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On the Qualitative Behavior of Third-Order Differential Equations with a Neutral Term

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Abstract: In this paper, we analyze the asymptotic behavior of solutions to a class of third-order neutral differential equations. Using different methods, we obtain some new results concerning the oscillation of this type of equation. Our new results complement related contributions to the subject. The symmetry plays a important and fundamental role in the study of oscillation of solutions to these equations. An example is presented in order to clarify the main results.

Keywords: oscillation; third order; neutral coefficients; differential equation

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1. Introduction

The importance of neutral differential equations is that they contribute to many applicationsin physics, engineering, chemistry and medicine, so third-order equations appear in computer engineering, networks, aeromechanical systems and others; see [1–3].

To the best of our knowledge, there is little research that has focused on the study of differential equations of third-order with neutral terms in the non-canonical case, while there are many studies discussing oscillation in the canonical case (see [4–11]).

This work is concerned with the oscillation of a class of third-order nonlinear neutral functional differential equations with delay term

$$\left(m(z)\left((\Psi(z)+p(z)\Psi(\tau(z)))''\right)^{\alpha}\right)'+\int_a^b \nu(z,s)f(\Psi(\sigma(z,s)))\mathrm{d}s=0, \tag{1}$$

where α is the quotient of odd positive integers, and we assume that the following hypotheses are satisfied:

(C$_1$) $m,p\in C([z_0,\infty),\mathbb{R})$, $\nu\in C([z_0,\infty)\times[a,b],\mathbb{R})$, $p(z)\le p_0<\infty, \nu(z,s)\ge 0$ and

$$\pi(z)=\int_{z_0}^{\infty} m^{-1/\alpha}(s)\mathrm{d}s<\infty;$$

(C$_2$) $\tau\in C^1([z_0,\infty),\mathbb{R}),\sigma\in C^1([z_0,\infty)\times[a,b],\mathbb{R}),\tau(z)<z$, $\tau'(z)<\tau_0$, $\sigma(z,s)<z$, σ has nonnegative partial derivatives and $\lim_{z\to\infty}\tau(z)=\infty$, $\lim_{z\to\infty}\sigma(z,s)=\infty$ and $\sigma\circ\tau=\tau\circ\sigma$;

(C$_3$) $f\in C(\mathbb{R},\mathbb{R})$ and satisfies $f(\Psi)>kx^{\alpha}$ for all $k>0$.

By a solution of (1), we mean a function Ψ defined on $[z_\Psi,\infty)$ for $z_\Psi\ge z_0$, which satisfies the property $m(z)\left((\Psi(z)+p(z)\Psi(\tau(z)))''\right)^{\alpha}\in C^1([z_\Psi,\infty),\mathbb{R})$ and Ψ satisfies (1) on $[z_\Psi,\infty)$. In what follows, we assume that solutions of (1) exist on some half-line $[z_\Psi,\infty)$ and satisfy the condition $\sup\{|\Psi(z)|: z\le z<\infty\}>0$ for any $z\ge z_\Psi$.

Definition 1. *A solution of* (1) *is said to be oscillatory if it has arbitrarily large zeros on* $[z_0, \infty)$. *Otherwise, a solution that is not oscillatory is said to be nonoscillatory.*

Definition 2. *Equation* (1) *is said to be oscillatory if every solution of it is oscillatory.*

Lately, great attention has been devoted to the theory of oscillation in DDEs. The works [12–15] develop techniques and methods for studying the oscillations of DDEs. This development was necessarily reflected in the study of the oscillation of third-order DDEs, and this can be seen through the works [16–18].

The motivation for this article is to complement the results reported in [19], which discussed the oscillatory properties of a third-order equation with a late term. Therefore, we discuss their findings and results below.

Grace et al. [20] considered the equation with delay term

$$\left(m(z)\left(\Psi''(z)\right)^{\alpha}\right)' + \nu(z) f(\Psi(\sigma(z))) = 0 \tag{2}$$

in canonical and noncanonical form.

Saker and Dzurina [19] obtained several criteria, which ensure that the third-order nonlinear delay differential Equation (2) with $f(\Psi(\sigma(z))) = \Psi^{\beta}(\sigma(z))$ is oscillatory or tends to zero.

Moaaz in [21] improved the results in [19] and established some criteria that ensure that all solutions of (2) are oscillatory.

Dzurina et al. [22] presented oscillation results for the general equation

$$\left(m_1(z)\left(\left(m_2(z)\left(((\Psi(z) + p(z)\Psi(\tau(z))))'\right)^{\alpha}\right)'\right)^{\beta}\right)' + \nu(z) f(\Psi(\sigma(z))) = 0,$$

where $\phi(z)$ is defined as in (1).

Vidhyaa et al. [23] discussed several criteria of equation

$$\left(m_1(z)\left(m_2(z)\left((\Psi(z) + p(z)\Psi(\tau(z)))'\right)'\right)^{\alpha}\right)' + \nu(z)\Psi^{\alpha}(\sigma(z)) = 0,$$

with $\alpha = 1$ and $f(\Psi(\sigma(z))) \geq kx^{\alpha}(\sigma(z))$.

In the remainder of the article, we will adopt the following notation:

$$\begin{aligned}
\phi(z) &:= \Psi(z) + p(z)\Psi(\tau(z)) \\
\nu(z,s) &:= \int_a^b \nu(z,s)(1 - p(\sigma(z,s)))^{\alpha} \mathrm{d}s, \\
\eta(z,s) &:= \min\{\nu(z,s), \nu(\tau(z),s)\}.
\end{aligned}$$

Our objective of this article is to find results that ensure oscillation of (1) by using the Riccati technique and comparison method. These results extend and complement some previous results. Compared to the conditions mentioned in the previous literature, we obtain new sufficient conditions that are less restrictive.

2. Auxiliary Results

We now present some lemmas that will be used later.

Lemma 1. *Assume* $\Psi(z)$ *is nonoscillatory solution of* (1). *Then,* $\phi(z) > 0$ *and there are three possible cases of* $\phi(z)$:

$\mathbf{N}_1$ $\phi'(z) > 0$, $\phi''(z) > 0$;
$\mathbf{N}_2$ $\phi'(z) > 0$, $\phi''(z) < 0$;
$\mathbf{N}_3$ $\phi'(z) < 0$, $\phi''(z) > 0$.

Lemma 2. *(Thandapani [15]) Assume that $b_1, b_2 \in [0, \infty)$ and $\gamma > 0$. Then*

$$(b_1 + b_1)^\gamma \leq \mu(b_1^\gamma + b_2^\gamma),$$

where

(i) $\mu = 1$ *when* $\gamma \leq 1$;
(ii) $\mu = 2^{\gamma - 1}$ *when* $\gamma > 1$.

Lemma 3. *(Zhang [18]) Let $g(v) = Cv - Dv^{\alpha+1/\alpha}$ where $C, D > 0$ and leg α be a ratios of two odd positive integers. Then g attains its maximum value on $\mathbb{R}$ at $v^* = \alpha C/(\alpha+1)D$ such that*

$$\max_{v \in \mathbb{R}} g(v) = g(v^*) = \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{C^{\alpha+1}}{D^\alpha}.$$

Lemma 4. *Let*

Ψ be a positive solution of (1).

