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Editorial

Integral Equations: Theories, Approximations, and Applications

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1. Introduction

Linear and nonlinear integral equations of the first and second kinds have many applications in engineering and real life problems. Thus, we try to find efficient and accurate methods to solve these problems. The aim of this editorial is to overview the content of the special issue "Integral Equations: Theories, Approximations and Applications". This special issue collects innovative contributions addressing the top challenges in integral equations, integro-differential equations, multi dimensional problems, and ill-posed and singular problems with modern applications. In response to our call, we had 15 submissions from 16 countries (Azerbaijan, China, Egypt, Germany, India, Indonesia, Iran, Jordan, Korea, Malaysia, Romania, Russia, Saudi Arabia, Taiwan, Vietnam, and Yemen), of which 10 were accepted and five were rejected. This issue contains 10 technical articles and one editorial. It covers linear and nonlinear integral equations of the first and second kinds, singular and ill-posed kernels, system of integral equations, high-dimensional problems and especially new numerical, analytical, and semi-analytical methods for solving the problems mentioned by focusing on modern applications.

This special issue focuses on linear and nonlinear integral equations of the first and second kinds, singular and ill-posed kernels, system of integral equations, and high-dimensional problems for solving challenging and applicable problems, especially using novel numerical, analytical, and semi-analytical methods.

2. Brief Overview of the Contributions

Ibrahimov and Imanova in "Multistep Methods of the Hybrid Type and Their Application to Solve the Second Kind Volterra Integral Equation" [1] have focused on solving the integral equations with variable boundaries. For this aim, they have applied the advanced and hybrid types of multi-step methods. They have tried to show the connection between the obtained methods and some applicable methods to solve the first order initial-value problems. Applying the methods mentioned, they can change the problem to a system of algebraic equations. They have extended the methods for solving Volterra integro-differential equations. The numerical results show the accuracy of the method.

"Integro-Differential Equation for the Non-Equilibrium Thermal Response of Glass-Forming Materials: Analytical Solutions" has been published by A.A. Minakov and C. Schick [2]. In this study, they have studied the non-equilibrium thermal response of glass-forming substances with a dynamic (time-dependent) heat capacity to fast thermal perturbations based on an integro-differential equation. They have found that the heat transfer problem can be solved analytically for a heat source with an arbitrary time dependence and different geometries. In addition, they showed that the method can be used to analyze the response to local thermal perturbations in glass-forming materials, as well as temperature fluctuations during subcritical crystal nucleation and decay. The importance of this paper is related to some applications of the thermal properties of glass-forming materials, polymers, and nanocomposites.

Zhu et al. in [3] have studied the paper titled "A Type of Time-Symmetric Stochastic System and Related Games". This paper has been concerned with a type of time-symmetric stochastic system, namely the so-called forward–backward doubly stochastic differential equations, in which the forward equations are delayed doubly stochastic differential equations and the backward equations are anticipated backward doubly stochastic differential equations. Under some monotonicity assumptions, the existence and uniqueness of measurable solutions to forward–backward doubly stochastic differential equations have been obtained. The future development of many processes depends on both their current state and historical state, and these processes can usually be represented by stochastic differential systems with time delay. Therefore, a class of nonzero sum differential game for doubly stochastic systems with time delay has been studied in this paper. A necessary condition for the open-loop Nash equilibrium point of the Pontriagin-type maximum principle has been established, and a sufficient condition for the Nash equilibrium point has been obtained. Furthermore, the above results have been applied to the study of nonzero sum differential games for linear quadratic backward doubly stochastic systems with delay.

"Effects of Second-Order Velocity Slip and the Different Spherical Nanoparticles on Nanofluid Flow" have been studied by Zhu [4]. The paper theoretically has investigated the heat transfer of nanofluids with different nanoparticles inside a parallel-plate channel. The second-order slip condition has been adopted due to the microscopic roughness in the microchannels. After proper transformation, they have tried to convert the system of nonlinear partial differential equations to the ordinary differential equations with unknown constants, and they have solved the problem using the homotopy analysis method. As we know, this method has some important applications to solve the integral equations. Several graphs have been plotted to show the convergence regions. The semi-analytical expressions between Nu_B and N_{BT} are acquired. The results show that both first-order slip parameter and second-order slip parameter have positive effects on Nu_B of the MHD flow.

