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The Comparative Study for Solving Fractional-Order Fornberg–Whitham Equation via ρ-Laplace Transform

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Abstract: In this article, we also introduced two well-known computational techniques for solving the time-fractional Fornberg–Whitham equations. The methods suggested are the modified form of the variational iteration and Adomian decomposition techniques by ρ-Laplace. Furthermore, an illustrative scheme is introduced to verify the accuracy of the available methods. The graphical representation of the exact and derived results is presented to show the suggested approaches reliability. The comparative solution analysis via graphs also represented the higher reliability and accuracy of the current techniques.

Keywords: ρ-Laplace variational iteration method; ρ-Laplace decomposition method; partial differential equation; caputo operator; fractional Fornberg–Whitham equation (FWE)

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1. Introduction

With engineering and science development, non-linear evolution models have been analyzed as the problems to define physical phenomena in plasma waves, fluid mechanics, chemical physics, solid-state physics, etc. For the last few years, therefore, a lot of interest has been paid to the result (both numerical and analytical) of these significant models [1–4]. Different methods are available in the literature for the approximate and exact results of these models. In current years, fractional calculus (FC) applied in many phenomena in applied sciences, fluid mechanics, physics and other biology can be described as very effective using mathematical tools of FC. The fractional derivatives have occurred in many applied sciences equations such as reaction and diffusion processes, system identification, velocity signal analysis, relaxation of damping behaviour fabrics and creeping of polymer composites [5–8].

The investigation of non-linear wave models and their application is significant in different areas of engineering. Travelling wave notions are between the most attractive results for non-linear fractional-order partial differential equations (NLFPDEs). NLFPDEs are usually identified as mechanical processes and complex physical. Therefore, it is important to get exact results for non-linear time-fractional partial differential equations [9–12]. Overall, travelling wave results are between the exciting forms of products for NFPDEs. On the other hand, other NLFPDEs, such as the Camassa–Holm or the Kortewegde–Vries equations, have been well-known to have some moving wave solutions. These are non-linear multi-directional dispersive waves in shallow water design problems [13–16].

The FWE study is of crucial significance in different areas of mathematical physics. The FWEs [15,16] is defined as

$$D_{\Im}^{\delta}\mu - D_{\varphi\varphi\Im}\mu + D_{\varphi}\mu = \mu D_{\varphi\varphi\varphi}\mu - \mu D_{\varphi}\mu + 3D_{\varphi}\mu D_{\varphi\varphi}\mu. \quad (1)$$

The quantities performance of wave deformation, a non-linear dispersive wave model, is shown in the investigation. The FWE is presented as a mathematical model for limiting wave heights and wave breaks, allowing peakon results as a numerical model. In 1978, Fornberg and Whitham achieved a measured outcome of the form $\mu(\varphi, \eta) = Ce^{\left(\frac{-\varphi}{2} - \frac{4\Im}{3}\right)}$, where C is constant. The investigation of FWEs has been carried out by several analytical and numerical techniques, such as Adomian decomposition transform method [17], variational iteration technique [18], Lie Symmetry [19], new iterative method [20], differential transformation method [21], homotopy analysis transformation technique [22] and homotopy-perturbation technique [23].

Recently, Abdeljawad and Fahd [24] introduced the Laplace transformation of the fractional-order Caputo derivatives. We suggested a new iterative technique with ρ-Laplace transformation to investigate fractional-order ordinary and partial differential equations with fractional-order Caputo derivative. We apply this novel method for solving many fractional-order differential equations such as linear and non-linear diffusion equation, fractional-order Zakharov–Kuznetsov equation and Fokker–Planck equations. We analyzed the impact of δ and ρ in the process. The Variational iteration method (VIM) was first introduced by He [25,26] and was effectively implemented to the autonomous ordinary differential equation in [27], to non-linear polycrystalline solids [28], and other areas. Similarly, this technique is modified with ρ-Laplace transformation, so the modified method is called the ρ-Laplace variational iteration method. Many types of differential equations and partial differential equations have solved VITM. For example, this technique is analyzed for solving the time-fractional differential equation (FDEs) in [27]. In [28], this technique is applied to solve non-linear oscillator models. Compared to Adomian's decomposition process, VITM solves the problem without the need to compute Adomian's polynomials. This scheme provides a quick result to the equation, whereas the [29] mesh point techniques provide an analytical solution. This method can also be used to get a close approximation of the exact result. G. Adomian, an American mathematician, developed the Adomian decomposition technique. It focuses on finding series-like results and decomposing the non-linear operator into a sequence, with the terms presently computed using Adomian polynomials [30]. This method is modified with ρ-Laplace transform, so the modified approach is the ρ-Laplace decomposition method. This technique is used for the non-homogeneous FDEs [31–36].

This paper has implemented the ρ-Laplace variational iteration method and ρ-Laplace decomposition method to solve the time-fractional Fornberg–Whitham equations with the Caputo fractional derivative operator. The ρ-LDM and ρ-LVIM achieve the approximate results in the form of series results.

2. Basic Definitions

In this section, the fractional generalized derivative, the fractional generalized integral, the Mittag-Lefller function the ρ-Laplace transform have been discussed.

Definition 1. *The generalized fractional-order integral δ of a continuous function $f : [0, +\infty] \to R$ is expressed as [24]*

$$(I^{\delta,\rho} f)(\zeta) = \frac{1}{\Gamma(\delta)} \int_0^{\zeta} \left(\frac{\zeta^{\rho} - s^{\rho}}{\rho} \right)^{\delta-1} \frac{f(s)ds}{s^{1-\rho}},$$

the gamma function denote by Γ, $\rho > 0$, $\zeta > 0$ and $0 < \delta < 1$.

Definition 2. *The generalized fractional-order derivative of δ of a continuous function $f : [0, +\infty] \to R$ is given as [24].*

$$(D^{\delta,\rho} f)(\zeta) = (I^{1-\delta,\rho} f)(\zeta) = \frac{1}{\Gamma(1-\delta)} \left(\frac{d}{d\zeta}\right) \int_0^{\zeta} \left(\frac{\zeta^\rho - s^\rho}{\rho}\right)^{-\delta} \frac{f(s)ds}{s^{1-\rho}}.$$

where define the gamma function Γ, $\rho > 0$, $\zeta > 0$ and $0 < \delta < 1$.

Definition 3. *The Caputo fractional-order derivative δ of a continuous function $f : [0, +\infty] \to R$ is expressed as [24]*

$$(D^{\delta,\rho} f)(\zeta) = \frac{1}{\Gamma(1-\delta)} \int_0^{\zeta} \left(\frac{\zeta^\rho - s^\rho}{\rho}\right)^{-\delta} \beta^n \frac{f(s)ds}{s^{1-\rho}}.$$

where $n = 1$, $\rho > 0$, $\zeta > 0$, $\beta = \zeta^{1-\rho}\frac{d}{d\zeta}$ and $0 < \delta < 1$.

Definition 4. *The ρ-Laplace transformation of a continuous function $f : [0, +\infty] \to R$ is given as [24]*

$$L_\rho\{f(\zeta)\} = \int_0^{\infty} e^{-s\frac{\zeta^\rho}{\rho}} f(\zeta) \frac{d\zeta}{\zeta^{1-\rho}}.$$

The Caputo generalized fractional-order ρ-Laplace transform derivative of a continuous function f is defined by [24].

$$L_\rho\{D^{\delta,\rho} f(\zeta)\} = s^\delta L_\rho\{f(\zeta)\} - \sum_{k=0}^{n-1} s^{\delta-k-1}(I^{\delta,\rho}\beta^n f)(0) \;\; n = 1.$$

3. The General Methodology of ρ-LDM

The ρ-LDM is a combination of the Laplace decomposition method and the ρ-Laplace transformation. In this section, we solve the ρ-LDM solution of fractional partial differential equation. The main steps of this method are described as follows:

$$D_{\Im}^{\delta,\rho}\omega(\varphi,\Im) + \bar{\mathcal{L}}(\varphi,\Im) + \mathcal{N}(\varphi,\Im) - \mathcal{H}(\varphi,\Im) = 0, \quad 0 < \delta \le 1, \tag{2}$$

where $\bar{L}$ and $\mathcal{N}$ are linear and nonlinear functions, $\mathcal{H}$ is the sources function.

