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Identification of the Domain of the Sturm–Liouville Operator on a Star Graph

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Abstract: This article is devoted to the unique recovering of the domain of the Sturm–Liouville operator on a star graph. The domain of the Sturm–Liouville operator is uniquely identified from the set of spectra of a finite number of specially selected canonical problems. In the general case, the domain of the definition of the original operator can be specified by integro-differential linear forms. In the case when the domain of the Sturm–Liouville operator on a star graph corresponds to the boundary value problem, it is sufficient to choose only finite parts of the spectra of canonical problems for a unique identification of the boundary form. Moreover, the above statement is valid only for a symmetric star graph.

Keywords: boundary conditions; boundary value problems; canonical problems

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1. Introduction

The following result was presented in the well-known work of Borg [1]. The eigenvalues of the problem:

$$-y''(x) + q(x)y(x) = \lambda y(x), \quad 0 < x < \pi, \tag{1}$$

$$y'(0) - hy(0) = 0, \quad y'(\pi) + Hy(\pi) = 0, \tag{2}$$

are denoted by $\lambda_1, \lambda_2, \ldots$, where $q(x)$ is a real-valued function that is continuous on the interval $[0, \pi]$ and h, H are real numbers. In a similar way, the eigenvalues of Equation (1) with boundary conditions:

$$y'(0) - h_1 y(0) = 0, \quad y'(\pi) + Hy(\pi) = 0, \tag{3}$$

are denoted by $\mu_1, \mu_2, \ldots$, where $h_1 \neq h$. Then, the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ uniquely define the function $q(x)$ and numbers h, h_1, and H. Thus, Borg introduced the spectra of two canonical problems E_1 and E_2. Here:

E_1 is the first canonical problem (1)–(3);
E_2 is the second canonical problem (1)–(2).

The canonical problem E_2 coincides with the original problem (1) and (2), which must be recovered from the set of spectra of the canonical problems E_1 and E_2. The necessary and sufficient conditions are formulated and proven in order for these sequences $\{\lambda_n\}$ and $\{\mu_n\}$ to be two spectra of problems E_1 and E_2 in [2]. Plaksina [3] studied on the interval $[0, \pi]$ inverse problems for operators generated by operation $l = (-\frac{d^2}{dx^2} + q(x))$ and general nonseparated self-adjoint boundary conditions, which have the following form:

$$\begin{cases} y'(0) + \beta y(0) + e^{i\alpha} y(\pi) = 0, \\ y'(\pi) - e^{-i\alpha} y(0) + \gamma y(\pi) = 0, \end{cases} \tag{4}$$

where α, β, γ are real numbers. In [3], three canonical problems E_1, E_2, and E_3 were introduced. By the spectra of these canonical problems, the function $q(x)$ and numbers α, β, γ can be identified. The canonical problem E_1 is given by Equation (1) and separated boundary conditions:

$$y(0) = 0, \quad y'(\pi) + \gamma y(\pi) = 0. \tag{5}$$

Canonical problem E_2 is defined by Equation (1) and nonseparated boundary conditions of the form (4), with parameter α in (4), which should be replaced by α_1, keeping all other boundary coefficients. Canonical problem E_3 coincides with the problem (1)–(4). It turns out that for a unique recovering function $q(x)$ and numbers α, β, γ, α_1, it is not sufficient to give three spectra of canonical problems E_1, E_2, and E_3. For a unique recovering $q(x)$, α, β, γ, α_1, we need to add to the set of spectra of canonical problems E_1, E_2, and E_3 a certain set of sign sequences $\{\delta_n\}$, where each of δ_n is either $+1$ or -1. We note that in [3], the necessary and sufficient conditions were given for the above four sequences to be three spectra of canonical problems E_1, E_2, and E_3. Similar inverse spectral problems for differential operators on graphs were studied in [4].

Along with the above-mentioned statements of inverse problems by a set of spectra of canonical problems, it is of interest to study the possibility of uniquely recovering only the boundary conditions of canonical problems. In this case, it is assumed that the coefficient $q(x)$ of the differential expression is defined on the entire interval $[0, \pi]$. Such problems are called problems of the identification of boundary conditions [5,6]. Sometimes, this problem is called the problem of identifying the domain of the Sturm–Liouville operator, since the domain of the operator can be specified by different (but equivalent) sets of boundary conditions.

The problems of identifying the boundary conditions of canonical problems usually require a unique recovering of a finite number of boundary coefficients. In the case of Borg, only three real numbers h, h_1, and H need to be recovered. In Plaksina's case, four numbers α, β, γ, and α_1 must be recovered. Hence, it can be understood that to recover a finite number of boundary coefficients, it is not necessary to indicate sets of complete spectra of canonical problems. In [7], it was proven that for a unique recovering of the boundary conditions for higher order differential operators, it is sufficient to indicate only a finite number of eigenvalues from each canonical problem. In this paper, a similar result was proven for the Sturm—Liouville operator on a star graph. Other inverse spectral problems for Dirac operators on a star graph were studied in [8]. In our case, special attention was paid to nonseparated boundary conditions. In brief, we note that Theorem 5 holds only for a symmetric star graph. A star graph with all edges of the same length is called a symmetric star. Consequently, the main result of this article is: uniquely recovering the domain of a second-order differential operator on a star graph is valid only if the graph is symmetric.

Let the ends of the $(m+1)$-th rod be elastically connected to each other at one node. The free ends of m rods are somehow fixed and inaccessible to visual observation. The free visible end of one rod can be hit with a hammer, and the eigenfrequencies of longitudinal vibrations of the coupling structures can be measured. The paper states that there exists a finite set of eigenfrequencies, which uniquely determines the anchoring of the ends of the rods that are inaccessible to visual observation. Such problems are related to the problems of acoustic diagnostics. The mathematical model of the elastic connection of the $(m+1)$-th rod is defined by a star graph on which the Sturm–Liouville operator is defined with some boundary conditions. The paper proved the possibility of uniquely recovering the domain of the Sturm–Liouville operator on a star graph by a set of spectra of special canonical problems. It was proven in the work that a finite number of eigenfrequencies is sufficient for uniquely recovering the fixings of the ends of the rods. Moreover, the total number of eigenfrequencies required for unambiguous restoration of boundary restraints does not exceed $2(m+1)^2$. In [9], the problem of recovering the coefficients of differential equations from a finite set of eigenvalues of a boundary value problem with nonseparated boundary conditions was considered.

