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Well-Posedness and Porosity for Symmetric Optimization Problems

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Abstract: In the present work, we investigate a collection of symmetric minimization problems, which is identified with a complete metric space of lower semi-continuous and bounded from below functions. In our recent paper, we showed that for a generic objective function, the corresponding symmetric optimization problem possesses two solutions. In this paper, we strengthen this result using a porosity notion. We investigate the collection of all functions such that the corresponding optimization problem is well-posed and prove that its complement is a σ-porous set.

Keywords: complete metric space; generic element; lower semi-continuous function; porous set

JEL Classification: 49J27; 90C31

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Citation: Zaslavski, A.J. Well-Posedness and Porosity for Symmetric Optimization Problems. *Symmetry* **2021**, *13*, 1253. https://doi.org/10.3390/sym13071253

Academic Editor: Juan Luis García Guirao

Received: 16 June 2021
Accepted: 12 July 2021
Published: 13 July 2021

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1. Introduction

In this paper, we study a class of symmetric minimization problems, which was studied recently in our paper [1]. The results of [1] and of the present paper have prototypes in [2,3], where some minimization problems arising in crystallography were considered. It was shown in [2,3] that a typical symmetric minimization problem possesses exactly two minimizers, and every minimizing sequence converges to them in some natural sense. In [1], we extend the results of [2,3] for a sufficiently large class of symmetric minimization problems by showing that for a generic objective function, the corresponding symmetric optimization problem possesses two solutions. In this paper, we strengthen this result using a porosity notion. We investigate the collection of all functions such that the corresponding optimization problem is well-posed and prove that its complement is a σ-porous set.

More precisely, we study an optimization problem

$$g(\xi) \to \min, \ \xi \in X,$$

where X is a complete metric space and g is a lower semi-continuous and bounded from below function.

It is well-known that the above problem possesses a minimizer when the space X is compact or when the objective function f possesses a growth property and all bounded subsets of the space X satisfy certain compactness assumptions. Without such assumptions, the existence problem becomes more difficult. This difficulty is overcome by applying the Baire category approach, which was used for many mathematical problems [4–9].

Namely, it is known that the minimization problem stated above can be solved for a generic objective function [8–10]. More precisely, there is a collection $\mathcal{F}$ in a complete metric space of objective functions, which is a countable intersection of open and everywhere dense sets such that for every objective function $f \in \mathcal{F}$, the corresponding minimization problem has a unique solution, which is a limit of every minimizing sequence. See [9], which contains this result and its several extensions and modifications. Note that the generic approach in nonlinear analysis is used in [11–15], generic solvability of best approximation problems are discussed in [4,11,13], while generic existence of fixed points for nonlinear operators is established in [7,12,13].

In our recent paper [1] the goal was to establish a generic solvability of optimization problems with symmetry. These results have applications in crystallography [2,3]. In this paper, we strengthen this result using a porosity notion. We investigate the set of all functions for which the corresponding minimization problem is well-posed and show that its complement is a σ-porous set.

2. The Main Result

We begin this section recalling the following notion of porosity [3,4,7,9,12,13].

Suppose that (Y, d) is a complete metric space and define

$$B_d(y, r) = \{\xi \in X : \ d(y, \xi) \leq r\}.$$

We say that a set $E \subset Y$ is porous with respect to d (or just porous if the metric is understood) if there are a real number $\alpha \in (0, 1]$ and a positive number r_0 such that for every positive number $r \leq r_0$ and every point $y \in Y$ there is a point $z \in Y$ such that

$$B_d(z, \alpha r) \subset B_d(y, r) \setminus E.$$

We say that a set in the complete metric space Y is σ-porous with respect to d (or just σ-porous if the metric is understood) if this set is a countable union of porous (with respect to d) subsets of Y.