The initial condition is

$$\omega(\varphi, 0) = f(\varphi), \tag{3}$$

Apply ρ-Laplace transform to Equation (2),

$$L_\rho[D_{\Im}^{\delta,\rho}\omega(\varphi,\Im)] + L_\rho[\bar{\mathcal{L}}(\varphi,\Im) + \mathcal{N}(\varphi,\Im) - \mathcal{H}(\varphi,\Im)] = 0. \tag{4}$$

Applying the ρ-Laplace transformation differentiation property, we get

$$L_\rho[\omega(\varphi,\Im)] = \frac{1}{s}\omega(\varphi,0) + \frac{1}{s^\delta} L_\rho[\mathcal{H}(\varphi,\Im)] - \frac{1}{s^\delta} L_\rho\{\bar{\mathcal{L}}(\varphi,\Im) + \mathcal{N}(\varphi,\Im)\}]. \tag{5}$$

ρ-LDM solution of infinite series $\omega(\varphi,\Im)$,

$$\omega(\varphi,\Im) = \sum_{j=0}^{\infty} \omega_m(\varphi,\Im). \tag{6}$$

The $\mathcal{N}$ is the nonlinear term defined as

$$\mathcal{N}(\varphi,\Im) = \sum_{j=0}^{\infty} \mathcal{A}_m. \tag{7}$$

So the with the help of Adomian polynomial we can define the nonlinear terms

$$\mathcal{A}_m = \frac{1}{j!}\left[\frac{\partial^m}{\partial\lambda^m}\left\{\mathcal{N}\left(\sum_{k=0}^{\infty}\lambda^k\omega_k\right)\right\}\right]_{\lambda=0}. \tag{8}$$

Putting Equations (6) and (7) into (5), we get

$$L_\rho\left[\sum_{j=0}^{\infty}\omega_m(\varphi,\Im)\right] = \frac{1}{s}\omega(\varphi,0) + \frac{1}{s^\delta}S\{\mathcal{H}(\varphi,\Im)\} - \frac{1}{s^\delta}L_\rho\left\{\bar{\mathcal{L}}(\sum_{j=0}^{\infty}\omega_m) + \sum_{j=0}^{\infty}\mathcal{A}_m\right\}. \tag{9}$$

Using the inverse ρ-Laplace transform with Equation (9),

$$\sum_{j=0}^{\infty}\omega_m(\varphi,\Im) = L_\rho^{-1}\left[\frac{1}{s}\omega(\varphi,0) + \frac{1}{s^\delta}L_\rho\{\mathcal{H}(\varphi,\Im)\} - \frac{1}{s^\delta}L_\rho\left\{\bar{\mathcal{L}}\left(\sum_{j=0}^{\infty}\omega_m\right) + \sum_{j=0}^{\infty}\mathcal{A}_m\right\}\right]. \tag{10}$$

we define the next terms,

$$\omega_0(\varphi,\Im) = L_\rho^{-1}\left[\frac{1}{s}\omega(\varphi,0) + \frac{1}{s^\delta}L_\rho\{\mathcal{H}(\varphi,\Im)\}\right], \tag{11}$$

$$\omega_1(\varphi,\Im) = -L_\rho^{-1}\left[\frac{1}{s^\delta}L_\rho\{\bar{\mathcal{L}}_1(\omega_0) + \mathcal{A}_0\}\right].$$

For $m \geq 1$, is expressed as

$$\omega_{j+1}(\varphi,\Im) = -L_\rho^{-1}\left[\frac{1}{s^\delta}L_\rho\{\bar{\mathcal{L}}(\omega_m) + \mathcal{A}_m\}\right].$$

4. Convergence Analysis

Theorem 1. *[37] (Uniqueness theorem) Equation has a unique solution whenever* $0 < \varepsilon < 1$ *where* $\varepsilon = \frac{(\mathrm{h}_1+\mathrm{h}_2+\mathrm{h}_3)\Im^{\delta+1}}{(\delta-1)!}$.

Theorem 2. *[37] (Convergence Theorem) The series solution (11) and (12) of the problem (3) using ρ-LTADM and ρ-LTVIM converges if* $0 < \varepsilon < 1$.

Proof. Let S_ℓ be the mth partial sum, i.e., $S_\ell = \sum_{j=0}^{m}\omega_\ell(\varphi,\Im)$. We shall prove that S_ℓ is a Cauchy sequence in Banach space E. By using a new formulation of Adomian polynomials we get [37]

$$R(S_\ell) = \widehat{A}_\ell + \sum_{j=0}^{m-1}\widehat{A_j}$$

$$\aleph(S_\ell) = \widehat{A_\ell} + \sum_{n=0}^{m-1}\widehat{A_n}$$

$$
\begin{aligned}
||S_\ell - S_{m-1}|| &= \max_{\Im\in I}|S_\ell - S_{m-1}| = \max_{\Im\in I}\left|\sum_{j=n+1}^{m}\widehat{\omega_j}(\varphi,\Im)\right|, \quad j=0,1,2\cdots \\
&\le \max_{\Im\in I}\left|\begin{array}{c} L_\rho^{-1}\{\frac{1}{s^\delta}L_\rho\{[\sum_{j=n+1}^{m}\mathrm{k}[\omega_{j-1}(\varphi,\Im)]]\} \\ +L_\rho^{-1}\{\frac{1}{s^\delta}L_\rho\{[\sum_{j=n+1}^{m}M[\omega_{j-1}(\varphi,\Im)]]\} \\ +L_\rho^{-1}\{\frac{1}{s^\delta}L_\rho\{[\sum_{j=n+1}^{m}[A_{j-1}(\varphi,\Im)]]\} \end{array}\right|, \\
&\le \max_{\Im\in I}\left|\begin{array}{c} L_\rho^{-1}\{\frac{1}{s^\delta}L_\rho\{[\sum_{j=n+1}^{m}\mathrm{k}[\omega_{j}(\varphi,\Im)]]\} \\ +L_\rho^{-1}\{\frac{1}{s^\delta}L_\rho\{[\sum_{j=n+1}^{m}M[\omega_{j}(\varphi,\Im)]]\} \\ +L_\rho^{-1}\{\frac{1}{s^\delta}L_\rho\{[\sum_{j=n+1}^{m}[A_{j}(\varphi,\Im)]]\} \end{array}\right|, \\
&\le \max_{\Im\in I}\left|\begin{array}{c} L_\rho^{-1}\{\frac{1}{s^\delta}L_\rho\{[\sum_{j=n+1}^{m}\mathrm{k}[S_{m_1-1}-S_{m_2-1}]]\} \\ +L_\rho^{-1}\{\frac{1}{s^\delta}L_\rho\{[\sum_{j=n+1}^{m}M[S_{m_1-1}-S_{m_2-1}]]\} \\ +L_\rho^{-1}\{\frac{1}{s^\delta}L_\rho\{[\sum_{j=n+1}^{m}[S_{m_1-1}-S_{m_2-1}]]\} \end{array}\right|, \\
&\le \max_{\Im\in I}\left|\begin{array}{c} L_\rho^{-1}\{\frac{1}{s^\delta}L_\rho\{[\mathrm{k}[S_{m_1-1}-S_{m_2-1}]]\} \\ +L_\rho^{-1}\{\frac{1}{s^\delta}L_\rho\{[M[S_{m_1-1}-S_{m_2-1}]]\} \\ +L_\rho^{-1}\{\frac{1}{s^\delta}L_\rho\{[[S_{m_1-1}-S_{m_2-1}]]\} \end{array}\right|, \\
&\le \mathrm{k}_1\max_{\Im\in I}\left|L_\rho^{-1}\{\frac{1}{s^\delta}L_\rho\{[S_{m_1-1}-S_{m_2-1}]\}\right|, \\
&+ \mathrm{k}_2\max_{\Im\in I}\left|L_\rho^{-1}\{\frac{1}{s^\delta}L_\rho\{[S_{m_1-1}-S_{m_2-1}]\}\right|, \\
&+ \mathrm{k}_3\max_{\Im\in I}\left|L_\rho^{-1}\frac{1}{s^\delta}L_\rho\{[S_{m_1-1}-S_{m_2-1}]\}\right|, \\
&= \frac{(\mathrm{k}_1+\mathrm{k}_2+\mathrm{k}_3)\Im^{\delta-1}}{(\delta-1)!}||S_{m_1-1}-S_{m_2-1}||.
\end{aligned}
$$

