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Lie Symmetry Analysis, Conservation Laws, Power Series Solutions, and Convergence Analysis of Time Fractional Generalized Drinfeld-Sokolov Systems

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Abstract: In this work, we investigate invariance analysis, conservation laws, and exact power series solutions of time fractional generalized Drinfeld–Sokolov systems (GDSS) using Lie group analysis. Using Lie point symmetries and the Erdelyi–Kober (EK) fractional differential operator, the time fractional GDSS equation is reduced to a nonlinear ordinary differential equation (ODE) of fractional order. Moreover, we have constructed conservation laws for time fractional GDSS and obtained explicit power series solutions of the reduced nonlinear ODEs that converge. Lastly, some figures are presented for explicit solutions.

Keywords: conservation laws; Lie symmetry analysis; generalized Drinfeld–Sokolov systems; symmetry reduction; Erdelyi–Kober operators

1. Introduction

Because of the great importance of nonlinear fractional partial differential equations (NFPDEs) in physics, mechanics, hydrology, viscoelasticity, image processing, electromagnetics, and other fields, researchers have long been aware of the solutions and applications of fractional partial differential equations [1–12]. In recent years, parallel to the increase in mathematical techniques and the use of computer programs, many authors have an increased desire to work on fractional analysis. Therefore, many methods are used to solve the NFPDEs—for example, the finite-difference method [13], the multiple exp-function method [14], the homotopy perturbation method [15,16], the variational-iteration method, the exp method [17], the fractional sub-equation method [18], and the Lie invariance method [19–21]. When we look through the literature, we realize Sophus Lie firstly put forward a methodology about symmetry analysis at the end of the nineteenth century [22]. After that, some impressive Lie group methods were considered in order to obtain symmetries, symmetry groups, and symmetry reduction. These are the classical and nonclasical Lie group approaches [23,24] and the Clarkson and Kruksal direct methods [25,26]. The main role of Lie symmetry methods is to construct invariance properties having partial equations as invariant forms. With the aid of these properties, we can reduce an NFPDE into a nonlinear ODE of fractional order with the help of the Riemann-Liouville (RL) derivative.

The link between Lie symmetry analysis and conservation laws of differential equations was revealed by Noether [27]. A generalized Noether theorem was used in [28] to construct conservation laws of NFPDEs with fractional Lagrangians. However, some differential equations do not arise from Lagrangians. To overcome this problem, İbragimov [29] put forward a new method without Lagrangians. Lukashchuk [30] showed important results towards obtaining conservation laws of NFPDEs.

In this study, we deal with exact solutions of the time fractional GDSS by using Lie symmetry analysis and conservation laws. The time fractional GDSS that models 1D nonlinear wave processes in two-component media has the form

$$
\begin{aligned}
D_t^\alpha u + a_1 u u_x + b_1 u_{xxx} + c(v^q)_x = 0, \\
D_t^\alpha v + a_2 u v_x + b_2 v_{xxx} = 0,
\end{aligned}
\tag{1}
$$

where a_1, a_2, b_1, b_2, c, and q are arbitrary constants, and D_t^α is the RL fractional derivative defined in (2). We could not take $b_1 = b_2 = 0$, because the time fractional GDSS may develop finite time singularities [31–33]. According to our research, Lie symmetry methods have not been applied to the time fractional GDSS until now. With some special choices of a_1, a_2, b_1, b_2, c, and q, the equation can be reduced to the time fractional Drinfeld-Sokolov-Satsuma-Hirota equations discussed in [34].

The RL fractional partial derivative [35] is defined by

$$
D_t^\alpha u = \begin{cases}
\frac{\partial^m u}{\partial t^m}, & \alpha = m \in N, \\
\frac{1}{\Gamma(m-\alpha)} \frac{\partial^m}{\partial t^m} \int_0^t \frac{u(z,x)}{(t-z)^{(\alpha+1-m)}} dz, & m-1 < \alpha < m, \quad m \in N,
\end{cases}
\tag{2}
$$

where $\Gamma(z)$ is a Gamma function defined by

$$
\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt,
$$

which converges in the complex plane when $Re(z) > 0$.

2. Preliminaries for Symmetry Analysis

In this chapter, we give the basic idea of the Lie symmetry method. Consider a system of time fractional partial differential equations with independent variables x and t as follows:

$$
D_t^\alpha u(x,t) = F_1(x,t,u,u_t,u_x,u_{xx},v,v_t,v_x,v_{xx,\ldots}),
\tag{3}
$$

$$
D_t^\alpha v(x,t) = F_2(x,t,u,u_t,u_x,u_{xx},v,v_t,v_x,v_{xx,\ldots}),
\tag{4}
$$

where the subscripts denote partial derivatives. Equations (3) and (4) are invariant under a one-parameter Lie group of point transformations given as follows:

$$
\begin{aligned}
&\bar{t} = t + \epsilon\tau(t,x,u,v) + O(\epsilon^2), && \bar{x} = x + \epsilon\zeta(t,x,u,v) + O(\epsilon^2), \\
&\bar{u} = u + \epsilon\mu(t,x,u,v) + O(\epsilon^2), && \bar{v} = v + \epsilon\rho(t,x,u,v) + O(\epsilon^2), \\
&\frac{\partial^\alpha \bar{u}}{\partial \bar{t}^\alpha} = \frac{\partial^\alpha u}{\partial t^\alpha} + \epsilon\mu_\alpha^0(t,x,u,v) + O(\epsilon^2), && \frac{\partial^\alpha \bar{v}}{\partial \bar{t}^\alpha} = \frac{\partial^\alpha v}{\partial t^\alpha} + \epsilon\rho_\alpha^0(t,x,u,v) + O(\epsilon^2), \\
&\frac{\partial \bar{u}}{\partial \bar{x}} = \frac{\partial u}{\partial x} + \epsilon\mu^x(t,x,u,v) + O(\epsilon^2), && \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} = \frac{\partial^2 u}{\partial x^2} + \epsilon\mu^{xx}(t,x,u,v) + O(\epsilon^2), \\
&\frac{\partial^3 \bar{u}}{\partial \bar{x}^3} = \frac{\partial^3 u}{\partial x^3} + \epsilon\mu^{xxx}(t,x,u,v) + O(\epsilon^2), && \frac{\partial \bar{v}}{\partial \bar{x}} = \frac{\partial v}{\partial x} + \epsilon\rho^x(t,x,u,v) + O(\epsilon^2), \\
&\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} = \frac{\partial^2 v}{\partial x^2} + \epsilon\rho^{xx}(t,x,u,v) + O(\epsilon^2), && \frac{\partial^3 \bar{v}}{\partial \bar{x}^3} = \frac{\partial^3 v}{\partial x^3} + \epsilon\rho^{xxx}(t,x,u,v) + O(\epsilon^2),
\end{aligned}
\tag{5}
$$