For every function $h : Y \to (-\infty, \infty]$, where the set Y is nonempty, put

$$\inf(h) = \inf\{h(\xi) : \ \xi \in Y\}$$

and

$$\mathrm{dom}(h) = \{y \in Y : \ h(y) < \infty\}.$$

Suppose that (X, ρ) is a complete metric space. For every $z \in X$ and every positive Δ put

$$B(z, \Delta) = \{\xi \in X : \ \rho(z, \xi) \leq \Delta\}.$$

For every $z \in X$ and every subset $D \neq \emptyset$ of the space X, define

$$\rho(z, C) = \inf\{\rho(z, \xi) : \ \xi \in C\}.$$

Denote by $\mathcal{M}_l$ the collection of all functions $f : X \to R^1 \cup \{\infty\}$, which are bounded from below, lower semi-continuous, and which are not identical infinity. For each $h_1, d_2 \in \mathcal{M}_l$, define

$$\tilde{d}(h_1, h_2) = \sup\{|h_1(z) - h_2(z)| : \ z \in X\}, \tag{1}$$

$$d(h_1, h_2) = \tilde{d}(h_1, h_2)(1 + \tilde{d}(h_1, h_2))^{-1}. \tag{2}$$

Note that by convention, $d(h_1, h_2) = 1$ when $\tilde{d}(h_1, h_2) = \infty$.

It is clear that $d : \mathcal{M}_l \times \mathcal{M}_l \to [0, \infty)$ is a complete metric. We denote by $\mathcal{M}_c$ the collection of all continuous finite-valued functions $f : X \to R^1$ which are bounded from below. Clearly, $\mathcal{M}_c$ is a closed set in the complete metric space $(\mathcal{M}_l, d)$. We endow the space $\mathcal{M}_c$ with the metric d too.

Suppose that $T : X \to X$ is a continuous operator such that

$$T^2(z) = z \text{ for every } z \in X.$$

We denote by $\mathcal{M}_{l,T}$ the collection of all functions $f \in \mathcal{M}_l$ for which

$$f(T(x)) = f(x) \text{ for every point } x \in X$$

and define

$$\mathcal{M}_{c,T} = \{f \in \mathcal{M}_c : \ f \circ T = f\}.$$

Evidently, $\mathcal{M}_{l,T}$ and $\mathcal{M}_{c,T}$ are closed subsets of the complete metric space $\mathcal{M}_l$. We endow them with the same metric d too.

We investigate the optimization problem

$$f(x) \to \min, \; x \in X,$$

where the objective function $f \in \mathcal{M}_{l,T}$.

Given $f \in \mathcal{M}_{l,T}$, we say that the problem of minimization for f on X is well-posed with respect to $(\mathcal{M}_l, d)$ if the following properties are true:

There exists $x_f \in X$, which satisfies

$$\{x \in X : \; f(x) = \inf(f)\} = \{x_f, \; T(x_f)\}$$

and for every $\epsilon > 0$ there are an open neighborhood $\mathcal{U}$ of f in $\mathcal{M}_l$ and a positive number δ such that if a function $g \in \mathcal{U}$ and if a point $z \in X$ satisfies $g(z) \le \inf(g) + \delta$, then

$$|g(z) - f(x_f)| \le \epsilon$$

and

$$\min\{\rho(z, \{x_f, T(x_f)\}), \; \rho(T(z), \{x_f, T(x_f)\})\} \le \epsilon.$$

This notion has an analog in the optimization theory [9], where the set of minimizers is a singleton. Here, since the problem is symmetric, the set of minimizers contains two points in general.

The next theorem is our sole main result.

