcal{A}^h_{r,C})$. Then

$$\|J\| = \sup_{z\neq 0;\ z\in\mathcal{A}^n_{r,C}} \frac{\|Jz\|}{\|z\|},$$

where $z = (z_1, \dots, z_n)$, $z_j \in \mathcal{A}_{r,C}$ for each $j \in \{1, \dots, n\}$, where

$$\|z\|^2 = \sum_{j=1}^{n} \|z_j\|^2,$$

$\|a\|^2 = 2|b|^2 + 2|c|^2$ for each $a = b + \mathbf{i}c$ in $\mathcal{A}_{r,C}$ with b and c in $\mathcal{A}_r$ (see also Remark 2.1 of [22]).

In particular, it is useful to consider the following case: $w = J\xi + p$, where ξ is a $\mathbf{R}^{2n}$-valued random variable on a measurable space $(\Omega_{[0]}, {}_{[0]}\mathcal{F})$ and with a probability measure $P_{[0]} : {}_{[0]}\mathcal{F} \to [0,1]$, where $p \in \mathcal{A}^n_{r,C}$, where $\mathbf{R}^{2n}$ is embedded into $\mathcal{A}^n_{r,C}$ as $i_0\mathbf{R}^n + i_0\mathbf{i}\mathbf{R}^n$, where $J \in L_r(\mathcal{A}^n_{r,C}, \mathcal{A}^n_{r,C})$. This means that ξ is $({}_{[0]}\mathcal{F}, \mathcal{B}(\mathbf{R}^{2n}))$-measurable, while w is $({}_{[r]}\mathcal{F}, \mathcal{B}(\mathcal{A}^n_{r,C}))$-measurable, where $(\Omega_{[r]}, {}_{[r]}\mathcal{F})$ is a measurable space, $P_{[r]} : {}_{[r]}\mathcal{F} \to \mathcal{A}_{r,C}$ is a measure.

Assume that there is an injection $\theta : (\Omega_{[0]}, {}_{[0]}\mathcal{F}) \to (\Omega_{[r]}, {}_{[r]}\mathcal{F})$ and $P_{[0]}$ has an extension $\mathsf{P} = P^\theta_{[0]}$ on $(\Omega_{[r]}, {}_{[r]}\mathcal{F})$ such that $P^\theta_{[0]}(\Omega_{[r]} \setminus \theta(\Omega_{[0]})) = 0$, $P^\theta_{[0]}(A) = P_{[0]}(\theta^{-1}(A \cap \theta(\Omega_{[0]}))$ for each $A \in {}_{[r]}\mathcal{F}$ and $|P_{[r]}|(\Omega_{[r]} \setminus \theta(\Omega_{[0]})) = 0$. Then, it may be the case that P and $P_{[r]}$ are related by Formulas 2.4(2), 2.4(3) of [22] with the use of $U = U_{[r]} = J^2$ and $U_{[0]} = I$ using the $\mathcal{A}_{r,C}$-analytic extension. If $f = F(w)$, where $F : \mathcal{A}^n_{r,C} \to \mathcal{A}^h_{r,C}$ is a Borel measurable function; then, there exists a Borel measurable function $G : \mathbf{R}^{2n} \to \mathcal{A}^h_{r,C}$ such that $G(\xi) = f$. Therefore, if $u : \mathcal{A}^h_{r,C} \to \mathbf{R}$ is a Borel measurable function, using Formulas 2.4(2), 2.4(3) of [22] we put

$$Eu(f) = \int_{\Omega_{[0]}} u(G(\xi(\omega)))P_{[0]}(d\omega).$$

If

$$\int_{A_{[0]}} u(G(\xi(\omega))P_{[0]}(d\omega) = \int_{A_{[0]}} g(\theta(\omega))P_{[0]}(d\omega)$$

for each $A \in \mathcal{G}$, where $g : \Omega_{[r]} \to \mathbf{R}$ is $(\mathcal{G}, \mathcal{B}(\mathbf{R}))$-measurable, $A_{[0]} = \theta^{-1}(A \cap \theta(\Omega_{[0]}))$, ${}_{[0]}\mathcal{G} = \theta^{-1}(\mathcal{G} \cap \theta(\Omega_{[0]}))$, then g is called the conditional expectation of $u(f)$ relative to $\mathcal{G}$ and denoted by $E(u(f)|\mathcal{G}) = g$, since $\mathsf{P}(\Omega_{[r]} \setminus \theta(\Omega_{[0]})) = 0$ and $|P_{[r]}|(\Omega_{[r]} \setminus \theta(\Omega_{[0]})) = 0$, where $\mathcal{G}$ is a σ-subalgebra in ${}_{[r]}\mathcal{F}$.

This convention is used if some other is not specified.

Let $L_{r,i}(\mathcal{A}^n_{r,C}, \mathcal{A}^h_{r,C})$ denote a family of all right $\mathcal{A}_{r,C}$-linear operators J from $\mathcal{A}^n_{r,C}$ into $\mathcal{A}^h_{r,C}$ fulfilling the condition

$$J(\mathcal{A}^n_r) \subset \mathcal{A}^h_r. \tag{23}$$

Theorem 1. *Suppose that either $X = C^0_b(T,H)$ or $X = H^T$, where $H = \mathcal{A}^n_{r,C}$ with $n \in \mathbf{N}$, $2 \le r < \infty$, either $T = [0,s]$ with $0 < s < \infty$ or $T = [0,\infty)$. Then, there exists a family Ψ of pairwise inequivalent Markovian random functions with $\mathcal{A}_{r,C}$-valued transition measures of the type $\mu_{Ut,pt}$ (see Definition 2.4 of [22]) on X of a cardinality card$(\Psi) = \mathsf{c}$, where $\mathsf{c} = 2^{\aleph_0}$, $0 < t \in T$.*

Proof. Naturally, the algebra $\mathcal{A}^n_{r,C} = \otimes^n_{j=1}\mathcal{A}_{r,C}$, if considered to be a linear space over $\mathbf{R}$, also possesses a structure of the $\mathbf{R}$-linear space isomorphic with $\mathbf{R}^{2^{r+1}n}$. Therefore, the Borel σ-algebra $\mathcal{B}(\mathcal{A}^n_{r,C})$ of the algebra $\mathcal{A}^n_{r,C}$ is isomorphic with $\mathcal{B}(\mathbf{R}^{2^{r+1}n})$. So, put $P(t,A) = \mu_{Ut,pt}(A)$ for each $0 < t \in T$ and $A \in \mathcal{B}(H)$, where an operator U and a vector p are marked, satisfying conditions of Definitions 2.4 and 2.3(α) of [22].

Naturally, an embedding of $\mathbf{R}^n$ into $\mathcal{A}^n_{r,C}$ exists as $i_0\mathbf{R}^n$, where $i_0 = 1$. If $\xi(t)$ is an $\mathbf{R}^n$-valued random function, J is a right $\mathcal{A}_{r,C}$-linear operator $J : \mathcal{A}^n_{r,C} \to \mathcal{A}^n_{r,C}$ satisfying the condition $J(\mathcal{A}^n_r) \subset \mathcal{A}^n_r$, $v \in \mathcal{A}^n_{r,C}$ (see (22), (23) in Remark 4), then generally, $w(t) = J\xi(t) + vt$ is an $\mathcal{A}^n_{r,C}$-valued random function, where $0 \le t \in T$, $w(t)$ is a shortened notation of $w(t,\omega)$.

Operators $B_j^{\pm 1/2}$ exist (see, for example, Chapter IX, Section 13 in [23].), since B_j is positive definite for each j. On the Cayley–Dickson algebra $\mathcal{A}_r$, function $\sqrt{a}$ exists (see §3.7 and Lemma 5.16 in [19]). It has an extension on $\mathcal{A}_{r,C}$ and its branch, such that $\sqrt{a} > 0$ for each $a > 0$ can be specified by the following. Take an arbitrary $a = a_0 + \mathbf{i}a_1 \in \mathcal{A}_{r,C}$ with $a_0 \in \mathcal{A}_r$ and $a_1 \in \mathcal{A}_r$. Put $a_{0,0} = Re(a_0)$, $a_{1,0} = Re(a_1)$, $a_0{}' = a_0 - a_{0,0}$, $a_1{}' = a_1 - Re(a_1)$. If $a_{0,0} \neq 0$ and $a_{1,0} \neq 0$, a can be presented in the form $a = (\alpha + \mathbf{i}\beta)(u + \mathbf{i}v')$ with $\alpha \in \mathbf{R}$, $\beta \in \mathbf{R}$, $u \in \mathcal{A}_r$, $v' \in \mathcal{A}_r$, $Re(v') = 0$. Therefore, in the latter case $\sqrt{a} = \sqrt{\alpha + \mathbf{i}\beta}\sqrt{u + \mathbf{i}v'}$, since $\mathbf{C} = Z(\mathcal{A}_{r,C})$. If a is such that $a_{0,0} = 0$ and $a_{1,0} \neq 0$; then, for $b = \mathbf{i}a$, there are $b_{0,0} = -a_{1,0} \neq 0$ and $b_{1,0} = 0$. On the other hand, for a with $a_{1,0} = 0$, equation $(\gamma + \mathbf{i}\delta)^2 = a_0 + \mathbf{i}a_1{}'$ has a solution with γ and δ in $\mathcal{A}_r$, since, by utilizing the standard basis of the complexified Cayley–Dickson algebra, this equation can be written as the quadratic system in 2^r complex variables $\gamma_0 + \mathbf{i}\delta_0, \dots, \gamma_{2^r-1} + \mathbf{i}\delta_{2^r-1}$. The latter system has a solution (γ,δ) in $\mathcal{A}^2_r$, since each polynomial over $\mathbf{C}$ has zeros in $\mathbf{C}$ by the principal algebra theorem. Therefore, the initial equation has a solution in $\mathcal{A}_{r,C}$. Thus, the operator $U^{1/2} = \bigoplus_{j=1}^m a_j^{1/2}B_j^{1/2}$ exists and it evidently belongs to $L_r(\mathcal{A}^n_{r,C}, \mathcal{A}^n_{r,C})$.

Particularly, J can be $J = U^{1/2}$, while as $\xi(t)$, it is possible to take a Wiener process with the zero expectation and the unit covariance operator.

If $f \in X$, then $T \ni t \mapsto f(t)$ defines a continuous $\mathbf{R}$-linear projection π_t from X into H. Therefore, $\pi_{t_n} \times (\pi_{t_{n-1}} \times \ldots \times \pi_{t_1})$ provides a continuous $\mathbf{R}$-linear projection π_q from X into H^q for each $0 < t_1 < \ldots < t_n \in T$, where $q = \{t_1, \ldots, t_n\}$. These projections and Borel σ-algebras $\mathcal{B}(H^q)$ on H^q for finite linearly ordered subsets q in T induce an algebra $\mathcal{R}(X)$ of X. Since H^T is supplied with the product Tychonoff topology, a minimal σ-algebra $\mathcal{R}_\sigma(H^T)$ generated by $\mathcal{R}(H^T)$ coincides with the Borel σ-algebra $\mathcal{B}(H^T)$. Topological spaces T and H are separable and relative to the norm topology on $C^0_b(T,H)$; $\mathcal{R}_\sigma(C^0_b(T,H)) = \mathcal{B}(C^0_b(T,H))$ is also obtained.

By virtue of Proposition 2.7 of [22] and Formulas 2.4(2) and 2.4(3) of [22], a characteristic functional of $P_{U,p}(t,A) := \mu_{Ut,pt}$ fulfils Condition (17). It is worth to associate with $P_{U,p}(t,A)$ a spatially homogeneous transition measure $P_{U,p}(t_1,x_1,t_2,A)$ according to Equation (12) in Remark 2. Representation 2.10(2) of [22] implies that a bijective correspondence exists between σ-additive norm-bounded $\mathcal{A}_{r,C}$-valued measures and their characteristic functionals, since it is valid for each real-valued addendum $\mu_{j,k}$ (see, for example, [1,7]) and $Z(\mathcal{A}_{r,C}) = \mathbf{C}$. Moreover, a characteristic functional of the ordered convolution $(\mu * \nu)$ of two σ-additive norm-bounded $\mathcal{A}_{r,C}$-valued measures μ and ν is the ordered product $\hat{\mu} \cdot \hat{\nu}$ of their characteristic functionals $\hat{\mu}$ and $\hat{\nu}$, respectively. Therefore, Conditions (1)–(4) in Definition 1 are satisfied.

Then, Formulas (5), (7) and (8) in Definition 1 together with the data above describe an $\mathcal{A}_{r,C}$-valued Markov cylindrical distribution $P_{U,p}$ on X (see Corollary 2.6 of [22] and

Definition 1), since $t = t_2 - t_1 > 0$ for each $0 < t_1 < t_2 \in T$. The space H is Radon by the Theorem I.1.2 of [1], since H is separable and complete as the metric space. From Theorem 2.3 and Proposition 2.7 of [22], it follows that $P_{U,p}$ is uniformly norm-bounded. In view of Theorem 2.15 and Corollary 2.17 [22], this cylindrical distribution has an extension to a norm-bounded measure $P_{U,p}$ on a completion $\mathcal{R}_P(X)$ of $\mathcal{R}(X)$, where $\mathcal{R}_\sigma(X) = \mathcal{B}(X)$.

Considering different operators U and vectors p, and utilizing the Kakutani theorem (see, for example, in [1]), we infer that there is a family of the cardinality c of pairwise nonequivalent and orthogonal measures of such type $P_{U,p}$ on X since each P has the representation 2.10(2) of [22].

Let $\Omega = \Omega_{[r]}$ be the set of all elementary events

$$\omega := \{f : f \in X(T, H; (t_0, t_1, \dots, t_n); (0, x_1, \dots, x_n))\},$$

where Λ_ω is a finite subset of $\mathbf{N}$, $x_i \in H$, $(t_i : i \in \Lambda_\omega) \in \Upsilon_T$ is a subset of $T \setminus \{t_0\}$ (see Remarks 1 and 3), where $t_0 = 0$, where $t_i < t_j$ for each $i < j$ in Λ_ω. Hence, an algebra $\tilde{\mathrm{U}}$ exists of cylindrical subsets of $X_0(T, H)$ induced by the projections $\pi_q : X_0(T, H) \to H^q$, where $q \in \Upsilon_T$ is a subset in $T \setminus \{0\}$. This procedure induces algebra $\mathcal{R}(\Omega)$ of Ω. So, one can consider a Markovian random function corresponding to $P_{U,p}$ (see Definition 3). $\square$

Corollary 1. *Let $w(t,\omega)$ be a random function given by Theorem 1 with the transitional measure $\mu_{Ut,pt}$ for each $t > 0$, then*

$$E(w(t_2, \omega) - w(t_1, \omega)) = (t_2 - t_1)p \tag{24}$$

and

$$E((w_k(t_2, \omega) - p_k t_2)(w_h(t_1, \omega) - p_h t_1)) = (t_2 - t_1) a_j b_{k-\beta_{j-1}, h-\beta_{l-1}; j} \delta_{j,l} \tag{25}$$

for each k and h in $\{1, \dots, n\}$, where $0 < t_1 < t_2 \in T$, $1 + \beta_{j-1} \le k \le \beta_j$ and $1 + \beta_{l-1} \le h \le \beta_l$, $j = 1, \dots, m$, $l = 1, \dots, m$, where E means the expectation relative to $P^L_{U,p}$.

