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An Existence Result for a Class of $p(x)-$Anisotropic Type Equations

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Abstract: In this paper, we study a class of anisotropic variable exponent problems involving the $\vec{p}(.)$-Laplacian. By using the variational method as our main tool, we present a result regarding the existence of solutions without the so-called Ambrosetti–Rabinowitz-type conditions.

Keywords: elliptic problem; anisotropic; weak solution; asymmetric behaviour

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1. Introduction

The investigation of anisotropic problems has drawn the attention of many authors; for example, see the works presented in [1–15] and the references therein. This particular interest in the study of such problems is the basis of many applications to the modeling of wave dynamics and mechanical processes in anisotropic elastic.

Meanwhile, in the early 1990s, the first anisotropic PDE model was proposed by the authors of [16], which was used for both image enhancement and denoising in terms of anisotropic PDEs as well as allowing the preservation of significant image features (for more details, see for example [17]). In this work, we show that the mathematical model of homogeneous anisotropic elastic media movement can be introduced by dynamic system equations of elasticity; it is presented as a symmetrical hyperbolic system of the first order in term of velocity.

In the current paper, we study the anisotropic nonlinear elliptic problem of the form

$$\begin{aligned} -\sum_{i=1}^{N}\partial_{x_i}\left(|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\right)+|u|^{p_M(x)-2}u=f(x,u), \quad \text{for } x\in\Omega \\ \sum_{i=1}^{N}|\partial_{x_i}u|^{p_i(x)-2}\,\partial_{x_i}u\,\nu_i=0 \text{ for } x\in\partial\Omega \end{aligned} \tag{1}$$

where $\Omega\subset\mathbb{R}^N$ ($N\geq 3$) is a bounded open set with a smooth boundary (which can be viewed as the graph of a smooth function locally; see [18]), ν_i represents the components of the outer normal unit vector, p_i, $i=1,\ldots N$ are continuous functions on $\overline{\Omega}$, $p_M(x)=\max\{p_1(x),\ldots,p_N(x)\}$ and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a continuous function with the potential

$$F(x,t)=\int_0^t f(x,s)\,ds.$$

This type of problem with variable exponent growth conditions allows us to deal with equations with other types of nonlinearities due to the fact that the operator $\Delta_{\vec{p}}(u)$ such that

$$\Delta_{\vec{p}}(u) := \sum_{i=1}^{N} \partial_{x_i} \left(|\partial_{x_i} u|^{p_i - 2} \partial_{x_i} u \right)$$

gives us another behavior for partial derivatives in several directions. This differential operator involving a variable exponent can be regarded as an extension of the $p(x)-$Laplace operator for the anisotropic case; as far as we are aware, $\Delta_p(x)$ is not homogeneous, and so the $p(x)-$Laplacian has more complicated properties than the $p-$Laplacian.

A host of publications have studied various types of nonlinear anisotropic elliptic equations from the point of view of the existence and qualitative properties of the data.

As a result of the preoccupation with nonhomogeneous materials that behave differently in different spatial directions, anisotropic spaces with variable exponents were introduced (for more details, see [19]).

In [9], using an embedding theorem involving the critical exponent of anisotropic type, the authors presented some results regarding the existence and nonexistence of the following anisotropic quasilinear elliptic problem:

$$\begin{gathered} -\sum_{i=1}^{N} \partial_{x_i} \left(|\partial_{x_i} u|^{p_i - 2} \partial_{x_i} u \right) = f(x, u), \quad \text{for } x \in \Omega \\ u = 0 \text{ for } x \in \partial\Omega, \end{gathered} \tag{2}$$

with $f(x, u) = \lambda u^{p-1}$

In [6], the authors studied the above problem when $f(x, u) = |u|^{q(x)} - |u|^{r(x)}$, $u \geq 0$, with the the condition

$$1 < q^- \leq q^+ < r^- \leq r^+ < p_m^- \leq p_M^+,$$

where

$$p_M(x) = \max\{p_1(x), \ldots, p_N(x)\}, \quad p_m(x) = \min\{p_1(x), \ldots, p_N(x)\}.$$

Using the variational approach—especially, the minimum principle and the mountain pass theorem—the author obtained the existence of at least two nonnegative nontrivial weak solutions.

In [14], the authors studied the spectrum of the problem when $f(x, u) = \lambda g(x)|u|^{r(x)-2}u$; they showed the existence of $\mu > 0$ such that λ is an eigenvalue for any $\lambda > \mu$.

In this article, we work on the so-called anisotropic variable exponent Sobolev spaces which were introduced for the first time by the authors in [20]. Motivated by the ideas accurately introduced in [21], our goal is to improve upon the existence results for problem (1) in the variable exponent case. The nonlinearity is assumed to be $(p_M^+ - 1)$ superlinear as $t \to \infty$,, which means that f exhibits asymmetric behavior. Further, it need not satisfy the Ambrosetti–Rabinowitz condition, as is usual for superlinear problems. We note that we may obtain infinitely many solutions by assuming some symmetry on the nonlinearity f; that is $f(x, -t) = -f(x, t)$ for $x \in \Omega$ and $t \in \mathbb{R}$ (see for example [22]).

