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Double Integral of Logarithmic and Quotient Rational Functions Expressed in Terms of the Lerch Function

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Abstract: In this manuscript, the authors derive a double integral whose kernel involves the logarithmic function a polynomial raised to a power and a quotient function expressed it in terms of the Lerch function. All the results in this work are new.

Keywords: double integral; Aprey's constant; Catalan's constant; Lerch function

MSC: Primary 30E20; 33-01; 33-03; 33-04; 33-33B; 33E20

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1. Significance Statement

Double integrals are used in a myriad of areas in science. These integrals are used extensively in mathematics in the areas of calculus and statistics. Double integrals featuring the logarithmic function and a polynomial raised to a power as their kernel are not widely used but we were able to find at least one application namely the logarithmic integral transformation [1].

In this manuscript we aim to expand upon the usage of the definite integral of a polynomial function by providing a new double integral expressed in terms of the Lerch function. The Lerch function being a special function has the property of analytic continuation which allows for a wider range of computation of the parameters involved. Another property of the Lerch function is almost all Lerch zeta-functions have an asymmetrical zero-distribution [2].

2. Introduction

In this paper we derive the double integral given by

$$\int_0^\infty \int_0^1 \frac{y^{2m} x^n \left(x^{-n-1}-1\right)^m \log^k\left(ay^2\left(x^{-n-1}-1\right)\right)}{y^2+1} dxdy \tag{1}$$

where the parameters k, a, m are general complex numbers and $-1 < Re(m) < 1$. The double integral will be used to derive special cases in terms of special functions and fundamental constants. The derivations follow the method used by us in [3]. This method involves using a form of the generalized Cauchy's integral formula given by

$$\frac{y^k}{\Gamma(k+1)} = \frac{1}{2\pi i} \int_C \frac{e^{wy}}{w^{k+1}} dw. \tag{2}$$

We multiply both sides by a function of x and y, then take a definite double integral of both sides. This yields a definite integral in terms of a contour integral. Then we multiply both sides of Equation (2) by another function of x and y and take the infinite sums of both sides such that the contour integral of both equations are the same.

3. Definite Integral of the Contour Integral

We use the method in [3]. The variable of integration in the contour integral is $\alpha = m + w$. The cut and contour are in the second quadrant of the complex α-plane. The cut approaches the origin from the interior of the second quadrant and the contour goes round the origin with zero radius and is on opposite sides of the cut. Using a generalization of Cauchy's integral formula we form two equations by replacing y by $\log\left(ay^2\left(x^{-n-1}-1\right)\right)$ and multiplying by $\frac{y^{2m}x^n\left(x^{-n-1}-1\right)^m}{y^2+1}$ then taking the definite integral with respect $x \in [0,1]$ and $y \in [0,\infty)$ to get

$$\begin{aligned}
&\tfrac{1}{k!}\int_0^\infty\int_0^1 \tfrac{y^{2m}x^n\left(x^{-n-1}-1\right)^m \log^k\left(ay^2\left(x^{-n-1}-1\right)\right)}{y^2+1}dxdy \\
&= \tfrac{1}{2\pi i}\int_0^\infty\int_0^1\int_C \tfrac{a^w w^{-k-1}x^n y^{2(m+w)}\left(x^{-n-1}-1\right)^{m+w}}{y^2+1}dwdxdy \\
&= \tfrac{1}{2\pi i}\int_C\int_0^\infty\int_0^1 \tfrac{a^w w^{-k-1}x^n y^{2(m+w)}\left(x^{-n-1}-1\right)^{m+w}}{y^2+1}dxdydw \\
&= \tfrac{1}{2\pi i}\int_C \tfrac{\pi^2 a^w w^{-k-1}(m+w)\csc(2\pi(m+w))}{n+1}dw
\end{aligned} \tag{3}$$

from equations (3.249.7), (3.2511), and (3.241.2) in [4] where $Re(w+m) > 0$. The logarithmic function is given for example in Section (4.1) in [5]. We are able to switch the order of integration over $w+m$, x and y using Fubini's theorem since the integrand is of bounded measure over the space $\mathbb{C} \times [0,1] \times [0,\infty)$.