Then

$$\left(m(z)(\phi''(z))^\alpha + \frac{p_0^\alpha}{\tau_0} m(\tau(z))(\phi''(\tau(z)))^\alpha \right)' + \frac{k}{\mu} \int_a^b \eta(z,s)\phi^\alpha(\sigma(z,s))\mathrm{d}s \leq 0. \tag{3}$$

Furthermore, if $\phi'(z) > 0$ and $p(z) \in (0,1)$. Then

$$\left(m(z)(\phi''(z))^\alpha \right)' + ky^\alpha(\sigma(z,a))\nu(z,s) \leq 0. \tag{4}$$

and

$$\phi(\sigma(z)) \geq \acute{c}\sigma(z), \tag{5}$$

where $\tilde{c} := ck$.

Proof. Let $\Psi(z)$ be a positive solution of (1). Then, there exists $z_1 \geq z_0$ such that $\Psi(\sigma(z,s)) > 0$ and $\Psi(\tau(z)) > 0$ for all $z \geq z_1 \geq z_0$. From hypothesis $(\mathbf{C}_3)$, it follows from (1) that

$$\left(m(z)(\phi''(z))^\alpha \right)' + k \int_a^b \nu(z,s)\Psi^\alpha(\sigma(z,s))\mathrm{d}s \leq 0. \tag{6}$$

By Lemma 2, we obtain

$$\phi^\alpha(\sigma(z,s)) \leq \mu(\Psi^\alpha(\sigma(z,s)) + p_0^\alpha \Psi^\alpha(\sigma(\tau(z,s)))). \tag{7}$$

In view of $(\mathbf{C}_2)$ and inequality (6), we see

$$\frac{p_0^\alpha}{\tau'(z)} \left(m(\tau(z))(\phi''(\tau(z)))^\alpha \right)' + kp_0^\alpha \int_a^b \nu(\tau(z),s)\Psi^\alpha(\sigma(\tau(z),s))\mathrm{d}s \leq 0.$$

Hence

$$\frac{p_0^\alpha}{\tau_0} \left(m(\tau(z))(\phi''(\tau(z)))^\alpha \right)' + kp_0^\alpha \int_a^b \nu(\tau(z),s)\Psi^\alpha(\sigma(\tau(z),s))\mathrm{d}s \leq 0.$$

Using (6) with above inequality, and taking into account (7), we have

$$\left(m(z)(\phi''(z))^\alpha \right)' + \frac{p_0^\alpha}{\tau_0} \left(m(\tau(z))(\phi''(\tau(z)))^\alpha \right)' \leq -k \int_a^b \nu(z,s)\Psi^\alpha(\sigma(z,s)) + p_0^\alpha \nu(\tau(z),s)\Psi^\alpha(\sigma(\tau(z),s))\mathrm{d}s.$$

It follows that

$$\left(m(z)\left(\phi''(z)\right)^{\alpha}\right)' + \frac{p_0^{\alpha}}{\tau_0}\left(m(\tau(z))\left(\phi''(\tau(z))\right)^{\alpha}\right)' \leq -k\int_a^b \hat{O}(z,s)(\Psi^{\alpha}(\sigma(z,s)) + p_0^{\alpha}\Psi^{\alpha}(\sigma(\tau(z),s)))\mathrm{d}s.$$

Thus,

$$\left(m(z)\left(\phi''(z)\right)^{\alpha} + \frac{p_0^{\alpha}}{\tau_0}m(\tau(z))\left(\phi''(\tau(z))\right)^{\alpha}\right)' + \frac{k}{\mu}\int_a^b \eta(z,s)\phi^{\alpha}(\sigma(z,s))\mathrm{d}s \leq 0.$$

Now, assume that $\Psi(z)$ is a positive solution of (1) and $\phi'(z) > 0$; we find

$$\Psi(z) = \phi(z) - p(z)\Psi(\tau(z)) \geq \phi(z)(1 - p(z)),$$

and

$$\Psi^{\alpha}(\sigma(z,s)) \geq \phi^{\alpha}(\sigma(z,s))(1 - p(\sigma(z,s)))^{\alpha}. \tag{8}$$

Combining (6) and (8), we have

$$\left(m(z)\left(\phi''(z)\right)^{\alpha}\right)' + k\int_a^b \nu(z,s)\phi^{\alpha}(\sigma(z,s))(1 - p(\sigma(z,s)))^{\alpha}\mathrm{d}s \leq 0. \tag{9}$$

By the fact that $\sigma(z,s)$ is nondecreasing with respect to s, we have

$$\phi^{\alpha}(\sigma(z,s)) > \phi^{\alpha}(\sigma(z,a)).$$

From (9), we get (4).
Furthermore, it is easy to see that

$$\phi'(z) \geq \phi'(z_1) \text{ for } c \in [z_1, \infty).$$

Integrating from $\sigma(z)$ to z_1, we get

$$\phi(\sigma(z)) \geq c(\sigma(z) - z_1).$$

Hence, for any $\kappa \in (0,1)$ and $z \geq z_2$, we see that

$$\phi(\sigma(z)) \geq \tilde{c}\sigma(z), \quad \text{where } \tilde{c} = c\kappa.$$

The proof is complete. □

Lemma 5. *Let $\Psi(z)$ be a positive solution of (1) and $\phi(z)$ satisfying case $\mathbf{N}_3$. If*

$$\int_{z_0}^{\infty}\int_a^b \eta(z,s)\mathrm{d}s\mathrm{d}u = \infty, \tag{10}$$

or

$$\int_{z_0}^{\infty}\frac{1}{m(\zeta)}\left(\int_z^{\infty}\int_a^b \eta(u,s)\mathrm{d}s\mathrm{d}u\right)^{\frac{1}{\alpha}}\mathrm{d}\zeta = \infty, \tag{11}$$

then $\lim_{z\to\infty}\phi(z) = 0$.

Proof. Assume that $\Psi(z)$ is a positive solution of (1); there exists $z_1 \geq z_0$ such that $\Psi(\tau(z)) > 0$ and $\Psi(\sigma(z)) > 0$ for all $z \geq z_1 \geq z_0$. Since $\phi(z) > 0$ and $\phi'(z) < 0$, there is $\lambda \geq 0$ such that $\lim_{z\to\infty} \phi(z) = \lambda$. Assume that $\lambda > 0$. Integrating (3) from z_2 to z, we have

$$\begin{aligned} m(z)\left(\phi''(z)\right)^{\alpha} + \frac{p_0^{\alpha}}{\tau_0} m\left(\phi''(\tau(z))\right)^{\alpha} &\leq m(z_1)\left(\phi''(z_1)\right)^{\alpha} + \frac{p_0^{\alpha}}{\tau_0} m(\tau(z_1))\left(\phi''(\tau(z_1))\right)^{\alpha} \\ &\quad - \frac{k}{\mu}\int_{z_2}^{z}\int_a^b \eta(u,s)\phi^{\alpha}(\sigma(u,s))\mathrm{d}s\mathrm{d}u \\ &\leq m(z_1)\left(\phi''(z_1)\right)^{\alpha} + \frac{p_0^{\alpha}}{\tau_0} m(\tau(z_1))\left(\phi''(\tau(z_1))\right)^{\alpha} \\ &\quad - \frac{k}{\mu}\lambda^{\alpha}\int_{z_2}^{z}\int_a^b \eta(u,s)\mathrm{d}s\mathrm{d}u. \end{aligned}$$

Which contradicts (10), hence $\lambda = 0$. The proof is complete. □

Lemma 6. *Let $\Psi(z)$ be a positive solution of (1). If*

$$\int_{z_0}^{\infty}\int_a^b \eta(u,s)\sigma^{\alpha}(u,s)\mathrm{d}s\mathrm{d}u = \infty, \tag{12}$$

then case $\mathbf{N}_1$ is impossible.

