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Rational Type Contractions in Extended *b*-Metric Spaces

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Abstract: In this paper, we establish the existence of fixed points of rational type contractions in the setting of extended *b*-metric spaces. Our results extend considerably several well-known results in the existing literature. We present some nontrivial examples to show the validity of our results. Furthermore, as applications, we obtain the existence of solution to a class of Fredholm integral equations.

Keywords: comparison function; α-admissible; rational type contraction; extended *b*-metric space; Fredholm integral equation

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1. Introduction and Preliminaries

The concept of distance between two abstract objects has received importance not only for mathematical analysis but also for its related fields. Bakhtin [1] introduced *b*-metric spaces as a generalization of metric spaces (see also Czerwik [2]). Recently, Kamran et al. [3] gave the notion of extended *b*-metric space and presented a counterpart of Banach contraction mapping principle. On the other hand, fixed point results dealing with general contractive conditions involving rational type expression are also interesting. Some well-known results in this direction are involved (see [4–10]).

First, of all, we recall some fixed point theorems for rational type contractions in metric spaces.

Theorem 1 ([5]). *Let T be a continuous self mapping on a complete metric space (X,d). If T is a rational type contraction, there exist $\alpha,\beta\in[0,1)$, where $\alpha+\beta<1$ such that*

$$d(Tx,Ty)\leq \alpha d(x,y)+\beta\frac{d(x,Tx)d(y,Ty)}{d(x,y)},$$

for all $x,y\in X$, $x\neq y$, then T has a unique fixed point in X.

Theorem 2 ([4]). *Let T be a continuous self mapping on a complete metric space (X,d). If T is a rational type contraction, there exist $\alpha,\beta\in[0,1)$, where $\alpha+\beta<1$ such that*

$$d(Tx,Ty)\leq \alpha d(x,y)+\beta\cdot\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)},$$

for all $x,y\in X$, then T has a unique fixed point in X.

Fisher [11] refined the result of Khan [6] in the following way.

Theorem 3 ([11]). *Let T be a self mapping on a complete metric space (X, d). If T is a rational type contraction, T satisfies the inequality*

$$d(Tx, Ty) \leq k \begin{cases} \frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Ty)+d(y,Tx)}, & \text{if } d(x,Ty) + d(y,Tx) \neq 0, \\ 0, & \text{if } d(x,Ty) + d(y,Tx) = 0, \end{cases}$$

for all $x, y \in X$, where $0 \leq k < 1$. Then, T has a unique fixed point in X.

Ahmad et al. [12] extended Theorem 3 from metric spaces to generalized metric spaces (see [13] for more details). Piri et al. [14] extended the result of Ahmad et al. [12] in the following way.

Theorem 4 ([14]). *Let T be a self mapping on a complete generalized metric space (X, d_g). If T is a rational type contraction, T satisfies the inequality*

$$d_g(Tx, Ty) \leq k \begin{cases} \max\left\{d_g(x,y), \frac{d_g(x,Tx)d_g(x,Ty)+d_g(y,Ty)d_g(y,Tx)}{\mathcal{A}_0(x,y)}\right\}, & \text{if } \mathcal{A}_0(x,y) \neq 0, \\ 0, & \text{if } \mathcal{A}_0(x,y) = 0, \end{cases}$$

for all $x, y \in X, x \neq y$, where $0 \leq k < 1$ and $\mathcal{A}_0(x,y) = \max\{d_g(x,Ty), d_g(y,Tx)\}$. Then, T has a unique fixed point in X.

Let us recall some basic concepts in b-metric spaces as follows.

Definition 1 ([1,2]). *Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d_b : X \times X \to [0, +\infty)$ is called a b-metric on X, if, for all $x, y, z \in X$, the following conditions hold:*

(d_b1) $d_b(x,y) = 0$ if and only if $x = y$;
(d_b2) $d_b(x,y) = d_b(y,x)$;
(d_b3) $d_b(x,y) \leq s[d_b(x,z) + d_b(z,y)]$.

In this case, the pair (X, d_b) is called a b-metric space.

It is well-known that any b-metric space will become a metric space if $s = 1$. However, any metric space does not necessarily be a b-metric space if $s > 1$. In other words, b-metric spaces are more general than metric spaces (see [15]).

The following example gives us evidence that b-metric space is indeed different from metric space.