This work is organized as follows. In Section 2, we give the necessary notations and some properties of anisotropic variable exponent Sobolev spaces, in order to facilitate the reading of the paper. In Section 3, we present the main results, and finally, we prove the existence of the solution.

2. Preliminaries

We introduce the setting of our problem with some auxiliary results. For convenience, we recall some basic facts which are used later, with reference to [19,23,24].

For $r \in C_+(\overline{\Omega})$, we introduce the Lebesgue space with the variable exponent defined by

$$L^{r(\cdot)}(\Omega) = \{u : u \text{ is a measurable real-valued function, } \int_\Omega |u(x)|^{r(x)}\, dx < \infty\},$$

where

$$C_{+}(\overline{\Omega}) = \{r \in C(\overline{\Omega}; \mathbb{R}) : \inf_{x\in\Omega} r(x) > 1\}.$$

The space $L^{r(x)}(\Omega)$ endowed with the Luxemburg norm

$$\|u\|_{r(.)} = \|u\|_{L^{r(\cdot)}(\Omega)} = \inf\Big\{\mu > 0 : \int_{\Omega} \Big|\frac{u(x)}{\mu}\Big|^{r(x)} dx \le 1\Big\},$$

is a separable and reflexive Banach space.

Concerning the embedding result, we make the following proposition:

Proposition 1 ([24]). *Assume that Ω is bounded and $r_1, r_2 \in C_{+}(\overline{\Omega})$ such that $r_1 \le r_2$ in Ω. Then, the embedding $L^{r_2(\cdot)}(\Omega) \hookrightarrow L^{r_1(\cdot)}(\Omega)$ is continuous.*

Furthermore, the Hölder-type inequality

$$\Big|\int_{\Omega} u(x)v(x)\,dx\Big| \le 2\|u\|_{L^{r(\cdot)}(\Omega)}\|v\|_{L^{r'(\cdot)}(\Omega)} \tag{3}$$

holds for all $u \in L^{r(\cdot)}(\Omega)$ and $v \in L^{r'(\cdot)}(\Omega)$, where $L^{r'(\cdot)}(\Omega)$ the conjugate space of $L^{r(\cdot)}(\Omega)$, with

$$1/r(x) + 1/r'(x) = 1.$$

Moreover, we denote

$$r^{+} = \sup_{x\in\Omega} r(x), \quad r^{-} = \inf_{x\in\Omega} r(x)$$

For $u \in L^{r(\cdot)}(\Omega)$, we have the following properties:

$$\|u\|_{L^{r(\cdot)}(\Omega)} < 1\ (=1; >1) \ \Leftrightarrow\ \int_{\Omega} |u(x)|^{r(x)}\,dx < 1\ (=1; >1); \tag{4}$$

$$\|u\|_{L^{r(\cdot)}(\Omega)} > 1 \ \Rightarrow\ \|u\|^{r^{-}}_{L^{r(\cdot)}(\Omega)} \le \int_{\Omega} |u(x)|^{r(x)}\,dx \le \|u\|^{r^{+}}_{L^{r(\cdot)}(\Omega)}; \tag{5}$$

$$\|u\|_{L^{r(\cdot)}(\Omega)} < 1 \ \Rightarrow\ \|u\|^{r^{+}}_{L^{r(\cdot)}(\Omega)} \le \int_{\Omega} |u(x)|^{r(x)}\,dx \le \|u\|^{r^{-}}_{L^{r(\cdot)}(\Omega)}; \tag{6}$$

$$\|u\|_{L^{r(\cdot)}(\Omega)} \to 0\ (\to \infty) \ \Leftrightarrow\ \int_{\Omega} |u(x)|^{r(x)}\,dx \to 0\ (\to \infty). \tag{7}$$

To recall the definition of the isotropic Sobolev space with a variable exponent, $W^{1,r(\cdot)}(\Omega)$, we set

$$W^{1,r(\cdot)}(\Omega) = \{u \in L^{r(\cdot)}(\Omega) : \partial_{x_i} u \in L^{r(\cdot)}(\Omega) \text{ for all } i \in \{1, \dots, N\}\},$$

endowed with the norm

$$\|u\|_{W^{1,r(\cdot)}(\Omega)} = \|u\|_{L^{r(\cdot)}(\Omega)} + \sum_{i=1}^{N} \|\partial_{x_i} u\|_{L^{r(\cdot)}(\Omega)}.$$

The space $\big(W^{1,r(\cdot)}(\Omega), \|\cdot\|_{W^{1,r(\cdot)}(\Omega)}\big)$ is a separable and reflexive Banach space.