Letting $m_1 = m_2 + 1$, we get

$$||S_{m_2+1} - S_{m_2}|| \le \varepsilon||S_{m_2} - S_{m_2-1}|| \le \varepsilon^2||S_{m_2-1} - S_{m_2-2}|| \le \cdots \le \varepsilon^{m_2}||S_1 - S_0||,$$

where $\varepsilon = \frac{(\mathrm{k}_1+\mathrm{k}_2+\mathrm{k}_3)\Im^{\delta-1}}{(\delta-1)!}$ similarly, we have from the triangle inequality we get

$$
\begin{aligned}
&||S_{m_1-1} - S_{m_2-1}|| \le ||S_{m_1+1} - S_{m_2}|| + ||S_{m_1+2} - S_{m_2+1}|| + \cdots + ||S_{m_1} - S_{m_1-1}||, \\
&\le \left[\varepsilon^{m_2} + \varepsilon^{m_2+1} + \cdots + \varepsilon^{m_1-1}\right] \le ||S_1 + S_0||, \\
&\le \varepsilon^{m_2}(\frac{1-\varepsilon^{m_1-m_2}}{\varepsilon})||\omega_1||.
\end{aligned}
$$

Since $0 < \varepsilon < 1$ we have $1 - \varepsilon^{m_1-m_2} < 1$

$$||S_{m_1} + S_{m_2}|| \le \frac{\varepsilon^{m_2}}{1-\varepsilon} \le \max_{\Im\in I}||\omega||.$$

However $|\omega| < \infty$ so, as $m_2 \to \infty$ then $||S_{m_1} - S_{m_2}|| \to 0$, hence S_{m_1} is a Cauchy sequence, the series $\sum_{m_1=0}^{\infty}\omega_{m_1}$ converges and the proof is complete. □

Theorem 3. *[37] (Error estimate) The maximum absolute error of the series solution can be given the following formula*

$$\max_{\Im\in I}\left|\omega(\varphi,\Im) - \sum_{\ell=1}^{\infty}\omega_\ell(\varphi,\Im)\right| \le \frac{\varepsilon^{m_2}}{1-\varepsilon}\max_{\Im\in I}||\omega_1||.$$

5. The General Methodology of ρ-Laplace Variational Iteration Method

In this section we show the general methodology of the ρ-Laplace variational iteration method solution for fractional partial differential equations.

$$D_{\Im}^{\delta,\rho}\omega(\varphi,\Im)+\bar{\mathcal{L}}(\varphi,\Im)+\mathcal{N}(\varphi,\Im)-\mathcal{H}(\varphi,\Im)=0,\quad 0<\delta\leq 1, \tag{12}$$

with the initial condition

$$\omega(\varphi,0)=f(\varphi), \tag{13}$$

The using ρ-Laplace transformation to Equation (12),

$$L_\rho[D_{\Im}^{\delta,\rho}\omega(\varphi,\Im)]+L_\rho[\bar{\mathcal{L}}(\varphi,\Im)+\mathcal{N}(\varphi,\Im)-\mathcal{H}(\varphi,\Im)]=0. \tag{14}$$

Applying the differentiation property of ρ-Laplace transform, we get

$$s^\delta L_\rho[\omega(\varphi,\Im)]-s^{\delta-1}\omega(\varphi,0)=-L_\rho\big[\bar{\mathcal{L}}(\varphi,\Im)+\mathcal{N}(\varphi,\Im)-\mathcal{H}(\varphi,\Im)\big]. \tag{15}$$

The Lagrange multiplier is used in the iterative method

$$\begin{aligned} L_\rho[\omega_{j+1}(\varphi,\Im)]=&L_\rho[\omega_j(\varphi,\Im)]+\lambda(s)\Big[s^\delta L_\rho[\omega_j(\varphi,\Im)]-s^{\delta-1}\omega_j(\varphi,0)\\ &-L_\rho\{\bar{\mathcal{L}}(\varphi,\Im)+\mathcal{N}(\varphi,\Im)\}-L_\rho[\mathcal{H}(\varphi,\Im)]\Big]. \end{aligned} \tag{16}$$

The Lagrange multiplier is

$$\lambda(s)=-\frac{1}{s^\delta}, \tag{17}$$

using inverse ρ-Laplace transform L^{-1}, Equation (16), we get

$$\omega_{j+1}(\varphi,\Im)=\omega_j(\varphi,\Im)-L_\rho^{-1}\left[\frac{1}{s^\delta}\left[-L_\rho\{\bar{\mathcal{L}}(\varphi,\Im)+\mathcal{N}(\varphi,\Im)\}\right]-L_\rho[\mathcal{H}(\varphi,\Im)]\right], \tag{18}$$

the initial value can be defined as

$$\omega_0(\varphi,\Im)=L_\rho^{-1}\left[\frac{1}{s^\delta}\left\{s^{\delta-1}\omega(\varphi,0)\right\}\right]. \tag{19}$$

6. Implementation of Techniques

We now proceed to derive an approximate solution to the time-fractional nonlinear FW equations using suggested techniques with generalized Caputo fractional derivative.

6.1. Problem

Consider the time-fractional nonlinear FWE is given as

$$D_{\Im}^{\delta,\rho}\omega-D_{\varphi\varphi\Im}\omega+D_\varphi\omega=\omega D_{\varphi\varphi\varphi}\omega-\omega D_\varphi\omega+3D_\varphi\omega D_{\varphi\varphi}\omega,\quad 0<\delta\leq 1, \tag{20}$$

the initial condition is

$$\omega(\varphi,0)=e^{\left(\frac{\varphi}{2}\right)}. \tag{21}$$

Taking ρ-Laplace transform of (20),

$$s^\delta L_\rho[\omega(\varphi,\Im)]-s^{\delta-1}\omega(\varphi,0)=L_\rho\big[D_{\varphi\varphi\Im}\omega-D_\varphi\omega+\omega D_{\varphi\varphi\varphi}\omega-\omega D_\varphi\omega+3D_\varphi\omega D_{\varphi\varphi}\omega\big].$$

Applying inverse ρ-Laplace transform

$$\omega(\varphi,\Im)=L_\rho^{-1}\left[\frac{\omega(\varphi,0)}{s}-\frac{1}{s^\delta}L_\rho\big[D_{\varphi\varphi\Im}\omega-D_\varphi\omega+\omega D_{\varphi\varphi\varphi}\omega-\omega D_\varphi\omega+3D_\varphi\omega D_{\varphi\varphi}\omega\big]\right].$$