Example 1 ([16]). *Let (X, d) be a metric space and $d_b(x,y) = (d(x,y))^p$ for all $x, y \in X$, where $p > 1$ is a real number. Then, (X, d_b) is a b-metric space with $s = 2^{p-1}$. However, (X, d_b) is not a metric space.*

Definition 2 ([17]). *Let $\{x_n\}$ be a sequence in a b-metric space (X, d_b). Then,*

(i) $\{x_n\}$ is called a convergent sequence, if, for each $\epsilon > 0$, there exists $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that $d_b(x_n, x) < \epsilon$, for all $n \geq n_0$, and we write $\lim_{n \to \infty} x_n = x$;

(ii) $\{x_n\}$ is called a Cauchy sequence, if, for each $\epsilon > 0$, there exists $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that $d_b(x_n, x_m) < \epsilon$, for all $n, m \geq n_0$;

(iii) (X, d_b) is said to be complete if every Cauchy sequence is convergent in X.

The following theorem is a basic theorem for Banach type contraction in b-metric space.

Theorem 5 ([18]). *Let T be a self mapping on a complete b-metric space (X, d_b). Then, T has a unique fixed point in X if*

$$d_b(Tx, Ty) \leq k d_b(x, y)$$

holds for all $x, y \in X$, where $k \in [0,1)$ is a constant. Moreover, for any $x_0 \in X$, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to the fixed point.

Note that the distance function d_b utilized in b-metric spaces is generally discontinuous (see [15,19]). For fixed point results and more examples in b-metric spaces, the readers may refer to [15–18].

In what follows, we recall the concept of extend b-metric space and some examples.

Definition 3 ([3])**.** *Let X be a nonempty set. Suppose that $\theta : X \times X \to [1,+\infty)$ and $d_\theta : X \times X \to [0,+\infty)$ are two mappings. If for all $x, y, z \in X$, the following conditions hold:*
$(d_\theta 1)$ $d_\theta(x,y) = 0$ *if and only if* $x = y$;
$(d_\theta 2)$ $d_\theta(x,y) = d_\theta(y,x)$;
$(d_\theta 3)$ $d_\theta(x,y) \leq \theta(x,y)[d_\theta(x,z) + d_\theta(z,y)]$,
then d_θ is called an extended b-metric, and the pair (X, d_θ) is called an extended b-metric space.

Note that, if $1 \leq \theta(x,y) = s$ (a finite constant), for all $x, y \in X$, then extended b-metric space reduces to a b-metric space. That is to say, b-metric space is a generalization of metric space, and extended b-metric space is a generalization of b-metric space.

In the following, we introduce some examples for extended b-metric spaces.

Example 2. *Let $X = [0,+\infty)$. Define two mappings $\theta : X \times X \to [1,+\infty)$ and $d_\theta : X \times X \to [0,+\infty)$ as follows: $\theta(x,y) = 1 + x + y$, for all $x, y \in X$, and*

$$d_\theta(x,y) = \begin{cases} x + y, & x, y \in X,\ x \neq y, \\ 0, & x = y. \end{cases}$$

Then, (X, d_θ) is an extended b-metric space.

Indeed, $(d_\theta 1)$ and $(d_\theta 2)$ in Definition 3 are clear. Let $x, y, z \in X$. We prove that $(d_\theta 3)$ in Definition 3 is satisfied.

(i) *If $x = y$, then $(d_\theta 3)$ is clear.*

(ii) *If $x \neq y$, $x = z$, then*

$$\begin{aligned} \theta(x,y)[d_\theta(x,z) + d_\theta(z,y)] &= (1 + x + y)[0 + (z + y)] \\ &= (1 + x + y)(x + y) \\ &\geq x + y = d_\theta(x,y). \end{aligned}$$

(iii) *If $x \neq y$, $y = z$, then*

$$\begin{aligned} \theta(x,y)[d_\theta(x,z) + d_\theta(z,y)] &= (1 + x + y)[(x + z) + 0] \\ &= (1 + x + y)(x + y) \\ &\geq x + y = d_\theta(x,y). \end{aligned}$$

(iv) *If $x \neq y$, $y \neq z$, $x \neq z$, then*

$$\begin{aligned} \theta(x,y)[d_\theta(x,z) + d_\theta(z,y)] &= (1 + x + y)[(x + z) + (z + y)] \\ &\geq x + 2z + y \\ &\geq x + y = d_\theta(x,y). \end{aligned}$$

Consider the above cases, it follows that $(d_\theta 3)$ holds. Hence, the claim holds.