Proof. By virtue of Theorem 1, random function $w(t,\omega)$ has the transitional measure $P(t_1, x, t_2, A) = \mu^{F_{t_2,t_1}}(A - x) = P^L_{(t_2-t_1)U,(t_2-t_1)p}$, where $x = w(t_1, \omega)$. Therefore, Formulas (24) and (25) follow from Proposition 2.8 and Theorem 2.9 of [22]. $\square$

Definition 4. *Let $(\Omega, \mathcal{F}, P)$ be a measure space with an $\mathcal{A}_{r,C}$-valued σ-additive norm-bounded measure P on a σ-algebra $\mathcal{F}$ of a set Ω with $P(\Omega) = 1$. There is a filtration $\{\mathcal{F}_t : t \in T\}$, if $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2} \subset \mathcal{F}$ for each $t_1 < t_2$ in T, where $\mathcal{F}_t$ is a σ-algebra for each $t \in T$, where either $T = [0, s]$ with $0 < s < \infty$ or $T = [0, \infty)$. A filtration $\{\mathcal{F}_t : t \in T\}$ is called normal if $\{B \in \mathcal{F} : |P|(B) = 0\} \subset \mathcal{F}_0$ and $\mathcal{F}_t = \bigcap_{T \ni v > t} \mathcal{F}_v$ for each $t \in T$.*

Then, if for each $t \in T$ a random variable $u(t) : \Omega \to X$ with values in a topological space X is $(\mathcal{F}_t, \mathcal{B}(X))$-measurable, random function $\{u(t) : t \in T\}$ and filtration $\{\mathcal{F}_t : t \in T\}$ are adapted, where $\mathcal{B}(X)$ denotes the minimal σ-algebra on X containing all open subsets of X (i.e., the Borel σ-algebra). Let $\mathcal{G}$ be a minimal σ-algebra on $T \times \Omega$ generated by sets $(v, t] \times A$ with $A \in \mathcal{F}_v$, also $\{0\} \times A$ with $A \in \mathcal{F}_0$. Let also μ be a σ-additive measure on $(T \times \Omega, \mathcal{G})$ induced by the measure product $\lambda \times P$, where λ is the Lebesgue measure on T. If $u : T \times \Omega \to X$ is $(\mathcal{G}_\mu, \mathcal{B}(X))$-measurable, then u is called a predictable random function, where $\mathcal{G}_\mu$ denotes the completion of $\mathcal{G}$ by $|\mu|$-null sets, where $|\mu|$ is the variation of μ (see Definition 2.10 in [22]).

The random function given by Corollary 1 is called an $\mathcal{A}^n_{r,C}$-valued (U, p)-random function or, in short, U-random function for $p = 0$.

Remark 5. *Random functions described in the proof of Theorem 1 are $\mathcal{A}_{r,C}$ generalizations of the classical Brownian motion processes and of the Wiener processes.*

Let $w(t)$ be the $\mathcal{A}^n_{r,C}$-valued (U, p)-random function provided by Theorem 1 and Corollary 1. Let a normal filtration $\{\mathcal{F}_t : t \in T\}$ on $(\Omega, \mathcal{F}, P)$ be induced by $w(t)$. Therefore, $w(t)$ is $(\mathcal{F}_t, \mathcal{B}(\mathcal{A}^n_{r,C}))$-measurable for all $t \in T$; $w(t_1 + t_2) - w(t_1)$ is independent of any $A \in \mathcal{F}_{t_1}$

for each t_1 and $t_1 + t_2$ in T with $t_2 > 0$. In view of Theorem 1 and Corollary 1, conditions $\mathsf{P}(\Omega \setminus \theta(\Omega_{[0]})) = 0$ and $|P_{[r]}|(\Omega \setminus \theta(\Omega_{[0]})) = 0$ are satisfied, where $\Omega = \Omega_{[r]}$, $\mathcal{F} = {}_{[r]}\mathcal{F}$ (see Remark 4).

Suppose that $\{S(t) : t \in T\}$ is an $L_r(\mathcal{A}^n_{r,C}, \mathcal{A}^h_{r,C})$ valued random function (that is, random operator), $S(t) = S(t, \omega)$, $\omega \in \Omega$ (see also the notation in Remark 4). It is called elementary if a finite partition $0 = t_0 < t_1 < \ldots < t_k = s$ exists, so that

$$S(t) = \sum_{l=0}^{k-1} S_l \cdot ch_{(t_l, t_{l+1}]}, \tag{26}$$

where $S_l : \Omega \to L_r(\mathcal{A}^n_{r,C}, \mathcal{A}^h_{r,C})$ is $(\mathcal{F}_l, \mathcal{B}(L_r(\mathcal{A}^n_{r,C}, \mathcal{A}^h_{r,C}))$-measurable for each $l = 0, \ldots, k-1$, where n and h are natural numbers, where $ch_{(t_l, t_{l+1}]}$ denotes the characteristic function of the segment $(t_l, t_{l+1}] = \{t \in \mathbf{R} : t_l < t \leq t_{l+1}\}$, $T = [0, s]$. A stochastic integral relative to $w(t)$ and the elementary random function $S(t)$ is defined by the formula:

$$\int_0^t S(\tau) dw(\tau) := \sum_{l=0}^{k-1} S_l(w(t_{l+1} \wedge t) - w(t_l \wedge t)), \tag{27}$$

where $t \wedge t' = \min(t, t')$ for each t and t' in T. Similarly, elementary $L_{r,i}(\mathcal{A}^n_{r,C}, \mathcal{A}^h_{r,C})$ random functions and their stochastic integrals are defined. Put

$$< x, y >= x_1 \tilde{y}_1 + \ldots + x_h \tilde{y}_h \tag{28}$$

*for each x and y in $\mathcal{A}^h_{r,C}$,
where $y = (y_1, \ldots, y_h)$ with $y_l \in \mathcal{A}_{r,C}$ for each l, $\tilde{z} = z_0 - z'$ for each $z = z_0 + z'$ in $\mathcal{A}_{r,C}$ with $z_0 \in \mathbf{R}$ and $z' \in \mathcal{A}_{r,C}$, $Re(z') = 0$.*

Q^* denotes an adjoint operator of an $\mathbf{R}$-linear operator $Q : \mathcal{A}^n_{r,C} \to \mathcal{A}^h_{r,C}$, such that

$$< Qx, y >=< x, Q^* y > \tag{29}$$

for each $x \in \mathcal{A}^n_{r,C}$ and $y \in \mathcal{A}^h_{r,C}$.

Then, we put for $Q = A + \mathbf{i}B$ with A and B in $L_{r,i}(\mathcal{A}^n_{r,C}, \mathcal{A}^h_{r,C})$

$$\|Q\|_2^2 = 2Tr(AA^*) + 2Tr(BB^*). \tag{30}$$

Lemma 1. *Let*

(i) $S(t)$ be an elementary $L_r(\mathcal{A}^n_{r,C}, \mathcal{A}^h_{r,C})$-valued random variable with $E(\|S(t)\| | \mathcal{F}_a) < \infty$ P-almost everywhere on $(\Omega, \mathcal{F})$ for each $t \in [a, b]$, and let

(ii) $w = w_0 + \mathbf{i}w_1$ be an $\mathcal{A}^n_{r,C}$-valued random function with U_0- and U_1- random functions w_0 and w_1, respectively, having values in $\mathcal{A}^n_r$, so that U_0 and U_1 belong to $L_{r,i}(\mathcal{A}^n_{r,C}, \mathcal{A}^n_{r,C})$, and operator $U = U_0 + \mathbf{i}U_1$ fulfils Conditions 2.3(α) and of Definition 2.4 of [22], where w_0 and w_1 are independent; $0 \leq a < b < \infty$, $[a, b] \subset T$ (see Definitions 2.10 of [22] , Remarks 4 and 5 above).

Then, $E(\int_a^b S(t) dw(t) | \mathcal{F}_a) = 0$ P-almost everywhere on $(\Omega, \mathcal{F})$.

Proof. This follows from Corollary 1(24), and Formulas (26) and (27), since $0 \leq (b - a)E(\sum_{l=0}^{k-1} \|S_l\| | \mathcal{F}_a) < \infty$ P-almost everywhere and $E(w(t_2, \omega) - w(t_1, \omega)) = 0$ for each $t_2 > t_1$ in $[a, b]$ for the U-random function w. $\square$

Lemma 2. *Let $S = A + \mathbf{i}B$, with A and B belonging to $L_{r,i}(\mathcal{A}^n_{r,C}, \mathcal{A}^h_{r,C})$, where $n \in \mathbf{N}$, $h \in \mathbf{N}$, $2 \leq r < \infty$. Then,*

$$\|S\|_2^2 = Tr[(A + \mathbf{i}B)((A^* - \mathbf{i}B^*)] + Tr[(A - \mathbf{i}B)((A^* + \mathbf{i}B^*)] < \infty \ \text{ and} \tag{31}$$

$$\|S\| \leq \|S\|_2. \tag{32}$$

Proof. Since A and B belong to $L_{r,i}(\mathcal{A}^n_{r,C}, \mathcal{A}^h_{r,C})$, then

$$\|A + \mathbf{i}B\|_2^2 = 2Tr(AA^*) + 2Tr(BB^*) < \infty \tag{33}$$

by Formula (30), where $Tr(AA^*)$ denotes the trace of operator AA^*, as usual. On the other side,

$$[(A + \mathbf{i}B)((A^* - \mathbf{i}B^*)] + [(A - \mathbf{i}B)((A^* + \mathbf{i}B^*)] = 2(AA^* + BB^*).$$

Since $A \in L_{r,i}(\mathcal{A}^n_{r,C}, \mathcal{A}^h_{r,C})$, then $< Ae_k, e_l > \in \mathcal{A}_r$ for each $k = 1, \dots, n$, $l = 1, \dots, h$, where $\{e_k : k = 1, \dots, m\}$ denotes the standard orthonormal base in the Euclidean space $\mathbf{R}^m$, where $m = \max(n, h)$; $\mathbf{R}^n$ is embedded into $\mathcal{A}^n_{r,C}$ as $i_0\mathbf{R}^n$. Therefore, we deduce using Formulas (28), (29), and (33) that

$$Tr(AA^*) = \sum_{l,k} | < e_l, Ae_k > |^2 \geq 0, \tag{34}$$

since $Tr(AA^*) = \sum_l < AA^*e_l, e_l >= \sum_{l,k} < A^*e_l, e_k >< e_k, A^*e_l >$.

This implies Formula (31). From the Cauchy–Bunyakovskii–Schwarz inequality, Remark 4, Formulas (31) and (34), one obtains Inequality (32). □

Theorem 2. *If $S(t)$ is an elementary random function with values in $L_{r,i}(\mathcal{A}^n_{r,C}, \mathcal{A}^h_{r,C})$ and $w(t)$ is an U-random function in $\mathcal{A}^n_r$ as in Definition 4 with $U \in L_{r,i}(\mathcal{A}^n_{r,C}, \mathcal{A}^n_{r,C})$, then*

$$E\left[< \int_a^t S(\tau)dw(\tau), \int_0^t S(\tau)dw(\tau) > |\mathcal{F}_a \right]$$

$$= E\left[\int_a^t Tr(\{S(\tau)U^{1/2}\}\{(U^{1/2})^* S^*(\tau)\})d\tau|\mathcal{F}_a \right] \tag{35}$$

P*-almost everywhere for each $0 \leq a < t \in T$.*

Proof. Since $Ew(t) = 0$ and $U : \mathcal{A}^n_{r,C} \to \mathcal{A}^n_{r,C}$, $U \in L_{r,i}(\mathcal{A}^n_{r,C}, \mathcal{A}^n_{r,C})$ by the conditions of this theorem, $a_j \in \mathcal{A}_r \setminus \{0\}$ for each j and hence $U^{1/2} : \mathcal{A}^n_{r,C} \to \mathcal{A}^n_{r,C}$ and $U^{1/2} \in L_{r,i}(\mathcal{A}^n_{r,C}, \mathcal{A}^n_{r,C})$, since U satisfies the conditions of Definition 2.4 and 2.3(α) [22] (see also Theorem 1). Therefore, $w(t, \omega) \in \mathcal{A}^n_r$; hence, $S(t, \omega)w(t, \omega) \in \mathcal{A}^h_r$ for each $t \in T$ and P-almost all $\omega \in \Omega$, where $w(t)$ is a shortening of $w(t, \omega)$, while $S(t)$ is that of $S(t, \omega)$. On the other hand,

$$< x, x >= |x|^2 = \sum_{j=1}^{h} x_j \tilde{x}_j = \sum_{j=1}^{h} |x_j|^2 \tag{36}$$

for each $x \in \mathcal{A}^h_r$, where $|z|^2 = z\tilde{z} = \sum_{l=0}^{2^r-1} z_l^2$ for each z in the Cayley–Dickson algebra $\mathcal{A}_r$, where $z = z_0 i_0 + \ldots + z_{2^r-1} i_{2^r-1}$ with $z_l \in \mathbf{R}$ for each l, $\{i_0, \dots, i_{2^r-1}\}$ is the standard basis of $\mathcal{A}_r$.

Let $e_l \in \mathcal{A}^n_{r,C}$ and $f_l \in \mathcal{A}^h_{r,C}$, where $e_l = (\delta_{l,k} : k = 1, \dots, n)$ and $f_l = (\delta_{l,k} : k = 1, \dots, h)$, where $\delta_{l,k}$ is the Kronecker delta. Then, for an operator J in $L_{r,i}(\mathcal{A}^n_{r,C}, \mathcal{A}^h_{r,C})$ and each $x \in \mathcal{A}^n_{r,C}$, the representation is valid:

$$Jx = \sum_{k=1}^{n} \sum_{l=1}^{h} J_{l,k} x_k f_l, \tag{37}$$

where $x = x_1 e_1 + \ldots + x_n e_n$, $x_k \in \mathcal{A}_{r,C}$ and $J_{l,k} \in \mathcal{A}_r$ for each k and l.