Proof. Let $\Psi(z)$ be a positive solution of (1). Then, there exists $z_1 \geq z_0$ such that $\Psi(\tau(z)) > 0$ and $\Psi(\sigma(z)) > 0$ for all $z \geq z_1 \geq z_0$. On the contrary, assume that $\phi(z)$ satisfies case $\mathbf{N}_1$. Integrating (3) from z_2 to z and using (5), we get

$$\begin{aligned} m(z)\left(\phi''(z)\right)^{\alpha} + \frac{p_0^{\alpha}}{\tau_0} m(\tau(z))\left(\phi''(\tau(z))\right)^{\alpha} &\leq m(z_2)\left(\phi''(z_2)\right)^{\alpha} + \frac{p_0^{\alpha}}{\tau_0} m(\tau(z_2))\left(\phi''(\tau(z_2))\right)^{\alpha} \\ &\quad - (\tilde{c})^{\alpha}\frac{k}{\mu}\int_{z_1}^{z}\int_a^b \eta(u,s)\sigma^{\alpha}(u,s)\mathrm{d}s\mathrm{d}u, \end{aligned} \tag{13}$$

which is a contradiction. □

Lemma 7. *Let $\phi(z)$ be a positive increasing solution of (1). If*

$$\int_{z_0}^{\infty}\frac{1}{m^{1/\alpha}(z)}\left(\int_{z_0}^{z}\left(\int_a^b \eta(u,s)\sigma^{\alpha}(u,s)\mathrm{d}s\right)\mathrm{d}u\right)^{1/\alpha}\mathrm{d}z = \infty, \tag{14}$$

then ϕ satisfies case $\mathbf{N}_2$.

Proof. Let $\Psi(z)$ be a positive solution of (1). Then, there exists $z_1 \geq z_0$ such that $\Psi(\tau(z)) > 0$ and $\Psi(\sigma(z)) > 0$ for all $z \geq z_1 \geq z_0$. Since ϕ is increasing, ϕ satisfies either case $\mathbf{N}_1$ or case $\mathbf{N}_2$. In view of $\pi(z) < \infty$ and (14), we see that (12) holds. By Lemma 6, $\phi(z)$ satisfies case $\mathbf{N}_2$. The proof is complete. □

Lemma 8. *Let $\phi(z)$ be a positive increasing solution of (1). If ϕ satisfies (14), then:*

(a) *$\phi(z) \geq ty'(z), \phi(z)/z$ is decreasing, eventually, and $\lim_{z\to\infty}\phi(z)/z = \phi' = 0$,*

(b) *$\phi'(z) \geq -\pi(z)m^{\frac{1}{\alpha}}(z)\phi''(z)$ and $\phi'(z)/\pi(z)$ is increasing eventually.*

Proof. Let $\Psi(z)$ be a positive solution of (1). Then, there exists $z_1 \geq z_0$ such that $\Psi(\tau(z)) > 0$ and $\Psi(\sigma(z)) > 0$ for all $z \geq z_1 \geq z_0$. Using the fact that $\phi'(z)$ is decreasing, we see that there exists constant $\lambda \geq 0$ such that $\lim_{z\longrightarrow\infty}\phi'(z) = \lambda \geq 0$. We claim that $\lambda = 0$. As the proof of Lemma 6; we have (13). Take into account $\left(m(z)(\phi''(z))^{\alpha}\right)' \leq 0$ and $\phi''(z) < 0$, we have

$$m(z)\left(\phi''(z)\right)^{\alpha}\left(1 + \frac{p_0^{\alpha}}{\tau_0}\right) \leq -(\tilde{c})^{\alpha}\frac{k}{\mu}\int_{z_1}^{z}\int_a^b \eta(u,s)\sigma^{\alpha}(u,s)\mathrm{d}s\mathrm{d}u.$$

It follows that

$$m(z)\left(\phi''(z)\right)^{\alpha} \leq -(\tilde{c})^{\alpha}\left(\frac{\tau_0}{\tau_0+p_0^{\alpha}}\right)\frac{k}{\mu}\int_{z_1}^{z}\int_a^b \eta(u,s)\sigma^{\alpha}(u,s)\mathrm{d}s\mathrm{d}u$$

and

$$\phi''(z) \leq -\tilde{c}\left(\frac{1}{m(z)}\left(\frac{k\tau_0}{\mu(\tau_0+p_0^{\alpha})}\right)\right)^{1/\alpha}\left(\int_{z_1}^{z}\int_a^b \eta(u,s)\sigma^{\alpha}(u,s)\mathrm{d}s\mathrm{d}u\right)^{1/\alpha}.$$

Integrating from z_2 to z, we get

$$\phi'(z) \leq \phi'(z_2) - \tilde{c}\left(\frac{k\tau_0}{\mu(\tau_0+p_0^{\alpha})}\right)^{\frac{1}{\alpha}}\int_{z_2}^{z}\frac{\left(\int_{z_2}^{u}\int_a^b \eta(u,s)\sigma^{\alpha}(u,s)\mathrm{d}s\mathrm{d}u\right)^{\frac{1}{\alpha}}}{m^{\frac{1}{\alpha}}(\zeta)}\mathrm{d}\zeta.$$

Which in view of (14) contradicts the positivity of $\phi'(z)$. Thus, $\lambda = 0$. By Hospital's rule, we see that

$$\lim_{z\to\infty}\frac{\phi(z)}{z} = \phi'(z) = 0.$$

Thus

$$\phi(z_1) - \phi'(z)z_1 > 0. \tag{15}$$

So

$$\phi(z) > \phi'(z)z_1,$$

for $z \geq z_2$. Hence, by monotonicity of $\phi'(z)$, one can obtain that

$$\phi(z) = \phi(z_1) + \int_{z_1}^{z}\phi'(s)\mathrm{d}s \geq \phi(z_1) + \phi'(z)(z-z_1).$$

By (15), we have

$$\left(\frac{\phi(z)}{z}\right)' = \frac{ty'-\phi}{z^2} < 0.$$

Now, it is easy to see that

$$\phi'(z) \geq -\int_u^{\infty}\frac{1}{m^{1/\alpha}(s)}m^{1/\alpha}(s)\phi''(s)\mathrm{d}s \geq -m^{1/\alpha}(z)\phi''(z)\pi(z).$$

Thus,

$$\left(\frac{\phi'(z)}{\pi(z)}\right)' = \frac{m^{1/\alpha}(z)\phi''(z)\pi(z)+\phi'(z)}{m^{1/\alpha}(z)\pi^2(z)} \geq 0.$$

The proof is complete. □

3. Main Results

Theorem 1. *If*

$$\int_{z_0}^{\infty}\left(\frac{1}{m(\xi)}\int_{z_0}^{\infty}\int_a^b \eta(u,s)\mathrm{d}s\mathrm{d}u\right)^{\frac{1}{\alpha}}\mathrm{d}\xi = \infty, \tag{16}$$

then (1) *satisfies case* $\mathbf{N}_3$.