Now, we consider $\vec{p} : \overline{\Omega} \to \mathbb{R}^N$ to be the vectorial function

$$\vec{p}(x) = (p_1(x), \dots, p_N(x))$$

with $p_i \in C_{+}(\overline{\Omega})$ for all $i \in \{1, \dots, N\}$ and we recall that

$$p_M(x) = \max\{p_1(x), \dots, p_N(x)\}, \quad p_m(x) = \min\{p_1(x), \dots, p_N(x)\}.$$

The anisotropic space with a variable exponent is

$$X = W^{1,\vec{p}(\cdot)}(\Omega) = \{u \in L^{p_M(\cdot)}(\Omega) : \partial_{x_i} u \in L^{p_i(\cdot)}(\Omega) \text{ for all } i \in \{1, \dots, N\}\}$$

and it is endowed with the norm

$$\| u \| = \|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)} = \|u\|_{L^{p_M(\cdot)}(\Omega)} + \sum_{i=1}^{N} \|\partial_{x_i} u\|_{L^{p_i(\cdot)}(\Omega)}.$$

We point out that $\left(W^{1,\vec{p}(\cdot)}(\Omega), \| \cdot \|_{W^{1,\vec{p}(\cdot)}(\Omega)}\right)$ is a reflexive Banach space. Let introduce the following notations:

$$\bar{p}(x) = \frac{N}{\sum_{i=1}^{N} 1/p_i(x)} \text{ and } \bar{p}^*(x) = \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i(x)} - 1}.$$

Proposition 2 ([19]). *If $q \in C_+(\overline{\Omega})$ satisfies $1 < q(x) < \max\{\overline{p}^*(x), p_M(x)\}$ for all $x \in \overline{\Omega}$ then the embedding $W^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact.*

3. Main Results

Proposition 3. *Putting*

$$I(u) = \int_{\Omega} \sum_{i=1}^{N} \frac{1}{p_i(x)} |\partial_{x_i} u)|^{p_i(x)}\, dx.$$

then $I \in C^1(X, \mathbb{R})$, and the derivative operator I' of I is

$$I'(u).v = \sum_{i=1}^{N} \int_{\Omega} |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \partial_{x_i} v\, dx.$$

(i) The functional I' is of the (S_+) type, where I' is the Gâteaux derivative of the functional I.
(ii) $I' : X \to X^*$ is a bounded homeomorphism and a strictly monotone operator.

The proof of the first assertion (i) is similar to that in [2]. The second assertion is well known (for example, see [19]).

Next, we give the mountain pass theorem of Ambrosetti–Rabinowitz (see [25]).

Proposition 4. *Let X be a real Banach space with its dual space X^* and suppose that $\phi \in C^1(X, \mathbb{R})$ satisfies the condition*

$$max\Big(\phi(0), \phi(u_*)\Big) \le \beta \le \inf_{\|u\|=\rho} \phi(u),$$

for some $\alpha < \beta$, $\rho > 0$ and $u_* \in X$ with $\|u_*\| > \rho$. Let $c \le \beta$ be characterized by

$$c = \inf_{\gamma \in \Gamma} \max_{\tau \in [0,1]} \phi(\gamma(\tau)),$$

where $\Gamma = \{\gamma \in C([0,1], X) : \ \gamma(0) = \gamma(1) = u_*$ is the set of continuous paths joining 0 and u_*. Then, there exists a sequence $(u_n)_n$ in X such that

$$\phi(u_n) \to c \ge \beta \ \ and \ \ \Big(1 + \|u_n\|\Big) \|\phi'(u_n)\|_* \to 0. \qquad \square$$

The problem (1) is considered in the case when $f \in C(\overline{\Omega} \times \mathbb{R})$ such that

Hypothesis 1 (H1). *There exist $C > 0$ and $q \in C_+(\overline{\Omega})$ with $p_M^+ < q^- \le q^+ < \overline{p}^*(x)$ for all $x \in \overline{\Omega}$, such that f verifies*

$$|f(x,s)| \le C\big(1 + |s|^{q(x)-1}\big)$$

for all $x \in \Omega$ and all $s \in \mathbb{R}$ and $f(x,t) = f(x,0) = 0\ \forall x \in \Omega,\ t \le 0$.

Hypothesis 2 (H2). $\lim_{t\to 0} \frac{f(x,t)}{|t|^{p_M^+-1}} = l_1 < \infty$, $\lim_{t\to\infty} \frac{f(x,t)t}{|t|^{p_M^+}} = \infty$, *uniformly for* $x \in \Omega$.

Hypothesis 3 (H3). *For a.e* $x \in \Omega$, $\frac{f(x,t)}{t^{p_M^+-1}}$ *is nondecreasing with respect to* $t \geq 0$.

Hypothesis 4 (H4). $\limsup_{|t|\to+\infty} \frac{p_M(x)F(x,t)}{|\ t\ |^{p_M(x)}} < \lambda_1$, *uniformly for a.e* $x \in \Omega$, *with*

$$\lambda_1 = \inf_{u\in W^{1,\overrightarrow{p}(x)}(\Omega), u\neq 0} \frac{\sum_{i=1}^{N} \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} u|^{p_i(x)}\, dx + \int_\Omega \frac{1}{p_M(x)} |u|^{p_M(x)}\, dx}{\int_\Omega \frac{1}{p_M(x)} |u|^{p_M(x)}\, dx} > 0.$$

Hypothesis 5 (H5). *There exist* $a_0 > 0$ *and* $\delta > 0$ *such that*

$$F(x,t) \geq a_0|t|^{q_0},\ \forall x \in \Omega,\ |t| < \delta,$$

where $q_0 \in C(\overline{\Omega})$ *with* $q_0 < p_m^-$.