Identification of the boundary damage for coupling structures consisting of solids remains a challenging topic due to the influence of the solids on each other and experimental conditions. Identification of the boundary damage is difficult if the ends of the coupling structure are not accessible for visual inspection. Therefore, in this work, the eigenfrequencies of longitudinal vibrations of coupling structures were used to identify boundary damage, since the eigenfrequencies of vibrations of connecting structures can be measured by engineering sensors.

2. Solution of the Cauchy Problem for the Sturm–Liouville Equation on a Star Graph

Let $\Gamma = \{V, E\}$ be a star graph, where V is the set of vertices, numbered from zero to $m+1$ and E is the set of edges $e_1, \ldots, e_{m+1}$ of the graph $\Gamma = \{V, E\}$ [10]. On each edge e_j, the following j-th differential equation:

$$-y_j''(x_j) + p_j(x_j)y_j(x_j) = f_j(x_j), \quad 0 < x_j < b_j \tag{6}$$

holds. Further, we assumed that $p_j(x)$, $j \geq 1$ are real-valued continuous functions on e_j. Vertex $(m+1) \in V$ is called the inner vertex of the star graph. At the inner vertex $(m+1)$, Kirchhoff's laws [11]:

$$\begin{cases} y_{m+1}(b_{m+1}) = y_1(0) = \cdots = y_m(0), \\ y'_{m+1}(b_{m+1}) = y'_1(0) + \cdots + y'_m(0) \end{cases} \tag{7}$$

hold.

Vertices $0, 1, \ldots, m$ are called boundary vertices of the star graph (Figure 1).

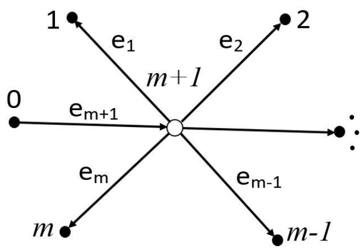

Figure 1. Star graph.

The set of boundary conditions:

$$U_k(y_1, \ldots, y_{m+1}) = \sum_{j=1}^{2} [\alpha_{kj} y_1^{(j-1)}(b_1) + \alpha_{k(2+j)} y_2^{(j-1)}(b_2) + \ldots$$
$$+ \alpha_{k(2m-2+j)} y_m^{(j-1)}(b_m) + \alpha_{k(2m+j)} y_{m+1}^{(j-1)}(0)] = 0 \tag{8}$$

holds at the boundary vertices $\{k \in 0, 1, \ldots, m\}$, where α_{ks} are complex numbers. Further, it is convenient to introduce the functions $c_j(x_j)$, $s_j(x_j)$, $j = 1, 2, \ldots, m+1$. Functions $c_j(x_j)$, $s_j(x_j)$ are solutions of the homogeneous differential equations:

$$-y_j''(x_j) + p_j(x_j)y_j(x_j) = 0, \quad 0 < x_j < b_j \tag{9}$$

with initial conditions:

$$\begin{cases} c_j(0) = s'_j(0) = 1, \quad c'_j(0) = s_j(0) = 0, \quad j = 1, 2, \ldots, m, \\ c_{m+1}(b_{m+1}) = s'_{m+1}(b_{m+1}) = 1, \quad c'_{m+1}(b_{m+1}) = s_{m+1}(b_{m+1}) = 0 \end{cases}$$

for each j. Since the star graph is a tree [12], there exists only one path connecting Vertex 0 to vertex j, where $j = 1, \ldots, m$. We denote this path by $S_j = e_{m+1} \cup e_j$. It is convenient to represent the indicated path S_j as a union of intervals $(0, b_{m+1})$ and $(b_{m+1}, b_{m+1} + b_j)$. We introduce the differential equations:

$$-\varphi''(x) + q_j(x)\varphi_j(x) = F_j(x), \quad x \in (0, b_{m+1}) \cup (b_{m+1}, b_{m+1} + b_j) \tag{10}$$

on the unions of intervals $S_j = (0, b_{m+1}) \cup (b_{m+1}, b_{m+1} + b_j)$, where:

$$\begin{aligned} q_j(x) &= p_{m+1}(x),\ F_j(x) = f_{m+1}(x), \quad \text{if} \quad 0 < x < b_{m+1}, \\ q_j(x) &= p_j(x), \quad F_j(x) = f_j(x), \quad \text{if} \quad b_{m+1} < x < b_{m+1} + b_j. \end{aligned}$$

We require the following condition:

$$\varphi_j(b_{m+1} - 0) = \varphi_j(b_{m+1} + 0), \quad A_j\varphi_j'(b_{m+1} - 0) = \varphi_j'(b_{m+1} + 0) \tag{11}$$

at the point $x = b_{m+1}$, where $\{A_j\}$ are arbitrary constants subject to a single requirement:

$$A_1 + A_2 + \cdots + A_m = 1. \tag{12}$$

Let $A_1, A_2, \ldots, A_m$ be fixed numbers that satisfy the equality (12). We also assume that the functions $\varphi_j(x)$, $\varphi_j'(x)$ are continuous from the left, that is $\varphi_j(b_{m+1} - 0) = \varphi_j(b_{m+1})$, $\varphi_j'(b_{m+1} - 0) = \varphi_j'(b_{m+1})$. By $\Phi_j(x)$ and $\Psi_j(x)$, we denote the solutions of Equation (10) with $F_j(x) \equiv 0$ on the path S_j, subject to conditions (11), as well as conditions:

$$\Phi_j(b_{m+1} - 0) = \Psi_j'(b_{m+1} - 0) = 1, \quad \Phi_j'(b_{m+1} - 0) = \Psi_j(b_{m+1} - 0) = 0. \tag{13}$$