Using ADM procedure, we get

$$\omega_0(\varphi,\Im) = L_\rho^{-1}\left[\frac{\omega(\varphi,0)}{s}\right] = L_\rho^{-1}\left[\frac{e^{\left(\frac{\varphi}{2}\right)}}{s}\right],$$

$$\omega_0(\varphi,t) = e^{\left(\frac{\varphi}{2}\right)}, \tag{22}$$

$$\sum_{\ell=0}^{\infty}\omega_{\ell+1}(\varphi,\Im) = L_\rho^{-1}\left[\frac{1}{s^\delta}L_\rho\left[\sum_{\ell=0}^{\infty}(D_{\varphi\varphi\Im}\omega)_\ell - \sum_{\ell=0}^{\infty}(D_\varphi\omega)_\ell + \sum_{\ell=0}^{\infty}A_\ell - \sum_{\ell=0}^{\infty}B_\ell + 3\sum_{\ell=0}^{\infty}C_\ell\right]\right], \quad \ell = 0,1,2,\cdots$$

$$\begin{aligned}
A_0(\omega D_{\varphi\varphi\varphi}\omega) &= \omega_0 D_{\varphi\varphi\varphi}\omega_0,\\
A_1(\omega D_{\varphi\varphi\varphi}\omega) &= \omega_0 D_{\varphi\varphi\varphi}\omega_1 + \omega_1 D_{\varphi\varphi\varphi}\omega_0,\\
A_2(\omega D_{\varphi\varphi\varphi}\omega) &= \omega_1 D_{\varphi\varphi\varphi}\omega_2 + \omega_1 D_{\varphi\varphi\varphi}\omega_1 + \omega_2 D_{\varphi\varphi\varphi}\omega_0,
\end{aligned}$$

$$\begin{aligned}
B_0(\omega D_\varphi\omega) &= \omega_0 D_\varphi\omega_0,\\
B_1(\omega D_\varphi\omega) &= \omega_0 D_\varphi\omega_1 + \omega_1 D_\varphi\omega_0,\\
B_2(\omega D_\varphi\omega) &= \omega_1 D_\varphi\omega_2 + \omega_1 D_\varphi\omega_1 + \omega_2 D_\varphi\omega_0,
\end{aligned}$$

$$\begin{aligned}
C_0(D_\varphi\omega D_{\varphi\varphi}\omega) &= D_\varphi\omega_0 D_{\varphi\varphi}\omega_0,\\
C_1(D_\varphi\omega D_{\varphi\varphi}\omega) &= D_\varphi\omega_0 D_{\varphi\varphi}\omega_1 + D_\varphi\omega_1 D_{\varphi\varphi}\omega_0,\\
C_2(D_\varphi\omega D_{\varphi\varphi}\omega) &= D_\varphi\omega_1 D_{\varphi\varphi}\omega_2 + D_\varphi\omega_1 D_{\varphi\varphi}\omega_1 + D_\varphi\omega_2 D_{\varphi\varphi}\omega_0,
\end{aligned}$$

for $\ell = 1$

$$\begin{aligned}
\omega_1(\varphi,\Im) &= L_\rho^{-1}\left[\frac{1}{s^\delta}L_\rho\left[D_{\varphi\varphi\Im}\omega_0 - D_\varphi\omega_0 + A_0 - B_0 + 3C_0\right]\right],\\
\omega_1(\varphi,t) &= -\frac{1}{2}L_\rho^{-1}\left[\frac{e^{\left(\frac{\varphi}{2}\right)}}{s^{\delta+1}}\right] = -\frac{1}{2}e^{\left(\frac{\varphi}{2}\right)}\frac{\left(\frac{\Im^\rho}{\rho}\right)^\delta}{\Gamma(\delta+1)}.
\end{aligned} \tag{23}$$

for $\ell = 2$

$$\begin{aligned}
\omega_2(\varphi,\Im) &= L_\rho^{-1}\left[\frac{1}{s^\delta}L_\rho\left[D_{\varphi\varphi\Im}\omega_1 - D_\varphi\omega_1 + A_1 - B_1 + 3C_1\right]\right],\\
\omega_2(\varphi,\Im) &= -\frac{1}{8}e^{\left(\frac{\varphi}{2}\right)}\frac{\left(\frac{\Im^\rho}{\rho}\right)^{2\delta-1}}{\Gamma(2\delta)} + \frac{1}{4}e^{\left(\frac{\varphi}{2}\right)}\frac{\left(\frac{\Im^\rho}{\rho}\right)^{2\delta}}{\Gamma(2\delta+1)},
\end{aligned} \tag{24}$$

for $\ell = 3$

$$\begin{aligned}
\omega_3(\varphi,\Im) =& L_\rho^{-1}\left[\frac{1}{s^\delta}L_\rho\left[D_{\varphi\varphi\Im}\omega_2 - D_\varphi\omega_2 + A_2 - B_2 + 3C_2\right]\right],\\
\omega_3(\varphi,\Im) =& -\frac{1}{32}e^{\left(\frac{\varphi}{2}\right)}\frac{\left(\frac{\Im^\rho}{\rho}\right)^{3\delta-2}}{\Gamma(3\delta-1)} + \frac{1}{8}e^{\left(\frac{\varphi}{2}\right)}\frac{\Im^{3\delta-1}}{\Gamma(3\delta)} - \frac{1}{8}e^{\left(\frac{\varphi}{2}\right)}\frac{\left(\frac{\Im^\rho}{\rho}\right)^{3\delta}}{\Gamma(3\delta+1)},
\end{aligned} \tag{25}$$

The ρ-LDM result of Example 1 is

$$\omega(\varphi,\Im) = \omega_0(\varphi,\Im) + \omega_1(\varphi,\Im) + \omega_2(\varphi,\Im) + \omega_3(\varphi,\Im) + \omega_4(\varphi,\Im) + \cdots,$$

$$\omega(\varphi,\Im)=e^{\left(\frac{\varphi}{2}\right)}-\frac{1}{2}e^{\left(\frac{\varphi}{2}\right)}\frac{\left(\frac{\Im^{\rho}}{\rho}\right)^{\delta}}{\Gamma(\delta+1)}-\frac{1}{8}e^{\left(\frac{\varphi}{2}\right)}\frac{\left(\frac{\Im^{\rho}}{\rho}\right)^{2\delta-1}}{\Gamma(2\delta)}+\frac{1}{4}e^{\left(\frac{\varphi}{2}\right)}\frac{\left(\frac{\Im^{\rho}}{\rho}\right)^{2\delta}}{\Gamma(2\delta+1)}-\frac{1}{32}e^{\left(\frac{\varphi}{2}\right)}\frac{\left(\frac{\Im^{\rho}}{\rho}\right)^{3\delta-2}}{\Gamma(3\delta-1)}$$
$$+\frac{1}{8}e^{\left(\frac{\varphi}{2}\right)}\frac{\Im^{3\delta-1}}{\Gamma(3\delta)}-\frac{1}{8}e^{\left(\frac{\varphi}{2}\right)}\frac{\left(\frac{\Im^{\rho}}{\rho}\right)^{3\delta}}{\Gamma(3\delta+1)}-\cdots. \tag{26}$$