From the conditions imposed on U (see Definition 2.4 of [22]), it follows that U and

$$U^{1/2} = \bigoplus_{l=1}^{m} a_j^{1/2} B_j^{1/2} \text{ and } (U^{1/2})^* = \bigoplus_{l=1}^{m} \tilde{a}_j^{1/2} B_j^{1/2} \tag{38}$$

belong to $L_{r,i}(\mathcal{A}^n_{r,C}, \mathcal{A}^n_{r,C})$, since the positive definite matrix $[B_j]^{1/2}$ with real matrix elements corresponds to the positive definite operator B_j for each j, and $z^{1/2} \in \mathcal{A}_r$ for each $z \in \mathcal{A}_r$.

By virtue of Proposition 2.5, and Formulas 2.8(2) and 2.8(3) in [22], $\mu_{Ut,0}$ is the $\mathcal{A}_r$-valued measure for each $t > 0$, since the Cayley–Dickson algebra $\mathcal{A}_r$ is power-associative and $\exp_l(z) = \exp(z)$ for each $z \in \mathcal{A}_r$.

Random function $S(t)w(t)$ is obtained from the standard Wiener process ξ in $\mathbf{R}^n$ with the zero expectation and the unit covariance operator with the use of operator $U^{1/2}$:

$$S(t)w(t) = S(t)U^{1/2}\xi(t) \tag{39}$$

according to Theorem 1. Therefore, the statement of this theorem follows from the Ito isometry theorem (see, for example, Proposition 1.2 in [1], Theorem 3.6 in [2], XII in Chapter VIII, Section 1 in [5]), Formulas (36)–(39) above and Remarks 4 and 5. □

Theorem 3. *Suppose that*

(i) $S(t)$ is an elementary $L_r(\mathcal{A}^n_{r,C}, \mathcal{A}^h_{r,C})$ valued random function and

(ii) $w = w_0 + \mathbf{i}w_1$ is an $\mathcal{A}^n_{r,C}$-valued random function satisfying Condition (ii) in Lemma 1. Then,

$$E\left[\left\| \int_a^t S(\tau)dw(\tau)\right\|^2 |\mathcal{F}_a\right] \leq \max(\|U_0^{1/2}\|_2^2, \|U_1^{1/2}\|_2^2)\; E\left[\int_a^t \|S(\tau)\|_2^2 d\tau|\mathcal{F}_a\right] \tag{40}$$

P-*almost everywhere for each* $0 \leq a < t \in T$.

Proof. We consider the following representation: $S(x + \mathbf{i}y) = (S_{0,0}x) + (S_{0,1}y) + \mathbf{i}(S_{1,0}x) + \mathbf{i}(S_{1,1}y)$ of S with $S_{l,k} \in L_{r,i}(\mathcal{A}^n_{r,C}, \mathcal{A}^h_{r,C})$ for every $l, k \in \{0, 1\}$ and $z = x + \mathbf{i}y \in \mathcal{A}^n_{r,C}$ with x and y in $\mathcal{A}^n_r$. For each $z = x + \mathbf{i}y \in \mathcal{A}^n_{r,C}$, we have $|Sz|^2 = |(S_{0,0}x) + (S_{0,1}y)|^2 + |(S_{1,0}x) + (S_{1,1}y)|^2$ (see Remark 2.1 of [22] and Formula (36) in Theorem 2 above). On the other hand, $|v|^2 =< v, v >$ for each $v \in \mathcal{A}^h_r$. For two operators G and H in $L_{r,i}(\mathcal{A}^n_{r,C}, \mathcal{A}^h_{r,C})$, the inequality is valid $|Tr(GH^*)|^2 \leq [Tr(GG^*)] \cdot [Tr(HH^*)]$ due to Representation (37). Applying Theorem 2 and Lemma 2 (see also Remarks 4 and 5) to $S_{0,0}w_0 + S_{0,1}w_1 = (S_{0,0} \oplus S_{0,1})\eta$ and $S_{1,0}w_0 + S_{1,1}w_1 = (S_{1,0} \oplus S_{1,1})\eta$, where $\eta = w_0 \oplus w_1$ and $U = U_0 \oplus U_1$, we infer that

$$E\left[\left\| \int_a^t S(\tau)dw(\tau)\right\|^2 |\mathcal{F}_a\right] =$$

$$2E\left[\int_0^t \left(\sum_{l,k=0}^{1} Tr(\{S_{l,k}(\tau)U_k^{1/2}\}\{(U_k^{1/2})^* S^*_{l,k}(\tau)\})\right)d\tau|\mathcal{F}_a\right]$$

$$\leq \max(\|U_0^{1/2}\|_2^2, \|U_1^{1/2}\|_2^2)\; E\left[\int_a^t \|S(\tau)\|_2^2 d\tau|\mathcal{F}_a\right] \tag{41}$$

P-almost everywhere for each $0 \leq a < t \in T$, since $|Tr(GH^*)| = |Tr(HG^*)|$ and $|a + b| \leq |a| + |b|$ for each a and b in $\mathcal{A}^h_r$. □

Lemma 3. *If conditions (i) in Theorem 3, (ii) in Lemma 1 are satisfied, then*

$$\mathsf{P}\left\{\left\| \int_a^b S(t)dw(t)\right\| > \beta \max(\|U_0^{1/2}\|_2, \|U_1^{1/2}\|_2)\right\} \leq$$

$$\alpha\beta^{-2} + \mathsf{P}\left\{\int_a^b \|S(t)\|_2^2 dt > \alpha\right\} \tag{42}$$

for each $\alpha > 0$, $\beta > 0$, $[a, b] \subset T$, $0 \leq a < b < \infty$.

Proof. According to Formula (26) $S(t) = S(t_l)$ for each $t_l < t \leq t_{l+1}$, where $a = t_0 < t_1 < \ldots < t_k = b$. Since $S(t)$ is $(\mathcal{F}_{t_l}, \mathcal{B}(L_r(\mathcal{A}^n_{r,C}, \mathcal{A}^h_{r,C}))$-measurable for each $t \in (t_l, t_{l+1}]$, then $\int_a^{t_{l+1}} \|S(t)\|_2^2 dt$ is $(\mathcal{F}_{t_l}, \mathcal{B}([0,\infty]))$-measurable. We consider a modified elementary random function $S_\alpha(t)$ such that $S_\alpha(t) = S(t)$ for each $t \leq t_l$ if $\int_a^{t_{l+1}} \|S(t)\|_2^2 dt \leq \alpha$; otherwise $S_\alpha(t) = 0$ for each $t \in (t_l, b]$ if $\int_a^{t_l} \|S(t)\|_2^2 dt \leq \alpha < \int_a^{t_{l+1}} \|S(t)\|_2^2 dt$ for some l. Therefore, $\int_a^t \|S_\alpha(t)\|_2^2 dt \leq \alpha$ for each $t \in [a,b]$; hence,

$$\mathsf{P}\{\sup_{t\in[a,b]} \|S_\alpha(t) - S(t)\|_2 > 0\} = \mathsf{P}\Big\{\int_a^b \|S(t)\|_2^2 dt > \alpha\Big\}. \tag{43}$$

Then, we deduce that

$$\mathsf{P}\Big\{\Big\|\int_a^b S(t)dw(t)\Big\| > \beta \max(\|U_0^{1/2}\|_2, \|U_1^{1/2}\|_2)\Big\} =$$

$$\mathsf{P}\Big\{\Big\|\int_a^b S_\alpha(t)dw(t) + \int_a^b (S(t) - S_\alpha(t))dw(t)\Big\| > \beta \max(\|U_0^{1/2}\|_2, \|U_1^{1/2}\|_2)\Big\} \leq$$

$$\mathsf{P}\Big\{\Big\|\int_a^b S_\alpha(t)dw(t)\Big\| > \beta \max(\|U_0^{1/2}\|_2, \|U_1^{1/2}\|_2)\Big\} + \mathsf{P}\Big\{\Big\|\int_a^b (S(t) - S_\alpha(t))dw(t)\Big\| > 0\Big\}$$

$$\leq \frac{E\Big[\Big\|\int_a^b S_\alpha(t)dw(t)\Big\|^2\Big]}{\beta^2 \max(\|U_0^{1/2}\|_2^2, \|U_1^{1/2}\|_2^2)} + \mathsf{P}\Big\{\int_a^b \|S(t)\|_2^2 dt > \alpha\Big\}$$

by Chebyshëv inequality (see, for example, in Section II.6 [7]), Equality (43) above, Formulas 2.10(1) and (2) in [22]. By virtue of Theorem 3 (see also Formulas (40) and (41))

$$E\Big[\Big\|\int_a^b S_\alpha(t)dw(t)\Big\|^2\Big] \leq \max(\|U_0^{1/2}\|_2^2, \|U_1^{1/2}\|_2^2)\ E\Big[\int_a^b \|S(t)\|_2^2 dt\Big],$$

since $E[E(\zeta|\mathcal{F}_a)] = E\zeta$ for a random variable $\zeta : \Omega \to [0,\infty]$ which is $(\mathcal{F}_a, \mathcal{B}([0,\infty]))$-measurable (Section II.7 [7]). This implies Inequality (42). □

Theorem 4. *If w is a U-random function and $\{S(t) : t \in T\}$ is an $L_{r,i}(\mathcal{A}^n_{r,C}, \mathcal{A}^h_{r,C})$-valued predictable random function satisfying the condition*

$$E[\int_a^t Tr(\{S(\tau)U^{1/2}\}\{(U^{1/2})^* S^*(\tau)\})d\tau] < \infty \tag{44}$$

for each $0 \leq a < t$ in T, where operator U is specified in Definition 2.4 [22], such that $U \in L_{r,i}(\mathcal{A}^n_{r,C}, \mathcal{A}^n_{r,C})$; then, a sequence $\{S_\kappa(t) : \kappa \in \mathbf{N}\}$ of elementary random functions exists with $t \in T$ such that

$$lim_{\kappa\to\infty} E[\int_a^t Tr(\{(S(\tau) - S_\kappa(\tau))U^{1/2}\}\{(U^{1/2})^*(S^*(\tau) - S_\kappa(\tau))\})d\tau] = 0 \tag{45}$$

for each $0 \leq a < t$ in T.

Proof. $Tr(\{S(\tau)U^{1/2}\}\{(U^{1/2})^* S^*(\tau)\} \geq 0$ for each $\tau \in T$, since $U \in L_{r,i}(\mathcal{A}^n_{r,C}, \mathcal{A}^n_{r,C})$ implying $a_j \in \mathcal{A}_r$; hence, $a_j^{1/2} \in \mathcal{A}_r$ for each j. In view of Formulas (35), (37) random function $S(\tau)U^{1/2}$ having values in $L_{r,i}(\mathcal{A}^n_r, \mathcal{A}^h_r)$ has the decomposition into a finite $\mathbf{R}$-linear combination

$$S(t)U^{1/2} = \sum_{l=1}^n \sum_{k=1}^h \sum_{j=0}^{2^r-1} \eta_{l,k;j} e_l \otimes f_k i_j \tag{46}$$

of real random functions $\eta_{l,k;j}$ using vectors e_l, f_k and the standard basis $\{i_0, i_1, \ldots, i_{2^r-1}\}$ of the Cayley–Dickson algebra $\mathcal{A}_r$ over $\mathbf{R}$. For each real-valued random function, the condition

$$E[\int_a^t \eta_{l,k;j}^2 d\tau] < \infty \tag{47}$$

is fulfilled for each $0 \leq a < t$ in T by (44); hence, a sequence of real-valued random functions $\eta_{l,k;j;\kappa}$ exists, such that

$$lim_{\kappa\to\infty} E[\int_a^t (\eta_{l,k;j} - \eta_{l,k;j;\kappa})^2 d\tau] = 0 \tag{48}$$

for each $t \in T$. Thus, Formulas (46), (47), and (48) imply (45). $\square$

Theorem 5. *If w fulfills Condition (ii) in Lemma 1 and $S(t)$ is a $L_r(\mathcal{A}_{r,C}^n, \mathcal{A}_{r,C}^h)$-valued predictable random function satisfying the following inequality:*

$$E\left[\int_a^b F(S; U_0, U_1)(\tau) d\tau\right] < \infty \tag{49}$$

for each $0 \leq a < b$ in T, where

$$F(S; U_0, U_1)(t) = \sum_{l,k=0}^{1} Tr(\{S_{l,k}(t) U_k^{1/2}\}\{(U_k^{1/2})^* S_{l,k}^*(t)\}). \tag{50}$$

Then, a sequence $\{S_\kappa(t) : \kappa \in \mathbf{N}\}$ of elementary random functions exists with $t \in T$, such that

$$lim_{\kappa\to\infty} E\left[\int_a^b F((S(\tau) - S_\kappa(\tau)); U_0, U_1)(\tau) d\tau\right] = 0 \tag{51}$$

for every $0 \leq a < b$ in T.

The proof is analogous to that of Theorem 4 with the use of Formula (41), using (49), (50) and (51), since $E(E(\zeta|\mathcal{F}_a)) = E\zeta$ with $\zeta = \int_a^b F(S; U_0, U_1)(\tau) d\tau$, $\zeta \geq 0$ P-almost everywhere.

Definition 5. *A sequence $\{S_\kappa(t) : \kappa \in \mathbf{N}\}$ of elementary $L_r(\mathcal{A}_{r,C}^n, \mathcal{A}_{r,C}^h)$-valued random functions with $t \in T$ is mean absolute square convergent to a predictable $L_r(\mathcal{A}_{r,C}^n, \mathcal{A}_{r,C}^h)$-valued random function $\{S(t) : t \in T\}$, where w satisfies Condition (ii) in Lemma 1, if Condition (51) in Theorem 5 is satisfied. The corresponding mean absolute square limit is induced by Formulas (41) and (51), and is denoted by l.i.m.. The family of all predictable $L_r(\mathcal{A}_{r,C}^n, \mathcal{A}_{r,C}^h)$-valued random functions $\{S(t) : t \in T\}$ satisfying Condition (49) is denoted by $V_{2,1}(U_0, U_1, a, b, n, h)$*

A stochastic integral of $S \in V_{2,1}(U_0, U_1, a, b, n, h)$ is:

$$\int_0^t S(\tau) dw(\tau) := l.i.m._{\kappa\to\infty} \int_0^t S_\kappa(\tau) dw(\tau), \tag{52}$$

where $w = w_0 + \mathbf{i}w_1$ is an $\mathcal{A}_{r,C}^n$-valued random function with U_0 and U_1 random functions w_0 and w_1, respectively, having values in $\mathcal{A}_r^n$, where $0 \leq a \leq t \leq b$ in T, where w satisfies Condition (ii) in Lemma 1.