4. The Lerch Function Contour Integral Representations

4.1. The Lerch Function

The Lerch function section (1.11) in [6] has a series representation given by

$$\Phi(z,s,v) = \sum_{n=0}^{\infty}(v+n)^{-s}z^n \tag{4}$$

where $|z| < 1, v \neq 0, -1, \ldots$ and is continued analytically by its integral representation given by

$$\Phi(z,s,v) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-(v-1)t}}{e^t - z}dt \tag{5}$$

where $Re(v) > 0$, and either $|z| \leq 1, z \neq 1, Re(s) > 0$, or $z = 1, Re(s) > 1$.

4.2. Derivation of the First Contour Integral

In this section we will again use Cauchy's integral formula (2) and take the infinite sum to derive equivalent sum representations for the contour integrals. We proceed using Equation (2) and replace y by $\log(a) + 2i\pi(2y+1)$ and multiply both sides by $-\frac{2i\pi^2 m e^{2i\pi m(2y+1)}}{n+1}$ and take the infinite sum over $y \in [0,\infty)$ simplifying in terms of the Lerch function to get

$$\begin{aligned}
&\tfrac{i^{k-1}2^{2k+1}\pi^{k+2}e^{2i\pi m}m\Phi\left(e^{4im\pi},-k,\frac{1}{2}-\frac{i\log(a)}{4\pi}\right)}{(n+1)\Gamma(k+1)} \\
&= -\tfrac{1}{2\pi i}\textstyle\sum_{y=0}^{\infty}\int_C \tfrac{2i\pi^2 m a^w w^{-k-1}e^{2i\pi(2y+1)(m+w)}}{n+1}dw \\
&= -\tfrac{1}{2\pi i}\int_C\textstyle\sum_{y=0}^{\infty} \tfrac{2i\pi^2 m a^w w^{-k-1}e^{2i\pi(2y+1)(m+w)}}{n+1}dw \\
&= \tfrac{1}{2\pi i}\int_C \tfrac{\pi^2 m a^w w^{-k-1}\csc(2\pi(m+w))}{n+1}dw
\end{aligned} \tag{6}$$

from Equation (1.232.3) in [4] where $\operatorname{csch}(ix) = -i\csc(x)$ from Equation (4.5.10) in [5] and $Im(w+m) > 0$ for the sum to converge.

4.3. Derivation of the Second Contour Integral

Use Equation (6) and multiply by $1/m$ and replace $k \to k-1$ and simplify to get

$$\begin{aligned} &-\frac{i^k 2^{2k-1}\pi^{k+1}e^{2i\pi m}\Phi\left(e^{4im\pi},1-k,\frac{1}{2}-\frac{i\log(a)}{4\pi}\right)}{(n+1)\Gamma(k)} \\ &= \frac{1}{2\pi i}\int_C \frac{\pi^2 a^w w^{-k}\csc(2\pi(m+w))}{n+1}dw \end{aligned} \tag{7}$$

Main results

In the proceeding section, we will derive integral formula in terms of Catalan's constant G, Polygamma function $\psi_n(z)$, and Hurwitz zeta function $\zeta(n,u)$ For $k,a,n \in \mathbb{C}, -1 < Re(m) < 1$,

$$\begin{aligned} &\int_0^\infty\int_0^1 \frac{y^{2m}x^n\left(x^{-n-1}-1\right)^m\log^k\left(ay^2\left(x^{-n-1}-1\right)\right)}{y^2+1}dxdy \\ &= \frac{1}{n+1}2^{2k-1}(i\pi)^{k+1}e^{2i\pi m}\left(ik\Phi\left(e^{4im\pi},1-k,\frac{1}{2}-\frac{i\log(a)}{4\pi}\right)\right. \\ &\qquad \left.-4\pi m\Phi\left(e^{4im\pi},-k,\frac{1}{2}-\frac{i\log(a)}{4\pi}\right)\right) \end{aligned} \tag{8}$$

Proof. Observe that the right-hand side of Equation (8) is equal the addition of the right-hand sides of Equations (6) and (29), so we can equate the left-hand sides to yield the stated result. □