The simplify we can write Equation (26), we get

$$\omega(\varphi,\Im)=e^{\left(\frac{\varphi}{2}\right)}\left[1-\frac{\left(\frac{\Im^{\rho}}{\rho}\right)^{\delta}}{2\Gamma(\delta+1)}-\frac{1}{8}\frac{\left(\frac{\Im^{\rho}}{\rho}\right)^{2\delta-1}}{\Gamma(2\delta)}+\frac{1}{4}\frac{\left(\frac{\Im^{\rho}}{\rho}\right)^{2\delta}}{\Gamma(2\delta+1)}-\frac{1}{32}\frac{\left(\frac{\Im^{\rho}}{\rho}\right)^{3\delta-2}}{\Gamma(3\delta-1)}+\frac{1}{8}\frac{\left(\frac{\Im^{\rho}}{\rho}\right)^{3\delta-1}}{\Gamma(3\delta)}-\frac{1}{8}\frac{\left(\frac{\Im^{\rho}}{\rho}\right)^{3\delta}}{\Gamma(3\delta+1)}+\cdots\right]. \tag{27}$$

The analytical result by ρ-LVIM.
The iteration method apply for Equation (20), we get

$$\omega_{\ell+1}(\varphi,\Im)=\omega_{\ell}(\varphi,\Im)-L_{\rho}^{-1}\left[\frac{1}{s^{\delta}}L_{\rho}\left\{s^{\delta}D_{\Im}\omega_{\ell}-D_{\varphi\varphi\Im}\omega_{\ell}+D_{\varphi}\omega_{\ell}-\omega_{\ell}D_{\varphi\varphi\varphi}\omega_{\ell}+\omega_{\ell}D_{\varphi}\omega_{\ell}\ -3D_{\varphi}\omega_{\ell}D_{\varphi\varphi}\omega_{\ell}\right\}\right], \tag{28}$$

where

$$\omega_0(\varphi,\Im)=e^{\left(\frac{\varphi}{2}\right)}. \tag{29}$$

For $\ell=0,1,2,\cdots$

$$\omega_1(\varphi,\Im)=\omega_0(\varphi,\Im)-L_{\rho}^{-1}\left[\frac{1}{s^{\delta}}L_{\rho}\left\{s^{\delta}D_{\Im}\omega_0-D_{\varphi\varphi\Im}\omega_0+D_{\varphi}\omega_0-\omega_0D_{\varphi\varphi\varphi}\omega_0\right.\right.$$
$$\left.\left.+\omega_0D_{\varphi}\omega_0-3D_{\varphi}\omega_0D_{\varphi\varphi}\omega_0\right\}\right],$$
$$\omega_1(\varphi,\Im)=-\frac{1}{2}e^{\left(\frac{\varphi}{2}\right)}\frac{\left(\frac{\Im^{\rho}}{\rho}\right)^{\delta}}{\Gamma(\delta+1)}, \tag{30}$$

$$\omega_2(\varphi,\Im)=\omega_1(\varphi,\Im)-L_{\rho}^{-1}\left[\frac{1}{s^{\delta}}L_{\rho}\left\{s^{\delta}D_{\Im}\omega_1-D_{\varphi\varphi\Im}\omega_1+D_{\varphi}\omega_1-\omega_1D_{\varphi\varphi\varphi}\omega_1\right.\right.$$
$$\left.\left.+\omega_1D_{\varphi}\omega_1-3D_{\varphi}\omega_1D_{\varphi\varphi}\omega_1\right\}\right],$$
$$\omega_2(\varphi,\Im)=-\frac{1}{8}e^{\left(\frac{\varphi}{2}\right)}\frac{\left(\frac{\Im^{\rho}}{\rho}\right)^{2\delta-1}}{\Gamma(2\delta)}+\frac{1}{4}e^{\left(\frac{\varphi}{2}\right)}\frac{\left(\frac{\Im^{\rho}}{\rho}\right)^{2\delta}}{\Gamma(2\delta+1)}, \tag{31}$$

$$\omega_3(\varphi,\Im)=\omega_2(\varphi,\Im)-L_{\rho}^{-1}\left[\frac{1}{s^{\delta}}L_{\rho}\left\{s^{\delta}D_{\Im}\omega_2-D_{\varphi\varphi\Im}\omega_2+D_{\varphi}\omega_2-\omega_2D_{\varphi\varphi\varphi}\omega_2\right.\right.$$
$$\left.\left.+\omega_2D_{\varphi}\omega_2-3D_{\varphi}\omega_2D_{\varphi\varphi}\omega_2\right\}\right],$$
$$\omega_3(\varphi,\Im)=-\frac{1}{32}e^{\left(\frac{\varphi}{2}\right)}\frac{\left(\frac{\Im^{\rho}}{\rho}\right)^{3\delta-2}}{\Gamma(3\delta-1)}+\frac{1}{8}e^{\left(\frac{\varphi}{2}\right)}\frac{\left(\frac{\Im^{\rho}}{\rho}\right)^{3\delta-1}}{\Gamma(3\delta)}-\frac{1}{8}e^{\left(\frac{\varphi}{2}\right)}\frac{\left(\frac{\Im^{\rho}}{\rho}\right)^{3\delta}}{\Gamma(3\delta+1)}, \tag{32}$$

$$\omega(\varphi,\Im)=\sum_{m=0}^{\infty}\omega_m(\varphi)=e^{\left(\frac{\varphi}{2}\right)}-\frac{1}{2}e^{\left(\frac{\varphi}{2}\right)}\frac{\left(\frac{\Im^{\rho}}{\rho}\right)^{\delta}}{\Gamma(\delta+1)}-\frac{1}{8}e^{\left(\frac{\varphi}{2}\right)}\frac{\left(\frac{\Im^{\rho}}{\rho}\right)^{2\delta-1}}{\Gamma(2\delta)}+\frac{1}{4}e^{\left(\frac{\varphi}{2}\right)}\frac{\left(\frac{\Im^{\rho}}{\rho}\right)^{2\delta}}{\Gamma(2\delta+1)}$$
$$-\frac{1}{32}e^{\left(\frac{\varphi}{2}\right)}\frac{\left(\frac{\Im^{\rho}}{\rho}\right)^{3\delta-2}}{\Gamma(3\delta-1)}+\frac{1}{8}e^{\left(\frac{\varphi}{2}\right)}\frac{\Im^{3\delta-1}}{\Gamma(3\delta)}-\frac{1}{8}e^{\left(\frac{\varphi}{2}\right)}\frac{\left(\frac{\Im^{\rho}}{\rho}\right)^{3\delta}}{\Gamma(3\delta+1)}-\cdots. \tag{33}$$

The exact result of Equation (20) at $\delta=1$,

$$\omega(\varphi,\Im)=e^{\left(\frac{\varphi}{2}-\frac{2\Im}{3}\right)}. \tag{34}$$

Figure 1 shows the ρ-LDM and ρ-LVIM solution of the fractional Fornberg–Whitham defined by generalized fractional-order Caputo derivative in the space coordinate and time $0 < \Im \leq 0.5$, $\rho = 1$ and $\delta = 1$. Figure 2, the 3D graph shows approximate and exact solutions graph at $\delta = 1$ and $\rho = 0.9$; the figure shows that different fractional-order at δ. Similarly, in Figure 3, the 2D graph of exact and approximate solutions plot at $\delta = 1$ and $\rho = 0.9$ the figure shows that different fractional-order at δ.