Proposition 1. *Let the conditions of Theorem 5 be satisfied, and let $S \in V_{2,1}(U_0, U_1, a, b, n, h)$, $0 \leq a < c < b \in T$. Then, there exists $\int_\beta^\gamma S(t) dw(t)$ for each $a \leq \beta \leq \gamma \leq b$ and*

$$\int_a^b S(t) dw(t) = \int_a^c S(t) dw(t) + \int_c^b S(t) dw(t). \tag{53}$$

Proof. In view of Theorem 5, Definitions 4 and 5, and Remark 5, there exists $\int_\beta^\gamma S(t)dw(t)$ for each $a \leq \beta \leq \gamma \leq b$. Formula (53) for elementary random functions S_κ for each $\kappa \in \mathbf{N}$ follows from Formula (27). Hence, taking $l.i.m._{k\to\infty}$, we infer Equality (53) for $S \in V_{2,1}(U_0, U_1, a, b, n, h)$ by Theorem 5. □

Proposition 2. *If $S \in V_{2,1}(U_0, U_1, a, b, n, h)$, $S_\kappa \in V_{2,1}(U_0, U_1, a, b, n, h)$ for each $\kappa \in \mathbf{N}$, w satisfies Condition (ii) in Lemma 1, and*

$$\lim_{\kappa\to\infty} E \int_a^b F(S - S_\kappa; U_0, U_1)(t)dt = 0, \tag{54}$$

where $0 \leq a < b \in T$, then there exists

$$l.i.m._{\kappa\to\infty} \int_a^b S_\kappa(t)dw(t) = \int_a^b S(t)dw(t). \tag{55}$$

Proof. In view of Proposition 1, stochastic integrals $\int_a^b S(t)dw(t)$ and $\int_a^b S_\kappa(t)dw(t)$ exist for each $\kappa \in \mathbf{N}$. Then, Equality (55) follows from Theorem 5, Equality (54), and Formula (52) in Definition 5. □

Proposition 3. *If $S \in V_{2,1}(U_0, U_1, a, b, n, h)$, and if w satisfies Condition (ii) in Lemma 1, where $0 \leq a < b \in T$, then*

$$E\left[\int_a^b S(t)dw(t)\bigg|\mathcal{F}_a\right] = 0 \quad \mathsf{P}\text{-}\textit{almost everywhere and} \tag{56}$$

$$\begin{aligned} E\left[\left\|\int_a^t S(\tau)dw(\tau)\right\|^2|\mathcal{F}_a\right] &= 2E\left[\int_0^t F(S; U_0, U_1)(\tau)d\tau|\mathcal{F}_a\right] \\ &\leq \max(\|U_0^{1/2}\|_2^2, \|U_1^{1/2}\|_2^2)\ E\left[\int_a^t \|S(\tau)\|_2^2 d\tau|\mathcal{F}_a\right] \end{aligned} \tag{57}$$

P-*almost everywhere for each* $0 \leq a < t \in T$.

Proof. From Lemmas 1 and 2, and Proposition 1, Identity (56) follows. Then, Theorem 3 and Proposition 1 imply Inequality (57), since $E(E(\zeta|\mathcal{F}_a)) = E\zeta$ with $\zeta = \int_a^b F(S; U_0, U_1)(t)dt$ and since

$$\mathsf{P}\left\{\omega \in \Omega : E\left[\int_a^b F(S; U_0, U_1)(t)dt\bigg|\mathcal{F}_a\right](\omega) = \infty\right\} = 0; \ \ \zeta \geq 0$$

P-almost everywhere. □

Remark 6. *Let $ch_{[0,\infty)}(t) = 1$ for each $t \geq 0$, and $ch_{[0,\infty)}(t) = 0$ for each $t < 0$ be a characteristic function of $[0, \infty)$, $[0, \infty) \subset \mathbf{R}$. Then, $G(\tau) \in V_{2,1}(U_0, U_1, a, b, n, h)$ for each $t \in [a, b]$, if $S(\tau) \in V_{2,1}(U_0, U_1, a, b, n, h)$, where $G(\tau) := S(\tau)ch_{[0,\infty)}(\tau - t)$. It is put*

$$\eta(t) = \int_a^t S(\tau)dw(\tau) := \int_a^b S(\tau)ch_{[0,\infty)}(t - \tau)dw(\tau) \tag{58}$$

for each $t \in [a, b]$. From Proposition 3, it follows that $\eta(t)$ is defined P-almost everywhere. By virtue of Theorem IV.2.1 in [5], $\eta(t)$ is the separable random function up to the stochastic equivalence since $(\mathcal{A}_{r,C}^h, |\cdot|)$ is the metric space. Therefore, $\eta(t)$ is considered to be the separable random function.

Definition 6. *Let $\zeta(t)$, $t \in T$, be a $L_{r,C}^h$-valued random function adapted to the filtration $\{\mathcal{F}_t : t \in T\}$ of σ-algebras $\mathcal{F}_t$ and let $E|\zeta(t)| < \infty$ for each $t \in T$. If $E(\zeta(t)|\mathcal{F}_s) = \zeta(s)$ for each $s < t$*

in T, then the family $\{\zeta(t), \mathcal{F}_t : t \in T\}$ is called a martingale. If $\zeta(t) \in \mathbf{R}$ for each $t \in T$ and $E(\zeta(t)|\mathcal{F}_s) \geq \zeta(s)$ for each $s < t$ in T, then $\{\zeta(t), \mathcal{F}_t : t \in T\}$ is called a sub-martingale.

Lemma 4. *Assume that $S(t) \in V_{2,1}(U_0, U_1, a, b, n, h)$ and w satisfies Condition (ii) in Lemma 1, $0 \leq a < b < \infty$, $[a, b] \subset T$ and*

$$E\left[\int_a^b F(S; U_0, U_1)(t)dt \middle| \mathcal{F}_a \right] < \infty \tag{59}$$

and $\eta(t)$ is provided by Formula (58); then, $\{\eta(t), \mathcal{F}_t : t \in [a, b]\}$ is a martingale and $\{|\eta(t)|^2, \mathcal{F}_t : t \in [a, b]\}$ is the submartingale.

Proof. By virtue of Proposition 3 $\eta(t)$ is $(\mathcal{F}_t, \mathcal{B}(\mathcal{A}^h_{r,C}))$-measurable and $E(\eta(t_2) - \eta(t_1)|\mathcal{F}_{t_1}) = E\left[\int_{t_1}^{t_2} S(\tau)d\tau \middle| \mathcal{F}_{t_1} \right] = 0$ for each $a \leq t_1 < t_2 \leq b$. Hence $\{\eta(t), \mathcal{F}_t : t \in [a, b]\}$ is the martingale.

Random function $\eta(t)$ has the decomposition:

$$\eta(t) = \sum_{k \in \{0,1\};\ j \in \{0,1,\ldots,2^r-1\};\ l \in \{1,\ldots,h\}} \eta_{k,j,l}(t) i_j \mathbf{i}^k e_l \tag{60}$$

with $\eta_{k,j,l}(t) \in \mathbf{R}$, for each k, j, l, where $\{e_l : l = 1, \ldots, h\}$ is the standard orthonormal basis of the Euclidean space $\mathbf{R}^h$, where $\mathbf{R}^h$ is embedded into $\mathcal{A}^h_{r,C}$ as $i_0\mathbf{R}^h$. Therefore, each random function $\eta_{k,j,l}(t)$ is the martingale. Then,

$$|\eta(t)|^2 = \sum_{k \in \{0,1\};\ j \in \{0,1,\ldots,2^r-1\};\ l \in \{1,\ldots,h\}} |\eta_{k,j,l}(t)|^2. \tag{61}$$

By virtue of Theorem 1 and Corollary 2 in Chapter III, Section 1 [5], Inequality (59) and Formula (57) above $\{|\eta_{k,j,l}(t)|^2, \mathcal{F}_t : t \in [a, b]\}$ is the submartingale for each k, j, l. Consequently, $\{|\eta(t)|^2, \mathcal{F}_t : t \in [a, b]\}$ is the submartingale by Formulas (60) and (61). □

Lemma 5. *Let $S(t) \in V_{2,1}(U_0, U_1, a, b, n, h)$ and w satisfy Condition (ii) in Lemma 1 such that*

$$E\left[\int_a^b F(S; U_0, U_1)(t)dt \middle| \mathcal{F}_a \right] < \infty \tag{62}$$

Then,

$$\mathsf{P}\{ \sup_{t \in [a,b]} \left| \int_a^t S(\tau)dw(\tau) \right| > \beta \, \max(\|U_0^{1/2}\|_2, \|U_0^{1/2}\|_2) \middle| \mathcal{F}_a\} \leq$$

$$\beta^{-2} E\left[\int_a^b F(S; U_0, U_1)(t)dt \middle| \mathcal{F}_a \right], \tag{63}$$

$$\mathsf{P}\{ \sup_{t \in [a,b]} \left| \int_a^t S(\tau)dw(\tau) \right| > \beta \, \max(\|U_0^{1/2}\|_2, \|U_0^{1/2}\|_2)\} \leq$$

$$\beta^{-2} E\left[\int_a^b F(S; U_0, U_1)(t)dt \right]. \tag{64}$$

Proof. Inequality (64) follows from Inequality (63). Therefore, (63) remains to be proven. We take an arbitrary partition $a = t_0 < t_1 < \ldots < t_n = b$ of $[a, b]$. Then, we consider $\eta_k := \int_a^{t_k} S(\tau)dw(\tau)$. In view of Lemma 4 $\{\eta_l, \mathcal{F}_{t_l} : l = 1, \ldots, n\}$ is the martingale and $\{|\eta_l|^2, \mathcal{F}_{t_l} : l = 1, \ldots, n\}$ is the submartingale.

Therefore, from Theorem 5 in Chapter III, Section 1 [5], Formulas 2.10(1), (2) of [22] and Inequality (62), we deduce that

$$\mathsf{P}\{\sup_{0\leq l\leq n}|\eta_l| > \beta(\max(\|U_0^{1/2}\|_2, \|U_1^{1/2}\|_2)|\mathcal{F}_a\} \leq \beta^{-2}E(|\eta_n|^2|\mathcal{F}_a)$$

(see also Remark 5). Together with Proposition 3 above and the Fubini theorem (II.6.8 [7]), this implies that

$$\mathsf{P}\{\sup_{0\leq l\leq n}\left|\int_a^{t_l} S(\tau)dw(\tau)\right| > \beta\ \max(\|U_0^{1/2}\|_2, \|U_0^{1/2}\|_2)|\mathcal{F}_a\} \leq$$

$$\beta^{-2}E\left[\int_a^b F(S;U_0,U_1)(t)dt|\mathcal{F}_a\right]. \tag{65}$$

Random function $\int_a^t S(\tau)dw(\tau)$ is separable (see Remark 6); hence Inequality (63) follows from Inequality (65). $\square$

Theorem 6. *Let $S \in V_{2,1}(U_0,U_1,a,b,n,h)$ be a predictable $L_r(\mathcal{A}^n_{r,C},\mathcal{A}^h_{r,C})$-valued random function, let w satisfy Condition (ii) in Lemma 1, $[a,b] \subset T$. Then, random function $\eta(t) = \int_a^t S(\tau)dw(\tau)$ is stochastically continuous, where $t \in [a,b]$.*

Proof. If $S_\kappa(\tau)$ is an elementary $L_r(\mathcal{A}^n_{r,C},\mathcal{A}^h_{r,C})$-valued random function, then $\eta_\kappa(t) = \int_a^t S_\kappa(\tau)dw(\tau)$ is stochastically continuous by Formula (27), since $w(t)$ is stochastically continuous.

For each $S \in V_{2,1}(U_0,U_1,a,b,n,h)$ according to Definition 5 and the Fubini theorem $\int_a^b E(F(S;U_0,U_1))(t)dt < \infty$. By virtue of Theorem 5, there exists a sequence $\{S_\kappa(t) : \kappa \in \mathbf{N}\}$ of elementary $L_r(\mathcal{A}^n_{r,C},\mathcal{A}^h_{r,C})$-valued random functions, such that Limit (51) is satisfied. From Lemma 5 and the Fubini theorem, we infer that

$$\mathsf{P}\{\sup_{t\in[a,b]}|\int_a^t S(\tau)dw(\tau) - \int_a^t S_\kappa(\tau)dw(\tau)| > \epsilon\max(\|U_0^{1/2}\|_2, \|U_1^{1/2}\|_2)\}$$

$$\leq \epsilon^{-2}\int_a^b E(F(S-S_\kappa;U_0,U_1))(t)dt.$$

Therefore, there exists a sequence $\{\epsilon_\kappa : \kappa \in \mathbf{N}\}$ with $\lim_{\kappa\to\infty}\epsilon_\kappa = 0$ and a sequence $\{n_k \in \mathbf{N} : k \in \mathbf{N}\}$, such that

$$\sum_{k=1}^{\infty}\epsilon_k^{-2}\int_a^b E(F(S-S_{n_k};U_0,U_1))(t)dt < \infty$$

Consequently,

$$\sum_{k=1}^{\infty}\mathsf{P}\{\sup_{t\in[a,b]}|\int_a^b S(\tau)dw(\tau) - \int_a^b S_{n_k}(\tau)dw(\tau)| > \epsilon_k\} < \infty.$$

In view of the Borel–Cantelli lemma (see, for example, Chapter II, Section 10 [7]) a natural number $k_0 \in \mathbf{N}$ exists, such that

$$\mathsf{P}\{\sup_{t\in[a,b]}|\int_a^b S(\tau)dw(\tau) - \int_a^b S_{n_k}(\tau)dw(\tau)| > \epsilon_k\} = 1$$

for each $k \geq k_0$. Hence, $\int_a^t S(\tau)dw(\tau)$ is stochastically continuous since $\int_a^t S_{n_k}(\tau)dw(\tau)$ is stochastically continuous for each $k \in \mathbf{N}$. $\square$

Definition 7. The generalized Cauchy problem over the complexified Cayley–Dickson algebra $\mathcal{A}_{r,C}$. *Let*

$$H : T \times \mathcal{A}^h_{r,C} \to L_r(\mathcal{A}^n_{r,C}, \mathcal{A}^h_{r,C}) \text{ and} \tag{66}$$

$$G : T \times \mathcal{A}^h_{r,C} \to \mathcal{A}^h_{r,C} \text{ be mappings ,} \tag{67}$$

$$w = w_0 + \mathbf{i}w_1 \text{ be a random function in } \mathcal{A}^n_{r,C} \tag{68}$$

satisfying Condition (ii) in Lemma 1, where n and h are natural numbers.