Example 3. *Let $X = \mathbb{R}$. Define two mappings $\theta : X \times X \to [1, +\infty)$ and $d_\theta : X \times X \to [0, +\infty)$ as follows: $\theta(x, y) = 1 + |x| + |y|$, for all $x, y \in X$ and*

$$d_\theta(x, y) = \begin{cases} x^2 + y^2, & x, y \in X,\ x \neq y, \\ 0, & x = y. \end{cases}$$

Then, (X, d_θ) is an extended b-metric space.

Indeed, $(d_\theta 1)$ and $(d_\theta 2)$ in Definition 3 are obvious. Let $x, y, z \in X$. We prove that $(d_\theta 3)$ in Definition 3 is satisfied.

(i) If $x = y$, then $(d_\theta 3)$ is obvious.

(ii) If $x \neq y$, $x = z$, then

$$\begin{aligned} \theta(x, y)[d_\theta(x, z) + d_\theta(z, y)] &= (1 + |x| + |y|)[0 + (z^2 + y^2)] \\ &= (1 + |x| + |y|)(x^2 + y^2) \\ &\geq x^2 + y^2 = d_\theta(x, y). \end{aligned}$$

(iii) If $x \neq y$, $y = z$, then

$$\begin{aligned} \theta(x, y)[d_\theta(x, z) + d_\theta(z, y)] &= (1 + |x| + |y|)[(x^2 + z^2) + 0] \\ &= (1 + |x| + |y|)(x^2 + y^2) \\ &\geq x^2 + y^2 = d_\theta(x, y). \end{aligned}$$

(iv) If $x \neq y$, $y \neq z$, $x \neq z$, then

$$\begin{aligned} \theta(x, y)[d_\theta(x, z) + d_\theta(z, y)] &= (1 + |x| + |y|)[(x^2 + z^2) + (z^2 + y^2)] \\ &\geq (1 + |x| + |y|)(x^2 + y^2) \\ &\geq x^2 + y^2 = d_\theta(x, y). \end{aligned}$$

Consider the above cases, it follows that $(d_\theta 3)$ holds. Hence, the claim holds.

Example 4. *Let $X = \mathbb{R}$. Define two mappings $d_\theta : X \times X \to [0, +\infty)$ and $\theta : X \times X \to [1, +\infty)$ as follows:*

$$d_\theta(x, y) = \begin{cases} \frac{|x| + |y|}{1 + |x| + |y|}, & x, y \in X,\ x \neq y, \\ 0, & x = y, \end{cases}$$

and $\theta(x, y) = 1 + |x| + |y|$, for all $x, y \in X$. Then, (X, d_θ) is an extended b-metric space.

Indeed, $(d_\theta 1)$ and $(d_\theta 2)$ in Definition 3 are valid. Let $x, y, z \in X$. We prove that $(d_\theta 3)$ in Definition 3 is satisfied.

(i) If $x = y$, then $(d_\theta 3)$ holds.

(ii) If $x \neq y$, $x = z$, then

$$\begin{aligned} \theta(x, y)[d_\theta(x, z) + d_\theta(z, y)] &= (1 + |x| + |y|)\left(0 + \frac{|z| + |y|}{1 + |z| + |y|}\right) \\ &= (1 + |x| + |y|) \cdot \frac{|x| + |y|}{1 + |x| + |y|} \\ &\geq \frac{|x| + |y|}{1 + |x| + |y|} \\ &= d_\theta(x, y). \end{aligned}$$

(iii) *If $x \neq y$, $y = z$, then*

$$\begin{aligned}
\theta(x,y)[d_\theta(x,z) + d_\theta(z,y)] &= (1+|x|+|y|)\left(\frac{|x|+|z|}{1+|x|+|z|} + 0\right) \\
&= (1+|x|+|y|) \cdot \frac{|x|+|y|}{1+|x|+|y|} \\
&\geq \frac{|x|+|y|}{1+|x|+|y|} \\
&= d_\theta(x,y).
\end{aligned}$$

(iv) *If $x \neq y$, $y \neq z$, $x \neq z$, then, by the fact that $f(t) = \frac{t}{1+t}$ is nondecreasing on $[0,+\infty)$ and $|x|+|y| \leq |x|+|z|+|y|$, it follows that*

$$\begin{aligned}
\theta(x,y)[d_\theta(x,z) + d_\theta(z,y)] &= (1+|x|+|y|)\left(\frac{|x|+|z|}{1+|x|+|z|} + \frac{|z|+|y|}{1+|z|+|y|}\right) \\
&\geq (1+|x|+|y|)\left(\frac{|x|+|z|}{1+|x|+|z|+|y|} + \frac{|z|+|y|}{1+|x|+|z|+|y|}\right) \\
&= (1+|x|+|y|) \cdot \frac{|x|+2|z|+|y|}{1+|x|+|z|+|y|} \\
&\geq \frac{|x|+|z|+|y|}{1+|x|+|z|+|y|} \\
&\geq \frac{|x|+|y|}{1+|x|+|y|} = d_\theta(x,y).
\end{aligned}$$

Consider the above cases, it follows that $(d_\theta 3)$ holds. Hence, the claim holds.