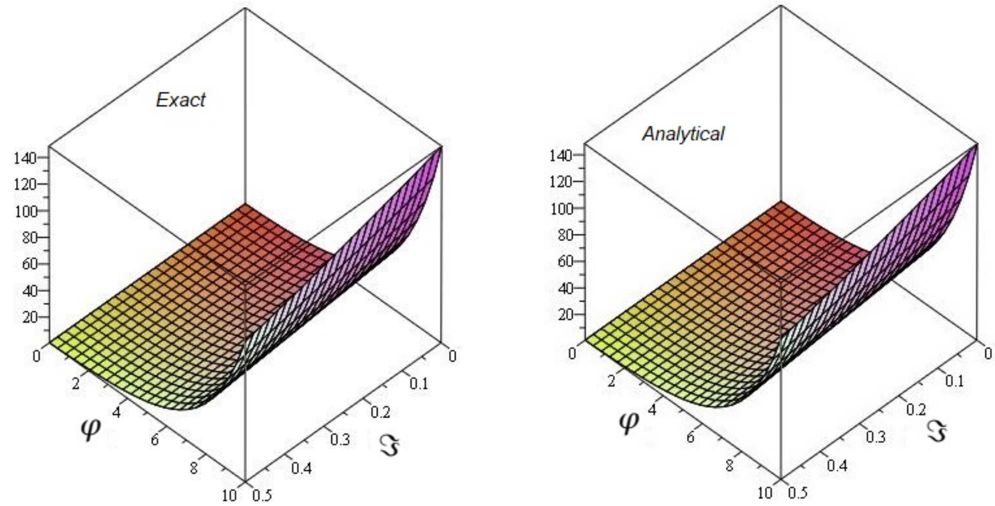


Figure 1. The graph of Exact and analytical solutions of $\delta = 1$ and $\rho = 1$ of problem 1.

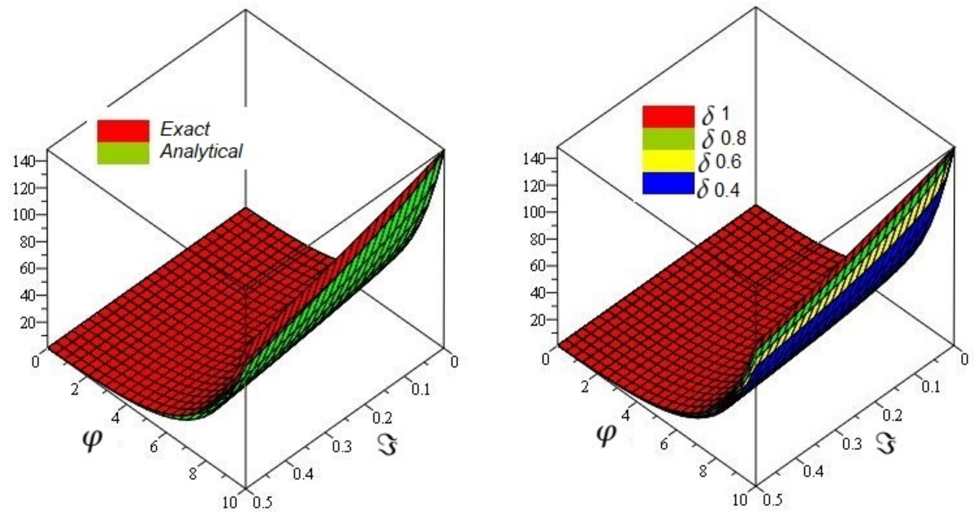
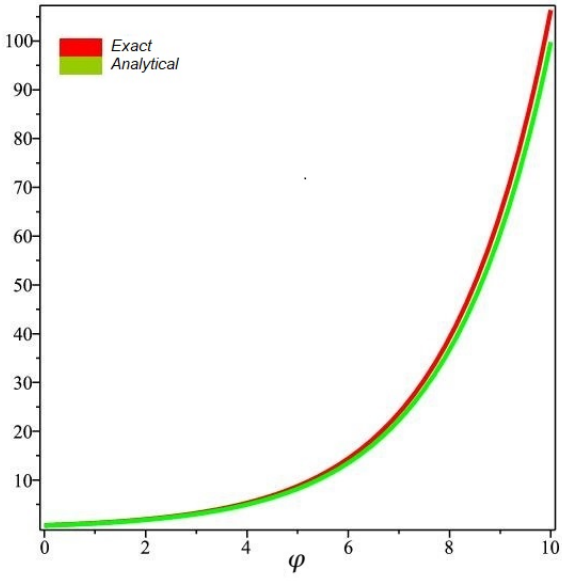


Figure 2. The first 3D graph of Exact and analytical solutions graph at $\delta = 1$ and $\rho = 0.9$ and second plot of the approximate different fractional-order of $\delta = 1$ of problem 1.



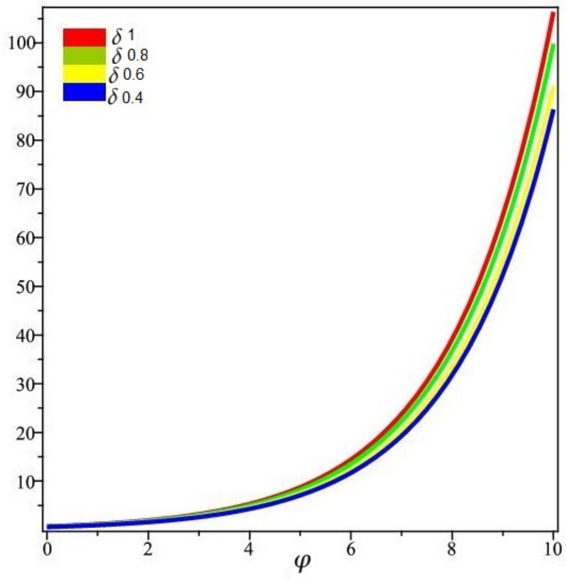


Figure 3. The first 2D graph of Exact and analytical solutions graph at $\delta = 1$ and $\rho = 0.9$ and second plot of the approximate different fractional-order of $\delta = 1$ of problem 1.

6.2. Problem

Consider the time-fractional non-linear FWE given as

$$D_{\Im}^{\delta,\rho}\omega - D_{\varphi\varphi\Im}\omega + D_{\varphi}\omega = \omega D_{\varphi\varphi\varphi}\omega - \omega D_{\varphi}\omega + 3D_{\varphi}\omega D_{\varphi\varphi}\omega, \quad \Im > 0, \quad 0 < \delta \leq 1, \tag{35}$$

with the initial condition

$$\omega(\varphi, 0) = \cosh^2\left(\frac{\varphi}{4}\right). \tag{36}$$

Taking ρ-Laplace transform of (35),

$$s^{\delta} L_{\rho}[\omega(\varphi,\Im)] - s^{\delta-1}\omega(\varphi,0) = L_{\rho}\left[D_{\varphi\varphi\Im}\omega - D_{\varphi}\omega + \omega D_{\varphi\varphi\varphi}\omega - \omega D_{\varphi}\omega + 3D_{\varphi}\omega D_{\varphi\varphi}\omega\right].$$

Applying inverse ρ-Laplace transform

$$\omega(\varphi,\Im) = L_{\rho}^{-1}\left[\frac{\omega(\varphi,0)}{s} - \frac{1}{s^{\delta}} L_{\rho}\left\{D_{\varphi\varphi\Im}\omega - D_{\varphi}\omega + \omega D_{\varphi\varphi\varphi}\omega - \omega D_{\varphi}\omega + 3D_{\varphi}\omega D_{\varphi\varphi}\omega\right\}\right].$$