A stochastic Cauchy problem over $\mathcal{A}_{r,C}$ is:

$$dY(t) = G(t, Y(t))dt + H(t, Y(t))dw(t) \text{ with } Y(a) = \zeta, \tag{69}$$

where $Y(t)$ is an $\mathcal{A}^h_{r,C}$-valued random function, ζ is an $\mathcal{A}^h_{r,C}$-valued random variable which is $\mathcal{F}_a$-measurable, $t \in [a, b] \subset T$, $0 \le a < b$, where H, G, w are as in (66)–(68). Problem (69) is understood as the following integral equation:

$$Y(t) = \zeta + \int_a^t G(\tau, Y(\tau))d\tau + \int_a^t H(\tau, Y(\tau))dw(\tau), \text{ where } t \in [a, b] \subset T. \tag{70}$$

Then, the random function $Y(t)$ is called a solution if it satisfies Conditions (71)–(73):

$$\text{random function } Y(t) \text{ is predictable,} \tag{71}$$

$$\forall t \in [a, b] \;\; \mathsf{P}\{Y(t) : \int_a^t \|G(\tau, Y(\tau))\| d\tau = \infty\} = 0 \text{ and} \tag{72}$$

$$\mathsf{P}\{\omega \in \Omega : \; \exists t \in [a, b],$$
$$Y(t) \neq \zeta + \int_a^t G(\tau, Y(\tau))d\tau + \int_a^t H(\tau, Y(\tau))dw(\tau)\} = 0, \tag{73}$$

where $Y(t)$ is a shortened notation of $Y(t, \omega)$.

Theorem 7. *Let $G(t, y)$ and $H(t, y)$ be Borel functions, w satisfy Condition (ii) in Lemma 1, and $K = const > 0$ be such that*

(i) $\|G(t, x) - G(t, y)\| + \|H(t, x) - H(t, y)\|_2 \le K\|x - y\|$ *and*
(ii) $\|G(t, y)\|^2 + \|H(t, y)\|_2^2 \le K^2(1 + \|y\|^2)$ *for each x and y in $\mathcal{A}^h_{r,C}$, $t \in [a, b] = T$, where $0 \le a < b < \infty$,*
(iii) $E[\|\zeta\|^2] < \infty$.

Then, a solution Y of Equation (70) exists (see Definition 7); if Y and Y_1 are two stochastically continuous solutions, then

$$\mathsf{P}\{\sup_{t \in [a,b]} \|Y(t) - Y_1(t)\| > 0\} = 0. \tag{74}$$

Proof. We consider a Banach space $B_{2,\infty} = B_{2,\infty}[a, b]$ consisting of all predictable random functions $X : [a, b] \times \Omega \to \mathcal{A}^h_{r,C}$ such that $X(t)$ is $(\mathcal{F}_t, \mathcal{B}(\mathcal{A}^h_{r,C}))$-measurable for each $t \in [a, b]$ and $\sup_{t \in [a,b]} E[\|X(t)\|^2] < \infty$ with the norm

$$\|X\|_{B_{2,\infty}} = (\sup_{t \in [a,b]} E[\|X(t)\|^2])^{1/2}. \tag{75}$$

In view of Proposition 2, there exists operator Q on $B_{2,\infty}$ such that

$$QX(t) = \zeta + \int_a^t G(\tau, X(\tau))d\tau + \int_a^t H(\tau, X(\tau))dw(\tau) \tag{76}$$

for each $t \in [a,b]$, since G and H satisfy Condition (ii) of this theorem. Then, $QX(t)$ is $(\mathcal{F}_t, \mathcal{B}(\mathcal{A}^h_{r,C}))$-measurable for each $t \in [a,b]$, since G and H are Borel functions and $X \in B_{2,\infty}$. By virtue of Proposition 3, using the inequality $(\alpha+\beta+\gamma)^2 \leq 3(\alpha^2+\beta^2+\gamma^2)$ for each α, β and γ in $\mathbf{R}$, the Cauchy–Bunyakovskii–Schwarz inequality, (75), (76), and Condition (ii) of this theorem, we infer that

$$E[\|QX(t)\|^2] \leq 3E[\|\zeta\|^2] + 3(b-a)\int_a^t K^2(1+\|X(\tau)\|^2)d\tau + 3E\int_a^t K^2(1+\|X(\tau)\|^2)d\tau$$

$$\leq 3E[\|\zeta\|^2] + 3K^2[(b-a)+1]E\int_a^b (1+\|X(\tau)\|^2)d\tau$$

$$\leq 3E[\|\zeta\|^2] + 3K^2(b-a)[(b-a)+1](1+\|X\|^2_{B_{2,\infty}}).$$

Thus, $Q : B_{2,\infty} \to B_{2,\infty}$. Then, using the Cauchy–Bunyakovskii–Schwarz inequality, 2.3(12) of [22], Proposition 3, Condition (i) of this theorem, and inequality $(\alpha+\beta)^2 \leq 2(\alpha^2+\beta^2)$ for each α and β in $\mathbf{R}$, we deduce that

$$E[\|QX(t) - X_1(t)\|^2] \leq 2(b-a)\int_a^t E[\|G(\tau, X(\tau)) - G(\tau, X_1(\tau))\|^2]d\tau$$

$$+ 2E[\| \int_a^t \{H(\tau, X(\tau)) - H(\tau, X_1(\tau))\}dw(\tau)\|^2]$$

$$\leq C_1 \int_a^t E[\|X(\tau) - X_1(\tau)\|]^2 d\tau \leq C_1(t-a)\|X - X_1\|^2_{B_{2,\infty}}$$

for each X and X_1 in $B_{2,\infty}$, $t \in [a,b]$, where $C_1 = 2K^2(b-a+1)$. Therefore, the operator $Q : B_{2,\infty} \to B_{2,\infty}$ is continuous. Then, we infer that

$$E[\|Q^m X(t) - Q^m X_1(t)\|^2] \leq C_1 \int_a^t E[\|Q^{m-1}X(\tau) - Q^{m-1}X_1(\tau)\|^2]d\tau$$

$$\leq \ldots \leq C_1^m \int \ldots \int_{a<t_1<\ldots<t_n<t} E[\|X(t_m) - X_1(t_m)\|^2]dt_1 \ldots dt_m$$

$$\leq C_1^m \|X - X_1\|^2_{B_{2,\infty}} (b-a)^m/m!$$

for each X and X_1 in $B_{2,\infty}$, $m = 1,2,3,\ldots$. Therefore,

$$\|Q^{m+1}X - Q^m X\|^2_{B_{2,\infty}} \leq C_1^m (b-a)^m \|QX - X\|^2_{B_{2,\infty}}/m!$$

for each $m = 1,2,3,\ldots$. Hence, the series $\sum_{m=1}^{\infty} \|Q^{m+1}X - Q^m X\|_{B_{2,\infty}}$ converges. Thus, the following limit exists $\lim_{m\to\infty} Q^m X(t) =: Y(t)$ in $B_{2,\infty}$. From the continuity of Q, it follows that $\lim_{m\to\infty} Q(Q^m X) = QY$, hence $QY = Y$. Thus,

$\|QY - Y\|_{B_{2,\infty}} = 0$. Consequently, $\mathsf{P}\{Y(t) = QY(t)\} = 1$ for each $t \in [a,b]$. This means that $Y(t)$ is the solution of Equation (70). In view of Theorem 6 and Condition (ii) of this theorem, solution $Y(t)$ is stochastically continuous up to the stochastic equivalence.

Now, let Y and Y_1 be two stochastically continuous solutions of Equation (70). We consider a random function $q_N(t)$, such that $q_N(t) = 1$ if $\|Y(\tau)\| \leq N$ and $\|Y_1(\tau)\| \leq N$ for each $\tau \in [a,t]$, $q_N(t) = 0$ in the opposite case where $t \in [a,b]$, $N > 0$. Therefore, $q_N(t)q_N(\tau) = q_N(t)$ for each $\tau < t$ in $[a,b]$; consequently,

$$q_N(t)[Y(t) - Y_1(t)] = q_N(t)[\int_a^t q_N(\tau)[G(\tau, Y(\tau)) - G(\tau, Y_1(\tau))]d\tau$$

$$+ \int_a^t q_N(\tau)[H(\tau, Y(\tau)) - H(\tau, Y_1(\tau))]dw(\tau)].$$

On the other hand,

$$q_N(\tau)[\|G(\tau,Y(\tau))-G(\tau,Y_1(\tau))\|+\|H(\tau,Y(\tau))-H(\tau,Y_1(\tau))\|]$$
$$\leq Kq_N(\tau)\|Y(\tau)-Y_1(\tau)\|\leq 2KN$$

by Condition (i). This implies that $E[q_N(t)\|Y(t)-Y_1(t)\|^2]<\infty$. Then, using the Fubini theorem, 2.3(12) of [22], Proposition 3, Lemma 5, we deduce that

$$E[q_N(t)\|Y(t)-Y_1(t)\|^2]\leq 2E[q_N(t)\|\int_a^t q_N(\tau)[G(\tau,Y(\tau))-G(\tau,Y_1(\tau))d\tau\|^2]+$$
$$2E[q_N(t)\|\int_a^t q_N(\tau)[H(\tau,Y(\tau))-H(\tau,Y_1(\tau))]dw(\tau)\|^2]$$
$$\leq 2(b-a)\int_a^t E[q_N(\tau)\|G(\tau,Y(\tau))-G(\tau,Y_1(\tau))\|^2]d\tau$$
$$+4\int_a^t E[q_N(\tau)F(H(\tau,Y(\tau))-H(\tau,Y_1(\tau));U_0,U_1)]d\tau$$
$$\leq 2K^2[b-a+\max(\|U_0^{1/2}\|_2^2,\|U_1^{1/2}\|_2^2)]\int_a^t E[q_N(\tau)\|Y(\tau)-Y_1(\tau)\|^2]d\tau.$$

Thus, a constant $C_2>0$ exists, such that

$$E[q_N(t)\|Y(t)-Y_1(t)\|^2]\leq C_2\int_a^t E[q_N(\tau)\|Y(\tau)-Y_1(\tau)\|^2]d\tau.$$

The Gronwall inequality (see Lemma 3.15 in [2], Lemma 1 in Chapter 8, Section 2 in [5]) implies that $E[q_N(t)\|Y(t)-Y_1(t)\|^2]=0$. Consequently,

$$\mathsf{P}\{Y(t)\neq Y_1(t)\}\leq \mathsf{P}\{\sup_{t\in[a,b]}\|Y(t)\|>N\}+\mathsf{P}\{\sup_{t\in[a,b]}\|Y_1(t)\|>N\}.$$

Random functions $Y(t)$ and $Y_1(t)$ are stochastically continuous and hence stochastically bounded. Consequently,

$$\lim_{N\to\infty}\mathsf{P}\{\sup_{t\in[a,b]}\|Y(t)\|>N\}=0 \text{ and}$$
$$\lim_{N\to\infty}\mathsf{P}\{\sup_{t\in[a,b]}\|Y_1(t)\|>N\}=0.$$

Therefore, random functions $Y(t)$ and $Y_1(t)$ are stochastically equivalent. This implies Equality (74). □

Corollary 2. *Let operators G and H be $G\in L_r(\mathcal{A}^h_{r,C},\mathcal{A}^h_{r,C})$ and $H\in L_r(\mathcal{A}^n_{r,C},\mathcal{A}^h_{r,C})$ such that G be a generator of a semigroup $\{S(t):t\in[0,\infty)\}$. Let $w(t)$ also be a random function fulfilling Condition (ii) in Lemma 1. Then, the Cauchy problem*

$$Y(t)=\zeta+\int_0^t GY(\tau)d\tau+\int_0^t Hdw(\tau), \tag{77}$$

where $t\in T$, $E[\|\zeta\|^2]<\infty$, has a solution

$$Y(t)=S(t)\zeta+\int_0^t S(t-\tau)Hdw(\tau) \tag{78}$$

for each $0\leq t\in T$.

Proof. Condition $G \in L_r(\mathcal{A}^h_{r,C}, \mathcal{A}^h_{r,C})$ implies that $\|G\| = \sup_{x \in \mathcal{A}^h_{r,C}, \|x\|=1} \|Gx\| < \infty$, where $\|x\|^2 = \|x_1\|^2 + \ldots + \|x_h\|^2$, $x = (x_1, \ldots, x_h) \in \mathcal{A}^h_{r,C}$, $x_k \in \mathcal{A}_{r,C}$ for each k. As a realization of the semigroup $S(t)$, it is possible to take $\{S(t) = \exp_l(Gt) : t \geq 0\}$ since G is a bounded operator and $\|\exp_l(Gt)\| \leq \exp(\|G\|t)$ for each $t \geq 0$ by Formulas 2.1(9) and 2.3(12) in [22]. Therefore, from Theorem 7 applied to Equation (77), Assertion (78) of this corollary follows. □

Theorem 8. *Let G, H, and w satisfy conditions of Theorem 7, and $Y_{t,z}(t)$ be an $\mathcal{A}^h_{r,C}$-valued random function satisfying the following equation:*

$$Y_{t,z}(t_1) = z + \int_t^{t_1} G(\tau, Y_{t,z}(\tau))d\tau + \int_t^{t_1} H(\tau, Y_{t,z}(\tau))dw(\tau), \tag{79}$$

where $z \in \mathcal{A}^h_{r,C}$, $t < t_1$ in $[a,b] \subset T$, $0 \leq a < b < \infty$. Then, random function Y satisfying Equation (70) is Markovian with the following transitional measure:

$$P(t, z, t_1, A) = P\{Y_{t,z}(t_1) \in A\} \tag{80}$$

for each $A \in \mathcal{B}(\mathcal{A}^h_{r,C})$.

Proof. Random function $Y(t)$ is $(\mathcal{F}_t, \mathcal{B}(\mathcal{A}^h_{r,C}))$-measurable for each $t \in [a,b]$. On the other hand, $Y_{t,z}(t_1)$ is induced by the random function $w(t_1) - w(t)$ for each $t_1 \in (t,b]$, where $w(t_1) - w(t)$ is independent of $\mathcal{F}_t$. Therefore, $Y_{t,z}(t_1)$ is independent of $Y(t)$ and each $A \in \mathcal{F}_t$ (see (79)). By virtue of Theorem 7, $Y(t_1)$ is the unique (up to stochastic equivalence) solution of the following equation:

$$Y(t_1) = Y(t) + \int_t^{t_1} G(\tau, Y(\tau))d\tau + \int_t^{t_1} H(\tau, Y(\tau))dw(\tau) \tag{81}$$

and $Y_{t,Y(t)}(t_1)$ is also its solution. Consequently, $\mathsf{P}\{Y(t_1) = Y_{t,Y(t)}(t_1)\} = 1$.