Example 5. *Let $X = [0,+\infty)$ and $\theta(x,y) = \frac{3+x+y}{2}$ be a function on $X \times X$. Define a mapping $d_\theta : X \times X \to [0,+\infty)$ as follows:*

$$\begin{aligned}
d_\theta(x,y) =& 0, \text{ for all } x,y \in X,\ x = y, \\
d_\theta(x,y) =& d_\theta(y,x) = 5, \text{ for all } x,y \in X \setminus \{0\}, x \neq y, \\
d_\theta(x,0) =& d_\theta(0,x) = 2, \text{ for all } x \in X \setminus \{0\}.
\end{aligned}$$

Then, (X, d_θ) is an extended-b metric space.
As a matter of fact, obviously, $(d_\theta 1)$ and $(d_\theta 2)$ hold. For $(d_\theta 3)$, we have the following cases:
(i) Let $x,y,z \in X \setminus \{0\}$ such that x,y and z are distinct each other, then

$$d_\theta(x,y) = 5 \leq 5(3+x+y) = \theta(x,y)[d_\theta(x,z) + d_\theta(z,y)].$$

(ii) Let $x,y \in X \setminus \{0\}, x \neq y$ and $z = 0$, then

$$d_\theta(x,y) = 5 \leq 2(3+x+y) = \theta(x,y)[d_\theta(x,0) + d_\theta(0,y)].$$

(iii) Let $x,z \in X \setminus \{0\}, x \neq z$ and $y = 0$, then

$$d_\theta(x,0) = 2 \leq \frac{7}{2}(3+x) = \theta(x,0)[d_\theta(x,z) + d_\theta(z,y)].$$

Therefore, $(d_\theta 3)$ in Definition 3 holds. Thus, the claims hold.

Remark 1. *Examples 2–5 are extended b-metric spaces but not b-metric spaces.*

Similar to Definition 2, we recall some concepts in extended b-metric spaces as follows.

Definition 4 ([3])**.** *Let $\{x_n\}$ be a sequence in an extended b-metric space (X, d_θ). Then,* *(i)* $\{x_n\}$ *is called a convergent sequence, if, for each* $\epsilon > 0$*, there exists* $n_0 = n_0(\epsilon) \in \mathbb{N}$ *such that* $d_\theta(x_n, x) < \epsilon$*, for all* $n \geq n_0$*, and we write* $\lim_{n\to\infty} x_n = x$*;*

(ii) $\{x_n\}$ *is called a Cauchy sequence, if, for each* $\epsilon > 0$*, there exists* $n_0 = n_0(\epsilon) \in \mathbb{N}$ *such that* $d_\theta(x_n, x_m) < \epsilon$*, for all* $n, m \geq n_0$*;*

(iii) (X, d_θ) *is said to be complete if every Cauchy sequence is convergent in* X*.*

As we know, the limit of convergent sequence in extended b-metric space (X, d_θ) is unique provided that d_θ is a continuous mapping (see [3]).

Definition 5 ([20,21])**.** *Let* T *be a self mapping on an extended b-metric space* (X, d_θ)*. For* $x_0 \in X$*, the set*

$$O(x_0, T) = \{x_0, Tx_0, T^2x_0, T^3x_0, \cdots\}$$

is said to be an orbit of T *at* x_0*.* T *is said to be orbitally continuous at* $\xi \in X$ *if* $\lim_{k\to\infty} T^k x_0 = \xi$ *implies* $\lim_{k\to\infty} TT^k x_0 = T\xi$*. Moreover, if every Cauchy sequence of the form* $\{T^k x_0\}_{k=1}^{\infty}$ *is convergent to some point in* X*, then* (X, d_θ) *is said to be a* T*-orbitally complete space.*

Note that, if (X, d_θ) is complete extended b-metric space, then X is T-orbitally complete for any self-mapping T on X. Moreover, if T is continuous, then it is obviously orbitally continuous in X. However, the converse may not be true.