We introduce a particular solution of Equation (10) by the formula:

$$\varphi_j(x) = \int_0^x \frac{\begin{vmatrix} \Phi_j(x) & \Psi_j(x) \\ \Phi_j(t) & \Psi_j(t) \end{vmatrix}}{\begin{vmatrix} \Phi_j'(t) & \Psi_j'(t) \\ \Phi_j(t) & \Psi_j(t) \end{vmatrix}} F_j(x)dt \tag{14}$$

for any j from the set $\{1, \ldots, m\}$ and $x \in S_j$. It is clear that the functions $\varphi_1(x), \varphi_2(x), \ldots, \varphi_m(x)$ defined by Formula (14) satisfy the Kirchhoff conditions (7) at the point $x = b_{m+1}$. Indeed, we can write:

$$\varphi_j(x) = \int_0^x \begin{vmatrix} c_{m+1}(x) & s_{m+1}(x) \\ c_{m+1}(t) & s_{m+1}(t) \end{vmatrix} f_{m+1}(t)dt, x \in (0, b_{m+1}),$$

$$\begin{aligned} \varphi_j(x + b_{m+1}) = &\int_0^{b_{m+1}} \begin{vmatrix} \Phi_j(x + b_{m+1}) & \Psi_j(x + b_{m+1}) \\ c_{m+1}(t) & s_{m+1}(t) \end{vmatrix} f_{m+1}(t)dt \\ &+ \int_0^x \begin{vmatrix} c_j(x) & s_j(x) \\ c_j(t) & s_j(t) \end{vmatrix} f_j(t)dt, \quad x \in (0, b_j) \end{aligned}$$

at $x \in (0, b_{m+1}) \cup (b_{m+1}, b_{m+1} + b_j)$. We note that the following representation:

$$\Phi_j(x + b_{m+1}) = c_j(x), \Psi_j(x + b_{m+1}) = A_j s_j(x)$$

holds for $x \in (0, b_j)$ and the following representation:

$$\Phi_j(x) = c_{m+1}(x), \Psi_j(x) = s_{m+1}(x)$$

holds for $x \in (0, b_{m+1})$. Thus, we state the following theorem.

Theorem 1. *The solution of the Cauchy problem for the Sturm–Liouville Equations (6) and (7) with Cauchy conditions:*

$$\theta_{m+1}(0) = 0, \quad \theta'_{m+1}(0) = 0 \tag{15}$$

at the point $x_{m+1} = 0$, we denote by $\Theta = (\theta_1(x_1), \theta_2(x_2), \dots, \theta_{m+1}(x_{m+1}))$, and it has the following representation:

$$\theta_{m+1}(x_{m+1}) = \int_0^{x_{m+1}} \begin{vmatrix} c_{m+1}(x_{m+1}) & s_{m+1}(x_{m+1}) \\ c_{m+1}(t) & s_{m+1}(t) \end{vmatrix} f_{m+1}(t)dt, \quad x_{m+1} \in e_{m+1},$$

$$\theta_j(x_j) = \int_0^{b_{m+1}} \begin{vmatrix} c_j(x_j) & A_j s_j(x_j) \\ c_{m+1}(t) & s_{m+1}(t) \end{vmatrix} f_{m+1}(t)dt$$
$$+ \int_0^{x_j} \begin{vmatrix} c_j(x_j) & s_j(x_j) \\ c_j(t) & s_j(t) \end{vmatrix} f_j(t)dt, \quad x_j \in e_j, \quad j = 1, 2, \dots, m,$$

where $A_1, A_2, \dots, A_m$ are arbitrary numbers subject to requirement (12).

Remark 1. *Theorem 1 implies that the solution to the Cauchy problem (6), (7), (15) depends on arbitrary $(m-1)$ constants. Constants $A_1, A_2, \dots, A_m$ that satisfy the requirement (12) determine at the point $x_{m+1} = b_{m+1}$ the portions of flow along the edges $e_1, e_2, \dots, e_m$ with respect to the flow on the edges e_{m+1}. The numbers $A_1, A_2, \dots, A_m$ we call the connecting constants.*

3. Construction of a Biorthogonal System of Solutions for a Set of Boundary Conditions

Let a set of boundary conditions (8) be given by boundary forms $U_0(\cdot), \dots, U_m(\cdot)$. The system of solutions $R_j = r_{1j}(x_1), \dots, r_{m+1,j}(x_{m+1})$ of the problem (7)–(9) for $j = 0, 1, \dots, m$ is called biorthogonal to the boundary forms $U_0(\cdot), \dots, U_m(\cdot)$, if the following requirement:

$$U_k(R_j) = \delta_{kj}, \quad k, j = 0, 1, \dots, m \tag{16}$$

holds, where δ_{kj} is the Kronecker delta.

In this section, we find sufficient conditions for the existence of a biorthogonal system of solutions. In other words, what conditions must the set of boundary forms $\{U_k$, k = 0,1,...,m$\}$ satisfy in order for a biorthogonal system of solutions to the problem (7)–(9) to exist? We introduce the following matrix:

$$T = \begin{bmatrix} U_0(c_1, c_2, \dots, c_{m+1}) & U_0^1(s_1) & U_0^2(s_2) & \dots & U_0^m(s_m) & U_0^{m+1}(s_{m+1}) \\ U_1(c_1, c_2, \dots, c_{m+1}) & U_1^1(s_1) & U_1^2(s_2) & \dots & U_1^m(s_m) & U_1^{m+1}(s_{m+1}) \\ \dots\dots\dots & \dots & \dots & \dots & \dots & \dots \\ U_m(c_1, c_2, \dots, c_{m+1}) & U_m^1(s_1) & U_m^2(s_2) & \dots & U_m^m(s_m) & U_m^{m+1}(s_{m+1}) \\ 0 & 1 & 1 & \dots & 1 & -1 \end{bmatrix},$$

where:

$$U_i^k(s_k) = \sum_{j=1}^{2} \alpha_{i(2k-2+j)} s_k^{(j-1)}(b_k), \quad k \geq 1, \quad i \geq 0,$$
$$U_i^{m+1}(s_{m+1}) = \sum_{j=1}^{2} \alpha_{i(2m+j)} s_{m+1}^{(j-1)}(0). \tag{17}$$

Theorem 2. *Let a set of boundary forms $U_0(\cdot), \dots, U_m(\cdot)$ be such that:*

$$detT \neq 0.$$

Then, there exists a unique system of solutions to the problem (7)–(9), which is biorthogonal to forms $\{U_0, \ldots, U_m\}$.