Definition 1. *We define the weak solution for problem* (1) *as a function* $u \in W^{1,\overrightarrow{p}(x)}(\Omega)$ *satisfying*

$$\sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u\, \partial_{x_i} v\, dx + \int_\Omega |u|^{p_M(x)-2} uv dx - \int_\Omega f(x,u) v\, dx = 0,$$

for all $v \in W^{1,\overrightarrow{p}(x)}(\Omega)$.

Our main result in this section is the following.

Theorem 1. *(a) Assume* $(H1), (H2)$ *and* $(H3)$, *then* (1) *has at least a nontrivial solution.*
(b) Under the assumptions $(H4) - (H5)$, *the problem* (1) *has at least a nontrivial solution in* $W^{1,\overrightarrow{p}(x)}(\Omega)$.

It is well known that the (AR) condition defined by

$$(AR)\ \ p_M^+ F(x,s) \leq f(x,s)s\ \ a.e\ x \in \Omega,$$

plays a crucial role in guaranteeing that every Palais Smale sequence of associated functionals is bounded in $W^{1,\overrightarrow{p}}(\Omega)$. Here, we avoid using the condition (AR) under various assumptions on f and by different methods. Notice that the condition $(H2)$ is weaker than the (AR) condition, and thus it is more interesting. Moreover, for instance, the function $f(x,t) = |t|^{p_M^+-2} t \log(1+|t|),\ t \in \mathbb{R}$ verifies our assumptions (H1)–(H3); however, it does not satisfy the (A-R) type condition.

4. Proofs

Firstly, related to problem (1.1), we have the associated functional $\phi:\ W^{1,\overrightarrow{p}(x)}(\Omega) \to \mathbb{R}$ is given by

$$\phi(u) = \int_\Omega \sum_{i=1}^{N} \frac{1}{p_i(x)} |\partial_{x_i} u|^{p_i(x)}\, dx + \int_\Omega \frac{1}{p_M(x)} |u|^{p_M(x)}\, dx - \int_\Omega F(x,u)\, dx.$$

From the continuous embedding

$$W^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega),\ \ \forall s(x) \in [1, \bar{p}^*(x)],$$

it follows that $\phi \in C^1(W^{1,\overrightarrow{p}(x)}(\Omega), \mathbb{R})$ (see [19]).

Lemma 1. *Assume that* $(H1), (H2)$ *and* $(H3)$ *hold; then,*
(i) There exists $v \in X$ *with* $v > 0$ *such that* $\phi(tv) \to -\infty$ *as* $t \to \infty$.
(ii) There exist $\alpha, \beta > 0$ *such that* $\phi(u) \geq \beta$ *for all* $u \in X$ *with* $\|u\| = \alpha$.

Proof. (i) In view of the condition $(H2)$, we may choose a constant $K > 0$ such that

$$F(x,s) > K|s|^{p_M^+} \text{uniformly in} x \in \Omega, |s| > C_K. \tag{8}$$

Let $t > 1$ large enough and $v \in X$ with $v > 0$, from (8) we get

$$\begin{aligned}
\phi(tv) &\leq t^{p_M^+} \sum_{i=1}^{N} \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} v|^{p_i(x)}\, dx + t^{p_M^+} \int_\Omega \frac{1}{p_M(x)} |v|^{p_M(x)}\, dx - \int_{|tv|>C_K} F(x,tv)dx - \int_{|tv|\leq C_K} F(x,tv)dx \\
&\leq t^{p_M^+} \frac{1}{p_m^-} \sum_{i=1}^{N} \int_\Omega |\partial_{x_i} v|^{p_i(x)}\, dx + t^{p_M^+} \frac{1}{p_M^-} \int_\Omega |v|^{p_M(x)}\, dx - Kt^{p_M^+} \int_\Omega |v|^{p_M^+} dx - \int_{|tv|\leq C_K} F(x,tv)dx \\
&\leq t^{p_M^+} \frac{1}{p_m^-} \sum_{i=1}^{N} \int_\Omega |\partial_{x_i} v|^{p_i(x)}\, dx + t^{p_M^+} \frac{1}{p_M^-} \int_\Omega |v|^{p_M(x)}\, dx - Kt^{p_M^+} \int_\Omega |v|^{p_M^+} dx + C_1,
\end{aligned}$$

where $C_1 > 0$ is a constant, taking K to be sufficiently large to ensure that

$$\frac{1}{p_m^-} \sum_{i=1}^{N} \int_\Omega |\partial_{x_i} v|^{p_i(x)}\, dx + \frac{1}{p_M^-} \int_\Omega |v|^{p_M(x)}\, dx - K \int_\Omega |v|^{p_M^+} dx < 0$$