where

$$
\begin{aligned}
\mu^x &= D_x(\mu) - u_x D_x(\zeta) - u_t D_x(\tau), \\
\mu^{xx} &= D_x(\mu^x) - u_{xx} D_x(\zeta) - u_{xt} D_x(\tau), \\
\mu^{xxx} &= D_x(\mu^{xx}) - u_{xxx} D_x(\zeta) - u_{xxt} D_x(\tau), \\
\rho^x &= D_x(\rho) - v_x D_x(\zeta) - v_t D_x(\tau), \\
\rho^{xx} &= D_x(\rho^x) - v_{xx} D_x(\zeta) - v_{xt} D_x(\tau), \\
\rho^{xxx} &= D_x(\rho^{xx}) - v_{xxx} D_x(\zeta) - v_{xxt} D_x(\tau), \\
\mu_\alpha^0 &= D_t^\alpha(\mu) + \zeta D_t^\alpha(u_x) - D_t^\alpha(\zeta u_x) + D_t^\alpha(u(D_t\rho)) - D_t^{\alpha+1}(\rho u) + \rho D_t^{\alpha+1}(u), \\
\rho_\alpha^0 &= D_t^\alpha(\rho) + \zeta D_t^\alpha(v_x) - D_t^\alpha(\zeta v_x) + D_t^\alpha(v(D_t\rho)) - D_t^{\alpha+1}(\rho v) + \rho D_t^{\alpha+1}(v),
\end{aligned}
\tag{6}
$$

wherein D_x and D_t are total differential operators given as follows:

$$
\begin{aligned}
D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + v_x \frac{\partial}{\partial v} + v_{xx} \frac{\partial}{\partial v_x} + ..., \\
D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + v_t \frac{\partial}{\partial v} + v_{tt} \frac{\partial}{\partial v_t} + ...
\end{aligned}
\tag{7}
$$

The infinitesimal generator of Equations (3) and (4) consists of a set of vector fields given by

$$
X = \zeta \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial u} + \rho \frac{\partial}{\partial v}.
\tag{8}
$$

One can prove that the infinitesimal generators defined above must satisfy the following invariance criterion for Equations (3) and (4):

$$
Pr^n X(\Delta_1)|_{\Delta_1=0} = 0 \quad and \quad Pr^n X(\Delta_2)|_{\Delta_2=0} = 0, \quad n = 1, 2, ..,
\tag{9}
$$

where

$$
\Delta_1 := D_t^\alpha u(x,t) - F_1(x,t,u,u_t,u_x,u_{xx},v,v_t,v_x,v_{xx,...}),
$$

and

$$
\Delta_2 := D_t^\alpha v(x,t) - F_2(x,t,u,u_t,u_x,u_{xx},v,v_t,v_x,v_{xx,...}).
$$

As the lower limit of integral in (1) is fixed, it is going to be invariant under the transformations given in (5), so the corresponding invariance condition becomes

$$
\tau(t,x,u,v) = 0.
\tag{10}
$$

Under the condition (9), the α-th infinitesimal related to RL fractional time derivative [36] is given by

$$
\mu_\alpha^0 = \frac{\partial^\alpha \mu}{\partial t^\alpha} + (\mu_u - \alpha D_t(\tau))\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \mu_u}{\partial t^\alpha} - v\frac{\partial^\alpha \mu_v}{\partial t^\alpha} + \mu_v \frac{\partial^\alpha v}{\partial t^\alpha} - \sum_{n=1}^\infty \binom{\alpha}{n} D_t^n(\zeta) D_t^{\alpha-n}(u_x)
$$

$$
+ \sum_{n=1}^\infty \left[\binom{\alpha}{n} \frac{\partial^n \mu_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) \right] D_t^{\alpha-n}(u) + \sum_{n=1}^\infty \binom{\alpha}{n} \frac{\partial^n \mu_v}{\partial t^n} D_t^{\alpha-n}(v) + \lambda,
$$

$$
\rho_\alpha^0 = \frac{\partial^\alpha \rho}{\partial t^\alpha} + (\rho_v - \alpha D_t(\tau))\frac{\partial^\alpha v}{\partial t^\alpha} - u\frac{\partial^\alpha \rho_u}{\partial t^\alpha} - v\frac{\partial^\alpha \rho_v}{\partial t^\alpha} + \rho_u \frac{\partial^\alpha u}{\partial t^\alpha} - \sum_{n=1}^\infty \binom{\alpha}{n} D_t^n(\zeta) D_t^{\alpha-n}(v_x)
$$

$$
+ \sum_{n=1}^\infty \left[\binom{\alpha}{n} \frac{\partial^n \rho_v}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) \right] D_t^{\alpha-n}(v) + \sum_{n=1}^\infty \binom{\alpha}{n} \frac{\partial^n \rho_u}{\partial t^n} D_t^{\alpha-n}(u) + \beta,
$$

where

$$\lambda = \sum_{n=2}^{\infty} \sum_{m_1+m_2=2}^{n} \sum_{k_1=0}^{m_1} \sum_{r_1=0}^{k_1} \sum_{k_2=0}^{m_2} \sum_{r_2=0}^{k_2} \binom{\alpha}{n} \binom{n}{m_1} \binom{n-m_1}{m_2} \binom{k_1}{r_1} \binom{k_2}{r_2} \frac{1}{k_1! k_2!} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}$$

$$\times (-u)^{r_1} (-v)^{r_2} \frac{\partial^{m_1} u^{k_1-r_1}}{t^{m_1}} \frac{\partial^{m_2} v^{k_2-r_2}}{t^{m_2}} \frac{\partial^{n-m_1-m_2+k_1+k_2} \mu}{\partial^{n-m_1-m_2} \partial u^{k_1} \partial v^{k_2}},$$