Proof of Theorem 2. Let us write down the general solution of the homogeneous system of differential Equation (9) subject to the Kirchhoff conditions (7):

$$y_{m+1}(x_{m+1}) = Dc_{m+1}(x_{m+1}) + Es_{m+1}(x_{m+1}), \quad x \in e_{m+1}, \tag{18}$$

$$y_j(x_j) = Dc_j(x_j) + EA_js_j(x_j), \quad x \in e_j, \quad j = 1, \ldots, m. \tag{19}$$

Here, D, E and $A_1, \ldots, A_m$ are arbitrary numbers subject to the condition (12). Let us prove that there exist R that satisfy the following equalities:

$$U_0(R) = 1, U_1(R) = 0, \ldots, U_m(R) = 0. \tag{20}$$

We seek function $R = (r_1(x), r_2(x), \ldots, r_{m+1}(x))$ in the forms (18) and (19). By substituting the expressions (18) and (19) into the equalities (20), we obtain a system of algebraic equations with respect to $D, E, A_1E, A_2E, \ldots, A_mE$:

$$Tz = l_1, \tag{21}$$

where:

$$z = [D, A_1E, A_2E, \ldots, A_mE, E]^T, \quad l_1 = [1, 0, 0, \ldots, 0]^T.$$

Since $detT \neq 0$, the numbers $E, A_1, A_2, \ldots, A_m, D$ are uniquely found from the system (21). Therefore, R_0 is determined from the conditions (16) for $j = 0$ in a unique way. It is also checked in the same way that $R_1, R_2, \ldots, R_m$ are determined from the conditions (16) for $j \geq 1$ in a unique way. The proof of Theorem 2 is complete. □

4. An Equivalent Set of Boundary Forms That Have an Integral Form

In this section, the set of boundary forms $\{U_0, \ldots, U_m\}$ from (8) is replaced by an equivalent set of boundary forms $\{W_0, \ldots, W_m\}$. Let the number $A_m = 1 - A_1 - \cdots - A_{m-1}$ and the numbers $A_1, A_2, \ldots, A_{m-1}$ be arbitrary numbers. To represent the explicit form of the boundary forms $\{W_0, \ldots, W_m\}$, we introduce the following functions:

$$\rho_k^{(1)}(t) = \sum_{i=0}^{m} r_{m+1\,i}(0) \begin{vmatrix} U_i^k(c_k) & U_i^k(s_k) \\ c_k(t) & s_k(t) \end{vmatrix}, \quad t \in e_k, k \leq m,$$

$$\rho_{m+1}^{(1)}(t) = \sum_{i=0}^{m} r_{m+1\,i}(0) \left(\begin{vmatrix} U_i^{m+1}(c_{m+1}) & U_i^{m+1}(s_{m+1}) \\ c_{m+1}(t) & s_{m+1}(t) \end{vmatrix} + \sum_{i=1}^{m} \begin{vmatrix} U_i^i(c_i) & A_iU_i^i(s_i) \\ c_{m+1}(t) & s_{m+1}(t) \end{vmatrix} \right), \quad t \in e_{m+1},$$

$$\rho_k^{(2)}(t) = \sum_{i=0}^{m} r'_{m+1\,i}(0) \begin{vmatrix} U_i^k(c_k) & U_i^k(s_k) \\ c_k(t) & s_k(t) \end{vmatrix}, \quad t \in e_k, \quad k \leq m,$$

$$\rho_{m+1}^{(2)}(t) = \sum_{i=0}^{m} r'_{m+1\,i}(0) \left(\begin{vmatrix} U_i^{m+1}(c_{m+1}) & U_i^{m+1}(s_{m+1}) \\ c_{m+1}(t) & s_{m+1}(t) \end{vmatrix} + \sum_{l=1}^{m} \begin{vmatrix} U_i^l(c_i) & A_iU_i^i(s_l) \\ c_{m+1}(t) & s_{m+1}(t) \end{vmatrix} \right), \quad t \in e_{m+1},$$

$$\rho_k^{(j+2)}(t) = \sum_{i=0}^{m} (r'_{j\,i}(0) - A_jr'_{m+1\,i}(b_{m+1})) \begin{vmatrix} U_i^k(c_k) & U_i^k(s_k) \\ c_k(t) & s_k(t), \end{vmatrix} \quad t \in e_{m+1}, \quad k \leq m,$$

$$cc\rho_{m+1}^{(j+2)}(t) = \sum_{i=0}^{m}(r'_{ji}(0) - A_j r'_{m+1\,i}(b_{m+1}))\left(\begin{vmatrix} U_i^{m+1}(c_{m+1}) & U_i^{m+1}(s_{m+1}) \\ c_{m+1}(t) & s_{m+1}(t) \end{vmatrix} + \sum_{i=1}^{m}\begin{vmatrix} U_i^i(c_i) & A_i U_i^i(s_i) \\ c_{m+1}(t) & s_{m+1}(t) \end{vmatrix}\right), \quad t \in e_{m+1}. \tag{22}$$

Now, we define new boundary forms by the following formulas:

$$W_0(y_1, y_2, \dots, y_{m+1}) = y_{m+1}(0) + \sum_{k=1}^{m+1}\int_0^{b_k}\rho_k^1(t)(-y''_k(t) + p_k(t)y_k(t))dt,$$

$$W_1(y_1, y_2, \dots, y_{m+1}) = y'_{m+1}(0) + \sum_{k=1}^{m+1}\int_0^{b_k}\rho_k^2(t)(-y''_k(t) + p_k(t)y_k(t))dt,$$

$$W_{j+1}(y_1, y_2, \dots, y_{m+1}) = y'_j(0) - A_j y'_{m+1}(b_{m+1}) + \sum_{k=1}^{m+1}\int_0^{b_k}\rho_k^{j+2}(t)(-y''_k(t) + p_k(t)y_k(t))dt, \quad j = 1, \dots, m-1.$$

Theorem 3. *Let the problem (6)–(8) have a unique solution $Y = (y_1(x), y_2(x), \dots, y_{m+1}(x))$. Then, the set of boundary conditions (8) is equivalent to the following boundary conditions:*

$$W_k(y_1, y_2, \dots, y_{m+1}) = 0, \quad k = 0, 1, \dots, m. \tag{23}$$

The boundary conditions defined by Theorem 1 are called canonical boundary conditions or normalized boundary conditions [7]. Therefore, instead of restoring the boundary conditions (8), we restore the boundary conditions (23).