which implies that

$$\phi(tv) \to -\infty \quad \text{as } t \to +\infty.$$

(ii) For $\| u \| < 1$ we have

$$\begin{aligned}
\phi(u) &\geq \frac{1}{p_M^+} \int_\Omega \sum_{i=1}^{N} |\partial_{x_i} u|^{p_i(x)}\, dx + \frac{1}{p_M^+} \int_\Omega |u|^{p_M(x)}\, dx - \int_\Omega F(x,u)\, dx \\
&\geq \frac{1}{p_M^+} \sum_{i=1}^{N} \|\partial_{x_i} u\|^{p_M^+}_{L^{p_i(\cdot)}(\Omega)} + \frac{1}{p_M^+} \|u\|^{p_M^+}_{L^{p_M(\cdot)}(\Omega)} - \int_\Omega F(x,u)\, dx \\
&\geq \frac{1}{p_M^+} \|u\|^{p_M^+} - \int_\Omega F(x,u)\, dx.
\end{aligned}$$

On the other side, from $(H1)$ and $(H2)$,

$$|f(x,u)| \leq \varepsilon |u|^{p_M^+ - 1} + C(\varepsilon)|u|^{q(x)-1}, \;\; \forall (x,u) \in \Omega \times \mathbb{R}.$$

By the continuous embedding from X into $L^{q(x)}(\Omega)$ and $L^{p_M^+}(\Omega)$ there exist $c_1, c_2 > 0$ such that

$$\|u\|_{L^{p_M^+}(\Omega)} \leq c_1 \|u\|, \quad \|u\|_{L^{q^+}(\Omega)}, \|u\|_{L^{q^-}(\Omega)} \leq c_2 \|u\| \tag{9}$$

for all $u \in X$. Thus,

$$\begin{aligned}
\int_\Omega F(x,u)\, dx &\leq \int_\Omega \frac{\varepsilon}{p_M^+} |u|^{p_M^+} dx + \int_\Omega \frac{C(\varepsilon)}{q(x)} |u|^{q(x)}\, dx \\
&\leq \varepsilon c_1^{p_M^+} \|u\|^{p_M^+} + c_2^{q^-} \frac{C(\varepsilon)}{q^-} \|u\|^{q^-}
\end{aligned} \tag{10}$$

for all $x \in \Omega$ and all $u \in X$.

Therefore,

$$\phi(u) \geq \left(\frac{1}{2(N+1)^{p_m^+} p_M^+} - C(\varepsilon) c_2^{q^-} \|u\|^{q^- - p_M^+} - \varepsilon c_1^{p_M^+} \right) \|u\|^{p_M^+},$$

since $1 < p_M^+ < q^-$, then for α sufficiently small we take $\beta > 0$ such that

$$\phi(u) \geq \beta, \forall u \in X \text{with} \|u\| = \alpha.$$

$\square$

Lemma 2. *Under the assumptions* $(H1)$ *and* $(H3)$, *for any* $(u_n)_n \subset X$ *such that*

$$\phi'(u_n).u_n \to 0, \quad as \quad n \to \infty,$$

then there is a subsequence, still denoted by $(u_n)_n$, *such that*

$$\phi(tu_n) \leq \frac{t^{p_m^-}}{p_m^-}\left[\frac{1}{n} + \int_\Omega \frac{1}{p_m^-} f(x,u_n)u_n \, dx\right] - \int_\Omega F(x,u_n)\, dx,$$

for all $t \in \mathbb{R}$ *and* $t > 0$.

Proof. Consider a function g such that

$$g(t) = \frac{1}{p_m^-} t^{p_m^-} f(x,u_n)u_n - F(x,tu_n),$$

then

$$\begin{aligned} g'(t) &= t^{p_m^- - 1} f(x,u_n)u_n - f(x,tu_n)u_n \\ &= t^{p_m^- - 1} u_n \Big(f(x,u_n) - \frac{f(x,tu_n)}{t^{p_m^- - 1}} \Big), \end{aligned}$$

which means that $g'(t) \geq 0$ for $t \in]0,1]$ and $g'(t) \leq 0$ when $t \geq 1$, it follows that

$$g(t) \leq g(1), \quad \forall t > 0. \tag{11}$$

From the hypothesis $\phi'(u_n).u_n \to 0$, for any $n > 1$, we have

$$|\phi'(u_n).u_n| < \frac{1}{n},$$

therefore

$$-\frac{1}{n} < \phi'(u_n).u_n = \int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)}\, dx + \int_\Omega |u_n|^{P_M(x)}\, dx - \int_\Omega f(x,u_n)u_n\, dx < \frac{1}{n}. \tag{12}$$

Using the formulas (11) and (12), we obtain

$$\begin{aligned} \phi(tu_n) &= \int_\Omega \sum_{i=1}^N \frac{1}{p_i(x)} |\partial_{x_i} tu_n|^{p_i(x)}\, dx + \int_\Omega \frac{1}{p_M(x)} |tu_n|^{P_M(x)}\, dx - \int_\Omega F(x,tu_n)\, dx \\ &< \frac{t^{p_m^-}}{p_m^-}\Big[-\frac{1}{n} + \int_\Omega \Big(f(x,u_n)u_n - F(x,tu_n) \Big)\, dx \Big]. \end{aligned} \tag{13}$$