Theorem 1. *Suppose that $\mathcal{A}$ is either $\mathcal{M}_{l,T}$ or $\mathcal{M}_{c,T}$. Then, there is a set $\mathcal{B} \subset \mathcal{A}$ such that its complement $\mathcal{A} \setminus \mathcal{B}$ is σ-porous in the metric space $(\mathcal{A}, d)$ and that for every function $f \in \mathcal{B}$ the minimization problem for f on the space X is well-posed with respect to $(\mathcal{M}_l, d)$.*

3. Auxiliary Results

Lemma 1. *For every positive number $r \le 1$, each $f, g \in \mathcal{M}_l$, which satisfy $d(f,g) \le 4^{-1}r$ and each $x \in X$,*

$$|g(x) - f(x)| \le r.$$

Proof. Let $r \in (0,1]$, $f, g \in \mathcal{M}_l$ satisfy

$$d(f,g) \le 4^{-1}r \tag{3}$$

and let $x \in X$ be given. By (2) and (3),

$$d(f,g) \le 4^{-1},$$

$$\tilde{d}(f,g) = d(f,g)(1 - d(f,g))^{-1} \le 2d(f,g) \le 2^{-1}r.$$

In view of (1) and the equation above,

$$|g(x) - f(x)| \le 2^{-1}r.$$

□

Lemma 2. *Suppose that $f \in \mathcal{M}_{l,T}$, $\epsilon \in (0,1)$, $r \in (0,1]$. Then there are $\bar{f} \in \mathcal{M}_{l,T}$ and $\bar{x} \in X$ such that $\bar{f} \in \mathcal{M}_{c,T}$ if $f \in \mathcal{M}_{c,T}$,*

$$f(x) \le \bar{f}(x) \le f(x) + r/2, \; x \in X \tag{4}$$

and that for each $y \in X$, which satisfies

$$\bar{f}(y) \leq \inf(\bar{f}) + \epsilon r/4 \tag{5}$$

the equation

$$\min\{\rho(y, \bar{x}),\ \rho(T(y), \bar{x})\} \leq \epsilon$$

is true.

Proof. There exists $\bar{x} \in X$ satisfying

$$f(\bar{x}) \leq \inf(f) + \epsilon r/4.$$

Define a function $\bar{f} \in \mathcal{M}_l$ as follows:

$$\bar{f}(x) = f(x) + 2^{-1} r \min\{\rho(x, \bar{x}), \rho(T(x), \bar{x}), 1\},\ x \in X. \tag{6}$$

Clearly, $\bar{f} \in \mathcal{M}_{l,T}$, and $\bar{f} \in \mathcal{M}_{c,T}$ if $f \in \mathcal{M}_{c,T}$ and (4) is true. Let $y \in X$ and (5) hold. By (5) and (6),

$$f(y) + 2^{-1} r \min\{\rho(y, \bar{x}), \rho(T(y), \bar{x}), 1\} = \bar{f}(y) \leq \inf(\bar{f}) + \epsilon r/4$$

$$\leq \bar{f}(\bar{x}) + \epsilon r/4 = f(\bar{x}) + \epsilon r/4 \leq f(y) + \epsilon r/2.$$

Therefore,

$$\min\{\rho(y, \bar{x}),\ \rho(T(y), \bar{x})\} \leq \epsilon.$$

□

4. Proof of Theorem 1

For every integer $n \geq 1$ let $\mathcal{A}_n$ be the collection of all functions $f \in \mathcal{A}$ such that:

(i) there are a point $\bar{x} \in X$ and $\delta > 0$ such that if $z \in X$ and $f(z) \leq \inf(f) + \delta$, then the inequality $\rho(\bar{x}, \{z, T(z)\}) \leq 1/n$ is valid.

Let a natural number n be given. We claim that the set $\mathcal{A} \setminus \mathcal{A}_n$ is porous.