and

$$\beta = \sum_{n=2}^{\infty} \sum_{m_1+m_2=2}^{n} \sum_{k_1=0}^{m_1} \sum_{r_1=0}^{k_1} \sum_{k_2=0}^{m_2} \sum_{r_2=0}^{k_2} \binom{\alpha}{n} \binom{n}{m_1} \binom{n-m_1}{m_2} \binom{k_1}{r_1} \binom{k_2}{r_2} \frac{1}{k_1! k_2!} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}$$

$$\times (-u)^{r_1} (-v)^{r_2} \frac{\partial^{m_1} u^{k_1-r_1}}{t^{m_1}} \frac{\partial^{m_2} v^{k_2-r_2}}{t^{m_2}} \frac{\partial^{n-m_1-m_2+k_1+k_2} \rho}{\partial^{n-m_1-m_2} \partial u^{k_1} \partial v^{k_2}}.$$

Theorem 1. *$u = \nu_1(x,t)$ and $v = \nu_2(x,t)$ are invariant solutions of Equations (3) and (4) if and only if*

(i) $u = \nu_1(x,t)$ and $v = \nu_2(x,t)$ satisfy the following expressions:

$$X\nu_1 = 0 \Leftrightarrow \left(\zeta \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial u} + \rho \frac{\partial}{\partial v} \right) \nu_1 = 0,$$

and

$$X\nu_2 = 0 \Leftrightarrow \left(\zeta \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial u} + \rho \frac{\partial}{\partial v} \right) \nu_2 = 0.$$

(ii) $u = \nu_1(x,t)$ and $v = \nu_2(x,t)$ are also solutions of (3) and (4), respectively.

3. Lie Symmetry Analysis and Reduction of Time Fractional GDSS

In this part of the work, we acquire an infinitesimal generator of the Generalized Drinfeld–Sokolov systems by applying Lie point symmetries. Assume that Equation (1) are invariant under one parameter transformations (5). We then have that

$$\begin{aligned} \bar{u}_{\bar{t}}^{\alpha} + a_1 \bar{u} \bar{u}_{\bar{x}} + b_1 \bar{u}_{\bar{x}\bar{x}\bar{x}} + c(\bar{v}^q)_{\bar{x}} &= 0, \\ \bar{v}_{\bar{t}}^{\alpha} + a_2 \bar{u} \bar{v}_{\bar{x}} + b_2 \bar{v}_{\bar{x}\bar{x}\bar{x}} &= 0. \end{aligned} \tag{11}$$

Applying the second prolongation Pr^2 transformation to Equation (1), then using transformation (6), we obtain invariant equations as follows:

$$\begin{aligned} \mu_{\alpha}^{0} + a_1 u_x \mu + a_1 u \mu^x + b_1 \mu^{xxx} + cq(q-1)v^{q-2} \rho v_x &= 0, \\ \rho_{\alpha}^{0} + a_2 v_x \mu + a_2 u \rho^x + c\rho^{xxxx} &= 0. \end{aligned} \tag{12}$$

Putting the values of ρ_{α}^0, $\mu_{\alpha}^0, \rho, \mu$ μ^x, μ^{xxx}, ρ^x, and ρ^{xxx} from Equation (6) into Equation (12) and then isolating the coefficients in partial derivatives with respect to u and v, we obtain a determined system of linear equations stated as

$$\mu_{uu} = \rho_{vv} = \zeta_u = \tau_u = \tau_t = \tau_x = \zeta_u = \tau_v = \tau_x = \mu_v = \tau_{xu} = \rho_{uuu} = \dots = \mu_{xvu} = 0,$$

$$\alpha\tau_t + 3\mu x = 0,$$

$$-a_1 u\tau_v + \alpha c q v^{q-1}\tau_u - 3b_1\tau_{xxv} = 0,$$

$$\alpha a_1 u\tau_t + cv^{q-1}q\rho_u - a_1 u\zeta_x = 0,$$

$$-a_2 u\zeta_x + a_2\mu - b_2\zeta_{xxx} + 3b_2\rho_{xxv} + \alpha\tau_t = 0,$$

$$\binom{\alpha}{n}\left(\frac{\partial^n \rho_v}{\partial t^n}\right)D_{t^{\alpha-n}}v - \frac{1}{n+1}\binom{\alpha}{n}\left[D_{t^{\alpha-n}}vD_{t^{1+n}}\alpha\tau - D_{t^{\alpha-n}}vD_{t^{1+n}}n\tau\right.$$

$$\left. + D_{t^{\alpha-n}}v_x D_{t^n}n\zeta + D_{t^{\alpha-n}}v_x D_{t^n}\zeta\right] = 0. \qquad n = 3, 4, \dots.$$

Solving all these determination equations, we obtain an explicit form of infinitesimal symmetry for Equation (1) as

$$\zeta = -c_1\alpha q x + c_2, \qquad \tau = -3tqc_1, \qquad \mu = 2uqc_1\alpha, \qquad \rho = 4v\alpha c_1,$$

where c_1 and c_2 are arbitrary constants. Thus, we can construct corresponding vector fields:

$$X = (-c_1\alpha q x + c_2)\frac{\partial}{\partial x} - 3tqc_1\frac{\partial}{\partial t} + 2uq\alpha c_1\frac{\partial}{\partial u} + 4v\alpha c_1\frac{\partial}{\partial v}. \tag{13}$$

It is stated that there are two vector fields spanning Equation (1):

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = -\alpha q x\frac{\partial}{\partial x} - 3tq\frac{\partial}{\partial t} + 2uq\alpha\frac{\partial}{\partial u} + 4v\alpha\frac{\partial}{\partial v}. \tag{14}$$

Case 1. *For the symmetry X_1, we can write characteristics equations as follows:*

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0} = \frac{dv}{0}.$$

Solving these characteristic equations, we can easily obtain a trivial solution.

Case 2. *Lastly we focus on symmetry X_2, so we can write characteristic equations as*

$$\frac{dx}{-\alpha q x} = \frac{dt}{-3qt} = \frac{du}{2q\alpha u} = \frac{dv}{4v\alpha}.$$

Solving these characteristic equations, we get both the similarity variable and the similarity transformation as follows:

$$\xi = xt^{\frac{-\alpha}{3}}, \qquad u = t^{\frac{-2\alpha}{3}}f(\xi), \qquad v = t^{\frac{-4\alpha}{3q}}g(\xi), \tag{15}$$

where f and g are arbitrary functions of ξ.