$\square$

Proof of Theorem 1. (a) Let $(u_n)_n \subset X$ satisfying the assertions of Proposition 4; then,

$$\phi(u_n) = \sum_{i=1}^N \int_\Omega \frac{1}{p_i(x)} |\partial_{x_i} u_n|^{p_i(x)} + \int_\Omega \frac{1}{p_M(x)} |u|^{p_M(x)}\, dx - \int_\Omega F(x,u_n)\, dx = c + o(1)$$

and

$$\Big(1 + \|u_n\|\Big)\|\phi'(u_n)\| \to 0$$

then

$$\|u_n\| - \int_\Omega f(x,u_n)u_n\,dx = o(1)$$

and also

$$\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}u_n|^{p_i(x)-2}\partial_{x_i}u_n.\partial_{x_i}\varphi + \int_\Omega |u|^{p_M(x)-2}\varphi\,dx - \int_\Omega f(x,u_n)\varphi = o(1)$$

$\forall \varphi \in X$.

Let us show that the sequence $(u_n)_n$ is bounded in $W^{1,\overrightarrow{p}(x)}(\Omega)$.

Define

$$t_n = \frac{(2p_M^+ c)^{1/p_M^+}}{\|u_n\|} > 0$$

and

$$\omega_n = t_n u_n.$$

Because $\|\omega_n\| = (2p_M^+ c)^{1/p_M^+}$ so ω_n is bounded in $W^{1,\overrightarrow{p}(x)}(\Omega)$, therefore, up to a subsequence still denoted by $(\omega_n)_n$, we have

$$\omega_n \rightharpoonup \omega \;\; in \;\; W^{1,\overrightarrow{p}(x)}(\Omega)$$

$$\omega_n \to \omega \;\; in \;\; L^{q(x)}(\Omega), \; for \;\; q(x) \in (1, \max\{\overline{p}^*(x), p_M(x)\})$$

and

$$\omega_n \to \omega \;\; a.e \;\; in \;\; \Omega.$$

Suppose that $\|u_n\| \to \infty$, we confirm that $\omega \equiv 0$. Indeed, putting

$$\Omega_1 = \{x \in \Omega \; : \;\; \omega(x) = 0\}$$

and

$$\Omega_2 = \{x \in \Omega : \;\; \omega(x) \neq 0\}.$$

Since $|u_n| = \|u_n\| \frac{|\omega_n|}{(2p_M^+ c)^{\frac{1}{p_M^+}}}$, we can easily see that $|u_n(x)| \to \infty \;\; a.e \;\; in \;\; \Omega_2$

From the assumption $(H2)$ and for a large enough n, we find that

$$\frac{f(x,u_n)u_n}{|u_n|^{p_M^+}} > k \;\; uniformly \;\; x \in \Omega_2$$

for a large enough k. Thus,

$$\begin{aligned}
2p_M^+ c &= \lim_{n\to\infty} \|\omega_n\|^{p_M^+} \\
&= \lim_{n\to\infty} |t_n|^{p_M^+}\|u_n\|^{p_M^+} \\
&= \lim_{n\to\infty} |t_n|^{p_M^+} \int_\Omega \frac{|f(x,u_n)u_n|^{p_M^+}}{|u_n|^{p_M^+}} |u_n|^{p_M^+}\,dx \\
&> k \lim_{n\to\infty} \int_{\Omega_2} |\omega_n|^{p_M^+}\,dx \\
&= k \int_{\Omega_2} |\omega|^{p_M^+}\,dx.
\end{aligned} \tag{14}$$

The fact that $2p_M^+ c$ is constant and k is sufficiently large allows us to infer that $|\Omega_2| = 0$ and then $\omega \equiv 0$ in Ω.

Furthermore, since $\omega = 0$ and in view of the continuity of the Nemitskii operator, we get

$$F(.,\omega_n) \to 0 \;\; in \;\; L^1(\Omega),$$

which implies that

$$\lim_n F(x,\omega_n) = 0;$$

then,

$$\begin{aligned}
\phi(\omega_n) &\geq \frac{1}{p_M^+} t_n^{p_M^+} \Big[\int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)}\, dx + \int_\Omega |u|^{p_M(x)}\, dx \Big] - o(1) \\
&\geq \frac{1}{p_M^+} 2p_M^+ c - o(1) = 2c - o(1) \\
&> c.
\end{aligned} \tag{15}$$

Once more, as in (12), for a certain $n > 1$ we have

$$\frac{-1}{n} < \frac{p_m^-}{p_M^+} \langle \phi'(u_n), u_n \rangle < \frac{1}{n}.$$