For $k,a,n \in \mathbb{C}, -1 < Re(m) < 1, -1 < Re(p) < 1$,

$$\begin{aligned} &\int_0^\infty\int_0^1 \frac{x^n\left(y^{2p}\left(x^{-n-1}-1\right)^p-y^{2m}\left(x^{-n-1}-1\right)^m\right)\log^k\left(ay^2\left(x^{-n-1}-1\right)\right)}{y^2+1}dxdy \\ &= \frac{1}{n+1}i^k 2^{2k-1}\pi^{k+1}\left(e^{2i\pi m}\left(k\Phi\left(e^{4im\pi},1-k,\frac{1}{2}-\frac{i\log(a)}{4\pi}\right)\right)\right) \\ &\qquad +4i\pi m\Phi\left(e^{4im\pi},-k,\frac{1}{2}-\frac{i\log(a)}{4\pi}\right) \\ &-e^{2i\pi p}\left(k\Phi\left(e^{4ip\pi},1-k,\frac{1}{2}-\frac{i\log(a)}{4\pi}\right)+4i\pi p\Phi\left(e^{4ip\pi},-k,\frac{1}{2}-\frac{i\log(a)}{4\pi}\right)\right) \end{aligned} \tag{9}$$

Proof. Use Equation (8) and form a second equation by replacing $m \to p$ and take their difference and simplify. □

$$\int_0^\infty\int_0^1 \frac{y^{2m}x^n\left(x^{-n-1}-1\right)^m}{y^2+1}dxdy = \frac{\pi^2 m\csc(2\pi m)}{n+1} \tag{10}$$

Proof. Use Equation (8) and set $k=0$ and simplify using entry (2) in Table below (64:12:7) in [7]. □

$$\begin{aligned} &\int_0^\infty\int_0^1 \frac{y^{2m}x^n\left(x^{-n-1}-1\right)^m\log\left(y^2\left(x^{-n-1}-1\right)\right)}{y^2+1}dxdy \\ &= \frac{\pi^2(1-2\pi m\cot(2\pi m))\csc(2\pi m)}{n+1} \end{aligned} \tag{11}$$

Proof. Use Equation (8) and set $k=1$ and simplify using entry (1) in Table below (64:12:7) in [7]. □

$$\int_0^\infty \int_0^1 \frac{x^n\left(y^{2p}\left(x^{-n-1}-1\right)^p - y^{2m}\left(x^{-n-1}-1\right)^m\right)}{(y^2+1)\log\left(y^2\left(x^{-n-1}-1\right)\right)}dxdy$$
$$= \frac{1}{8n+8}\left(ie^{2i\pi m}\Phi\left(e^{4im\pi},2,\tfrac{1}{2}\right) - ie^{2i\pi p}\Phi\left(e^{4ip\pi},2,\tfrac{1}{2}\right) + 8\pi m\tanh^{-1}\left(e^{2i\pi m}\right)\right.$$
$$\left. - 8\pi p\tanh^{-1}\left(e^{2i\pi p}\right)\right) \tag{12}$$

Proof. Use Equation (8) and form a second equation by $m \to p$ and take their difference and set $k = -1, a = 1$ and simplify using entry (1) in Table below (64:12:7) in [7]. □

$$\int_0^\infty \int_0^1 \frac{x^n\left(y\sqrt{x^{-n-1}-1}-1\right)}{\sqrt{y}(y^2+1)\sqrt[4]{x^{-n-1}-1}\log(y^2(x^{-n-1}-1))}dxdy = \frac{G}{n+1} \tag{13}$$

Proof. Use Equation (12) and set $m = 1/4, p = -1/4$ and simplify in terms of Catalan's constant G, using entry (4) in Table below (64:12:7) in [7] and Equation (2.3) in [8]. □

$$\int_0^\infty \int_0^1 \frac{y\left(\frac{1}{x}-1\right)^{1/2}-1}{\sqrt{y}(y^2+1)\log\left(\left(\frac{1}{x}-1\right)y^2\right)\sqrt[4]{\frac{1}{x}-1}}dxdy = G \tag{14}$$