Theorem 2. *The similarity transformations (15) reduce Equation (1) to a nonlinear ODE with a fractional order of the form*

$$\left(P_{\frac{3}{\alpha}}^{1-\frac{5\alpha}{3},\alpha}f\right)(\xi) + a_1 ff_\xi + b_1 f_{\xi\xi\xi} + cqg^{q-1}g_\xi = 0,$$

$$\left(P_{\frac{3}{\alpha}}^{1-\alpha-\frac{4\alpha}{3q},\alpha}g\right)(\xi) + a_2 fg_\xi + b_2 g_{\xi\xi\xi} = 0, \tag{16}$$

where the Erdelyi–Kober (EK) fractional differential operators [37] are given as

$$\left(P_\beta^{\tau,\alpha}f\right)(\xi) := \prod_{j=0}^{n-1}\left(\tau+j-\frac{1}{\beta}\xi\frac{d}{d\xi}\right)\left(K_\beta^{\tau+\alpha,n-\alpha}f\right)(\xi), \tag{17}$$

$$\left(P_\beta^{\tau,\alpha}g\right)(\xi) := \prod_{j=0}^{n-1}\left(\tau+j-\frac{1}{\beta}\xi\frac{d}{d\xi}\right)\left(K_\beta^{\tau+\alpha,n-\alpha}g\right)(\xi), \tag{18}$$

and with the EK Fractional integral operators [38,39] defined as

$$\left(K_\beta^{\tau+\alpha,n-\alpha}f\right)(\xi) := \begin{cases} \frac{1}{\Gamma(\alpha)}\int_1^\infty (u-1)^{\alpha-1}u^{-(\tau+\alpha)}f\left(\xi u^{\frac{1}{\beta}}\right)du, & \alpha>0, \\ f(\xi), & \alpha=0, \end{cases} \tag{19}$$

$$\left(K_\beta^{\tau+\alpha,n-\alpha}g\right)(\xi) := \begin{cases} \frac{1}{\Gamma(\alpha)}\int_1^\infty (u-1)^{\alpha-1}u^{-(\tau+\alpha)}g\left(\xi u^{\frac{1}{\beta}}\right)du, & \alpha>0, \\ g(\xi), & \alpha=0, \end{cases} \tag{20}$$

and

$$n = \begin{cases} [\alpha]+1, & \alpha\in N, \\ \alpha, & \alpha\notin N. \end{cases} \tag{21}$$

Proof. Let $n-1<\alpha<n, n=1,2,3,....$ Using the transformations defined in (15) and the definition of the RL fractional derivative, we have

$$D_t^\alpha u(x,t) = \frac{\partial n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}s^{\frac{-2\alpha}{3}}f(xs^{\frac{-\alpha}{3}})ds\right]. \tag{22}$$

Let $v=\frac{t}{s}$, so $ds=\frac{-t}{v^2}dv$. Equation (22) becomes

$$D_t^\alpha u(x,t) = \frac{\partial n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_1^\infty t^{n-\frac{5\alpha}{3}}(v-1)^{n-1-\alpha}v^{n+1-\frac{5\alpha}{3}}f(\xi v^{\frac{\alpha}{3}})dv\right], \tag{23}$$

$$D_t^\alpha u(x,t) = \frac{\partial n}{\partial t^n}\left[t^{n-\frac{5\alpha}{3}}\left(K_{\frac{3}{\alpha}}^{1-\frac{2\alpha}{3},n-\alpha}f\right)(\xi)\right], \tag{24}$$

$$D_t^\alpha u(x,t) = \frac{\partial n-1}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\frac{5\alpha}{3}}\left(K_{\frac{3}{\alpha}}^{1-\frac{2\alpha}{3},n-\alpha}f\right)(\xi)\right)\right], \tag{25}$$

$$D_t^\alpha u(x,t) = \frac{\partial n-1}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-1-\frac{5\alpha}{3}}\left(n-\frac{5\alpha}{3}-\frac{\alpha}{3}\xi\frac{d}{d\xi}\right)\left(K_{\frac{3}{\alpha}}^{1-\frac{2\alpha}{3},n-\alpha}f\right)(\xi)\right)\right]. \tag{26}$$

Applying the above procedure $n-1$ times, we obtain

$$D_t^\alpha u(x,t) = t^{-\frac{5\alpha}{3}}\left(\prod_{j=0}^{n-1}\left(1-\frac{5\alpha}{3}+j-\frac{\alpha}{3}\xi\frac{d}{d\xi}\right)\left(K_{\frac{3}{\alpha}}^{1-\frac{2\alpha}{3},n-\alpha}f\right)(\xi)\right). \tag{27}$$

Now, by using Equation (17), we can immediately write

$$D_t^\alpha u(x,t) = \frac{\partial n}{\partial t^n}\left[t^{n-\frac{5\alpha}{3}}\left(K_{\frac{3}{\alpha}}^{1-\frac{2\alpha}{3},n-\alpha}f\right)(\xi)\right] = t^{\frac{-5\alpha}{3}}\left(P_{\frac{3}{\alpha}}^{1-\frac{5\alpha}{3},\alpha}f\right)(\xi). \tag{28}$$

Consequently, we prove that the first equation defined in (1) reduces to an ODE of a fractional order:

$$\left(P_{\frac{3}{\alpha}}^{1-\frac{5\alpha}{3},\alpha} f \right)(\xi) + a_1 f f_\xi + b_1 f_{\xi\xi\xi} + cq g^{q-1} g_\xi = 0. \tag{29}$$

Similarly, using transformations (15) and the definition of the RL fractional derivative, we easily reduced the second equation of (1) into an ODE:

$$\left(P_{\frac{3}{\alpha}}^{1-\alpha-\frac{4\alpha}{3q},\alpha} g \right)(\xi) + a_2 f g_\xi + b_2 g_{\xi\xi\xi} = 0. \tag{30}$$

The proof is completed. □

4. Conservation Laws for the Time Fractional GDSS

Before obtaining the conserved vector for the time fractional GDSS, some important definitions should be given. We can start with the RL right-sided time fractional derivative, defined as follows:

$$oD_t^n(oI^{n-\alpha}f), \tag{31}$$

where D_t is a total differential operator with respect to t, and the $oI^{n-\alpha}$ is the right-sided time-fractional integral of $n - \alpha$ [40] given by

$$oI^{n-\alpha}f(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_t^p \frac{f(x,\theta)}{(\theta-t)^{1-n+\alpha}} d\theta, \tag{32}$$

where $n = [\alpha] + 1$.