Let $f \in C^0_b(\mathcal{A}^h_{r,C}, \mathcal{A}_{r,C})$, where $C^0_b(\mathcal{A}^h_{r,C}, \mathcal{A}_{r,C})$ denotes the family of all bounded continuous functions from $\mathcal{A}^h_{r,C}$ into $\mathcal{A}_{r,C}$. Let $g \in R_b(\Omega, \mathcal{A}_{r,C})$, where $R_b(\Omega, \mathcal{A}_{r,C})$ denotes the family of all random variables $g : \Omega \to \mathcal{A}_{r,C}$ such that there exists $C_g = const > 0$ for which $\mathsf{P}\{\|g\| < C_g\} = 1$, where C_g may depend on g. We put

$$q(z, \omega) = f(Y_{t,z}(t_1, \omega)). \tag{82}$$

Hence, $f(Y(t_1, \omega)) = q(Y(t_1), \omega)$, where $Y(t)$ is a shortening of $Y(t, \omega)$ as above, $\omega \in \Omega$ (see (81)). Assume first that q has the following decomposition:

$$q(z, \omega) = \sum_{k=1}^m q_k(z) u_k(\omega), \tag{83}$$

where $q_k : \mathcal{A}^h_{r,C} \to \mathcal{A}_{r,C}$, $u_k : \Omega \to \mathcal{A}_{r,C}$, $m \in \mathbf{N}$. This implies that $u_k(\omega)$ is independent of $\mathcal{F}_t$ for each k. Therefore, using (82), we deduce that

$$E[g \sum_{k=1}^m q_k(Y(t)) u_k(\omega)] = \sum_{k=1}^m E[g q_k(Y(t))] E u_k(\omega)$$

$$= E[\sum_{k=1}^m g q_k(Y(t))] E u_k(\omega) \text{ and}$$

$$E[\sum_{k=1}^m q_k(Y(t)) u_k(\omega) | Y(t)] = \sum_{k=1}^m q_k(Y(t)) E u_k(\omega). \text{ Consequently,}$$

$$Egf(Y(t_1)) = EgE[f(Y(t_1)) | Y(t)] \tag{84}$$

for q of the form (83). This implies that

$$E[f(Y(t_1))|\mathcal{F}_t] = v(Y(t)), \tag{85}$$

where $v(z) = Ef(Y_{t,z}(t_1))$.

Then, $E[\|gq(Y(\tau),\omega)\|^2] \leq C_g^2\|f\|_C^2$ for each $\tau \in [a,b]$ by 2.3(12) [22], since g and f are bounded, where $\|f\|_C := \sup_{z\in\mathcal{A}^h_{r,C}}\|f(z)\| < \infty$. Therefore, for each $\epsilon > 0$, there exists $f_{(\epsilon)} \in C^0_b(\mathcal{A}^h_{r,C}, \mathcal{A}_{r,C})$ for which $q_{(\epsilon)}(z,\omega) = f_{(\epsilon)}(Y_{t,z}(t_1,\omega))$ has the decomposition of type (83) and such that $E[\|q_{(\epsilon)}(Y(t),\omega) - q(Y(t),\omega)\|^2] < \epsilon/C_g^2$. Taking $\epsilon \downarrow 0$, one obtains that Formulas (84) and (85) are accomplished for each $f \in C^0_b(\mathcal{A}^h_{r,C}, \mathcal{A}_{r,C})$. Therefore, $P\{Y(t_1) \in A|\mathcal{F}_t\} = P\{Y(t_1) \in A|Y(t)\}$ for each $A \in \mathcal{B}(\mathcal{A}^h_{r,C})$, $t < t_1$ in $[a,b]$, since the families $R_b(\Omega, \mathcal{A}_{r,C})$ and $C^0_b(\mathcal{A}^h_{r,C}, \mathcal{A}_{r,C})$ of all such g and f are separate points in $\mathcal{A}^h_{r,C}$. This implies that $P\{Y(t_1) \in A|\mathcal{F}_t\} = P_{t,Y(t)}(t_1, A)$ for each $A \in \mathcal{B}(\mathcal{A}^h_{r,C})$, where $P_{t,z}(t_1, A) = P\{Y_{t,z}(t_1) \in A\}$. Thus, Equality (80) is proven. □

3. Conclusions

The results obtained in this paper, namely, random functions and measures in modules over the complexified Cayley–Dickson algebras, and the integration of the generalized diffusion PDE, open new opportunities for the integration of PDEs of an order higher than 2. Indeed, a solution of a stochastic PDE with real or complex coefficients of an order higher than 2 can be decomposed into a solution of a sequence of PDEs of order 1 or 2 with $\mathcal{A}_{r,C}$ coefficients [9,24]. They can be used for further studies of random functions and integration of stochastic differential equations over octonions and the complexified Cayley–Dickson algebra $\mathcal{A}_{r,C}$. Equations of the type (70) are related with generalized diffusion PDEs of the second order. For example, this approach can be applied to PDEs describing nonequilibrium heat transfer, fourth order Schrödinger- or Klein-Gordon-type PDEs.

Another application of obtained results is for the implementation of the plan described in [22]. It is related with investigations of analogs of Feynman integrals over the complexified Cayley–Dickson algebra $\mathcal{A}_{r,C}$ for solutions of PDEs of orders higher than 2.

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Appendix A. Basics on Hypercomplex Numbers

Remark A1. *Quaternions and octonions (over the real field* $\mathbf{R}$*) are the particular cases of hypercomplex numbers. The algebra* $\mathbf{O}$ *of octonions (octaves, the Cayley algebra) is defined as an eight-dimensional nonassociative algebra over* $\mathbf{R}$ *with a basis, for example,*

$$\mathbf{b}_3 := \mathbf{b} := \{1, i, j, k, l, il, jl, kl\} \text{ such that} \tag{A1}$$

$$i^2 = j^2 = k^2 = l^2 = -1,\ ij = k,\ ji = -k,\ jk = i,\ kj = -i,\ ki = j,\ ik = -j,\ li = -il,$$
$$jl = -lj,\ kl = -lk; \tag{A2}$$

$$(\alpha + \beta l)(\gamma + \delta l) = (\alpha\gamma - \tilde{\delta}\beta) + (\delta\alpha + \beta\tilde{\gamma})l \tag{A3}$$

is the multiplication law in the octonion algebra $\mathbf{O}$ *for each* $\alpha, \beta, \gamma, \delta \in \mathbf{H}$*, where* $\mathbf{H}$ *denotes the quaternion skew field,* $\xi := \alpha + \beta l \in \mathbf{O}$*,* $\eta := \gamma + \delta l \in \mathbf{O}$*,* $\tilde{z} := v - wi - xj - yk$ *for* $z = v + wi + xj + yk \in \mathbf{H}$ *with* $v, w, x, y \in \mathbf{R}$*.*

The octonion algebra is neither commutative nor associative, since $(ij)l = kl$, $i(jl) = -kl$, but it is distributive and $\mathbf{R}1$ *is its center. If $\xi := \alpha + \beta l \in \mathbf{O}$, then*

$$\tilde{\xi} := \tilde{\alpha} - \beta l \tag{A4}$$

is called the adjoint element of ξ, where $\alpha, \beta \in \mathbf{H}$. Then

$$(\xi\eta)\tilde{} = \tilde{\eta}\tilde{\xi},\ \tilde{\xi} + \tilde{\eta} = (\xi + \eta)\tilde{}\ \text{and}\ \xi\tilde{\xi} = |\alpha|^2 + |\beta|^2, \tag{A5}$$

where $|\alpha|^2 = \alpha\tilde{\alpha}$ such that

$$\xi\tilde{\xi} =: |\xi|^2\ \text{and}\ |\xi| \tag{A6}$$

is the norm in $\mathbf{O}$*. Therefore,*

$$|\xi\eta| = |\xi||\eta| \tag{A7}$$

Consequently, $\mathbf{O}$ *does not contain divisors of zero (see also [12,25–27]). The multiplication of octonions satisfies Equations (A8) and (A9) below:*

$$(\xi\eta)\eta = \xi(\eta\eta), \tag{A8}$$

$$\xi(\xi\eta) = (\xi\xi)\eta, \tag{A9}$$

which forms the alternative system. In particular, $(\xi\xi)\xi = \xi(\xi\xi)$. Put $\tilde{\xi} = 2a - \xi$, where $a = Re(\xi) := (\xi + \tilde{\xi})/2 \in \mathbf{R}$. Since $\mathbf{R}1$ *is the center of the octonion algebra* $\mathbf{O}$ *and $\tilde{\xi}\xi = \xi\tilde{\xi} = |\xi|^2$. Then, from (A8) and (A9) by induction, it follows that for each $\xi \in \mathbf{O}$ and each n-tuplet (product), $n \in \mathbf{N}$, $\xi(\xi(\ldots\xi\xi)\ldots) = (\ldots(\xi\xi)\xi\ldots)\xi$ the result does not depend on an order of brackets (order of multiplications). Hence, the definition of $\xi^n := \xi(\xi(\ldots\xi\xi)\ldots)$ does not depend on the order of brackets. This also shows that $\xi^m\xi^n = \xi^n\xi^m$, $\xi^m\tilde{\xi}^m = \tilde{\xi}^m\xi^n$ for each $n, m \in \mathbf{N}$ and $\xi \in \mathbf{O}$.*

Apart from the quaternions, the octonion algebra can not be realized as the subalgebra of the algebra $\mathbf{M}_8(\mathbf{R})$ of all 8×8-matrices over $\mathbf{R}$*, since* $\mathbf{O}$ *is not associative, but the matrix algebra $\mathbf{M}_8(\mathbf{R})$ is associative (see, for example, [10,25–28]). There are the natural embeddings $\mathbf{C} \hookrightarrow \mathbf{O}$ and $\mathbf{H} \hookrightarrow \mathbf{O}$, but neither* $\mathbf{O}$ *over* $\mathbf{C}$*, nor* $\mathbf{O}$ *over* $\mathbf{H}$*, nor* $\mathbf{H}$ *over* $\mathbf{C}$ *are algebras, since the centers of them are $Z(\mathbf{H}) = Z(\mathbf{O}) = \mathbf{R}$ equal to the real field.*

We consider also the Cayley–Dickson algebras $\mathcal{A}_n$ over $\mathbf{R}$*, where 2^n is its dimension over* $\mathbf{R}$*. They are constructed by induction starting from* $\mathbf{R}$ *such that $\mathcal{A}_{n+1}$ is obtained from $\mathcal{A}_n$ with the help of the doubling procedure, in particular, $\mathcal{A}_0 := \mathbf{R}$, $\mathcal{A}_1 = \mathbf{C}$, $\mathcal{A}_2 = \mathbf{H}$, $\mathcal{A}_3 = \mathbf{O}$ and $\mathcal{A}_4$ is known as the sedenion algebra [10,28]. The Cayley–Dickson algebras are $*$-algebras, that is, there is a real-linear mapping $\mathcal{A}_n \ni a \mapsto a^* \in \mathcal{A}_n$ such that*

$$a^{**} = a, \tag{A10}$$

$$(ab)^* = b^*a^* \tag{A11}$$

for each $a, b \in \mathcal{A}_n$. Then, they are nicely normed, that is,

$$a + a^* =: 2Re(a) \in \mathbf{R}\ \text{and} \tag{A12}$$

$$aa^* = a^*a > 0\ \text{for each}\ 0 \neq a \in \mathcal{A}_n. \tag{A13}$$

The norm in it is defined by the equation:

$$|a|^2 := aa^*. \tag{A14}$$

We also denote a^* by $\tilde{a}$. Each nonzero Cayley–Dickson number $0 \neq a \in \mathcal{A}_n$ has a multiplicative inverse given by $a^{-1} = a^*/|a|^2$.

The doubling procedure is as follows. Each $z \in \mathcal{A}_{n+1}$ is written in the form $z = a + bl$, where $l^2 = -1$, $l \notin \mathcal{A}_n$, $a, b \in \mathcal{A}_n$. The addition is component-wise. The conjugate of a Cayley–Dickson number z is prescribed by the formula:

$$z^* := a^* - bl. \tag{A15}$$

Multiplication is given by:

$$(a + bl)(c + dl) = (ac - \tilde{d}b) + (da + b\tilde{c})l \tag{A16}$$

for each a, b, c, d in $\mathcal{A}_n$.

Remark A2. *By $\{i_0, i_1, \ldots, i_{2^r-1}\}$, the standard basis of the Cayley–Dickson algebra $\mathcal{A}_r = \mathcal{A}_{r,\mathbf{R}}$ is denoted over the real field $\mathbf{R}$ such that $i_0 = 1$, $i_l^2 = -1$ and $i_l i_k = -i_k i_l$ for each $l \neq k$ with $1 \leq l$ and $1 \leq k$. For $r \geq 3$ the multiplication of them is generally nonassociative (see also Remark A1). In particular $\mathcal{A}_3$ is the octonion algebra. Henceforth, the complexified Cayley–Dickson algebra $\mathcal{A}_{r,\mathbf{C}} = \mathcal{A}_r \oplus (\mathcal{A}_r\mathbf{i})$ is also considered where $\mathbf{i}^2 = -1$, $\mathbf{i}b = b\mathbf{i}$ for each $b \in \mathcal{A}_r$, $2 \leq r < \infty$. This means that each complexified Cayley–Dickson number $z \in \mathcal{A}_{r,C}$ can be written in the form $z = x + \mathbf{i}y$ with x and y in $\mathcal{A}_r$, $x = x_0 i_0 + x_1 i_1 + \ldots + x_{2^r-1} i_{2^r-1}$, while $x_0, \ldots, x_{2^r-1}$ are in $\mathbf{R}$. The real part of z is $Re(z) = x_0 = (z + z^*)/2$, the imaginary part of z is defined as $Im(z) = z - Re(z)$, where the conjugate of z is $z^* = \tilde{z} = Re(z) - Im(z)$. Thus, $z^* = x^* - \mathbf{i}y$ with $x^* = x_0 i_0 - x_1 i_1 - \ldots - x_{2^r-1} i_{2^r-1}$. Then, $|z|^2 = |x|^2 + |y|^2$, where $|x|^2 = xx^* = x_0^2 + \ldots + x_{2^r-1}^2$.*

Clearly, the $\mathbf{F}$-algebra structure on $\mathcal{A}_{r,\mathbf{F}}$ induces the $\mathbf{F}$-algebra structure on $\mathcal{A}_{r,\mathbf{F}}^l$ such that $w + z = (w_1 + z_1, \ldots, w_l + z_l)$ and $\kappa(w, z) = (w_1 z_1, \ldots, w_l z_l)$ for each w and z in $\mathcal{A}_{r,\mathbf{F}}^l$, $w = (w_1, \ldots, w_l)$ with $w_k \in \mathcal{A}_{r,\mathbf{F}}$ for each $k = 1, \ldots, l$, where $\kappa : \mathcal{A}_{r,\mathbf{F}}^l \times \mathcal{A}_{r,\mathbf{F}}^l \to \mathcal{A}_{r,\mathbf{F}}^l$, where $2 \leq l$. This also induces the $\mathcal{A}_{r,\mathbf{F}}$-bimodule structure on $\mathcal{A}_{r,\mathbf{F}}^l$ such that $\kappa_1(b, w) = (bw_1, \ldots, bw_l)$ and $\kappa_2(b, w) = (w_1 b, \ldots, w_l b)$ for each $b \in \mathcal{A}_{r,\mathbf{F}}^l$ and $w \in \mathcal{A}_{r,\mathbf{F}}^l$, where $\kappa_j : \mathcal{A}_{r,\mathbf{F}} \times \mathcal{A}_{r,\mathbf{F}}^l \to \mathcal{A}_{r,\mathbf{F}}^l$ for each $j \in \{1, 2\}$.