By Lemma 1, for every positive number $r \leq 1$, each $f, g \in \mathcal{M}_l$ satisfying $d(f, g) \leq 4^{-1} r$ and each $x \in X$,

$$|g(x) - f(x)| \leq r. \tag{7}$$

By Lemma 2 applied with $\epsilon = (2n)^{-1}$, the following property is valid:

(ii) for each function $f \in \mathcal{A}$ and every positive number $r \leq 1$, there exist $\bar{f} \in \mathcal{A}$ and $\bar{x} \in X$ such that

$$\tilde{d}(f, \bar{f}) \leq r/4 \tag{8}$$

and that for each $y \in X$ satisfying

$$\bar{f}(y) \leq \inf(\bar{f}) + 16^{-1} r n^{-1} \tag{9}$$

the equation

$$\min\{\rho(y, \bar{x}),\ \rho(T(y), \bar{x})\} \leq (2n)^{-1}. \tag{10}$$

is valid.

Fix

$$\bar{r} = 4^{-1},\ \alpha = 80^{-1} n^{-1}. \tag{11}$$

Let $f \in \mathcal{A}$ and a positive number $r \leq \bar{r}$ be given. By property (ii), there exist $\bar{f} \in \mathcal{A}$ and $\bar{x} \in X$ such that

$$\tilde{d}(f, \bar{f}) \leq r/4 \tag{12}$$

and that the next property is true:

(iii) for every point $y \in X$ satisfying (9), Equation (10) is true.

Let a function $g \in \mathcal{A}$ satisfy

$$d(g, \bar{f}) \leq \alpha r. \tag{13}$$

By (2) and (11)–(13),

$$d(g, f) \leq \alpha r + r/4 \leq r/2. \tag{14}$$

By (2), (11) and (13),

$$\begin{aligned}\tilde{d}(g, \bar{f}) &\leq d(g, \bar{f})(1 - d(g, \bar{f}))^{-1} \\ &\leq \alpha r(1 - \alpha r)^{-1} \leq 2\alpha r\end{aligned} \tag{15}$$

and

$$|\inf(\bar{f}) - \inf(g)| \leq 2\alpha r. \tag{16}$$

Let a point $z \in X$ satisfy the inequality

$$g(z) \leq \inf(g) + \alpha r. \tag{17}$$

By (15),

$$|g(z) - \bar{f}(z)| \leq 2\alpha r. \tag{18}$$

By (11) and (16)–(18),

$$\begin{aligned}\bar{f}(z) &\leq g(z) + 2\alpha r \leq \inf(g) + 3\alpha r \leq \inf(\bar{f}) + 5\alpha r \\ &\leq \inf(\bar{f}) + 16^{-1} r n^{-1}.\end{aligned} \tag{19}$$

Property (iii), (9), (10) and (19) imply that

$$\min\{\rho(z, \bar{x}),\ \rho(T(z), \bar{x})\} \leq (2n)^{-1}.$$

Thus

$$g \in \mathcal{A}_n$$

by definition. Together with (14), this implies that

$$\{g \in \mathcal{A} :\ d(g, \bar{f}) \leq \alpha r\} \subset \{g \in \mathcal{A} :\ d(g, f) \leq r\} \cap \mathcal{A}_n.$$

Thus, the set $\mathcal{A} \setminus \mathcal{A}_n$ is σ-porous. Then the set

$$\mathcal{A} \setminus \cap_{n=1}^{\infty} \mathcal{A}_n = \cup_{n=1}(\mathcal{A} \setminus \mathcal{A}_n)$$

is σ-porous.

Let

$$f \in \cap_{n=1}^{\infty} \mathcal{A}_n. \tag{20}$$

By (20), for every integer $n \geq 1$, there are $x_n \in X$ and $\delta_n > 0$ such that the following property is valid:

(iv) if a point $z \in X$ satisfies the inequality $f(z) \leq \inf(f) + \delta_n$, then the equation $\rho(x_n, \{z, T(z)\}) \leq 1/n$ holds.