All solutions of $u(x,t)$ and $v(x,t)$ provide the following conservation equation:

$$D_t(N^t) + D_x(N^x) = 0, \tag{33}$$

where $N^t = N^t(x,t,u,..)$ and $N^x = N^x(x,t,u,..)$.

We use the İbragimov method [29] to construct conservation laws of Equation (1). The formal Langrangian of Equation (1) is defined as the following:

$$\mathcal{L} = \phi(x,t)(D_t^\alpha u + a_1 u u_x + b_1 u_{xxx} + cqv^{q-1}v_x) + \sigma(x,t)(D_t^\alpha v + a_2 u v_x + b_2 v_{xxx}), \tag{34}$$

where σ and ϕ define the new dependent variable of x and t.

Now, we express Euler-Langrangian operators [41] as follows:

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (D_t^\alpha)^* \frac{\partial}{\partial D_t^\alpha u} - D_x \frac{\partial}{\partial u_x} - D_x^3 \frac{\partial}{\partial u_{xxx}}, \tag{35}$$

and

$$\frac{\delta}{\delta v} = \frac{\partial}{\partial v} + (D_t^\alpha)^* \frac{\partial}{\partial D_t^\alpha v} - D_x \frac{\partial}{\partial v_x} - D_x^3 \frac{\partial}{\partial v_{xxx}}, \tag{36}$$

where $(D_t^\alpha)^*$ is the adjoint operator of D_t^α.

Using the Euler Lagrange Equations (35) and (36), we can write the following expression:

$$\frac{\delta\mathcal{L}}{\delta u} = 0, \quad and \quad \frac{\delta\mathcal{L}}{\delta v} = 0. \tag{37}$$

Thus, we have

$$\bar{X} + D_t(\tau)\mathcal{I} + D_x(\zeta)\mathcal{I} = W_1 \frac{\delta}{\delta u} + W_2 \frac{\delta}{\delta v} + D_t N^t + D_x N^x, \tag{38}$$

where $\mathcal{I}$ denotes the identity operator, $\frac{\delta}{\delta u}$ and $\frac{\delta}{\delta v}$ represents the Euler Lagrange operators, C^t and C^x are the Noether operators, and $\bar{X}$ is defined as

$$
\begin{aligned}
\bar{X} = {}& \zeta \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial u} + \rho \frac{\partial}{\partial v} + \mu_\alpha^0 \frac{\partial}{\partial D_t^\alpha u} + \rho_\alpha^0 \frac{\partial}{\partial D_t^\alpha v} + \mu^x \frac{\partial}{\partial u_x} \\
& + \mu^{xxx} \frac{\partial}{\partial u_{xxx}} + \rho^x \frac{\partial}{\partial v_x} + \rho^{xxx} \frac{\partial}{\partial v_{xxx}},
\end{aligned}
\tag{39}
$$

and we can give Lie characteristics functions W for vector field X_2 as

$$
W_1 = \mu - \tau u_t - \zeta u_x = 2uq\alpha + 3tqu_t - \alpha qx u_x,
\tag{40}
$$

and

$$
W_2 = \rho - \tau v_t - \zeta v_x = 4v\alpha + 3tqv_t - \alpha qx v_x.
\tag{41}
$$

Using the RL time fractional derivative in Equation (1), we now write components of conserved vectors [40,41], as follows:

$$
\begin{aligned}
N^t = {}& \tau \mathcal{I} + \sum_{k=0}^{n-1} oD_t^{\alpha-1-k}(W_1) D_t^k \frac{\partial \mathcal{L}}{\partial o D_t^\alpha u} - (-1)^n J\left(W1, D_t^n \frac{\partial \mathcal{L}}{\partial o D_t^\alpha u} \right) \\
& + \sum_{k=0}^{n-1} oD_t^{\alpha-1-k}(W_2) D_t^k \frac{\partial \mathcal{L}}{\partial o D_t^\alpha v} - (-1)^n J\left(W2, D_t^n \frac{\partial \mathcal{L}}{\partial o D_t^\alpha v} \right),
\end{aligned}
\tag{42}
$$

where J is defined as follows:

$$
J(r,s) = \frac{1}{\Gamma(n-1)} \int_0^t \int_t^T \frac{r(\tau,x)s(\nu,x)}{(\nu-\tau)^{\alpha+1-n}} d\nu dt,
\tag{43}
$$

and the explicit form of N^x for Equation (1) is given as

$$
\begin{aligned}
N^x = {}& \zeta \mathcal{I} + W_1 \left[\frac{\partial}{\partial u_x} - D_x \frac{\partial}{\partial u_{xx}} + D_x D_x \frac{\partial}{\partial u_{xxx}} \right] + W_2 \left[\frac{\partial}{\partial v_x} - D_x \frac{\partial}{\partial v_{xx}} + D_x D_x \frac{\partial}{\partial v_{xxx}} \right] \\
& + D_x(W_1) \left[\frac{\partial}{\partial u_{xx}} - D_x \frac{\partial}{\partial u_{xxx}} \right] + D_x(W_2) \left[\frac{\partial}{\partial v_{xx}} - D_x \frac{\partial}{\partial v_{xxx}} \right] + D_x D_x \left[\frac{\partial}{\partial u_{xxx}} \right] \\
& + D_x D_x \left[\frac{\partial}{\partial v_{xxx}} \right].
\end{aligned}
$$