Proof. Assume that $\Psi(z)$ is a positive solution of (1). Then, there exists $z_1 \geq z_0$ such that $\Psi(\tau(z)) > 0$ and $\Psi(\sigma(z)) > 0$ for all $z \geq z_1$. Suppose to the contrary that $\phi(z)$ satisfies case $\mathbf{N}_1$ or $\mathbf{N}_2$. Since ϕ is increasing, it follows that

$$\phi(z) \geq \phi(z_1) = \varrho \ \text{ for } \ z \geq z_2.$$

Hence,

$$\phi(\sigma(z,s)) \geq \phi(z_1,s) = \varrho. \tag{17}$$

By $\left(m(z)(\phi''(z))^{\alpha}\right)' \leq 0$, we have

$$m(z)(\phi''(z))^{\alpha} + \frac{p_0^{\alpha}}{\tau_0} m(\tau(z))(\phi''(\tau(z)))^{\alpha} \geq m(z)(\phi''(z))^{\alpha}\left(1 + \frac{p_0^{\alpha}}{\tau_0}\right). \tag{18}$$

In (3), we get

$$\left(m(z)(\phi''(z))^{\alpha} + \frac{p_0^{\alpha}}{\tau_0} m(\tau(z))(\phi''(\tau(z)))^{\alpha}\right)' \leq -\frac{k}{\mu}\int_a^b \eta(z,s)\phi^{\alpha}(\sigma(z,s))\mathrm{d}s. \tag{19}$$

Integrating (19) from z_2 to z and using (17), we obtain

$$\begin{aligned} m(z)(\phi''(z))^{\alpha} + \frac{p_0^{\alpha}}{\tau_0} m(\tau(z))(\phi''(\tau(z)))^{\alpha} \leq{}& m(z_1)(\phi''(z_1))^{\alpha} + \frac{p_0^{\alpha}}{\tau_0} m(\tau(z_1))(\phi''(\tau(z_1)))^{\alpha} \\ &- \frac{\varrho^{\alpha} k}{\mu}\int_{z_2}^{z}\int_a^b \eta(u,s)\mathrm{d}s\mathrm{d}u. \end{aligned} \tag{20}$$

First, assume that $\phi(z)$ satisfies case $\mathbf{N}_1$. We note that

$$\left(m(z)(\phi''(z))^{\alpha} + \frac{p_0^{\alpha}}{\tau_0} m(\tau(z))(\phi''(\tau(z)))^{\alpha}\right) > 0.$$

Using the fact $\pi(z) < \infty$ together with (16) yields that $\int_{z_2}^{z}\int_a^b \eta(u,s)\mathrm{d}s\mathrm{d}u = \infty$, and this contradicts the positivity of $m(z)(\phi''(z))^{\alpha}$.

If $\phi(z)$ satisfies case $\mathbf{N}_2$, using (18) in (20) becomes

$$m(z)(\phi''(z))^{\alpha}\left(1 + \frac{p_0^{\alpha}}{\tau_0}\right) \leq -\frac{\varrho^{\alpha} k}{\mu}\int_{z_2}^{z}\int_a^b \eta(u,s)\mathrm{d}s\mathrm{d}u,$$

that is,

$$\phi''(z) \leq -\varrho\left(\frac{k\tau_0}{\mu(\tau_0 + p_0^{\alpha})}\right)^{\frac{1}{\alpha}}\left(\frac{1}{m(z)}\int_{z_2}^{z}\int_a^b \eta(u,s)\mathrm{d}s\mathrm{d}u\right)^{\frac{1}{\alpha}}.$$

Integrating from z_2 to z, we have

$$\phi'(z) \leq \phi'(z_2) - \varrho\left(\frac{k\tau_0}{\mu(\tau_0 + p_0^{\alpha})}\right)^{\frac{1}{\alpha}}\int_{z_2}^{z}\left(\frac{1}{m(\xi)}\int_{z_2}^{z}\int_a^b \eta(u,s)\mathrm{d}s\mathrm{d}u\right)^{\frac{1}{\alpha}}\mathrm{d}\xi.$$

This contradicts the positivity of $\phi'(z)$. The proof of the lemma is complete. □

Theorem 2. *If*

$$\liminf_{z\to\infty}\int_{\sigma(z)}^{z}\left(\frac{1}{m(\zeta)}\int_{z_2}^{u}\int_a^b \eta(u,s)\sigma^{\alpha}(u,s)\mathrm{d}s\mathrm{d}u\right)^{\frac{1}{\alpha}}\mathrm{d}\zeta > \left(\frac{\mu(\tau_0 + p_0^{\alpha})}{k\tau_0 e^{\alpha}}\right)^{\frac{1}{\alpha}}, \tag{21}$$

then (1) *satisfies case* $\mathbf{N}_3$.

Proof. Let $\Psi(z)$ be a positive solution of (1). Then, there exists $z_1 \geq z_0$ such that $\Psi(\tau(z)) > 0$ and $\Psi(\sigma(z)) > 0$ for all $z \geq z_1$. Suppose to the contrary that ϕ satisfies case $\mathbf{N}_1$ or $\mathbf{N}_2$. In view of (21), (14) holds. Hence, by Lemma 8, $\phi(z)$ satisfies case $\mathbf{N}_2$ and properties (**a**) and (**b**). This implies that

$$\phi(\sigma(z)) \geq \sigma(z)\phi'(\sigma(z)).$$

Combining the above inequality along with (3), we get

$$\left(m(\phi''(z))^{\alpha}+\frac{p_0^{\alpha}}{\tau_0}m(\phi''(\tau(z)))^{\alpha}\right)'\leq-\frac{k}{\mu}\int_a^b\eta(z,s)\sigma^{\alpha}(z,s)\left(\phi'(\sigma(z,s))\right)^{\alpha}\mathrm{d}s$$
$$\leq-\frac{k}{\mu}\left(\phi'(\sigma(z,b))\right)^{\alpha}\int_a^b\eta(z,s)\sigma^{\alpha}(z,s)\mathrm{d}s. \quad (22)$$

Integrating from z_2 to z and using (18), we have

$$-m(z)\left(\phi''(z)\right)^{\alpha}\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\geq\frac{k}{\mu}\left(\phi'(\sigma(z,b))\right)^{\alpha}\int_{z_2}^{z}\int_a^b\eta(u,s)\sigma^{\alpha}(u,s)\mathrm{d}s\mathrm{d}u.$$

Using the fact that $\phi''(z)<0$, we see that

$$-\phi''(z)\geq\left(\frac{k\tau_0}{\mu(\tau_0+p_0^{\alpha})}\right)^{\frac{1}{\alpha}}\phi'(\sigma(z))\left(\frac{1}{m(z)}\int_{z_2}^{z}\int_a^b\eta(u,s)\sigma^{\alpha}(u,s)\mathrm{d}s\mathrm{d}u\right)^{\frac{1}{\alpha}}.$$

Now, set $\chi(z)=\phi'(z)$; we obtain

$$\chi'(z)+\left(\frac{k\tau_0}{\mu(\tau_0+p_0^{\alpha})}\right)^{\frac{1}{\alpha}}\left(\frac{1}{m(z)}\int_{z_2}^{z}\int_a^b\eta(u,s)\sigma^{\alpha}(u,s)\mathrm{d}s\mathrm{d}u\right)^{\frac{1}{\alpha}}\chi(\sigma(z))\leq 0. \quad (23)$$

In view of ([13], Theorem 1), we see that, the associated delay Equation (23) has positive solution, which is a contradiction. The proof is complete. □

Theorem 3. *Assume that (14) hold. If*

$$\limsup_{z\to\infty}\pi^{\alpha}(z)\int_{z_2}^{z}\int_a^b\eta(u,s)\sigma^{\alpha}(u,s)\mathrm{d}s\mathrm{d}u>\frac{\mu(\tau_0+p_0^{\alpha})}{k\tau_0}, \quad (24)$$

then (1) satisfies case $\mathbf{N}_3$.