If U is a domain in $\mathbf{F}^{l2^r}$, then to each vector $u = (u_0, \ldots, u_{2^r-1}) \in U$ a unique Cayley–Dickson number $z = \hat{z}(u) = u_0 i_0 + u_1 i_1 + \ldots + u_{2^r-1} i_{2^r-1}$ is posed, where either $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$, $u_j \in \mathbf{F}^l$ for each $j = 0, \ldots, 2^r - 1$; $l \in \mathbf{N}$. This gives a domain $V = \{z : z = \hat{z}(u), u \in U\}$ in $\mathcal{A}_{r,\mathbf{F}}^l$. Vice versa, to each Cayley–Dickson number $z \in, V$, a unique vector $\pi(z) = \hat{u}(z) = u \in U$, corresponds to:

$$u_j = \pi_j(z) \text{ for each } j, \tag{A17}$$

where $\pi_j : \mathcal{A}_{r,\mathbf{F}}^l \to \mathbf{F}^l$ is a $\mathbf{F}$-linear operator given by the formulas:

$$\pi_j(z) = (-zi_j + i_j(2^r - 2)^{-1}\{-z + \sum_{k=1}^{2^r-1} i_k(zi_k^*)\})/2 \tag{A18}$$

for each $j = 1, 2, \ldots, 2^r - 1$,

$$\pi_0(z) = (z + (2^r - 2)^{-1}\{-z + \sum_{k=1}^{2^r-1} i_k(zi_k^*)\})/2, \tag{A19}$$

where $2 \leq r \in \mathbf{N}$, z is a Cayley–Dickson vector (or a number for $l = 1$) presented as follows.

$$z = z_0 i_0 + z_1 i_1 + \ldots + z_{2^r-1} i_{2^r-1} \in \mathcal{A}_{r,\mathbf{F}},\ z_j \in \mathbf{F}^l \tag{A20}$$

for each $j = 0, \ldots, 2^r - 1$; $z^* = z_0 i_0 - z_1 i_1 - \ldots - z_{2^r-1} i_{2^r-1}$ (see Formulas II(1.1)–(1.3) in [29]).

Appendix B. Hypercomplex Measures

In Appendix B, basic facts on hypercomplex measures from [22] are given.

Remark A3. PDEs. *λ_n denotes the Lebesgue measure on the Euclidean space $\mathbf{R}^n$. Consider a domain U in $\mathbf{R}^n$, such that $U \subseteq cl(Int(U))$, where $Int(U)$ denotes the interior of U, while $cl(U)$ notates the closure of U in $\mathbf{R}^n$.*

Let $\{i_0, i_1, \dots, i_{2^r-1}\}$ notate the standard basis of the Cayley–Dickson algebra $\mathcal{A}_r$ over the real field $\mathbf{R}$, such that $i_0 = 1$, $i_l^2 = -1$ and $i_l i_k = -i_k i_l$ for each $l \neq k$ with $1 \leq l$ and $1 \leq k$ (see also Remark 1 and Definition 2 in Introduction). The Cayley–Dickson algebra $\mathcal{A}_r$ is nonassociative for each $r \geq 3$ and nonalternative for each $r \geq 4$, for example, $(i_1 i_2) i_4 = -i_1(i_2 i_4)$, etc. Then, $\mathcal{A}_{r,C}$ stands for the complexified Cayley–Dickson algebra $\mathcal{A}_{r,C} = \mathcal{A}_r \oplus (\mathcal{A}_r \mathbf{i})$, where $\mathbf{i}^2 = -1$, $\mathbf{i}b = b\mathbf{i}$ for each $b \in \mathcal{A}_r$, $2 \leq r < \infty$ Therefore, each complexified Cayley–Dickson number $z \in \mathcal{A}_{r,C}$ has the form $z = x + \mathbf{i}y$ with x and y in $\mathcal{A}_r$, $x = x_0 i_0 + x_1 i_1 + \ldots + x_{2^r-1} i_{2^r-1}$, while $x_0, \dots, x_{2^r-1}$ are in $\mathbf{R}$. The real part of z is $Re(z) = x_0 = (z + z^*)/2$, the imaginary part of z is defined as $Im(z) = z - Re(z)$, where the conjugate of z is $z^* = \tilde{z} = Re(z) - Im(z)$, that is $z^* = x^* - \mathbf{i}y$ with $x^* = x_0 i_0 - x_1 i_1 - \ldots - x_{2^r-1} i_{2^r-1}$. Then $|z|^2 = |x|^2 + |y|^2$, where $|x|^2 = xx^* = x_0^2 + \ldots + x_{2^r-1}^2$. It is useful also to put $\|z\| = |z|\sqrt{2}$.

Each function $f : U \to \mathcal{A}_{r,C}$ has a decomposition

$$f(x) = \sum_{s=0}^{2^r-1} f_s(x) i_s,$$

where $f_s : U \to \mathbf{C}$ for each s, $\mathcal{A}_{r,C}$ denotes the complexified Cayley–Dickson algebra (see above). Function $f(x)$ is differentiable (in real variables) at x in U if and only if $f_s(x)$ is differentiable at x for each $s = 0, 1, \dots, 2^r - 1$.

Sobolev space $H^k(U, \lambda_n, \mathcal{A}_{r,C})$ is the completion by a norm $\|f\|_k$ of the space of all k times continuously differentiable (in real variables) functions $f : U \to \mathcal{A}_{r,C}$ with compact support, where

$$\|f\|_k^2 := \sum_{j=0}^{k} \int_U \|f^{(j)}(x)\|^2 \lambda_n(dx), \tag{A21}$$

$f^{(j)}(x) = D_x^j f(x)$ denotes the j-th derivative poly-$\mathbf{R}$-linear operator on $\mathbf{R}^n$ at a point x, where n is a natural number. Particularly, it may be $U = \mathbf{R}^n$.

Suppose that an operator B_j is realized as an elliptic PDO $\hat{B}_j$ of the second order on the Sobolev space $H^2(\mathbf{R}^{m_j}, \lambda_{m_j}, \mathbf{R})$ by real variables $x_{1+\beta_{j-1}}, \dots, x_{\beta_j}$, where $m_0 = 0$, $\beta_0 = 0$, $\beta_j = m_0 + \ldots + m_j$, $m_j \in \mathbf{N}$ for each $j = 1, 2, \ldots$.

We consider a second-order PDO of the form

$$\hat{B} = -\frac{1}{2} \sum_{j=1}^{m} a_j \hat{B}_j, \tag{A22}$$

where $a_j = a_{j,0} + \mathbf{i} a_{j,1}$ are nonzero coefficients, $a_j \in \mathcal{A}_{r,C}$, $a_{j,0}$ and $a_{j,1}$ belong to $\mathcal{A}_r$, where $\hat{B}_j$ is an elliptic PDO of the second order on $H^2(\mathbf{R}^{m_j}, \lambda_{m_j}, \mathbf{R})$ by real variables $x_{1+\beta_{j-1}}, \dots, x_{\beta_j}$, where $2 \leq r < \infty$.

There are the natural embeddings $H^2(\mathbf{R}^{m_j}, \lambda_{m_j}, \mathcal{A}_{r,C}) \hookrightarrow H^2(\mathbf{R}^n, \lambda_n, \mathcal{A}_{r,C})$, where $n = m_1 + \ldots + m_m = \beta_m$. Thus, $\hat{B}$ and all $\hat{B}_j$ are defined on $H^2(\mathbf{R}^n, \lambda_n, \mathcal{A}_{r,C})$. Let σ^* also be a first-order PDO

$$\sigma^* f(x) = \sum_{j=1}^{m} \sigma_j^* f(x) \text{ and} \tag{A23}$$

$$\sigma_j^* f(x) = \sum_{k=\beta_{j-1}+1}^{\beta_j} \psi_{k;j} \frac{\partial f(x)}{\partial x_k} \tag{A24}$$

for each $f \in H^1(\mathbf{R}^n, \lambda_n, \mathcal{A}_{r,C})$, where $\beta_j = m_0 + \ldots + m_j$ for each j, $m_0 = 0$, $\beta_0 = 0$; $\psi_{k;j} \in \mathcal{A}_{r,C}$ for each k and j. Then, the operator

$$\hat{S} = \frac{\partial}{\partial t} + \hat{B} + \sigma^* \tag{A25}$$

is defined on a Sobolev space $H^{2,1}(\mathbf{R}^n \times \mathbf{R}, \lambda_{n+1}, \mathcal{A}_{r,C})$, where $H^{k,l}(U \times V, \lambda_{n+1}, \mathcal{A}_{r,C})$ is the completion relative to a norm $\|f\|_{k,l}$ of the space of all functions $f(x,t) : U \times V \to \mathcal{A}_{r,C}$ continuously differentiable k times in x and l times in t with compact support, where $V \subseteq cl(Int(V)) \subset \mathbf{R}$,

$$\|f\|^2_{k,l} := \sum_{j=0}^{k} \sum_{s=0}^{l} \int_{U \times V} \|D^j_x D^s_t f(x,t)\|^2 \lambda_{n+1}(dx), \tag{A26}$$

where $x \in U$, $t \in V$. Evidently, $H^{k,l}(U \times V, \lambda_{n+1}, \mathcal{A}_{r,C})$ has a structure of a Hilbert space over $\mathbf{R}$, also of a two-sided $\mathcal{A}_{r,C}$-module. Particularly, $H^0(U, \lambda_n, \mathcal{A}_{r,C}) = L^2(U, \lambda_n, \mathcal{A}_{r,C})$.

Using the change in variables, we consider operators with constant coefficients

$$\hat{B}_j f(x) = \sum_{u,k=1}^{m_j} b_{u,k;j} \frac{\partial^2 f(x)}{\partial x_{u+\beta_{j-1}} \partial x_{k+\beta_{j-1}}}, \tag{A27}$$

for each $f \in H^2(\mathbf{R}^n, \lambda_n, \mathcal{A}_{r,C})$, where $b_{u,k;j} \in \mathbf{R}$ for every u, k, j, $\beta_j = m_0 + \ldots + m_j$, $m_0 = 0$, $\beta_0 = 0$. $[B_j]$ denotes a matrix with matrix elements $b_{u,k;j} \in \mathbf{R}$ for every u and k in $\{1, \ldots, m_j\}$, where $j = 1, \ldots, m$. B_j notates a linear operator $B_j : \mathbf{R}^{m_j} \to \mathbf{R}^{m_j}$ prescribed by its matrix $[B_j]$. Since the operator $\hat{B}_j$ is elliptic, then without loss of generality, matrix $[B_j]$ is symmetric and positive definite. Then, using a variable change, it is also frequently possible to impose the condition $Re(\psi_{k;j}\psi^*_{i;l}) = 0$ if either $k \neq i$ or $j \neq l$.

Let A be a unital normed algebra over $\mathbf{R}$, where A may be nonassociative, and let its center $Z(\mathsf{A})$ contain the real field $\mathbf{R}$. Then, by ${}_l\prod_{k=1}^{m} u_k$, we denote an ordered product from right to left, such that

$${}_l\prod_{k=1}^{m} u_k = u_m({}_l\prod_{k=1}^{m-1} u_k) \tag{A28}$$

for each $m \geq 2$, where ${}_l\prod_{k=1}^{1} u_k = u_1$; $u_1, \ldots, u_m$ are elements of A. Then, we put

$$\exp_l(z) = 1 + \sum_{n=1}^{\infty} \frac{{}_l(z^n)}{n!}, \tag{A29}$$

where ${}_l(z^n) = {}_l\prod_{k=1}^{n} z$, which corresponds to the ordered product from right to left (see above (A28)), $z \in \mathsf{A}$, that is, for the particular case $u_1 = z, \ldots, u_n = z$.

Definition A1. *Let X be a right module over $\mathcal{A}_{r,C}$ such that*

$$X = X_0 \oplus X_1 i_1 \oplus \ldots \oplus X_{2^r-1} i_{2^r-1},$$

where $X_0, \ldots, X_{2^r-1}$ are pairwise isomorphic vector spaces over $\mathbf{C}$. If an addition $x + y$ in X is jointly continuous in x and y and a right multiplication xb is jointly continuous in $x \in X$, and $b \in \mathcal{A}_{r,C}$ and X_j is a topological vector space for each $j \in \{0, 1, \ldots, 2^r - 1\}$, then X is a topological right module over $\mathcal{A}_{r,C}$.

For the right module X over $\mathcal{A}_{r,C}$ an operator h from X into $\mathcal{A}_{r,C}$ is called right $\mathcal{A}_{r,C}$-linear in a weak sense if and only if it $h(fb) = (h(f))b$ for each $f \in X_0$ and $b \in \mathcal{A}_{r,C}$. Then, X^*_r denotes a family of all continuous right $\mathcal{A}_{r,C}$-linear operators $h : X \to \mathcal{A}_{r,C}$ in the weak sense on the topological right module X over $\mathcal{A}_{r,C}$.

An operator $h : X \to \mathcal{A}_{r,C}$ is right $\mathcal{A}_{r,C}$-linear if and only if $h(fb) = (h(f))b$ for each $f \in X$ and $b \in \mathcal{A}_{r,C}$.

Symmetrically, on a left module Y over $\mathcal{A}_{r,C}$ such that

$$Y = Y_0 \oplus i_1 Y_1 \oplus \ldots \oplus i_{2^r-1} Y_{2^r-1},$$

where $Y_0, \ldots, Y_{2^r-1}$ are pairwise isomorphic vector spaces over $\mathbf{C}$ are defined left $\mathcal{A}_{r,C}$-linear operators and left $\mathcal{A}_{r,C}$-linear in a weak sense operators. A family of all continuous left $\mathcal{A}_{r,C}$-linear operators $g : Y \to \mathcal{A}_{r,C}$ on the topological left module Y over $\mathcal{A}_{r,C}$ in the weak sense is denoted by Y_l^*.

X is a two-sided module over the complexified Cayley–Dickson algebra $\mathcal{A}_{r,C}$ if and only if it is a left and right module over $\mathcal{A}_{r,C}$ and $i_j x_j = x_j i_j$ for each $x_j \in X_j$ and $j \in \{0, 1, \ldots, 2^r - 1\}$.