We can now derive the corresponding conserved vectors respectively as follows:

i: $\alpha \in (0,1)$

$$
\begin{aligned}
N_1^t = {}& \phi_0 D_t^{\alpha-1}(2uq\alpha + 3tqu_t - \alpha qx u_x) + J((2uq\alpha + 3tqu_t - \alpha qx u_x), \phi_t) \\
& + \sigma_0 D_t^{\alpha-1}(4v\alpha + 3tqu_t - \alpha qx v_x) + J((4v\alpha + 3tqv_t - \alpha qx v_x), \sigma_t),
\end{aligned}
$$

ii: $\alpha \in (1,2)$

$$
\begin{aligned}
N_2^t = {}& \phi_0 D_t^{\alpha-1}(2uq\alpha + 3tqu_t - \alpha qx u_x) + J((2uq\alpha + 3tqu_t - \alpha qx u_x), \phi_t) \\
& - \phi_t oD_t^{\alpha-2}(2uq\alpha + 3tqu_t - \alpha qx u_x) - J((2uq\alpha + 3tqu_t - \alpha qx u_x), \phi_{tt}) \\
& + \sigma_0 D_t^{\alpha-1}(4v\alpha + 3tqu_t - \alpha qx v_x) + J((4v\alpha + 3tqv_t - \alpha qx v_x), \sigma_t) \\
& - \sigma_t oD_t^{\alpha-2}(4v\alpha + 3tqu_t - \alpha qx v_x) - J((4v\alpha + 3tqu_t - \alpha qx v_x), \sigma_{tt}),
\end{aligned}
$$

and

$$N_1^x = W_1(\phi a_1 u + b_1 \phi_{xx}) + W_2(\phi c q v^{q-1} + \sigma a_2 u + b_2 \sigma_{xx}) - b_1 \phi_x D_x(W_1)$$
$$- b_2 \sigma_x D_x(W_2) + b_1 \phi D_x^2(W_1) + b_2 \sigma D_x^2(W_2),$$

where
$W_1 = 2uq\alpha + 3tqu_t - \alpha qxu_x$ and $W_2 = 4v\alpha + 3tqv_t - \alpha qxv_x$.

5. Series Solutions of Equations (29) and (30)

In this section, we examine the power series solution of the system. This method is more accurate and efficient for obtaining an exact analytical solution. The procedures of the method are given in [42]. We can construct

$$f(\xi) = \sum_{n=0}^{\infty} r_n \xi^n, \qquad g(\xi) = \sum_{n=0}^{\infty} s_n \xi^n, \tag{44}$$

$$f_\xi = \sum_{n=1}^{\infty} n r_n \xi^{n-1}, \qquad f_{\xi\xi\xi} = \sum_{n=3}^{\infty} n(n-1)(n-2) r_n \xi^{n-3}, \tag{45}$$

and

$$g_\xi = \sum_{n=1}^{\infty} n s_n \xi^{n-1}, \qquad g_{\xi\xi\xi} = \sum_{n=3}^{\infty} n(n-1)(n-2) s_n \xi^{n-3}. \tag{46}$$

Substituting Equations (44)–(46) into Equations (29) and (30), we have

$$\sum_{n=0}^{\infty} \frac{\Gamma(2 - \frac{2\alpha}{3} + \frac{n\alpha}{3})}{\Gamma(2 - \frac{5\alpha}{3} + \frac{n\alpha}{3})} r_n \xi^n + a_1 \sum_{n=0}^{\infty} r_n \xi^n \sum_{n=0}^{\infty} (n+1) r_{n+1} \xi^n$$
$$+ b_1 \sum_{n=0}^{\infty} (n+3)(n+2)(n+1) r_{n+3} \xi^n + cq \left(\sum_{n=0}^{\infty} s_n \xi^n \right)^{q-1} \sum_{n=0}^{\infty} (n+1) s_{n+1} \xi^n = 0, \tag{47}$$

and

$$\sum_{n=0}^{\infty} \frac{\Gamma(2 - \frac{4\alpha}{3q} + \frac{n\alpha}{3})}{\Gamma(2 - \alpha - \frac{4\alpha}{3q} + \frac{n\alpha}{3})} s_n \xi^n + a_2 \sum_{n=0}^{\infty} r_n \xi^n \sum_{n=0}^{\infty} (n+1) s_{n+1} \xi^n$$
$$+ b_2 \sum_{n=0}^{\infty} (n+3)(n+2)(n+1) s_{n+3} \xi^n = 0. \tag{48}$$

When $n = 0$ in Equations (47) and (48), we obtain the following coefficients:

$$r_3 = \frac{1}{6b_1} \left(\frac{\Gamma(2 - \frac{2\alpha}{3})}{\Gamma(2 - \frac{5\alpha}{3})} r_0 + a_1 r_0 r_1 + cq s_1^{q-1} s_0 \right), \tag{49}$$

and

$$s_3 = \frac{1}{6b_2} \left(\frac{\Gamma(2 - \frac{4\alpha}{3q})}{\Gamma(2 - \alpha - \frac{4\alpha}{3q})} s_0 + a_2 r_0 s_1 \right). \tag{50}$$

When $n \geqslant 1$, we have the following coefficients

$$r_{n+3} = \frac{1}{b_1(n+3)(n+2)(n+1)} \left(\frac{\Gamma(2 - \frac{2\alpha}{3} + \frac{n\alpha}{3})}{\Gamma(2 - \frac{5\alpha}{3} + \frac{n\alpha}{3})} r_n + a_1 \sum_{k=0}^{n} (n-k+1) r_{n-k+1} r_k \right.$$
$$\left. + cq \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \dots \sum_{k_{q-1}=0}^{k_{q-2}} (n-k+1) s_{n-k+1} s_{k_1-k_2} \dots s_{k_{q-2}-k_{q-1}} s_{k_{q-1}} \right), \tag{51}$$

and

$$s_{n+3} = \frac{1}{b_2(n+3)(n+2)(n+1)} \left(\frac{\Gamma(2 - \frac{4\alpha}{3q} + \frac{n\alpha}{3})}{\Gamma(2 - \alpha - \frac{4\alpha}{3q} + \frac{n\alpha}{3})} s_n + a_2 \sum_{k=0}^{n} (n-k+1)s_{n-k+1}r_k \right). \tag{52}$$