Proof. Use Equation (13) and set $n = 0$ and simplify. □

$$\int_0^\infty \int_0^1 \frac{\sqrt{y}x^n\sqrt[4]{x^{-n-1}-1}\log^k\left(ay^2\left(x^{-n-1}-1\right)\right)}{y^2+1}dxdy$$
$$= \frac{1}{n+1}i^k 2^{3k-2}\pi^{k+1}\left(2\pi\left(\zeta\left(-k,\tfrac{1}{4}-\tfrac{i\log(a)}{8\pi}\right)-\zeta\left(-k,\tfrac{3}{4}-\tfrac{i\log(a)}{8\pi}\right)\right)\right.$$
$$\left. - ik\left(\zeta\left(1-k,\tfrac{1}{4}-\tfrac{i\log(a)}{8\pi}\right)-\zeta\left(1-k,\tfrac{3}{4}-\tfrac{i\log(a)}{8\pi}\right)\right)\right) \tag{15}$$

Proof. Use Equation (8) and set $m = 1/4$ and simplify in terms the Hurwitz zeta function using entry (4) in Table below (64:12:7) in [7]. □

$$\int_0^\infty \int_0^1 \frac{\sqrt{y}x^n\sqrt[4]{x^{-n-1}-1}\log^k\left(y^2\left(x^{-n-1}-1\right)\right)}{y^2+1}dxdy$$
$$= \frac{i^k 2^{3k-2}\pi^{k+1}\left(2\pi\left(\zeta\left(-k,\tfrac{1}{4}\right)-\zeta\left(-k,\tfrac{3}{4}\right)\right)-ik\left(\zeta\left(1-k,\tfrac{1}{4}\right)-\zeta\left(1-k,\tfrac{3}{4}\right)\right)\right)}{n+1} \tag{16}$$

Proof. Use Equation (15) and set $a = 1$ and simplify. □

$$\int_0^\infty \int_0^1 \frac{\sqrt{y}x^n\sqrt[4]{x^{-n-1}-1}}{(y^2+1)\log\left(y^2\left(1-x^{-n-1}\right)\right)}dxdy$$
$$= \frac{2i\pi\left(\psi^{(0)}\left(\frac{3}{8}\right)-\psi^{(0)}\left(\frac{7}{8}\right)\right)+\psi^{(1)}\left(\frac{3}{8}\right)-\psi^{(1)}\left(\frac{7}{8}\right)}{32(n+1)} \tag{17}$$

Proof. Use Equation (15) and set $a = -1$ then apply L'Hopital's rule to the right-hand side as $k \to -1$ and simplify using Equation (64:4:1) in [7]. □

$$\int_0^\infty \int_0^1 \frac{\sqrt{y}\sqrt[4]{\frac{1}{x}-1}}{(y^2+1)\log\left(\frac{(x-1)y^2}{x}\right)}dxdy = \frac{1}{32}\left(2i\pi\left(\psi^{(0)}\left(\tfrac{3}{8}\right)-\psi^{(0)}\left(\tfrac{7}{8}\right)\right)\right.$$
$$\left. + \psi^{(1)}\left(\tfrac{3}{8}\right)-\psi^{(1)}\left(\tfrac{7}{8}\right)\right) \tag{18}$$

Proof. Use Equation (17) and set $n = 0$ and simplify. □

$$\int_0^\infty \int_0^1 \frac{x\sqrt{y}\log\left(\log\left(\left(1-\frac{1}{x^2}\right)y^2\right)\right)\sqrt[4]{\frac{1}{x^2}-1}}{y^2+1}dxdy$$
$$= \frac{1}{16}\pi\left(2i\left(\psi^{(0)}\left(\frac{3}{8}\right) - \psi^{(0)}\left(\frac{7}{8}\right)\right) + \pi\left(\log\left(\frac{64\pi^2\Gamma\left(-\frac{1}{8}\right)^4}{625\Gamma\left(-\frac{5}{8}\right)^4}\right) + i\pi\right)\right) \tag{19}$$