Thus,

$$\begin{aligned}
\phi(u_n) &= \int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)}\, dx + \int_\Omega \frac{1}{p_M(x)} |t_n u_n|^{p_M(x)}\, dx - \int_\Omega F(x,u_n)\, dx \\
&\geq \frac{1}{p_M^+} \frac{p_M^+}{p_m^-} \Big(\frac{-1}{n} + \int_\Omega f(x,u_n) u_n\, dx \Big) - \int_\Omega F(x,u_n)\, dx
\end{aligned} \tag{16}$$

that is,

$$\phi(u_n) + \frac{1}{np_m^-} \geq \int_\Omega \Big(\frac{1}{p_m^-} f(x,u_n) u_n - F(x,u_n) \Big)\, dx. \tag{17}$$

Meanwhile, from Lemma 2,

$$\phi(tu_n) \leq \frac{t^{p_m^-}}{np_m^-} + \int_\Omega \Big(\frac{1}{p_m^-} f(x,u_n) u_n - F(x,u_n) \Big)\, dx. \tag{18}$$

By virtue of (17) and (18), we have

$$\phi(\omega_n) \leq \frac{t^{p_m^-}+1}{np_m^-} + \phi(u_n) \to c,$$

which is contradictory with (15). Therefore, $(u_n)_n$ is bounded in $W^{1,\overrightarrow{p(x)}}(\Omega)$.

Now, regarding the boundedness of $(u_n)_n$ in X and the fact that X is reflexive, there exists $u \in X$ such that $u_n \rightharpoonup u$. Since ϕ' is a (S_+) type map (because I' is of the (S_+) type (see Proposition 3)), thus $u_n \to u$ in X, thereby $(u_n)_n$ converges strongly to a nontrivial critical point of ϕ and the proof of Theorem 1 is achieved.

(b) Recall that by applying Jensen's inequality to the convex function

$$t\ (\geq 0) \to t^{p_m^-},$$

we obtain

$$\sum_{i=1}^N \|\partial_{x_i} u\|_{p_i(.)}^{p_m^-} \geq \frac{1}{N^{p_m^- - 1}} \Big[\sum_{i=1}^N \Big[\|\partial_{x_i} u\|_{p_i(.)} \Big] \Big]^{p_m^-}.$$

Let us make the following notations:

$$\mathcal{F}_1 = \{i \in \{1,\ldots,N\} : \|\partial_{x_i} u_n\|_{L^{p_i}(\Omega)} \leq 1\},$$
$$\mathcal{F}_2 = \{i \in \{1,\ldots,N\} : \|\partial_{x_i} u_n\|_{L^{p_i}(\Omega)} > 1\}.$$

Then, we get

$$\begin{aligned}\sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u_n|^{p_i(x)} dx &= \sum_{i\in\mathcal{F}_1} \int_\Omega |\partial_{x_i} u_n|^{p_i(.)} dx + \sum_{i\in\mathcal{F}_2} \int_\Omega |\partial_{x_i} u_n|^{p_i(.)}\, dx \\ &\geq \sum_{i\in\mathcal{F}_1} \|\partial_{x_i} u_n\|^{p_M}_{L^{p_i(.)}} + \sum_{i\in\mathcal{F}_2} \|\partial_{x_i} u_n\|^{p_m^-}_{L^{p_i(.)}} \\ &\geq \sum_{i=1}^{N} \|\partial_{x_i} u_n\|^{p_m^-}_{L^{p_i(.)}} - \sum_{i\in\mathcal{F}_1} \|\partial_{x_i} u_n\|^{p_m^-}_{L^{p_i(.)}},\end{aligned}$$

so,

$$\sum_{i=1}^{N} \int_\Omega |\partial_{x_i} u_n|^{p_i(x)} dx \geq \sum_{i=1}^{N} \|\partial_{x_i} u_n\|^{p_m^-}_{L^{p_i(.)}} - N. \tag{19}$$

and

$$\begin{aligned}\sum_{i=1}^{N} \frac{1}{p_i(x)} \left\|\frac{\partial u}{\partial x_i}\right\|_{p_i(.)} &\geq \frac{1}{p_M^+} \sum_{i\in\mathcal{F}_1} \left\|\frac{\partial u}{\partial x_i}\right\|^{p_i^+}_{p_i(.)} + \frac{1}{p_M^+} \sum_{i\in\mathcal{F}_2} \left\|\frac{\partial u}{\partial x_i}\right\|^{p_i^-}_{p_i(.)} \\ &\geq \frac{1}{p_M^+}\left(\frac{1}{p_M^+}\sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|^{p_m^-}_{p_i(.)} - N\right) \\ &\geq \frac{1}{p_M^+}\left[\frac{1}{N^{p_m^-}-1}\left(\sum_{i=1^N} \left\|\frac{\partial u}{\partial x_i}\right\|_{p_i(.)}\right)^{p_m^-} - N\right].\end{aligned} \tag{20}$$