Using ADM procedure, we get

$$\omega_0(\varphi,\Im) = L_{\rho}^{-1}\left[\frac{\omega(\varphi,0)}{s}\right] = L_{\rho}^{-1}\left[\frac{\cosh^2(\frac{\varphi}{4})}{s}\right],$$

$$\omega_0(\varphi,\Im) = \cosh^2\left(\frac{\varphi}{4}\right), \tag{37}$$

$$\sum_{\ell=0}^{\infty}\omega_{\ell+1}(\varphi,\Im) = L_{\rho}^{-1}\left[\frac{1}{s^{\delta}} L_{\rho}\left[\sum_{\ell=0}^{\infty}(D_{\varphi\varphi\Im}\omega)_{\ell} - \sum_{\ell=0}^{\infty}(D_{\varphi}\omega)_{\ell} + \sum_{\ell=0}^{\infty}A_{\ell} - \sum_{\ell=0}^{\infty}B_{\ell} + 3\sum_{\ell=0}^{\infty}C_{\ell}\right]\right], \quad \ell = 0,1,2,\cdots$$

for $\ell = 0$

$$\omega_1(\varphi, \Im) = L_\rho^{-1}\left[\frac{1}{s^\delta} L_\rho\left[D_{\varphi\varphi\Im}\omega_0 - D_\varphi\omega_0 + A_0 - B_0 + 3C_0\right]\right],$$
$$\omega_1(\varphi, \Im) = -\frac{11}{32} L_\rho^{-1}\left[\frac{\sinh\left(\frac{x}{2}\right)}{s^{\delta+1}}\right] = -\frac{11}{32}\sinh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^\delta}{\Gamma(\delta+1)}. \tag{38}$$

for $\ell = 1$

$$\omega_2(\varphi, \Im) = L_\rho^{-1}\left[\frac{1}{s^\delta} L_\rho\left[D_{\varphi\varphi\Im}\omega_1 - D_\varphi\omega_1 + A_1 - B_1 + 3C_1\right]\right],$$
$$\omega_2(\varphi, \Im) = -\frac{11}{28}\sinh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^\delta}{\Gamma(\delta+1)} + \frac{121}{1024}\cosh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^{2\delta}}{\Gamma(2\delta+1)}, \tag{39}$$

for $\ell = 2$

$$\omega_3(\varphi, \Im) = L_\rho^{-1}\left[\frac{1}{s^\delta} L_\rho\left[D_{\varphi\varphi\Im}\omega_2 - D_\varphi\omega_2 + A_2 - B_2 + 3C_2\right]\right],$$
$$\omega_3(\varphi, \Im) = -\frac{11}{512}\sinh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^\delta}{\Gamma(\delta+1)} + \frac{121}{2048}\cosh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^{2\delta}}{\Gamma(2\delta+1)} - \frac{1331}{49152}\sinh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^{3\delta}}{\Gamma(3\delta+1)}, \tag{40}$$

The ρ-LDM result for problem 2 is

$$\omega(\varphi, \Im) = \omega_0(\varphi, \Im) + \omega_1(\varphi, \Im) + \omega_2(\varphi, \Im) + \omega_3(\varphi, \Im) + \omega_4(\varphi, \Im) + \cdots,$$

$$\begin{aligned}\omega(\varphi, \Im) = \cosh^2\left(\frac{\varphi}{4}\right) &- \frac{11}{32}\sinh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^\delta}{\Gamma(\delta+1)} - \frac{11}{28}\sinh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^\delta}{\Gamma(\delta+1)} + \frac{121}{1024}\cosh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^{2\delta}}{\Gamma(2\delta+1)} \\ &- \frac{11}{512}\sinh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^\delta}{\Gamma(\delta+1)} + \frac{121}{2048}\cosh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^{2\delta}}{\Gamma(2\delta+1)} - \frac{1331}{49152}\sinh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^{3\delta}}{\Gamma(3\delta+1)}\cdots.\end{aligned} \tag{41}$$

The analytical solution by ρ-LVIM.
The iteration method is apply by Equation (35), we get

$$\omega_{\ell+1}(\varphi, \Im) = \omega_\ell(\varphi, \Im) - L_\rho^{-1}\left[\frac{1}{s^\delta} L_\rho\left\{s^\delta D_\Im\omega_\ell - D_{\varphi\varphi\Im}\omega_\ell + D_\varphi\omega_\ell - \omega_\ell D_{\varphi\varphi\varphi}\omega_\ell + \omega_\ell D_\varphi\omega_\ell - 3D_\varphi\omega_\ell D_{\varphi\varphi}\omega_\ell\right\}\right], \tag{42}$$

where

$$\omega_0(\varphi, t) = \cosh^2\left(\frac{\varphi}{4}\right). \tag{43}$$

For $\ell = 0, 1, 2, \cdots$

$$\omega_1(\varphi, \Im) = \omega_0(\varphi, \Im) - L_\rho^{-1}\left[\frac{1}{s^\delta} L_\rho\left\{s^\delta D_\Im\omega_0 - D_{\varphi\varphi\Im}\omega_0 + D_\varphi\omega_0 - \omega_0 D_{\varphi\varphi\varphi}\omega_0 + \omega_0 D_\varphi\omega_0 - 3D_\varphi\omega_0 D_{\varphi\varphi}\omega_0\right\}\right],$$
$$\omega_1(\varphi, \Im) = \cosh^2\left(\frac{\varphi}{4}\right) - \frac{11}{32}\sinh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^\delta}{\Gamma(\delta+1)}, \tag{44}$$

$$\omega_2(\varphi,\Im) = \omega_1(\varphi,\Im) - L_\rho^{-1}\left[\frac{1}{s^\delta}L_\rho\left\{s^\delta D_\Im\omega_1 - D_{\varphi\varphi\Im}\omega_1 + D_\varphi\omega_1 - \omega_1 D_{\varphi\varphi\varphi}\omega_1 + \omega_1 D_\varphi\omega_1 - 3D_\varphi\omega_1 D_{\varphi\varphi}\omega_1\right\}\right],$$
$$\omega_2(\varphi,\Im) = \cosh^2\left(\frac{\varphi}{4}\right) - \frac{11}{32}\sinh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^\delta}{\Gamma(\delta+1)} - \frac{11}{28}\sinh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^\delta}{\Gamma(\delta+1)} + \frac{121}{1024}\cosh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^{2\delta}}{\Gamma(2\delta+1)}, \tag{45}$$

$$\omega_3(\varphi,\Im) = \omega_2(\varphi,\Im) - L_\rho^{-1}\left[\frac{1}{s^\delta}L_\rho\left\{s^\delta D_\Im\omega_2 - D_{\varphi\varphi\Im}\omega_2 + D_\varphi\omega_2 - \omega_2 D_{\varphi\varphi\varphi}\omega_2 + \omega_2 D_\varphi\omega_2 - 3D_\varphi\omega_2 D_{\varphi\varphi}\omega_2\right\}\right],$$

$$\omega_3(\varphi,\Im) = \cosh^2\left(\frac{\varphi}{4}\right) - \frac{11}{32}\sinh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^\delta}{\Gamma(\delta+1)} - \frac{11}{28}\sinh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^\delta}{\Gamma(\delta+1)} + \frac{121}{1024}\cosh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^{2\delta}}{\Gamma(2\delta+1)}, \tag{46}$$
$$- \frac{11}{512}\sinh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^\delta}{\Gamma(\delta+1)} + \frac{121}{2048}\cosh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^{2\delta}}{\Gamma(2\delta+1)} - \frac{1331}{49152}\sinh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^{3\delta}}{\Gamma(3\delta+1)},$$

$$\omega(\varphi,\Im) = \sum_{m=0}^{\infty}\omega_m(\varphi) = \cosh^2\left(\frac{\varphi}{4}\right) - \frac{11}{32}\sinh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^\delta}{\Gamma(\delta+1)} - \frac{11}{28}\sinh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^\delta}{\Gamma(\delta+1)} + \frac{121}{1024}\cosh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^{2\delta}}{\Gamma(2\delta+1)}, \tag{47}$$
$$- \frac{11}{512}\sinh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^\delta}{\Gamma(\delta+1)} + \frac{121}{2048}\cosh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^{2\delta}}{\Gamma(2\delta+1)} - \frac{1331}{49152}\sinh\left(\frac{\varphi}{4}\right)\frac{\left(\frac{\Im^\rho}{\rho}\right)^{3\delta}}{\Gamma(3\delta+1)} - \cdots.$$

The exact result of Equation (35) at $\delta = 1$,

$$\omega(\varphi,\Im) = \cosh^2\left(\frac{\varphi}{4} - \frac{11\Im}{24}\right). \tag{48}$$

Figure 4 shows the ρ-LDM and ρ-LVIM solution of the fractional Fornberg–Whitham defined by generalized Caputo fractional-order derivative in the space coordinate and time $0 < \Im \leq 0.5$, $\rho = 1$ and $\delta = 1$. Figure 5, the 3D graph shows exact and approximate solutions plot at $\delta = 1$ and $\rho = 0.9$; the figure shows that different fractional-order at δ. Similarly, in Figure 6, the 2D graph of exact and approximate solutions plot at $\delta = 1$ and $\rho = 0.9$ the figure shows that different fractional-order at δ.