Thus, the coefficients r_n ($n \geqslant 1$) and s_n ($n \geqslant 1$) can be calculated for Equation (44) as arbitrary constants. This shows that there are power series solutions to the Equations (29) and (30). We then acquire the explicit power series solutions of Equation (1) (Figures 1 and 2) as

$$u(x,t) = r_0 t^{\frac{-2\alpha}{3}} + r_1 xt^{-\alpha} + r_2 x^2 t^{\frac{-4\alpha}{3}} + \frac{1}{6b_1} \left(\frac{\Gamma(2 - \frac{2\alpha}{3})}{\Gamma(2 - \frac{5\alpha}{3})} r_0 + a_1 r_0 r_1 + cq s_0^{q-1} s_1 \right) x^3 t^{\frac{-5\alpha}{3}}$$

$$+ \sum_{n=1}^{\infty} \frac{1}{b_1(n+3)(n+2)(n+1)} \left(\frac{\Gamma(2 - \frac{2\alpha}{3} + \frac{n\alpha}{3})}{\Gamma(2 - \frac{5\alpha}{3} + \frac{n\alpha}{3})} r_n + a_1 \sum_{k=0}^{n} (n-k+1)r_{n-k+1}r_k \right. \tag{53}$$

$$+ cq \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{q-1}=0}^{k_{q-2}} (n-k+1)s_{n-k+1}s_{k_1-k_2}\cdots s_{k_{q-2}-k_{q-1}}s_{k_{q-1}} \right) x^{n+3} t^{\frac{-\alpha(n+5)}{3}},$$

$$v(x,t) = s_0 t^{\frac{-4\alpha}{3q}} + s_1 xt^{\frac{-\alpha(q+4)}{3q}} + s_2 x^2 t^{\frac{-\alpha(2q+4)}{3q}} + \frac{1}{6b_2} \left(\frac{\Gamma(2 - \frac{4\alpha}{3q})}{\Gamma(2 - \alpha - \frac{4\alpha}{3q})} s_0 + a_2 r_0 s_1 \right) x^3 t^{\frac{-\alpha(3q+4)}{3q}}$$

$$+ \sum_{n=1}^{\infty} \frac{1}{b_2(n+3)(n+2)(n+1)} \left(\frac{\Gamma(2 - \frac{4\alpha}{3q} + \frac{n\alpha}{3})}{\Gamma(2 - \alpha - \frac{4\alpha}{3q} + \frac{n\alpha}{3})} s_n + a_2 \sum_{k=0}^{n} (n-k+1)s_{n-k+1}r_k \right) \tag{54}$$

$$\times x^{n+3} t^{\frac{-\alpha(nq+3q+4)}{3q}}.$$
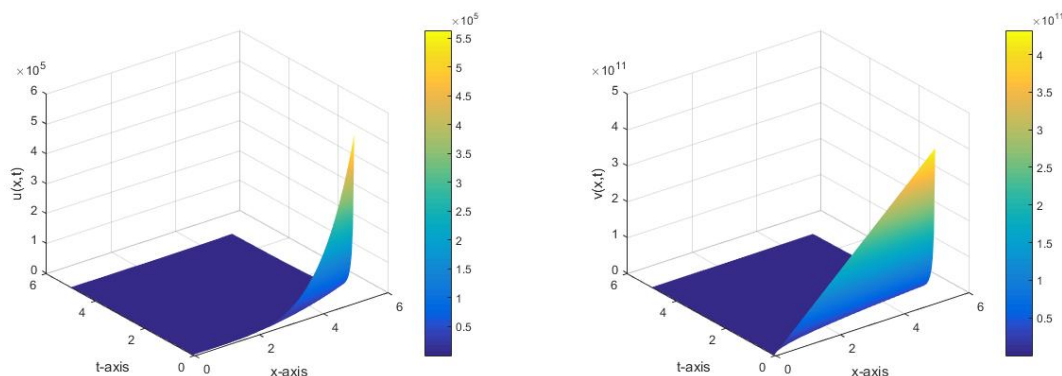
Figure 1. 3-D plots of Equations (53) and (54) with $r_0 = 1$, $r_1 = 2$, $r_2 = 3$, $s_0 = s_1 = s_2 = 2$, $b_1 = b_2 = 1$, $a_1 = a_2 = 2$, $q = 2$, $c = 2$ and $\alpha = 0.5$.
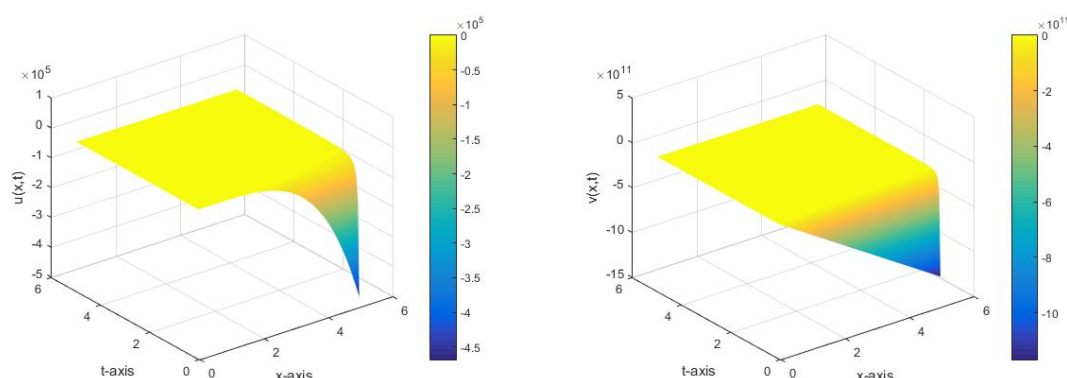
Figure 2. 3-D plots of Equations (53) and (54) with $r_0 = 2$, $r_1 = 2$, $r_2 = 2$, $s_0 = s_1 = s_2 = 1$, $b_1 = 1$, $b_2 = 6$, $a_1 = -6$, $a_2 = -2$, $q = 1$, $c = -6$ and $\alpha = 0.75$.