Proof. Suppose to the contrary that ϕ satisfies case $\mathbf{N}_1$ or $\mathbf{N}_2$. We see that (12) holds due to $\pi(z)<\infty$ (this mean that $\lim_{z\to\infty}\pi(z)=0$) and condition (24). Hence, by Lemma 8, $\phi(z)$ satisfies case $\mathbf{N}_2$ in addition to properties (**a**) and (**b**). As in the proof of Theorem 2 with the fact that $m(z)(\phi''(z))^{\alpha}$ is nonincreasing and integrating (22) from z_2 to z, we obtain

$$-m(z)\left(\phi''(z)\right)^{\alpha}\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\geq-\frac{k}{\mu}\pi^{\alpha}(z)m(z)\left(\phi''(z)\right)^{\alpha}\int_{z_2}^{z}\int_a^b\eta(u,s)\sigma^{\alpha}(u,s)\mathrm{d}s\mathrm{d}u.$$

That is,

$$\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\geq\pi^{\alpha}(z)\int_{z_2}^{z}\int_a^b\eta(u,s)\sigma^{\alpha}(u,s)\mathrm{d}s\mathrm{d}u.$$

This contradicts (24). The proof is complete. □

Theorem 4. *Assume that (14) holds. If there exists a nondecreasing function* $\rho\in C^1([z_0,\infty),(0,\infty))$ *and* $\sigma'(z)>0$, *such that*

$$\limsup_{z\to\infty}\int_z^z\left(\left(\frac{k\tau_0}{\mu(\tau_0+p_0^{\alpha})}\right)^{\frac{1}{\alpha}}\frac{\rho(\zeta)}{\sigma(\zeta)m^{\frac{1}{\alpha}}(\zeta)}\int_{z_2}^{u}\int_a^b\eta(u,s)\sigma^{\alpha}(u,s)\mathrm{d}s\mathrm{d}u-\frac{\rho'^2(\zeta)}{4p(\zeta)\sigma'(\zeta)}\right)\mathrm{d}\zeta=\infty, \quad (25)$$

for any $z\in[z_0,\infty)$, *then (1) satisfies case* $\mathbf{N}_3$.

Proof. Let $\Psi(z)$ be a positive solution of (1). Then there exists $z_1 \geq z_0$ such that $\Psi(\tau(z)) > 0$ and $\Psi(\sigma(z)) > 0$ for all $z \geq z_1$. Suppose to the contrary that ϕ satisfies case $\mathbf{N}_1$ or $\mathbf{N}_2$. By Lemma 8, $\phi(z)$ satisfies case $\mathbf{N}_2$ and the properties (**a**) and (**b**). Define the function $w(z)$ by

$$w(z) := \rho(z)\frac{\phi'(z)}{\phi(\sigma(z))}\,. \tag{26}$$

Then $w(z) > 0$, and

$$w'(z) = \frac{\rho'(z)}{\rho(z)}w(z) + \frac{\rho(z)\phi''(z)}{\phi(\sigma(z))} - \frac{\rho(z)\phi'(z)\phi'(\sigma(z))\sigma'(z)}{\phi^2(\sigma(z))}.$$

Using the fact $\phi'(z)$ is decreasing, we have

$$\begin{aligned} w'(z) &\leq \frac{\rho'(z)w(z)}{\rho(z)} + \rho(z)\frac{\phi''(z)}{\phi(\sigma(z))} - \rho(z)\frac{(\phi'(z))^2\sigma'(z)}{\phi^2(\sigma(z))} \\ &= \frac{\rho'(z)w(z)}{\rho(z)} + \rho(z)\frac{\phi''(z)}{\phi(\sigma(z))} - \frac{\sigma'(z)}{\rho(z)}\left(\rho(z)\frac{\phi'(z)}{\phi(\sigma(z))}\right)^2. \end{aligned}$$

By (26), we obtain

$$w'(z) \leq \frac{\rho'(z)w(z)}{\rho(z)} + \rho(z)\frac{\phi''(z)}{\phi(\sigma(z))} - \frac{\sigma'(z)}{\rho(z)}w^2(z). \tag{27}$$

Integrating (3) from z_2 to z and $(\phi(\sigma(z))/\sigma(z))' < 0$, we have

$$\begin{aligned} -\left(m(z)\left(\phi''(z)\right)^\alpha + \frac{p_0^\alpha}{\tau_0}m(\tau(z))\left(\phi''(\tau(z))\right)^\alpha\right) &\geq -\left(m(z_2)\left(\phi''(z_2)\right)^\alpha + \frac{p_0^\alpha}{\tau_0}m(z_2)\left(\phi''(\tau(z_2))\right)^\alpha\right) \\ &\quad + \frac{k}{\mu}\int_{z_2}^{z}\int_a^b \eta(u,s)\phi^\alpha(\sigma(u,s))\mathrm{d}s\mathrm{d}u \\ &\geq -\left(m(z_2)\left(\phi''(z_2)\right)^\alpha + \frac{p_0^\alpha}{\tau_0}m(z_2)\left(\phi''(\tau(z_2))\right)^\alpha\right) \\ &\quad + \frac{k}{\mu}\left(\frac{\phi(\sigma(z))}{\sigma(z)}\right)^\alpha\int_{z_2}^{z}\int_a^b \eta(u,s)\sigma^\alpha(u,s)\mathrm{d}s\mathrm{d}u. \end{aligned} \tag{28}$$

Since $\lim_{z\to\infty}\phi(z)/z = 0$, there exists $z_3 > z_2$ such that

$$-\left(m(z_2)\left(\phi''(z_2)\right)^\alpha + \frac{p_0^\alpha}{\tau_0}m(z_2)\left(\phi''(\tau(z_2))\right)^\alpha\right) - \frac{k}{\mu}\left(\frac{\phi(\sigma(s))}{\sigma(s)}\right)^\alpha\int_{z_2}^{z}\int_a^b \eta(u,s)\sigma^\alpha(u,s)\mathrm{d}s\mathrm{d}u > 0.$$