Remark 2. *We note that the functions $\rho_k^{(j)}(t)$ for a fixed k from the set $\{1, \dots, m+1\}$ are defined on the edge e_k and represent the solutions of the homogeneous Equation (9).*

Proof of Theorem 3. By Theorem 1, we define a solution to the Cauchy problem as follows:

$$\theta_j(x_j) = \int_0^{b_{m+1}}\begin{vmatrix} c_j(x_j) & A_j s_j(x_j) \\ c_{m+1}(t) & s_{m+1}(t) \end{vmatrix} f_{m+1}(t)dt + \int_0^{x_j}\begin{vmatrix} c_j(x_j) & s_j(x_j) \\ c_j(t) & s_j(t) \end{vmatrix} f_j(t)dt, x_j \in e_j, \quad j = 1, 2, \dots, m,$$

$$\theta_{m+1}(x_{m+1}) = \int_0^{x_{m+1}}\begin{vmatrix} c_{m+1}(x_{m+1}) & s_{m+1}(x_{m+1}) \\ c_{m+1}(t) & s_{m+1}(t) \end{vmatrix} f_{m+1}(t)dt, \quad x_{m+1} \in e_{m+1}. \tag{24}$$

By direct verification, we can check that the following functions:

$$\vartheta_j(x_j) = \theta_j(x_j) - \sum_{i=0}^{m} U_i(\theta) r_{ji}(x_j), \quad j = 1, 2, \dots, m+1 \tag{25}$$

satisfy the boundary condition (8). Let us introduce the notation $V = (\vartheta_1(x_1), \ldots, \vartheta_{m+1}(x_{m+1}))$. Indeed, the identity $U_i(Y) = \sum_{k=1}^{m+1} U_i^k(y_k)$ implies that:

$$\sum_{j=1}^{m+1} U_0^j(\vartheta_j) = \sum_{j=0}^{m+1} U_0^j(\theta_j) - \sum_{i=0}^{m} U_0(\theta) \sum_{j=1}^{m+1} U_0^j(r_{ji}).$$

The quantity $U_i^k(y_k)$ is defined in a similar way as (17). Hence, (16) implies that:

$$U_0(V) = U_0(\theta) - \sum_{i=0}^{m} U_i(\theta) U_0(R_i) = 0,$$

which confirms that the boundary conditions (8) hold. It is not hard to understand that V is a solution to the problem (6)–(8). The uniqueness of the solution to the problem (6)–(8) implies that:

$$Y = V \text{ or } y_j(x_j) = \vartheta_j(x_j), \quad j = 1, 2, \ldots, m+1. \tag{26}$$

Formulas (25) and (26) imply the following expressions:

$$\theta_j(x_j) = y_j(x_j) + \sum_{i=0}^{m} U_i(\Theta) r_{ji}(x_j), \quad j = 1, 2, \ldots, m+1, \tag{27}$$

since Θ satisfies the following conditions:

$$\theta_{m+1}(0) = 0,\ \theta'_{m+1}(0) = 0,\ \theta'_j(0) - A_j \theta'_{m+1}(b_{m+1}) = 0,\ j = 1, \ldots, m-1. \tag{28}$$

The substitution on the right-hand side of the relations (27) into the conditions (28) gives:

$$\begin{aligned} & y_{m+1}(0) + \sum_{i=0}^{m} U_i(\Theta) r_{m+1i}(0) = 0, \\ & y'_{m+1}(0) + \sum_{i=0}^{m} U_i(\Theta) r'_{m+1i}(0) = 0, \\ & y'_j(0) - A_j y'_{m+1}(b_{m+1}) + \sum_{i=0}^{m} U_m(\Theta)(r'_{ji}(0) \\ & \qquad - A_j r'_{m+1i}(b_{m+1})) = 0, \quad j = 1, \ldots, m-1. \end{aligned} \tag{29}$$

Now, we calculate the values $U_0(\Theta), \ldots, U_m(\Theta)$ by applying (24). As a result, we have:

$$\begin{aligned} U_i(\Theta) = \sum_{k=1}^{m+1} U_i^k(\Theta_k) = \sum_{k=1}^{m} \int_0^{b_{m+1}} & \begin{vmatrix} U_i^k(c_k) & A_k U_i^k(s_k) \\ c_{m+1}(t) & s_{m+1}(t) \end{vmatrix} f_{m+1}(t) dt \\ & + \sum_{k=1}^{m+1} \int_0^{b_k} \begin{vmatrix} U_i^k(c_k) & U_i^k(s_k) \\ c_k(t) & s_k(t) \end{vmatrix} f_k(t) dt \end{aligned} \tag{30}$$

for $i = 0, 1, \ldots, m$. We substitute the right-hand side of the relations (30) into the equalities (29), then we have:

$$\begin{cases} y_{m+1}(0) + \sum\limits_{k=1}^{m+1} \int\limits_{0}^{b_k} \rho_k^{(1)}(t) f_k(t) dt = 0, \\ y'_{m+1}(0) + \sum\limits_{k=1}^{m+1} \int\limits_{0}^{b_k} \rho_k^{(2)}(t) f_k(t) dt = 0, \\ y'_j(0) - A_j y'_{m+1}(b_{m+1}) \\ \qquad + \sum\limits_{k=1}^{m+1} \int\limits_{0}^{b_k} \rho_k^{(j+2)}(t) f_k(t) dt = 0, \quad j = 1, \ldots, m-1. \end{cases} \tag{31}$$

Since $f_k(t) = -y''_k(t) + p_k(t) y_k(t)$ for $k \geq 1$, then the relations (31) imply the proof of Theorem 3. □

5. Selection of Canonical Problems and the Statement of the Inverse Problem

In this section, we present the method for the selection of canonical problems, the spectra of which allow uniquely finding the boundary conditions of the original boundary value problem or the boundary conditions that are equivalent to them. Therefore, it is sufficient to determine the functions:

$$\{\rho_k^{(j)}(t), \quad k = 1, \ldots, m+1, \quad j = 1, \ldots, m+2\}$$

by the spectra of the canonical problems. In fact, to determine the boundary coefficients we use not the entire spectrum of the auxiliary canonical problem, but only its finite part.

The number of auxiliary canonical problems is equal to the number of edges of the graph-star. Therefore, we build $(m+1)$ canonical problems. As the first canonical problem, we chose the problem (6) and (7) with the following boundary conditions:

$$\begin{aligned} &W_0(y) = 0, \quad y'_{m+1}(0) = 0, \\ &y'_j(0) - A_j y'_{m+1}(b_{m+1}) = 0, \quad j = 1, \ldots, m-1. \end{aligned} \tag{32}$$

As a second canonical problem, we chose the problem (6) and (7) with the following boundary conditions:

$$\begin{aligned} &W_0(y) = 0, \quad W_1(y) = 0, \\ &y'_j(0) - A_j y'_{m+1}(b_{m+1}) = 0, \quad j = 1, \ldots, m-1. \end{aligned} \tag{33}$$

In a similar way, we chose the $3\text{rd}, 4\text{th}, \ldots, (m+1) - \text{th}$ canonical problems. As the $(m+1) - \text{th}$ canonical problem, we chose the problem (6), (7), and (23), which is equivalent to the problem (6)–(8).

We clarify the statement of the problem of recovering the boundary conditions.

The statement of the first problem:

We need to uniquely recover the first boundary vector-function $\{\rho_k^{(1)}(t), k = 1, \ldots, m+1\}$ by the given differential Equation (6) and by the spectrum of the first canonical problem.