Suppose that a sequence $\{z_i\}_{i=1}^{\infty} \subset X$ satisfies

$$\lim_{i \to \infty} f(z_i) = \inf(f). \tag{21}$$

Let a natural number n be given. By (21) and property (iv), for every large enough positive integer i,

$$\{\rho(x_n, \{z_i,\ T(z_i)\}) \leq n^{-1}.$$

Since n is an arbitrary positive integer, there is a sub-sequence $\{z_{i_p}\}_{p=1}^{\infty}$ such that at least one of the sequences $\{z_{i_p}\}_{p=1}^{\infty}$ and $\{T(z_{i_p})\}_{p=1}^{\infty}$ converges. Since T is continuous and T^2 is the identity operator, they both converge and

$$T(\lim_{p\to\infty} z_{i_p}) = \lim_{p\to\infty} T(z_{i_p}). \tag{22}$$

Set

$$x_f = \lim_{p\to\infty} z_{i_p}. \tag{23}$$

By (21), (23) and the lower semi-continuity of f,

$$f(x_f) = f(T(x_f)) = \inf(f). \tag{24}$$

Applying property (iv) with $z_i = x_f$ for every natural number i, we obtain that

$$\rho(x_n, \{x_f,\ T(x_f)\}) \leq n^{-1} \text{ for every natural number } n \geq 1. \tag{25}$$

Let $\xi \in X$ be such that

$$f(\xi) = \inf(f). \tag{26}$$

By (26) and property (iv) applied with $z_i = \xi$ for every integer $i \geq 1$ we obtain that

$$\rho(x_n, \{\xi,\ T(\xi)\}) \leq n^{-1} \text{ for every natural number } n. \tag{27}$$

Equations (25) and (27) imply that

$$\min\{\rho(\xi, x_f),\ \rho(T(\xi), x_f),\ \rho(\xi, T(x_f)), \rho(T(\xi), T(x_f))\} \leq 2n^{-1}.$$

Since n is an arbitrary positive integer, we conclude that and at least one of the following equalities is true:

$$\xi = x_f,\ \xi = T(x_f).$$

Thus

$$\{x \in X :\ f(x) = \inf(f)\} = \{x_f, T(x_f)\}. \tag{28}$$

Let $\epsilon > 0$. Fix a natural number n such that

$$4n^{-1} < \epsilon. \tag{29}$$

Property (iv) and (25) imply that for every $z \in X$ which satisfies the inequality

$$f(z) \leq \inf(f) + \delta_n,$$

we have

$$\rho(x_n, \{z, T(z)\}) \leq 1/n \tag{30}$$

and

$$\min\{\rho(z, x_f),\ \rho(T(z), x_f),\ \rho(z, T(x_f)),\ \rho(T(z), T(x_f))\} \leq 2n^{-1}. \tag{31}$$

Fix a positive number

$$\delta < \min\{3^{-1}\delta_n,\ 8^{-1}\epsilon\}. \tag{32}$$

Let a function $g \in \mathcal{M}_l$ satisfy

$$\tilde{d}(g, f) \leq \delta \tag{33}$$

and let a point $z \in X$ be such that

$$g(z) \leq \inf(g) + \delta. \tag{34}$$

By Equations (32)–(34), we have

$$f(z) \le g(z) + \delta \le \inf(g) + 2\delta \le \inf(f) + 3\delta \le \inf(f) + \delta_n. \tag{35}$$

It follows from (29), (31) and (35) that

$$\min\{\rho(z, x_f),\ \rho(T(z), x_f),\ \rho(z, T(x_f)),\ \rho(T(z), T(x_f))\} \le 2n^{-1} < \epsilon. \tag{36}$$

Thus, (28) holds and for each function $g \in \mathcal{M}_l$ which satisfies (33) and every point $z \in X$ satisfying (34) Equation (36) holds. By Equations (33) and (34), we have

$$|g(z) - \inf(f)| \le 2\delta < \epsilon.$$

Thus, the minimization problem for f on X is well-posed with respect to $(\mathcal{M}_l, d)$ for all $f \in \cap_{n=1}^{\infty} \mathcal{A}_n$. Theorem 1 is proved.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

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