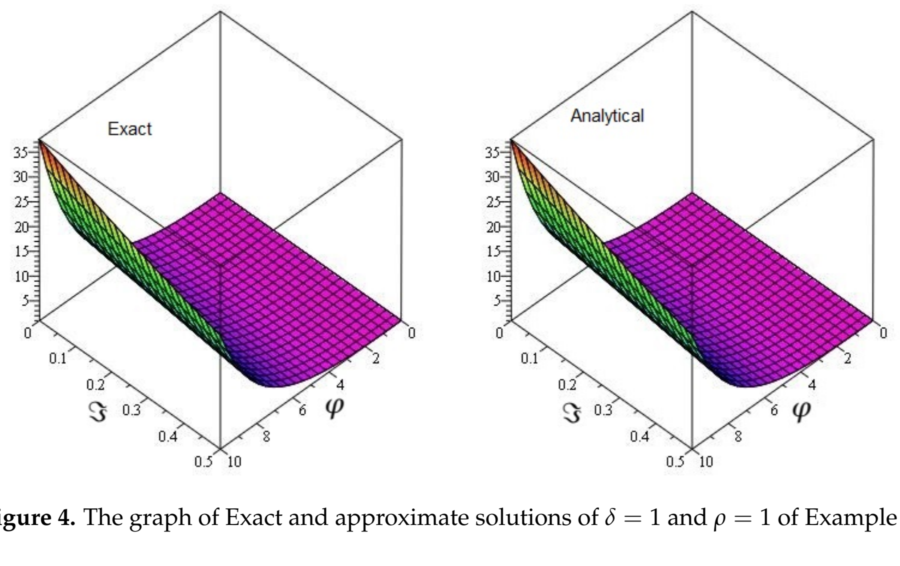


Figure 4. The graph of Exact and approximate solutions of $\delta = 1$ and $\rho = 1$ of Example 2.

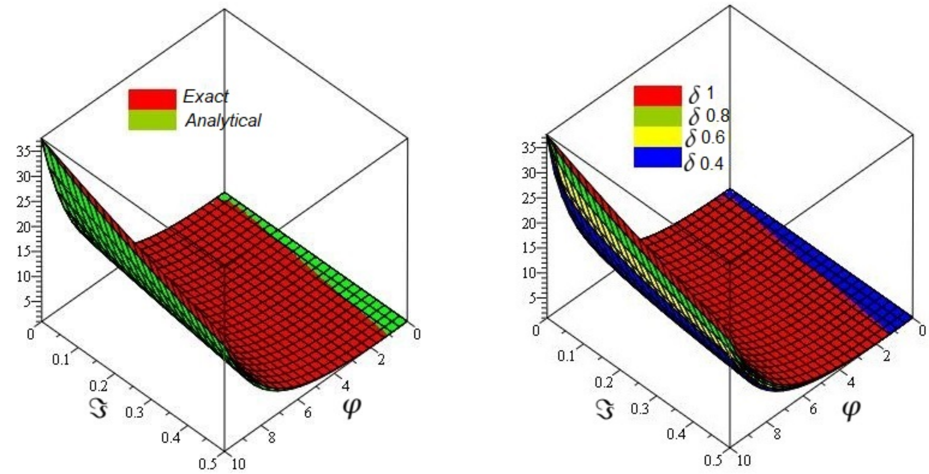


Figure 5. The first 3D graph of Exact and approximate solutions plot at $\delta = 1$ and $\rho = 0.9$ and second plot of the approximate different fractional-order of $\delta = 1$ of Example 2.

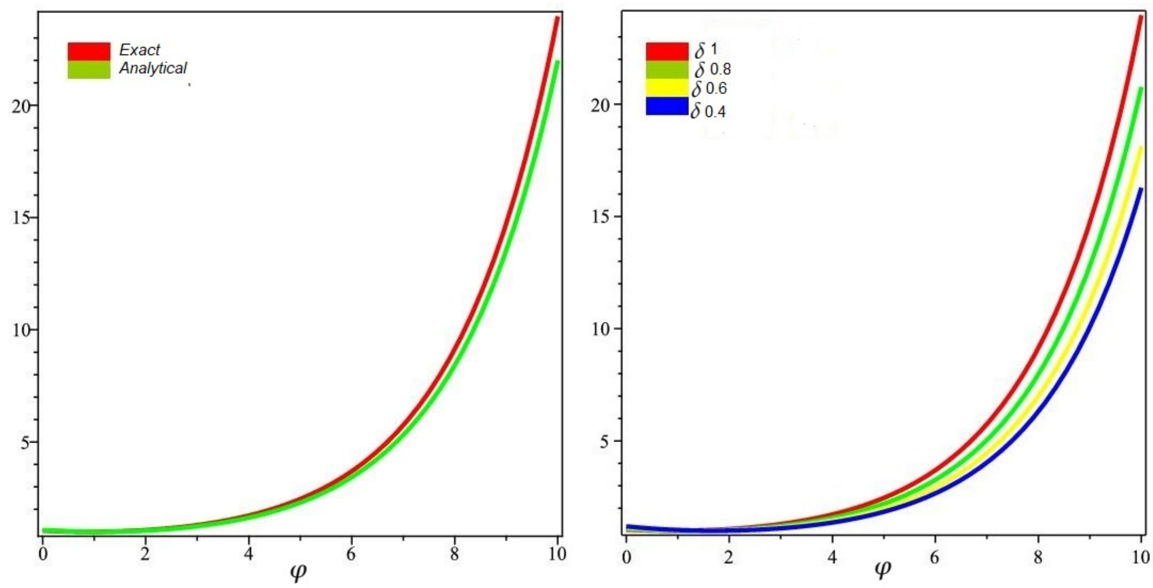


Figure 6. The first 2D graph of Exact and approximate solutions plot at $\delta = 1$ and $\rho = 0.9$ and second plot of the approximate different fractional-order of $\delta = 1$ of Example 2.

7. Conclusions

In this article, different semi-analytical techniques are implemented to solve time-fractional Fornberg–Whitham equation. The approximate solution of the equations is evaluated to confirm the validity and reliability of the proposed methods. Graphs of the solutions are plotted to display the closed relation between the obtained and exact results. In addition, the suggested techniques provide easily computable components for the series-form tests. It is investigated that the results achieved in the series form have a higher convergence rate towards the exact results. The proposed methods have a small number of calculations to achieve the approximate solution. In conclusion, it is found that the proposed technique is a sophisticated method for solving other NLFPDEs. In the future, the analytical result of non-linear fractional-order boundary values problems achieved using this technique is in the form of uniform convergence series.

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Conflicts of Interest: The authors declare no conflict of interest.

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