The statement of second inverse problem:

We need to uniquely recover the second boundary vector-function $\{\rho_k^{(2)}(t), k = 1, \ldots, m+1\}$ by the given differential Equation (6), by boundary vector-function $\{\rho_k^{(1)}(t), k = 1, \ldots, m+1\}$, and by the spectrum of the second canonical problem.

In a similar way, we state the $3\text{rd}, 4\text{th}, \ldots, m\text{th}$ inverse problems.

The statement of the $(m+1)$th inverse problem:

We need to uniquely recover the $(m+1)$th boundary vector-function $\{\rho_k^{(m+1)}(t), k = 1, \ldots, m+1\}$ by the given differential Equation (6), by boundary vector-functions $\{\rho_k^{(1)}(t)\}, \ldots, \{\rho_k^{(m)}(t)\}$, $k = 1, \ldots, m+1$, and by the spectrum of the $(m+1)$th canonical problem. In fact, not the entire spectrum of the canonical problem will be used, but only its end part. This idea is worked out in more detail in the following sections.

6. A Uniqueness Theorem for Recovering Boundary Functions

The transition from the boundary conditions (8) to equivalent canonical boundary forms allows us to prove the uniqueness theorem for the recovery of boundary vector-functions $\{\rho_k^{(1)}(t), \quad k = 1, \dots, m+1\}, \dots, \{\rho_k^{(m+1)}(t), \quad k = 1, \dots, m+1\}$. Further, the s-th canonical problem is called problem E_s. We denote by $\widetilde{E}_s$ the problem of the type E_s with the same Equation (6), but with different parameters in the boundary conditions (7). Further, we assume that if some symbol denotes an object related to the problem E_s, then the same symbol with a "wave" at the top denotes a similar object of the problem $\widetilde{E}_s$.

Theorem 4. *We fix an integer number s in the set* $\{1, \dots, m+1\}$. *Assume that the spectra of the problems* E_s *and* $\widetilde{E}_s$ *coincide. If a* $\rho_k^{(1)}(t) = \widetilde{\rho}_k^{(1)}(t), \dots, \rho_k^{(s-1)}(t) = \widetilde{\rho}_k^{(s-1)}(t)$, $k = 1, \dots, m+1$ *in* $L_2(\Gamma)$ *and the systems of the root functions of the problems* E_s *and* $\widetilde{E}_s$ *are complete in* $L_2(\Gamma)$, *then* $\rho_k^{(s)}(t) = \widetilde{\rho}_k^{(s)}(t)$ *on the space* $L_2(\Gamma)$.

We introduce [13] in a natural way the metric topology and Lebesgue measure on the graph Γ. Space $L_2(\Gamma)$ is understood as the L_2-space with respect to this measure. In other words, in $L_2(\Gamma)$, we introduce the inner product by the formula:

$$(y, z) = \sum_{j=1}^{m+1} (y_j, z_j) = \sum_{j=1}^{m+1} \int_0^{b_j} y_j(x_j)\overline{z_j(x_j)}dx_j, \quad y, z \in L_2(\Gamma). \tag{34}$$

Then, by (34), the conditions (31) take the following form:

$$\begin{cases} y_{m+1}(0) + <\rho^{(1)}, f> = 0, \\ y'_{m+1}(0) + <\rho^{(2)}, f> = 0, \\ y'_j(0) - A_j y'_{m+1}(b_{m+1}) + <\rho^{(j+2)}, f> = 0, \quad j = 1, \dots, m-1, \end{cases}$$

where:

$$\rho^{(1)} = (\rho_1^{(1)}, \dots, \rho_{m+1}^{(j)}) \in L_2(\Gamma),$$
$$f = (f_1, \dots, f_{m+1}) \in L_2(\Gamma).$$

We note that $f_t = -y''_j(t) + p_j(t)y_j(t), \quad j = 1, \dots, m+1$. Thus, by our notation, we have:

$$\begin{cases} W_0(y) = y_{m+1}(0) + <\rho^{(1)}, -y'' + py>, \\ W_1(y) = y'_{m+1}(0) + <\rho^{(2)}, -y'' + py>, \\ W_{j+1}(y) = y'_j(0) - A_j y'_{m+1}(b_{m+1}) + <\rho^{(j+2)}, -y'' + py>, \quad j = 1, \dots, m-1, \end{cases} \tag{35}$$

where $\quad py = (p_1y_1, p_2y_2, \dots, p_{m+1}y_{m+1})$.

Proof of Theorem 4 for $s = 1$. By $u^{(1)} = (u_1^{(1)}(x_1), u_2^{(1)}(x_2), \dots, u_{m+1}^{(1)}(x_{m+1}))$, we denote the solution to the Cauchy problem:

$$-\frac{d^2}{dx_j^2}u_j^{(1)}(x_j) + p_j(x_j)u_j^{(1)}(x_j) = \lambda u_j^{(1)}(x_j), \quad x_j \in (0, b_j), \quad j = 1, \dots, m+1,$$

$$u_{m+1}^{(1)}(0) = 1, \quad \frac{du_{m+1}^{(1)}}{dx_{m+1}}(0) = 0,$$

$$\frac{du_k^{(1)}(x_k)}{dx_k}\Big|_{x_k=0} - A_k \frac{du_{m+1}^{(1)}(x_{m+1})}{dx_{m+1}}\Big|_{x_{m+1}=b_{m+1}} = 0, \quad k = 1, \dots, m-1.$$

We note that the functions $u_j^{(1)}(x_j, \lambda)$ are entire functions of the parameter λ, since, by Theorem 1, the Cauchy problem is uniquely solvable for all complex λ. Let $\lambda = \lambda^{(1)}$ be the arbitrary eigenvalue of the problem E_1. Then,

$$u^{(1)}(\lambda^{(1)}) = (u_1^{(1)}(x_1, \lambda^{(1)}), u_2^{(1)}(x_2, \lambda^{(1)}), \dots, u_{m+1}^{(1)}(x_{m+1}, \lambda^{(1)}))$$

is the eigenfunction of the problem E_1 corresponding to eigenvalue $\lambda^{(1)}$. The first boundary condition of the problem E_1 has the following form:

$$W_0(u^1(\lambda^{(1)})) = u_{m+1}^{(1)}(0, \lambda^{(1)}) + \lambda^{(1)} < \rho^{(1)}, u^{(1)}(\lambda^{(1)}) >= 0.$$