Proof. Use Equation (15) and take the first partial derivative with respect to k and set $k = 0, a = 1, n = 1$ and simplify using Equations (64:4:1) and (64:10:2) in [7]. □

$$\int_0^\infty \int_0^1 \frac{x\sqrt{y}\log\left(\left(\frac{1}{x^2}-1\right)y^2\right)\log\left(\log\left(\left(\frac{1}{x^2}-1\right)y^2\right)\right)\sqrt[4]{\frac{1}{x^2}-1}}{y^2+1}dxdy$$
$$= \frac{1}{4}\pi^2\left(-4iG + 2 + i\pi + \log\left(\frac{64\pi^2\Gamma\left(-\frac{1}{4}\right)^4}{81\Gamma\left(-\frac{3}{4}\right)^4}\right)\right) \tag{20}$$

Proof. Use Equation (15) and take the first partial derivative with respect to k and set $k = 1, a = 1, n = 1$ and simplify using Equations (64:4:1) and (64:10:2) in [7]. □

$$\int_0^\infty \int_0^1 \frac{\sqrt[4]{\frac{1}{x}-1}\sqrt{y}}{(y^2+1)\log\left(\frac{(x-1)y^2}{x}\right)}dxdy = \frac{1}{32}\left(\psi^{(1)}\left(\frac{3}{8}\right) - \psi^{(1)}\left(\frac{7}{8}\right)\right)$$
$$- 2i\sqrt{2}\pi\left(\pi + 2\log\left(\tan\left(\frac{\pi}{8}\right)\right)\right) \tag{21}$$

Proof. Use Equation (8) set $k = -1, a = -1, m = 1/4, n = 0$ and simplify using entry (4) in Table below (64:12:7) and Equation (64:4:1) in [7]. □

$$\int_0^\infty \int_0^1 \frac{\sqrt[4]{\frac{1}{x}-1}\sqrt{y}}{(y^2+1)\log\left(i\left(\frac{1}{x}-1\right)y^2\right)}dxdy$$
$$= \frac{1}{32}\left(\psi^{(1)}\left(\frac{5}{16}\right) - \psi^{(1)}\left(\frac{13}{16}\right)\right) \tag{22}$$
$$- 4i\pi\left(\sin\left(\frac{\pi}{8}\right)\left(\sqrt{2}\pi + 2\log\left(\tan\left(\frac{\pi}{16}\right)\right)\right) + 2\cos\left(\frac{\pi}{8}\right)\log\left(\cot\left(\frac{3\pi}{16}\right)\right)\right)$$

Proof. Use Equation (8) set $k = -1, a = i, m = 1/4, n = 0$ and simplify using entry (4) in Table below (64:12:7) and Equation (64:4:1) in [7]. □

$$\int_0^\infty \int_0^1 \frac{\sqrt[4]{\frac{1}{x}-1}\sqrt{y}}{(y^2+1)\sqrt{\log\left(\frac{(x-1)y^2}{x}\right)}}dxdy = \left(\frac{1}{32} + \frac{i}{32}\right)\sqrt{\pi}\left(-4i\pi\left(\zeta\left(\frac{1}{2}, \frac{3}{8}\right)\right)\right)$$
$$- \zeta\left(\frac{1}{2}, \frac{7}{8}\right)\Bigg) + \zeta\left(\frac{3}{2}, \frac{3}{8}\right) - \zeta\left(\frac{3}{2}, \frac{7}{8}\right)\Bigg) \tag{23}$$

Proof. Use Equation (8) set $k = -1/2, a = -1, m = 1/4, n = 0$ and simplify using entry (4) in Table below (64:12:7) in [7]. □

$$\int_0^\infty \int_0^1 \frac{\sqrt[4]{\frac{1}{x}-1}\sqrt{y}\log^i\left(\frac{(x-1)y^2}{x}\right)}{y^2+1}dxdy = i^i 2^{-2+3i}\pi^{1+i}\left(2\pi\left(\zeta\left(-i, \frac{3}{8}\right)\right)\right)$$
$$- \zeta\left(-i, \frac{7}{8}\right)\Bigg) + \zeta\left(1-i, \frac{3}{8}\right) \tag{24}$$
$$- \zeta\left(1-i, \frac{7}{8}\right)$$