Combining the above inequality with (28) implies

$$\begin{aligned} -\left(m(z)\left(\phi''(z)\right)^\alpha + \frac{p_0^\alpha}{\tau_0}m(\tau(z))\left(\phi''(\tau(z))\right)^\alpha\right) &\geq -\left(m\left(\phi''(z_2)\right)^\alpha + \frac{p_0^\alpha}{\tau_0}m\left(\phi''(\tau(z_2))\right)\right)^\alpha \\ &\quad + \frac{k}{\mu}\left(\frac{\phi(\sigma(z))}{\sigma(z)}\right)^\alpha\int_{z_2}^{z}\int_a^b \eta(u,s)\sigma^\alpha(u,s)\mathrm{d}s\mathrm{d}u \\ &\quad - \frac{k}{\mu}\left(\frac{\phi(\sigma(z))}{\sigma(z)}\right)^\alpha\int_{z_2}^{z}\int_a^b \eta(u,s)\sigma^\alpha(u,s)\mathrm{d}s\mathrm{d}u \\ &\geq \frac{k}{\mu}\left(\frac{\phi(\sigma(z))}{\sigma(z)}\right)^\alpha\int_{z_2}^{z}\int_a^b \hat{O}(u,s)\sigma^\alpha(u,s)\mathrm{d}s\mathrm{d}u. \end{aligned}$$

Using (18), we have

$$-m(z)\left(\phi''(z)\right)^{\alpha}\left(1+\frac{p_0^{\alpha}}{\tau_0}\right) \geq \frac{k}{\mu}\left(\frac{\phi(\sigma(z))}{\sigma(z)}\right)^{\alpha}\int_{z_2}^{z}\int_a^b \eta(u,s)\sigma^{\alpha}(u,s)\mathrm{d}s\mathrm{d}u,$$

that is,

$$\frac{\phi''(z)}{\phi(\sigma(z))} \leq -\left(\frac{k\tau_0}{\mu(\tau_0+p_0^{\alpha})}\right)^{\frac{1}{\alpha}}\frac{1}{\sigma(z)m^{\frac{1}{\alpha}}(z)}\int_{z_2}^{z}\int_a^b \eta(u,s)\sigma^{\alpha}(u,s)\mathrm{d}s\mathrm{d}u. \tag{29}$$

Substituting (29) in (27), yields that

$$\begin{aligned} w'(z) &\leq \frac{\rho'(z)}{\rho(z)}w(z) - \frac{\rho(z)}{\sigma(s)}\left(\frac{k\tau_0}{\mu(\tau_0+p_0^{\alpha})}\right)^{\frac{1}{\alpha}}\frac{1}{m^{\frac{1}{\alpha}}(z)}\int_{z_2}^{z}\int_a^b \eta(u,s)\sigma^{\alpha}(u,s)\mathrm{d}s\mathrm{d}u - \frac{\sigma'(z)}{\rho(z)}w^2(z) \\ &= -\frac{\sigma'(z)}{\rho(z)}\left(w(z)-\frac{\rho'(z)}{2\sigma'(z)}\right)^2 + \frac{\rho'^2(z)}{4p(z)\sigma'(z)} \\ &\quad -\frac{\rho(z)}{\sigma(s)}\left(\frac{k\tau_0}{\mu(\tau_0+p_0^{\alpha})}\right)^{\frac{1}{\alpha}}\frac{1}{m^{\frac{1}{\alpha}}(z)}\int_{z_2}^{z}\int_a^b \eta(u,s)\sigma^{\alpha}(u,s)\mathrm{d}s\mathrm{d}u. \end{aligned}$$

Hence,

$$w'(z) \leq -\frac{\rho(z)}{\sigma(s)}\left(\frac{k\tau_0}{\mu(\tau_0+p_0^{\alpha})}\right)^{\frac{1}{\alpha}}\frac{1}{m^{\frac{1}{\alpha}}(z)}\int_{z_2}^{z}\int_a^b \eta(u,s)\sigma^{\alpha}(u,s)\mathrm{d}s\mathrm{d}u + \frac{\rho'^2(z)}{4p(z)\sigma'(z)}.$$

Integrating from z_3 to z, we have

$$w(z) \leq w(z_3) - \int_{z_3}^{z}\left(\left(\frac{k\tau_0}{\mu(\tau_0+p_0^{\alpha})}\right)^{\frac{1}{\alpha}}\frac{\rho(\zeta)}{\sigma(\zeta)m^{\frac{1}{\alpha}}(\zeta)}\int_{z_2}^{u}\int_a^b \eta(u,s)\sigma^{\alpha}(u,s)\mathrm{d}s\mathrm{d}u - \frac{\rho'^2(\zeta)}{4p(\zeta)\sigma'(\zeta)}\right)\mathrm{d}\zeta,$$

which is a contradiction. The proof is complete. □

Theorem 4 can be used in a wide range of applications by choosing $\rho(z) = \frac{1}{\pi}$; we conclude the following corollary

Corollary 1. *Assume that (14) holds. If there exists a nondecreasing function $\rho \in C^1([z_0,\infty),(0,\infty))$ and $\sigma'(z) > 0$ such that*

$$\limsup_{z\to\infty}\int_{z}^{z}\left(\frac{\left(\frac{k\tau_0}{\mu(\tau_0+p_0^{\alpha})}\right)^{\frac{1}{\alpha}}}{\pi(\zeta)\sigma(\zeta)m^{\frac{1}{\alpha}}(\zeta)}\int_{z_2}^{u}\int_a^b \eta(z,s)\sigma^{\alpha}(z,s)\mathrm{d}s\mathrm{d}u - \frac{m^{-2/\alpha}(\zeta)}{4\sigma'(\zeta)\pi^3(\zeta)}\right)\mathrm{d}\zeta = \infty, \tag{30}$$

for any $z \in [z_0,\infty)$, then (1) satisfies case $\mathbf{N}_3$.

Theorem 5. *Assume that (14) hold. If there exists a nondecreasing function $\delta \in C^1([z_0,\infty),(0,\infty))$, such that*

$$\limsup_{z\to\infty}\int_{z_2}^{z}\left(\delta(u)k\frac{\int_a^b \eta(u,s)\mathrm{d}s}{\mu} - \frac{(\delta'(u))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\sigma'(u))^{\alpha}\pi^{\alpha}(u)\delta^{\alpha}(u)}\right)\mathrm{d}u > \frac{\delta(z)}{\pi^{\alpha}(z)\sigma^{\alpha}(z)}, \tag{31}$$

then (1) satisfies case $\mathbf{N}_3$.

Proof. Let $\Psi(z)$ be a positive solution of (1); there exists $z_1 \geq z_0$ such that $\Psi(\tau(z)) > 0$ and $\Psi(\sigma(z)) > 0$ for all $z \geq z_1$. Suppose to the contrary that ϕ satisfies case $\mathbf{N}_1$ or $\mathbf{N}_2$. By Lemma 8, $\phi(z)$ satisfies case $\mathbf{N}_2$ and the properties (**a**) and (**b**).