6. Convergence Analysis of the Power Series Solution

In this part of the work, we will prove that the power series solutions (53) and (54) are convergent. Considering Equations (51) and (52), we can write

$$
\mid r_{n+3} \mid \leq \left(\frac{\mid \Gamma(2 - \frac{2\alpha}{3} + \frac{n\alpha}{3}) \mid}{\mid \Gamma(2 - \frac{5\alpha}{3} + \frac{n\alpha}{3}) \mid} \mid r_n \mid + \sum_{k=0}^{n} \mid r_{n-k+1} \mid \mid r_k \mid \right.
$$
$$
\left. + \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{q-1}=0}^{k_{q-2}} \mid s_{n-k+1} \mid \mid s_{k_1-k_2} \mid \cdots \mid s_{k_{q-2}-k_{q-1}} \mid \mid s_{k_{q-1}} \mid \right),
\tag{55}
$$

and

$$
\mid s_{n+3} \mid \leq \left(\frac{\mid \Gamma(2 - \frac{4\alpha}{3q} + \frac{n\alpha}{3}) \mid}{\mid \Gamma(2 - \alpha - \frac{4\alpha}{3q} + \frac{n\alpha}{3}) \mid} \mid s_n \mid + \sum_{k=0}^{n} \mid s_{n-k+1} \mid \mid r_k \mid \right).
\tag{56}
$$

Using the properties of Γ, we can easily show that $\frac{\mid \Gamma(2 - \frac{2\alpha}{3} + \frac{n\alpha}{3}) \mid}{\mid \Gamma(2 - \frac{5\alpha}{3} + \frac{n\alpha}{3}) \mid} \leq 1$ and $\frac{\mid \Gamma(2 - \frac{4\alpha}{3q} + \frac{n\alpha}{3}) \mid}{\mid \Gamma(2 - \alpha - \frac{4\alpha}{3q} + \frac{n\alpha}{3}) \mid} \leq 1$.

Therefore, we can write

$$
\mid r_{n+3} \mid \leq M \left(\mid r_n \mid + \sum_{k=0}^{n} \mid r_{n-k+1} \mid \mid r_k \mid \right.
$$
$$
\left. + \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{q-1}=0}^{k_{q-2}} \mid s_{n-k+1} \mid \mid s_{k_1-k_2} \mid \cdots \mid s_{k_{q-2}-k_{q-1}} \mid \mid s_{k_{q-1}} \mid \right),
\tag{57}
$$

and

$$
\mid s_{n+3} \mid \leq N \left(\mid s_n \mid + \sum_{k=0}^{n} \mid s_{n-k+1} \mid \mid r_k \mid \right),
\tag{58}
$$

where $M = max\left(\mid \frac{1}{b_1} \mid, \mid \frac{a_1}{b_1} \mid, \mid \frac{cq}{b_1} \mid \right)$ and $N = max\left(\mid \frac{1}{b_2} \mid, \mid \frac{a_2}{b_2} \mid \right)$ for an arbitrary n.

We can now define another power series form as

$$
K(\xi) = \sum_{n=0}^{\infty} y_n \xi^n \qquad and \qquad L(\xi) = \sum_{n=0}^{\infty} z_n \xi^n,
\tag{59}
$$

and let $y_i = |r_i|$ and $z_i = |s_i|$, $i = 0, 1, \dots$. We can then easily obtain

$$
\begin{aligned}
y_{n+3} \leq M\Bigg(& y_n + \sum_{k=0}^{n} y_{n-k+1} y_k \\
& + \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{q-1}=0}^{k_{q-2}} z_{n-k+1} z_{k_1-k_2} \cdots z_{k_{q-2}-k_{q-1}} z_{k_{q-1}} \Bigg),
\end{aligned}
\tag{60}
$$

and

$$
z_{n+3} \leq N\left(z_n + \sum_{k=0}^{n} z_{n-k+1} y_k \right),
\tag{61}
$$

Thus, it is obvious that $|y_n| \leq r_n$ and $|z_n| \leq s_n$ for $n = 0, 1, 2, \dots$. This also confirms that the series Equation (59) are the majorant series of Equation (44). We now have to prove that the series $K(\xi)$ and $L(\xi)$ have positive radius of convergence. By elementary calculations, we have the following:

$$
\begin{aligned}
K(\xi) = y_0 + y_1 \xi + y_2 \xi^2 + y_3 \xi^3 + M\Bigg(& \sum_{n=0}^{\infty} y_n + \sum_{n=0}^{\infty} \sum_{k=0}^{n} y_{n-k+1} y_k \\
& + \sum_{n=0}^{\infty} \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{q-1}=0}^{k_{q-2}} z_{n-k+1} z_{k_1-k_2} \cdots z_{k_{q-2}-k_{q-1}} z_{k_{q-1}} \Bigg) \xi^{n+3},
\end{aligned}
\tag{62}
$$

and

$$
L(\xi) = z_0 + z_1 \xi + z_2 \xi^2 + z_3 \xi^3 + N\left(\sum_{n=0}^{\infty} z_n + \sum_{n=0}^{\infty} \sum_{k=0}^{n} z_{n-k+1} y_k \right) \xi^{n+3}.
\tag{63}
$$

We can take into account an implicit functional system with respect to the independent variable ξ as follows:

$$
\mathcal{K}(\xi, K) = K - y_0 - y_1 \xi - y_2 \xi^2 - y_3 \xi^3 - M\left(\xi^2 (K - y_0) + \xi^2 K (K - y_0) + \xi^2 L (L - z_0)^{q-2} \right),
\tag{64}
$$

and

$$
\mathcal{L}(\xi, L) = L - z_0 - z_1 \xi - z_2 \xi^2 - z_3 \xi^3 - N\left(\xi^2 (L - z_0) + \xi^2 L (K - y_0) \right).
\tag{65}
$$

It is clear that $\mathcal{K}$ and $\mathcal{L}$ are analytical in the neighborhood of $(0, y_0)$ and $(0, z_0)$. Therefore, $\mathcal{K}(0, y_0) = 0$, $\mathcal{L}(0, z_0) = 0$, and $\frac{\partial}{\partial K}\mathcal{K}(0, y_0) \neq 0$, $\frac{\partial}{\partial L}\mathcal{L}(0, z_0) \neq 0$ According to the implicit function theorem [43], the convergence is proved.

7. Conclusions

In this paper, we used a Lie point symmetry method in order to reduce a time fractional generalized coupled Drinfeld-Sokolov system to a time fractional coupled ODE system with the aid of the Riemann Liouville derivative and the fractional EK differential operator. By using the İbragimov conservation theorem, we obtained conservation vectors of the system. We then acquired explicit exact solutions of the reduced time fractional coupled ODE system by using a power series expansion method and proved that the series solutions are convergent.

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