If $\|u\|_{p_M}(.) \geq 1$, the same lines as in [26], and [11] hold, then we have

$$\begin{aligned} I(u) &\geq \frac{1}{p_M^+}\left[\frac{1}{N^{p_m^-}-1}\left(\sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_{p_i(.)}\right)^{p_m^-} - N + \frac{1}{p_M^+}\|u\|^{p_m^-}_{p_M(.)}\right] \\ &\geq \frac{1}{p_M^+}\left[\frac{1}{N^{p_m^-}-1}\left(\sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_{p_i(.)}\right)^{p_m^-} + \|u\|^{p_m^-}_{p_M(.)}\right] - \frac{N}{p_M^+} \\ &\geq \frac{1}{2^{p_m^-}-1\, p_M^+}\inf\left\{1, \frac{1}{N^{p_m^-}-1}\right\}\left[\sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_{p_i(.)} + \|u\|_{p_M(.)}\right]^{p_m^-} - \frac{N}{p_M^+} \\ &\geq \frac{1}{2^{p_m^-}-1\, p_M^+}\inf\left\{1, \frac{1}{N^{p_m^-}-1}\right\}\|u\|^{p_m^-} - \frac{N}{p_M^+}. \end{aligned} \tag{21}$$

If $\|u\|_{p_M}(.) < 1$ we have

$$\begin{aligned} I(u) &\geq \frac{1}{p_M^+}\left[\frac{1}{N^{p_m^-}-1}\left(\sum_{i=1}^{N} \|\partial_{x_i} u\|_{p_i(.)}\right)^{p_m^-} + \|u\|^{p_m^-}_{p_M(.)} - 1 - N\right] \quad (because \|u\|_{p_M(.)} - 1 < 0) \\ &\geq \frac{1}{p_M^+}\left[\frac{1}{N^{p_m^-}-1}\left(\sum_{i=1}^{N} \|\partial_{x_i} u\|_{p_i(.)}\right)^{p_m^-} + \|u\|^{p_m^-}_{p_M(.)}\right] - \frac{N-1}{p_M^+} \\ &\geq \frac{1}{2^{p_m^-}-1\, p_M^+}\inf\left\{1, \frac{1}{N^{p_m^-}-1}\right\}\|u\|^{p_m^-} - \frac{N-1}{p_M^+}. \end{aligned} \tag{22}$$

Now, let us prove that ϕ is coercive: for $\| u \| > 1$, by $(H4)$, in either case (22) or (21) it yields

$$\begin{aligned}
\phi(u) &= \sum_{i=1}^{N} \int_{\Omega} \frac{1}{p_i(x)} |\partial_{x_i} u|^{p_i(x)}\, dx + \int_{\Omega} \frac{1}{p_M(x)} |u|^{p_M(x)}\, dx - \int_{\Omega} F(x,u) dx \\
&\geq \sum_{i=1}^{N} \int_{\Omega} \frac{1}{p_i(x)} |\partial_{x_i} u|^{p_i(x)}\, dx + \int_{\Omega} \frac{1}{p_M(x)} |u|^{p_M(x)}\, dx - (\lambda_1 - \epsilon) \int_{\Omega} \frac{| u |^{p_M(x)}}{p_M(x)} dx \\
&\geq \sum_{i=1}^{N} \int_{\Omega} \frac{1}{p_i(x)} |\partial_{x_i} u|^{p_i(x)} + \int_{\Omega} \frac{1}{p_M(x)} |u|^{p_M(x)}\, dx \\
&\quad - \frac{(\lambda_1 - \epsilon)}{\lambda_1} \sum_{i=1}^{N} \int_{\Omega} \frac{1}{p_i(x)} |\partial_{x_i} u|^{p_i(x)}\, dx + \int_{\Omega} \frac{1}{p_M(x)} |u|^{p_M(x)}\, dx \\
&\geq \frac{1}{2^{p_m^- - 1} p_M^+} \inf\left\{1, \frac{1}{N^{p_m^- - 1}}\right\} \left(1 - \frac{(\lambda_1 - \epsilon)}{\lambda_1}\right) \| u \|^{p_m^-} - c.
\end{aligned}$$

Thus, ϕ is coercive and has a global minimizer u_1 that means $\phi'(u_1)u_1 = 0$, which is nontrivial. Indeed, fixing $v_0 \in X \setminus \{0\}$ and $t > 0$ to be small enough, from $(H5)$, we obtain that

$$\begin{aligned}
\phi(tv_0) &\leq C_2 \left(\int_{\Omega} \frac{t^{p_i(x)}}{p_i(x)} |v_0|^{p_i(x)} dx + \int_{\Omega} \frac{t^{p_M(x)}}{p_M(x)} |v_0|^{p_M(x)} dx \right) - \int_{\Omega} F(x, tv_0) dx \\
&\leq C_3 t^{p_m^-} - C_4 t^{q_0} < 0, \qquad (23)
\end{aligned}$$

because $q_0 < p_m^-$. $\square$

5. Conclusions

In this work, we studied a kind of elliptic problem in an anisotropic form concerning the Sobolev space with variable exponents. By using the variational approach, and without assuming the Ambrosetti–Rabinowitz type conditions, we proved the existence of a nontrivial solution.

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