Define the function $w(z)$ by

$$w(z) := \delta(z)\left(\frac{m(\phi''(z))^{\alpha}}{\phi^{\alpha}(\sigma(z))} + \frac{1}{\pi^{\alpha}\sigma^{\alpha}(z)}\right). \tag{32}$$

From Lemma 8, it is easy to see that

$$\phi(\sigma(z)) \geq \sigma(z)\phi'(\sigma(z)) \geq \sigma(z)\phi'(z) \geq -\sigma(z)\pi(z)m^{\frac{1}{\alpha}}(z)\phi''(z). \tag{33}$$

That is, $w(z) > 0$ and

$$-\frac{\delta(z)m(\phi''(z))^{\alpha}}{\phi^{\alpha}(\sigma(z))} \leq \frac{\delta(z)}{\pi^{\alpha}\sigma^{\alpha}(z)}. \tag{34}$$

Using (19) and the fact $\phi'(z)$ is decreasing, we have

$$\left(m(z)(\phi''(z))^{\alpha}\right)' \leq -\frac{k}{\mu}\phi^{\alpha}(\sigma(z))\int_a^b \eta(z,s)\mathrm{d}s,$$

hence,

$$\frac{\left(m(z)(\phi''(z))^{\alpha}\right)'}{\phi^{\alpha}(\sigma(z))} \leq -\frac{k}{\mu}\int_a^b \eta(z,s)\mathrm{d}s.$$

That is,

$$\begin{aligned}
w'(z) &= \frac{\delta'(z)}{\delta(z)}w(z) + \frac{\delta(z)\left(m(z)(\phi''(z))^{\alpha}\right)'}{\phi^{\alpha}(\sigma(z))} - \frac{\alpha\delta(z)m(z)(\phi''(z))^{\alpha}\phi'(\sigma(z))\sigma'(z)}{\phi^{\alpha+1}(\sigma(z))} \\
&\quad + \frac{\alpha\delta(z)}{(\pi(z)\sigma(z))^{\alpha+1}}\left(\frac{\sigma(z)}{m^{\frac{1}{\alpha}}(z)} - \sigma'(z)\pi(z)\right) \\
&\leq \frac{\delta'(z)}{\delta(z)}w(z) - \delta(z)\frac{k}{\mu}\int_a^b \eta(z,s)\mathrm{d}s \\
&\quad - \frac{\alpha\sigma'(z)\phi'(z)}{-\delta^{\frac{1}{\alpha}}(z)m^{\frac{1}{\alpha}}(z)\phi''(z)}\left(w(z) - \frac{\delta(z)}{\pi(z)\sigma(z)}\right)^{\frac{1}{\alpha}+1} \\
&\quad + \frac{\alpha\delta(z)}{(\pi(z)\sigma(z))^{\alpha+1}}\left(\frac{\sigma(z)}{m^{\frac{1}{\alpha}}(z)} - \sigma'(z)\pi(z)\right).
\end{aligned}$$

In view of property (**b**) in Lemma 8, we find

$$\begin{aligned}
w'(z) &\leq \frac{\delta'(z)}{\delta(z)}w(z) - \delta(z)k\frac{\int_a^b \eta(z,s)\mathrm{d}s}{\mu} \\
&\quad - \frac{\alpha\sigma'(z)\pi(z)}{\delta^{\frac{1}{\alpha}}(z)}\left(w(z) - \frac{\delta(z)}{\pi^{\alpha}(z)\sigma^{\alpha}(z)}\right)^{\frac{1}{\alpha}+1} + \frac{\alpha\delta(z)}{(\pi(z)\sigma(z))^{\alpha+1}}\left(\frac{\sigma(z)}{m^{\frac{1}{\alpha}}(z)} - \sigma'(z)\pi(z)\right).
\end{aligned}$$

Set

$$A := \frac{\delta'(z)}{\delta(z)}, \quad B := \frac{\alpha\sigma'(z)\pi(z)}{\delta^{\frac{1}{\alpha}}(z)}, \quad C := \frac{\delta(z)}{\pi^{\alpha}(z)\sigma^{\alpha}(z)}.$$

Using Lemma 3, we obtain

$$w'(z) = -\delta(z)k\frac{\int_a^b \eta(z,s)\mathrm{d}s}{\mu} + \frac{\delta'(z)}{\pi^\alpha(z)\sigma^\alpha(z)} + \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\delta'(z))^{\alpha+1}}{(\sigma'(z))^\alpha \pi^\alpha(z)\delta^\alpha(z)} + \frac{\alpha\delta(z)}{(\pi(z)\sigma(z))^{\alpha+1}}\left(\frac{\sigma(z)}{m^{\frac{1}{\alpha}}(z)} - \sigma'(z)\pi(z)\right). \tag{35}$$

It is clear that

$$\left(\frac{\delta(z)}{\pi^\alpha(z)\sigma^\alpha(z)}\right)' = \frac{\delta'(z)}{\pi^\alpha(z)\sigma^\alpha(z)} + \frac{\alpha\delta(z)}{(\pi(z)\sigma(z))^{\alpha+1}}\left(\frac{\sigma(z)}{m^{\frac{1}{\alpha}}(z)} - \sigma'(z)\pi(z)\right).$$

In (35), we get

$$w'(z) = -\delta(z)k\frac{\int_a^b \eta(z,s)\mathrm{d}s}{\mu} + \left(\frac{\delta(z)}{\pi^\alpha(z)\sigma^\alpha(z)}\right)' + \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\delta'(z))^{\alpha+1}}{(\sigma'(z))^\alpha \pi^\alpha(z)\delta^\alpha(z)}.$$

Integrating the above inequality from z_2 to z yields

$$\int_{z_2}^z \left(\delta(u)k\frac{\int_a^b \eta(z,s)\mathrm{d}s}{\mu} - \frac{(\delta'(u))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\sigma'(u))^\alpha \pi^\alpha(u)\delta^\alpha(u)}\right)\mathrm{d}u + \frac{\delta(z)}{\pi^\alpha(z)\sigma^\alpha(z)} - \frac{\delta(z_2)}{\pi^\alpha(z_2)\sigma^\alpha(z_2)} \le w(z_2) - w(z).$$

From (32), we are led to

$$\int_{z_2}^z \left(\delta(u)k\frac{\int_a^b \eta(z,s)\mathrm{d}s}{\mu} - \frac{(\delta'(u))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\sigma'(u))^\alpha \pi^\alpha(u)\delta^\alpha(u)}\right)\mathrm{d}u + \frac{\delta(z)}{\pi^\alpha(z)\sigma^\alpha(z)} - \frac{\delta(z_2)}{\pi^\alpha(z_2)\sigma^\alpha(z_2)} \le \delta(z_2)\left(\frac{m(\phi''(z_2))^\alpha}{\phi^\alpha(\sigma(z_2))}\right) - \delta(z)\left(\frac{m(\phi''(z))^\alpha}{\phi^\alpha(\sigma(z))}\right). \tag{36}$$

By (34), (36) becomes

$$\int_{z_2}^z \left(\delta(u)k\frac{\int_a^b \eta(z,s)\mathrm{d}s}{\mu} - \frac{(\delta'(u))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\sigma'(u))^\alpha \pi^\alpha(u)\delta^\alpha(u)}\right)\mathrm{d}u \le \frac{\delta(z)}{\pi^\alpha\sigma^\alpha(z)}.$$

The proof is complete. □

4. Corollaries of the Main Theorems

Corollary 2. *Assume that* (16) *holds. Then* (1) *is almost oscillatory.*

Corollary 3. *Assume that* (21) *and* (10) *or* (11) *hold. Then* (1) *is almost oscillatory.*