Hence, it follows that:

$$< \rho^{(1)}, u^{(1)}(\lambda^{(1)}) >= -\frac{1}{\lambda^{(1)}}.$$

Therefore, the eigenvalues of the problem E_1 determine the Fourier coefficients of the function $\rho^{(1)}$ in the system of root functions that is conjugate to the problem E_1. Since the system of the root functions of the problem E_1 is complete in the space $L_2(\Gamma)$, then the system of the root functions of the conjugate problem is also complete in $L_2(\Gamma)$. Thus, if the spectra of the problems E_s and $\widetilde{E}_s$ coincide, then the Fourier coefficients of the functions $\rho^{(1)}$ and $\widetilde{\rho}^{(1)}$ coincide by the same complete system of the space $L_2(\Gamma)$. Therefore, in the space $L_2(\Gamma)$, functions $\rho^{(1)}$ and $\widetilde{\rho}^{(1)}$ coincide. This proof corresponds to the case of the simple eigenvalues of the problems E_s and $\widetilde{E}_s$. In the case of multiple eigenvalues, the reasoning requires a slight modification. □

Proof of Theorem 4 for $s = 2$. By $u^{(2)} = (u_1^{(2)}(x_1), u_2^{(2)}(x_2), \dots, u_{m+1}^{(2)}(x_{m+1}))$, we denote the solution to the Cauchy problem:

$$-\frac{d^2}{dx_j^2}u_j^{(2)}(x_j) + p_j(x_j)u_j^{(2)}(x_j) = \lambda u_j^{(2)}(x_j), \quad x_j \in (0, b_j), \quad j = 1, \dots, m+1,$$

$$W_0(u^{(2)}) = 0, \quad \frac{du_{m+1}^{(2)}}{dx_{m+1}}(0) = 0,$$

$$\frac{du_k^{(2)}(x_k)}{dx_k}\Big|_{x_k=0} - A_k \frac{du_{m+1}^{(2)}(x_{m+1})}{dx_{m+1}}\Big|_{x_{m+1}=b_{m+1}} = 0, \quad k = 1, \dots, m-1.$$

Let $\lambda = \lambda^{(2)}$ be an arbitrary eigenvalue of the problem E_2. Then,

$$u^{(2)}(\lambda^{(2)}) = (u_1^{(2)}(x_1, \lambda^{(2)}), u_2^{(2)}(x_2, \lambda^{(2)}), \dots, u_{m+1}^{(2)}(x_{m+1}, \lambda^{(2)}))$$

is the eigenvalue of the problem E_2 corresponding to eigenvalue $\lambda^{(2)}$. The second boundary condition of the problem E_2 can be represented as follows:

$$W_1(u^{(2)}(\lambda^{(2)})) = \frac{d}{dx_{m+1}} u^{(2)}_{m+1}(x_{m+1}, \lambda^{(2)})\Big|_{x_{m+1}=0} + \lambda^{(2)} < \rho^{(2)}, u^{(2)}(\lambda^{(2)}) >= 0$$

Hence, it follows that:

$$< \rho^{(2)}, u^{(2)}(\lambda^{(2)}) >= -\frac{1}{\lambda^{(2)}}.$$

The further reasoning repeats the proof of Theorem 4 for $s = 1$. The case, when problems E_1 and E_2 have common eigenvalues, requires a slight modification in the reasoning. The proofs of Theorem 4 for other s are similar. The proof of Theorem 4 is complete. □

7. Refinement of the Uniqueness Theorem in the Case of Boundary Value Problems

In this section, we refine Theorem 4 for boundary value problems. In Section 4 of this article, we presented Formula (22), which connects the functions $\rho^{(j)}_k(x_k)$ for $j, k = 1, \dots, m+1$ with the values of the boundary forms $\{U^k_i(c_k), U^k_i(s_k), k = 1, \dots, m+1, i = 1, \dots, m+1\}$. Formula (22) implies that the functions $\rho^{(j)}_k(x_k)$ for all $j = 1, \dots, m+1$ are solutions to the homogeneous equations $-y''_k(x_k) + p_k(x_k)y_k(x_k) = 0$, $k = 1, \dots, m+1$. Since the coefficients $p_1(x_1), \dots, p_{m+1}(x_{m+1})$ are given, then the solutions $\{c_k(x_k), s_k(x_k), k = 1, \dots, m+1\}$ are also known. We fix j in the set $\{1, \dots, m+1\}$.

Let:

$$\rho^{(j)}_k(x_k) = h^{(j)}_{1k} c_k(x_k) + h^{(j)}_{2k} s_k(x_k), \quad x_k \in (0, b_k), \quad k = 1, \dots, m+1.$$

with unknown constants $h^{(j)}_{1k}$, $h^{(j)}_{2k}$. Section 6 of this article provided a connection between the Fourier coefficients of the boundary function $\rho^{(j)} = (\rho^{(j)}_1(x_1), \dots, \rho^{(j)}_{m+1}(x_{m+1}))$ and the given eigenvalues of the problem E_j. Recall that:

$$< \rho^{(j)}, u^{(j)}(\lambda^{(j)}) >= -\frac{1}{\lambda^{(j)}},$$

where $u^{(j)}(\lambda^{(j)})$ is the eigenfunction of problem E_j corresponding to the eigenvalue $\lambda^{(j)}$.

Consequently, we obtain a system of equations:

$$\sum_{k=1}^{m+1} \left(h^{(j)}_{1k} \int_0^b c_k(x_k) \overline{u}^{(j)}_k(x_k, \lambda^{(j)}) + h^{(j)}_{2k} \int_0^b s_k(x_k) \overline{u}^{(j)}_k(x_k, \lambda^{(j)}) \right) = -\frac{1}{\lambda^{(j)}}$$

for unknown constants $h^{(j)}_{11}, h^{(j)}_{12}, h^{(j)}_{1m+1}, h^{(j)}_{21}, \dots, h^{(j)}_{2m+1}$. Thus, to uniquely determine the unknown constants $h^{(j)}_{11}, h^{(j)}_{12}, h^{(j)}_{1m+1}, h^{(j)}_{21}, \dots, h^{(j)}_{2m+1}$, it is sufficient to choose eigenvalues $\{\lambda^{(j)}_1, \lambda^{(j)}_2, \dots, \lambda^{(j)}_{2m+2}\}$ of problem E_j so that the determinant $Z = (Z_{ij})$ is nonzero, where:

$$Z_{k\,l+m+1} = \int_0^{b_1} s_l(x_l) \overline{u}^{(j)}(x_l, \lambda^{(j)}_k) dx_l,$$

$$Z_{kl} = \int_0^{b_l} c_l(x_l) \overline{u}^{(j)}(x_l, \lambda^{(j)}_k) dx_l, \quad l = 1, \dots, m+1, \quad k = 1, \dots, 2m+2.$$

Here, $u^{(j)}(\lambda^{(j)}_1), \dots, u^{(j)}(\lambda^{(j)}_{2m+2})$ are eigenfunctions of the problem E_j corresponding to the chosen eigenvalues.