Theorem A1. *Let a PDO $\hat{S}$ be of the form (A25), fulfilling the condition*

(α) $Re(a_{j,0}) > |q_j| \cdot |\sin \phi_j|$ with $q_j^2 = |Im(a_{j,0})|^2 - |Im(a_{j,1})|^2 - 2\mathbf{i}Re(a_{j,0}a_{j,1})$, $q_j \in \mathbf{C}$, $\phi_j = arg(q_j)$ for each j, where $2 \leq r < \infty$. Then, a fundamental solution $\mathcal{K}$ of the equation

$$\hat{S}\mathcal{K} = \delta(x,t) \text{ is} \tag{A30}$$

$$\mathcal{K}(x,t) = \frac{\theta(t)}{(2\pi)^n} \int_{\mathbf{R}^n} \exp_l(-\sum_{j=1}^{m} \{\frac{1}{2} a_j (B_j \mathbf{y}_j, \mathbf{y}_j) + \mathbf{i}(\mathbf{s}_j, \mathbf{y}_j)\} t) \exp(-\mathbf{i}(y,x)) \lambda_n(dy), \tag{A31}$$

where $\theta(t) = 0$ for each $t < 0$, while $\theta(t) = 1$ for each $t \geq 0$, where $2 \leq r < \infty$, $\mathbf{y}_j = (y_{\beta_{j-1}+1}, \ldots, y_{\beta_j})$ with $y_k \in \mathbf{R}$ for each k,

$$(\mathbf{s}_j, \mathbf{y}_j) = \sum_{k=\beta_{j-1}+1}^{\beta_j} s_k y_k, \tag{A32}$$

where $s_{\beta_{j-1}+k} = \psi_{k;j}$ for each $k = 1, \ldots, m_j$ and each $j = 1, \ldots, m$.

Definition A2. *Let $a_j \in \mathcal{A}_{r,C}$ satisfy Condition (α) of Theorem A1 for each j, $B_j : \mathbf{R}^{m_j} \to \mathbf{R}^{m_j}$ be a positive definite operator for each $j = 1, \ldots, m$, $p \in \mathcal{A}_{r,C}^n$, where $n = \beta_m = m_1 + \ldots + m_m$, $2 \leq r < \infty$. Let also $U : \mathcal{A}_{r,C} \to \mathcal{A}_{r,C}$ be an operator such that $U = \bigoplus_{j=1}^m a_j B_j$. We define*

$$(y,z)_s = \sum_{k=1}^{n} y_k z_k \tag{A33}$$

for each y and z in $\mathcal{A}_{r,C}^n$, where $z = (z_1, \ldots, z_n)$, $z_k \in \mathcal{A}_{r,C}$ for each k. (y,z) is also briefly written instead of $(y,z)_s$ when a situation is specified. Then,

$$\hat{\vartheta}_{U,p}(y) := \exp_l(-\frac{1}{2}(Uy,y) + \mathbf{i}(p,y)) \tag{A34}$$

is called a characteristic functional of an $\mathcal{A}_{r,C}$-valued measure $\vartheta_{U,p}$ on a Borel σ-algebra $\mathcal{B}(\mathbf{R}^n)$ of the Euclidean space $\mathbf{R}^n$, where $y \in \mathbf{R}^n$. We define a measure $\mu_{U,p}$ on the Borel σ-algebra $\mathcal{B}(\mathcal{A}_{r,C}^n)$ of the two-sided $\mathcal{A}_{r,C}$-module $\mathcal{A}_{r,C}^n$ by the formula:

$$\mu_{U,p}(p + U^{1/2}dh) = \vartheta_{U,p}(dx)\delta_{U,p}(p' + U^{1/2}dg), \tag{A35}$$

where $h = x + g$, $h \in \mathcal{A}_{r,C}^n$, $x \in \mathbf{R}^n$, $g \in X'$, $X' = (\mathcal{A}_{r,C} \ominus \mathbf{R}i_0)^n$, $\mathbf{R}^n$ is embedded into $\mathcal{A}_{r,C}^n$ as $\mathbf{R}^n i_0$, $p = p_0 + p'$ with $p_0 \in \mathbf{R}^n$ and $p' \in X'$,

$$\int_{Y'} f(y')\delta_{U,p}(dy') = f(p') \tag{A36}$$

for each $f \in C_b^0(Y', \mathcal{A}_{r,C})$, where $C_b^0(Y', \mathcal{A}_{r,C})$ denotes the family of all continuous bounded functions f from Y' into $\mathcal{A}_{r,C}$, $Y' = p' + U^{1/2}X'$.

Proposition A1. *The measure $\mu_{U,p}$ (see Definition A2) is σ-additive on $\mathcal{B}(\mathcal{A}_{r,C}^n)$.*

Corollary A1. *If conditions of Theorem A1 are fulfilled, $t > 0$, $p = -s$, measure $\mu_{Ut,pt}$ is σ-additive on $\mathcal{B}(\mathcal{A}^n_{r,C})$.*

Proposition A2. *For each $z \in \mathcal{A}_{r,C}$, function $\chi_z(t) = \exp_l(zt)$ is a character from $\mathbf{R}$ considered as the additive group into the algebra $\mathcal{A}_{r,C}$, such that*

$$\exp_l(zt) = (\exp_l(z))^t \text{ and} \tag{A37}$$

$$\chi_z(t + t_1) = \chi_z(t)\chi_z(t_1) \tag{A38}$$

for each $t \in \mathbf{R}$ and $t_1 \in \mathbf{R}$.

Definition A3. *Let Ω be a set with an algebra $\mathcal{R}$ of its subsets and an $\mathcal{A}_{r,C}$-valued measure $\mu : \mathcal{R} \to \mathcal{A}_{r,C}$, where $2 \leq r$, $\Omega \in \mathcal{R}$. Then*

$$|\mu| := \sum_{j=0}^{2^r-1} (|\mu_{j,0}| + |\mu_{j,1}|) \tag{A39}$$

is called a variation, and $|\mu|(\Omega)$ is a norm of the measure μ, where

$$\mu = \sum_{j=0}^{2^r-1} (\mu_{j,0} i_j + \mu_{j,1} i_j \mathbf{i}) \tag{A40}$$

is the decomposition of the measure μ.

$\mu_{j,k} : \mathcal{R} \to \mathbf{R}$, $|\mu_{j,k}|$ denotes the variation of a real-valued measure $\mu_{j,k}$ for each $j = 0, 1, \dots, 2^r - 1$ and $k = 0, 1$, $|\mu| : \mathcal{R} \to [0, \infty)$.

A class $\mathcal{G}$ of subsets of a set Ω is compact if, for any sequence G_k of its elements fulfilling $\bigcap_{k=1}^{\infty} G_k = \emptyset$, a natural number l exists so that $\bigcap_{k=1}^{l} G_k = \emptyset$.

An $\mathcal{A}_{r,C}$-valued measure μ (not necessarily σ-additive, i.e., a premeasure in another terminology) on an algebra $\mathcal{R}$ of subsets of the set Ω is approximated from below by a class $\mathcal{H}$, where $\mathcal{H} \subset \mathcal{R}$, if for each $A \in \mathcal{R}$ and $\epsilon > 0$ a subset B belonging to the class $\mathcal{H}$ exists, such that $B \subset A$ and $|\mu|(A \setminus B) < \epsilon$ (see Formula (A39)).

The $\mathcal{A}_{r,C}$-valued measure μ on the algebra $\mathcal{R}$ is called Radon if it is approximated from below by the compact class $\mathcal{H}$. In this case, the measure space $(\Omega, \mathcal{R}, \mu)$ is called Radon.

Remark A4. *Different forms of the diffusion PDE.*

In the classical case over the real field $\mathbf{R}$, different forms of the diffusion PDE such as backward Kolmogorov, Fokker–Planck–Kolmogorov, and stochastic are provided by Theorems 6 and 7 in Chapter I, Section 4, Theorem 4 in Chapter VIII, Section 2 in [5], or by Theorems 3.7, 3.11 in Chapter 3, Section 3.8 in [2]. The stochastic PDE

$$\xi_{t,x}(s) = x + \int_t^s a(u, \xi_{t,x}(u))du + \sum_{k=1}^{m} \int_t^s b_k(u, \xi_{t,x}(u))dw_k(u)$$

is considered to be the diffusion PDE with m variables in Equation (14) in Chapter VIII, Section 2 in [5], where $(b_1, \dots, b_m)$ denotes the diffusion operator reduced to the diagonal form, a is the transition (generally may be nonlinear shift) operator, $(w_1, \dots, w_k)$ denotes the Gaussian–Wiener process with values in the Euclidean space $\mathbf{R}^m$. Solutions of the diffusion PDE in its stochastic form provide evolutionary operators and their generators serving for solutions of backward Kolmogorov or Fokker–Planck–Kolmogorov PDEs (see [1,2,5]).

Following this terminology, a generalized analog of the Fokker–Planck–Kolmogorov PDE or backward Kolmogorov is obtained by substituting their partial differential operator by the partial differential operator $\hat{S}$ given by Formula (A25) in Remark A3. The generalized diffusion PDE itself (in the stochastic form) is Equation (70) in Definition 7 above.

In more details see also [30–34].

References

1. Dalecky, Y.L.; Fomin, S.V. *Measures and Differential Equations in Infinite-Dimensional Space*; Kluwer: Dordrecht, The Netherlands, 1991.
2. Gulisashvili, A.; van Casteren J.A. *Non-Autonomous Kato Classes and Feynman-Kac Propagators*; World Scientific: Hackensack, NJ, USA, 2006.
3. Johnson, G.W.; Lapidus, M.L. *The Feynman Integral and Feynman's Operational Calculus*; Oxford Univ. Press: Oxford, UK; Clarendon Press: New York, NY, USA, 2000.
4. Kim, B.J.; Kim B.S. Integration by parts formulas for analytic Feynman integrals of unbounded functionals. *Integr. Transform. and Spec. Funct.* **2009**, *20*, 45–57. [CrossRef]
5. Gihman, I.I.; Skorohod, A.V. *The Theory of Stochastic Processes*; Springer: New York, NY, USA, 1975; Nauka: Moscow, Russia, 1977.
6. Pap, E. *Handbook of Measure Theory*; Elsevier: Amsterdam, The Netherlands, 2002; Volume 1–2.
7. Shiryaev, A.N. *Probability*; MTsNMO: Moscow, Russia, 2011.
8. *Dirac, P.A.M. Die Prinzipen der Quantenmechanik*; Hirzel: Leipzig, Germany, 1930.
9. Ludkowski, S.V. Decompositions of PDE over Cayley-Dickson algebras. Rendic. dell'Ist. di Math. dell'Università di Trieste. *Nuova Ser.* **2014**, *46*, 1–23.
10. Baez, J.C. The octonions. *Bull. Am. Math. Soc.* **2002**, *39*, 145–205. [CrossRef]
11. Dickson L.E. *The Collected Mathematical Papers*; Chelsea Publishing Co.: New York, NY, USA, 1975; Volume 1–5.
12. Kantor, I.L.; Solodovnikov, A.S. *Hypercomplex Numbers*; Springer: New York, NY, USA, 1989.
13. Emch, G. Mèchanique quantique quaternionienne et Relativitè restreinte. *Helv. Phys. Acta* **1963**, *36*, 739–788.
14. Gürlebeck, K.; Sprössig, W. *Quaternionic and Clifford Calculus for Physicists and Engineers*; John Wiley and Sons: Chichester, UK, 1997.
15. Gürlebeck, K.; Sprössig, W. *Quaternionic Analysis and Elliptic Boundary Value Problem*; Birkhäuser: Basel, Switzerland, 1990.
16. Gürsey, F.; Tze, C.-H. *On the Role of Division, Jordan and Related Algebras in Particle Physics*; World Scientific Publ. Co.: Singapore, 1996.
17. Lawson, H.B.; Michelson, M.-L. *Spin Geometry*; Princeton University Press: Princeton, NJ, USA, 1989.
18. Ludkovsky, S.V.; van Oystaeyen, F. Differentiable functions of quaternion variables. *Bull. Sci. Math.* **2003**, *127*, 755–796 [CrossRef]
19. Ludkovsky, S.V. Differentiable functions of Cayley-Dickson numbers and line integration. *J. Math. Sci.* **2007**, *141*, 1231–1298. [CrossRef]
20. Ludkovsky, S.V. Functions of several Cayley-Dickson variables and manifolds over them. *J. Math. Sci.* **2007**, *141*, 1299–1330. [CrossRef]
21. Ludkovsky, S.V. Geometric loop groups and diffeomorphism groups of manifolds, stochastic processes on them, associated unitary representations. In *Focus on Groups Theory Research*; Ying, L.M., Ed.; Nova Science Publishers, Inc.: New York, NY, USA, 2006; pp. 59–136.
22. Ludkowski, S.V. Octonion measures for solutions of PDEs. *Adv. Appl. Clifford Algebras.* **2020**, *30*, 1–23. [CrossRef]
23. Gantmacher, F.R. *Theory of Matrices*; Nauka: Moscow, Russia, 1988.
24. Ludkovsky, S.V. Line integration of Dirac operators over octonions and Cayley-Dickson algebras. *Comput. Methods Funct. Theory* **2012**, *12*, 279–306. [CrossRef]
25. Kurosh, A.G. *Lectures on General Algebra*; Nauka: Moscow, Russia, 1973.
26. van der Waerden, B.L. *A History of Algebra*; Springer: Berlin/Heidelberg, Germany, 1985.
27. Ward, J.P. *Quaternions and Cayley Numbers: Algebra & Applications (Mathematics & Its Applications/403*; Kluwer: Dordrecht, The Netherlands, 1997; Volume 403.
28. Schafer, R.D. *An Introduction to Nonassociative Algebras*; Academic Press: New York, NY, USA, 1966.
29. Ludkovsky, S.V. Differential equations over octonions. *Adv. Appl. Clifford Alg.* **2011**, *21*, 773–797.
30. Freidlin, M. *Functional Integration and Partial Differential Equations*; Princeton University Press: Princeton, NJ, USA, 1985.
31. Ludkovsky, S.V. Feynman integration over octonions with application to quantum mechanics. *Math. Methods Appl. Sci.* **2010**, *33*, 1148–1173. [CrossRef]
32. Ludkovsky, S.V. Noncommutative quasi-conformal integral transforms over quaternions and octonions. *J. Math. Sci.* **2009**, *157*, 199–251. [CrossRef]
33. Ludkovsky, S.V. Two-sided Laplace transform over Cayley-Dickson algebras and its applications. *J. Math. Sci.* **2008**, *151*, 3372–3430. [CrossRef]
34. Ludkovsky, S.V. Residues of functions of octonion varaibles. *Far East J. Math. Sci. (FJMS)* **2010**, *39*, 65–104.