Proof. Use Equation (8) set $k = i, a = -1, m = 1/4, n = 0$ and simplify using entry (4) in Table below (64:12:7) and Equation (64:4:1) in [7]. □

$$\begin{aligned}&\int_0^\infty \int_0^1 \frac{\sqrt[4]{\frac{1}{x}-1}\sqrt{y}\left(\log\left(\frac{1}{x}-1\right)+2\log(y)\right)}{(y^2+1)\log\left(i\left(\frac{1}{x}-1\right)y^2\right)}dxdy \\ &= \frac{1}{64}\pi\left(4\pi\left(4-\sin\left(\frac{\pi}{8}\right)\left(\sqrt{2}\pi+2\log\left(\tan\left(\frac{\pi}{16}\right)\right)\right)\right)\right) \\ &+2\cos\left(\frac{\pi}{8}\right)\log\left(\tan\left(\frac{3\pi}{16}\right)\right)\Bigg)-i\left(\psi^{(1)}\left(\frac{5}{16}\right)-\psi^{(1)}\left(\frac{13}{16}\right)\right)\Bigg)\end{aligned} \tag{25}$$

Proof. Use Equation (8) take the first partial derivative with respect to k and set $k = -1$, $a = i, m = 1/4, n = 0$ and simplify using entry (4) in Table below (64:12:7) and Equation (64:4:1) in [7]. □

$$\int_0^\infty \int_0^1 \frac{\left(1-\sqrt{\frac{1}{x}-1}y\right)\log^2\left(\left(\frac{1}{x}-1\right)y^2\right)\log\left(\log\left(\left(\frac{1}{x}-1\right)y^2\right)\right)}{\sqrt[4]{\frac{1}{x}-1}\sqrt{y}(y^2+1)}dxdy = 32i\pi^2 G \tag{26}$$

Proof. Use Equation (8) take the first partial derivative with respect to k and set $k = 2$, $a = 1, m = 1/4, p = -1/4, n = 0$ and simplify in terms of Catalan's constant G, using entries (3) and (4) in Table below (64:12:7) and Equations (1:7:4) and (64:12:4) in [7]. □

$$\int_0^\infty \int_0^1 \frac{\log^k\left(\left(\frac{1}{x}-1\right)y^2\right)}{y^2+1}dxdy = -i^k 2^{2k-1}\left(2^{1-k}-1\right)k\pi^{k+1}\zeta(1-k) \tag{27}$$

Proof. Use Equation (8) and set $a = 1, m = 0, n = 0$ and simplify using Equation (64:12:1) in [7]. Note the integrand is highly oscillatory when $Re(k) < \pi$. □

$$\int_0^\infty \int_0^1 \frac{i\log^3\left(\left(\frac{1}{x}-1\right)y^2\right)\log\left(\log\left(\left(\frac{1}{x}-1\right)y^2\right)\right)}{18\pi^2(y^2+1)}dxdy = \zeta(3) \tag{28}$$

Proof. Use Equation (27) take the first partial derivative with respect to k and set $k = 3$ and simplify in terms of Aprey's constant $\zeta(3)$. □

$$\begin{aligned}&\int_0^\infty \int_0^1 \frac{15\log^4\left(\left(\frac{1}{x}-1\right)y^2\right)\log\left(\log\left(\left(\frac{1}{x}-1\right)y^2\right)\right)}{2\pi^5(y^2+1)}dxdy = \\ &-3360\zeta'(-3)+7+14i\pi+60\log(2)+28\log(\pi)\end{aligned} \tag{29}$$

Proof. Use Equation (27) take the first partial derivative with respect to k and set $k = 4$ and simplify. □

5. Discussion

In this work, we derived and evaluated a definite double integral involving the logarithmic function, a polynomial raised to a general power and a quotient rational function and expressed it in terms of the Lerch functions. We will be using our contour integral method for other integrals and derive other multiple integrals. We used Wolfram Mathematica to numerically verify our results for complex values of the parameters.

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