Hashemizadeh et al. have presented the paper "Matrix Method by Genocchi Polynomials for Solving Nonlinear Volterra Integral Equations with Weakly Singular Kernels" in [5]. In this study, they have worked on the spectral method for solving nonlinear Volterra integral equations with weakly singular kernels based on the Genocchi polynomials. Many other interesting results concerning nonlinear equations with discontinuous symmetric kernels with the application of group symmetry have remained beyond the scope of this paper. In the proposed approach, relying on the useful properties of Genocchi polynomials, they have produced an operational matrix and a related coefficient matrix to convert nonlinear Volterra integral equations with weakly singular kernels into a system of algebraic equations. This method is very fast and gives high-precision solutions with good accuracy in a low number of repetitions compared to other methods that are available. The error boundaries for this method have also been presented. Some illustrative examples have been provided to demonstrate the capability of the proposed method. In addition, the results derived from the new method are compared to Euler's method to show the superiority of the proposed method.

Micula in [6] has focused on the paper titled "A Numerical Method for Weakly Singular Nonlinear Volterra Integral Equations of the Second Kind". This paper presents a numerical iterative method for the approximate solutions of nonlinear Volterra integral equations of the second kind, with weakly singular kernels. In this study, the existence and uniqueness conditions of the solution have been proved using the unique fixed point of an integral operator. Iterative application of that operator to an initial function yields a sequence of functions converging to the true solution. Finally, an appropriate numerical integration scheme (a certain type of product integration) has been used to produce the approximations of the solution at given nodes. The convergence of the method and the error estimates have been illustrated by the author. The proposed method has been applied to some numerical examples.

"Matrix Expression of Convolution and Its Generalized Continuous Form" has been published by Y.H. Geum [7]. In this paper, they have considered the matrix expression of convolution and its generalized continuous form. The matrix expression of convolution is effectively applied in convolutional neural networks, and, in this study, we correlate the concept of convolution in mathematics to that in the convolutional neural network. Of course, convolution is one of the main processes of deep learning, the learning method of deep neural networks, as a core technology. In addition to this, the generalized continuous form of convolution has been expressed as a new variant of Laplace-type transform that encompasses almost all existing integral transforms.

Chaharborj et al. in [8] have studied the paper titled "Detecting Optimal Leak Locations Using Homotopy Analysis Method for Isothermal Hydrogen-Natural Gas Mixture in an Inclined Pipeline". The aim of this article is to use the homotopy analysis method to pinpoint the optimal location of leakage in an inclined pipeline containing hydrogen-natural gas mixture by obtaining quick and accurate analytical solutions for nonlinear transportation equations. Because of important applications of the homotopy analysis method for solving different kinds of integral equations, we have accepted to publish this paper on this issue. The homotopy analysis method utilizes a simple and powerful technique to adjust and control the convergence region of the infinite series solution using auxiliary parameters. The auxiliary parameters provide a convenient way of controlling the convergent region of series solutions. Numerical results have indicated that the approach is highly accurate, computationally very attractive, and easy to implement.

Noeiaghdam et al. have focused on "Error Estimation of the Homotopy Perturbation Method to Solve Second Kind Volterra Integral Equations with Piecewise Smooth Kernels: Application of the CADNA Library" in [9]. In this paper, they have studied the second kind of linear Volterra integral equations with a discontinuous kernel obtained from the load leveling and energy system problems. For solving this problem, they have proposed the homotopy perturbation method. They have discussed the convergence theorem and the error analysis of the formulation to validate the accuracy of the obtained solutions. In this study, the Controle et Estimation Stochastique des Arrondis de Calculs method (CESTAC) and the Control of Accuracy and Debugging for Numerical Applications (CADNA) library have been used to control the rounding error estimation. The advantage of the discrete stochastic arithmetic has been taken to find the optimal iteration, optimal error, and optimal approximation of the homotopy perturbation method. The comparative graphs between exact and approximate solutions show the accuracy and efficiency of the method.

Ameer et al. in [10] have published the paper titled "On (ϕ, ψ)-Metric Spaces with Applications". The aim of this article is to introduce the notion of a (ϕ, ψ)-metric space, which extends the metric space concept. In these spaces, the symmetry property has been preserved. They have presented a natural topology $\tau(\phi, \psi)$ in such spaces and discuss their topological properties. They also have established the Banach contraction principle in the context of (ϕ, ψ)-metric spaces, and they have illustrated the significance of their main theorem by examples. Ultimately, as applications, the existence of a unique solution of Fredholm type integral equations in one and two dimensions have been ensured.

3. Conclusions and Outlook

The Special Issue Book "Integral Equations: Theories, Approximations, and Applications" presents a collection of articles dealing with relevant topics in the field of integral equation. Various mathematical and computational techniques and approaches were presented to solve the linear and nonlinear problems. The success of this Special Issue has motivated the editors to propose a new Special Issue "Integral Equations: Theories, Approximations, and Applications II" that will complement the present one with a focus on modern applications of integral equations. We invite the research community to submit novel contributions covering numerical, analytical, and semi-analytical methods to solve the multi-dimensional linear and nonlinear integral equations.

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