Corollary 4. *Assume that* (14), (24), *and either* (10) *or* (11) *hold. Then* (1) *is almost oscillatory.*

Corollary 5. *Assume that* (14) *holds and if there exists a nondecreasing function* $\rho \in C^1([z_0,\infty),(0,\infty))$ *and* $\sigma'(z) > 0$, *such that* (25) *and either* (10) *or* (11) *hold. Then* (1) *is almost oscillatory.*

Corollary 6. *Assume that* (14) *holds and if there exists a nondecreasing function* $\delta \in C^1([z_0,\infty),(0,\infty))$, *such that* (31) *and either* (10) *or* (11) *hold. Then* (1) *is almost oscillatory.*

Example 1. *Consider the equation*

$$\left(z^2\left(\left(\Psi(z)+p_0\Psi(\epsilon t)\right)''\right)^{\alpha}\right)' + \frac{\nu_0}{z}\phi^{\alpha}(\lambda t) = 0,\ z \geq 1, \lambda \in (0,1), \tag{37}$$

where $m = z^2$, $\nu(z) = \frac{\nu_0}{z} > 0$ *and* $\sigma(z) = \lambda t$, $\tau(z) = \epsilon t$. *Set* $\delta(z) = \pi(z)\tau(z) = \epsilon$. *We conclude that Theorem 3 fails.*

By Theorem 2, the same conclusion holds for (37) if

$$\nu_0\lambda \ln\left(\frac{1}{\lambda}\right) > \frac{\tau_0 + p_0}{(\tau_0 k)e}. \tag{38}$$

By Theorem 3, the same conclusion holds for (37) if

$$\nu_0\lambda > \frac{\mu\left(\tau_0 + p_0^{\alpha}\right)}{k\tau_0}. \tag{39}$$

Furthermore, Corollary 1 if

$$\nu_0 > \frac{1}{4}\frac{\left(\tau_0 + p_0^{\alpha}\right)}{\lambda\tau_0}. \tag{40}$$

That is, since Lemma 5 is satisfied, and we see that (37) is almost oscillatory if conditions (38)–(40) hold.

5. Conclusions

In this article, we are interested in studying the asymptotic behavior of third-order neutral differential equations. We found more than one criterion to check the oscillation by Riccati method. Our results are an extension and complement to some results published in the literature. We supported the results obtained in this paper with examples. An interesting problem for further research could be to study the problem of oscillation when

$$\int_{z_0}^{\infty} m^{-1/(p-1)}(s)\mathrm{d}s = \infty,$$

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References

1. Agarwal, R.P.; Grace, S.R.; O'Regan, D. *Oscillation Theory for Second Order Linear, Half-Linear, Superlinear and Sublinear Dynamic Equations*; Kluwer Academic: Dordrecht, The Netherlands, 2002.
2. Agarwal, R.P.; Grace, S.R.; O'Regan, D. *Oscillation Theory for Second Order Dynamic Equations*; Taylor & Francis: London, UK, 2003.
3. Bainov, D.D.; Mishev, D.P. *Oscillation Theory for Neutral Differential Equations with Delay*; Adam Hilger: New York, NY, USA, 1991.

4. Bazighifan, O.; Postolache, M. Improved conditions for oscillation of functional nonlinear differential equations. *Mathematics* **2020**, *8*, 552. [CrossRef]
5. Bazighifan, O. On the oscillation of certain fourth-order differential equations with p-Laplacian like operator. *Appl. Math. Comput.* **2020**, *386*, 125475. [CrossRef]
6. Muhib, A. On oscillation of second-order noncanonical neutral differential equations. *J. Inequal. Appl.* **2021**, *2021*, 79. [CrossRef]
7. Dzurina, J.; Thandapani, E.; Tamilvanan, S. Oscillation of solutions to third-order half-linear neutral differential equations. *Electron. J. Differ. Equ.* **2012**, *29*, 1–9.
8. Graef, J.; Tunc, E.; Grace, S. Oscillatory and asymptotic behavior of a third-order nonlinear neutral differential equation. *Opuscula Math.* **2017**, *6*, 839–852. [CrossRef]
9. Jiang, Y.; Li, T. Asymptotic behavior of a third-order nonlinear neutral delay differential equation. *J. Inequal. Appl.* **2014**, *2014*, 512. [CrossRef]
10. Moaaz, O.; Awrejcewicz, J.; Muhib, A. Establishing new criteria for oscillation of odd-order nonlinear differential equations. *Mathematics* **2020**, *8*, 937. [CrossRef]
11. Elabbasy, E.M.; Qaraad, B.; Abdeljawad, T.; Moaaz, O. Oscillation criteria for a class of third-order damped neutral differential equations. *Symmetry* **2020**, *12*, 1988. [CrossRef]
12. Kumar, M.S.; Bazighifan, O.; Almutairi, A.; Chalishajar, D.N. Philos-type oscillation results for third-order differential equation with mixed neutral terms. *Mathematics* **2021**, *9*, 1021. [CrossRef]
13. Philos, C. On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays. *Arch. Math.* **1981**, *36*, 168–178. [CrossRef]
14. Agarwal, R.P.; Bohner, M.; Li, T.; Zhang, C. Oscillation of third-order nonlinear delay differential equations. *Taiwan Math. Soc.* **2013**, *17*, 545–558. [CrossRef]
15. Thandapani, E.; Li, T. On the oscillation of third-order quasi-linear neutral functional differential equations. *Arch. Math. (BRNO) Tomus.* **2011**, *47*, 181–199.
16. Tunc, E. Oscillatory and asymptotic behavior of third-order neutral differential equations with distributed deviating arguments. *Electron. J. Differ. Equ.* **2017**, *16*, 1–12.
17. Tiryaki, A.; Aktas, M.F. Oscillation criteria of a certain class of third-order nonlinear delay differential equations with damping. *J. Math. Anal. Appl.* **2007**, *325*, 54–68. [CrossRef]
18. Zhang, S.Y.; Wang, Q. Oscillation of second-order nonlinear neutral dynamic equations on time scales. *Appl. Math. Comput.* **2010**, *216*, 2837–2848. [CrossRef]
19. Saker, S.H.; Dzurina, J. On the oscillation of certain class of third-order nonlinear delay differential equations. *Math. Bohemica* **2010**, *135*, 225–237. [CrossRef]
20. Grace, S.R.; Agarwal, R.P.; Pavani, R.; Thandapani, E. On the oscillation of certain third order nonlinear functional differential bequations. *Appl. Math. Comput.* **2008**, *202*, 102–112.
21. Moaaz, O.; Qaraad, B.; El-Nabulsi, R.; Bazighifan, O. New results for Kneser solutions of third-order nonlinear neutral differential equations. *Mathematics* **2020**, *8*, 686. [CrossRef]
22. Dzurina, J.; Grace, S.R.; Jadlovska, I. On nonexistence ofKneser solutions of third-order neutral delay differential equations. *Appl. Math. Lett.* **2019**, *88*, 193–200. [CrossRef]
23. Vidhyaa, K.S.; Graef, J.R.; Thandapani, E. New oscillation results for third-order half-linear neutral differential equations. *Mathematics* **2020**, *8*, 325. [CrossRef]