Lemma 1. *Let $b_1 = b_2 = \cdots = b_{m+1} = b$. Suppose that the coefficients of the differential expressions $p_1(x), \ldots, p_{m+1}(x)$ are chosen so that the system of functions*
$\{c_1(x), s_1(x), c_2(x), s_2(x), \ldots, c_{m+1}(x), s_{m+1}(x)\}$ is linear independent on the interval $[0, b]$. Let problem E_j have infinitely many eigenvalues. Then, there exists a set of eigenvalues $\{\lambda_n^{(j)}, n = n_1, \ldots, n_{2m+2}\}$ of the problem E_j such that the determinant Z is nonzero. Here,

$$\lambda_1^{(j)} = \lambda_{n_1}^{(j)}, \ldots, \lambda_{n_{2m+2}}^{(j)} = \lambda_{n_{2m+2}}^{(j)}$$

are some eigenvalues of the canonical problem E_j.

Proof of Lemma 1. We prove by contradiction. Suppose that for any set of eigenvalues $\{\lambda_n^{(j)}, n = n_1, \ldots, n_{2m+2}\}$ of problem E_j, the determinant is equal to zero: $Z = 0$. We put $n_1 = 1, \ldots, n_{2m+1} = 2m + 1$. Since the problem E_j has infinitely many eigenvalues, assume that $\lambda_n^{(j)}, n = n_{2m+2}$ runs through the entire spectrum of the problem E_j. Since $Z = 0$, then the system of homogeneous linear algebraic equations $Z\overrightarrow{h} = 0$ has a nonzero solution, where:

$$\overrightarrow{h} = [h_{11}^{(j)}, \ldots, h_{1m+1}^{(j)} \quad h_{21}^{(j)}, \ldots, h_{2m+1}^{(j)}]$$

has a nonzero solution. We assume that $h_{11}^{(j)} \neq 0$. Consequently, the Fourier coefficients of the function $c_1(x)$ are linearly expressed in terms of the Fourier coefficients of the functions $\{s_1(x), c_2(x), s_2(x), \ldots, c_{m+1}(x), s_{m+1}(x)\}$. If all the Fourier coefficients of the function $c_1(x)$ can be linearly expressed in terms of the Fourier coefficients of the functions $\{s_1(x), c_2(x), s_2(x), \ldots, c_{m+1}(x), s_{m+1}(x)\}$, then the system of functions $\{c_1(x), s_1(x), c_2(x), s_2(x), \ldots, c_{m+1}(x), s_{m+1}(x)\}$ represents the linear dependent system. We have a contradiction. If $h_{11}^{(j)} \neq 0$ is not satisfied, then some modification in the reasoning is required. The proof of the lemma is complete. □

Lemma 1 and Theorem 4 imply the following theorem.

Theorem 5. *We fix j in the set $\{1, \ldots, m+1\}$. Let $b_1 = b_2 = \cdots = b_{m+1} = b$. Suppose that the coefficients $p_1(x), \ldots, p_{m+1}(x)$ of the differential expressions are chosen so that the system of functions $\{c_1(x), s_1(x), c_2(x), s_2(x), \ldots, c_{m+1}(x), s_{m+1}(x)\}$ is linear independent on the interval $[0, b]$. Let finite sets of eigenvalues of problems E_j and $\widetilde{E}_j$ from Lemma 1 coincide. If $\rho^{(1)} = \widetilde{\rho}^{(1)}, \ldots, \rho^{(j-1)} = \widetilde{\rho}^{(j-1)}$ in $L_2(\Gamma)$, then $\rho^{(j)} = \widetilde{\rho}^{(j)}$ in $L_2(\Gamma)$.*

In the proof of Theorem 4, it was established that the eigenfunctions E_j and $\widetilde{E}_j$ coincide if the corresponding eigenvalues coincide. This fact plays an essential role in the proof of Theorem 5. In conclusion, note that some of the constructions presented here can be found in [5,6]. In [14], examples of uniform beams with different boundary anchors having infinitely many identical eigenfrequencies of transverse vibrations were given. In Theorem 5, it was stated that a finite number of eigenfrequencies is sufficient for the unique recovering of boundary anchors. This does not contradict the above results of [14], since Theorem 5 states that the eigenfrequencies must be specially selected for the unique recovering of the boundary anchors of the beam.

8. Conclusions

Boundary value problems for differential equations on compact graphs can be specified by integro-differential conditions. The problem of determining the functions included in the integro-differential conditions is related to inverse problems. The recovery of the functions included in the integro-differential conditions was divided into three steps. In the first step, the integro-differential conditions were reduced to a normalized form. In the second step, we chose the canonical problems, by the spectra of which the integro-differential conditions would be restored. In the final step, a procedure was proposed to restore the functions included in the integro- differential conditions.

The paper proved the possibility of uniquely recovering the domain of the Sturm–Liouville operator on a star graph by the set of spectra of special canonical problems. In a particular case, when the domain of the operator is specified by boundary conditions, then for the unique recovery of the boundary coefficients, it is sufficient to specify only a finite set of its eigenvalues for each canonical problem. The total number of eigenvalues required to uniquely recover the boundary coefficients on a star graph with an $(m+1)$ edge does not exceed $2(m+1)^2$. The question of determining the minimum number of eigenvalues for the unique determination of the boundary coefficients seems to be interesting.

In this paper, the first two steps of the recovering of functions included in the integro-differential conditions were described in detail. The third step of the constructive restoration of functions requires its development. Only the recovery scheme was specified here. The modification to the recovery procedure is a question of interest, since interest is growing in the problem connected with identifying the boundary conditions of the differential operators on symmetric graph-like spaces.

The result of the work can be used to detect boundary defects in structures consisting of rods elastically connected in one node. The eigenfrequencies of the longitudinal vibrations of such structures can be measured by technical sensors. Based on the found eigenfrequencies, applying the results of this article, it is possible to identify boundary damage. The results of this article were theoretical, but in the future, they can be brought to constructively realizable algorithms for engineers.

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