In the sequel, unless otherwise specified, we always denote $Fix(T) = \{x \in X | Tx = x\}$.

Definition 6 ([22])**.** *Let* X *be a nonempty set and* $\alpha : X \times X \to \mathbb{R}$ *be a mapping. A mapping* $T : X \to X$ *is called* α*-admissible, if for all* $x, y \in X$*,* $\alpha(x, y) \geq 1$ *implies* $\alpha(Tx, Ty) \geq 1$*.*

Definition 7 ([23])**.** *Let* X *be a nonempty set and* $\alpha : X \times X \to \mathbb{R}$ *be a mapping. Then,* $T : X \to X$ *is called* α^**-admissible if it is a* α*-admissible mapping and* $\alpha(x, y) \geq 1$ *holds for all* $x, y \in Fix(T) \neq \emptyset$*.*

Example 6. *Let* $X = [0, +\infty)$ *and* $T : X \to X$ *be a mapping defined by* $Tx = \frac{x(1+x)}{2}$*. Let* $\alpha : X \times X \to \mathbb{R}$ *be a function defined by*

$$\alpha(x, y) = \begin{cases} 1, & x, y \in [0, 1], \\ 0, & otherwise. \end{cases}$$

Then, T *is* α*-admissible and* $Fix(T) = \{0, 1\}$*. Moreover,* $\alpha(x, y) \geq 1$ *is satisfied for all* $x, y \in Fix(T)$*. Consequently,* T *is* α^**-admissible.*

Example 7 ([23])**.** *Let* $X = [0, +\infty)$ *and* $T : X \to X$ *be a mapping defined by* $Tx = \sqrt{\frac{x(x^2+2)}{3}}$*. Let* $\alpha : X \times X \to [0, +\infty)$ *be a function defined by*

$$\alpha(x, y) = \begin{cases} 1, & x, y \in [0, 1], \\ 0, & otherwise. \end{cases}$$

Then, T *is a* α*-admissible mapping and* $Fix(T) = \{0, 1, 2\}$*. However,* $\alpha(x, 2) = \alpha(2, x) = 0$ *is satisfied for* $x \in \{0, 1\}$*. Thus,* T *is not* α^**-admissible.*

Definition 8 ([24])**.** *Let* T *be a self mapping on a nonempty set* X*. Then,* T *is called* α*-orbitally admissible if, for all* $x \in X$*,* $\alpha(x, Tx) \geq 1$ *leads to* $\alpha(Tx, T^2x) \geq 1$*.*

It is mentioned that each α-admissible mapping must be an α-orbitally admissible mapping (for more details, see [24]). For the uniqueness of fixed point, we will use the following definition frequently.

Definition 9. *An α-orbitally admissible mapping T is called α^*-orbitally admissible if $x, x^* \in Fix(T) \neq \emptyset$ implies $\alpha(x, x^*) \geq 1$.*

Definition 10 ([17,25])**.** *A function $\psi : [0, +\infty) \to [0, +\infty)$ is said to be a comparison function, if it is nondecreasing and $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$, where ψ^n denotes the n^{th} iteration of ψ.*

In what follows, the set of all comparison functions is denoted by Ψ. Some examples for comparison functions, the reader may refer to [26].

Lemma 1 ([27])**.** *Let $\psi \in \Psi$. Then, $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$.*

The following lemmas will be used in the sequel.

Lemma 2 ([28])**.** *Let (X, d_θ) be an extended b-metric space, $x_0 \in X$ and $\{x_n\}$ be a sequence in X. If $\psi \in \Psi$ satisfies*

$$\lim_{n,m\to\infty} \frac{\theta(x_n, x_m)\psi^n\big(d_\theta(x_0, x_1)\big)}{\psi^{n-1}\big(d_\theta(x_0, x_1)\big)} < 1 \tag{1}$$

and

$$0 < d_\theta(x_n, x_{n+1}) \leq \psi\big(d_\theta(x_{n-1}, x_n)\big)$$

for all $m > n \geq 2$, $n, m \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence in X.

Proof. From the given conditions, we get

$$0 < d_\theta(x_n, x_{n+1}) \leq \psi\big(d_\theta(x_{n-1}, x_n)\big) \leq \cdots \leq \psi^n\big(d_\theta(x_0, x_1)\big).$$

On taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} d_\theta(x_n, x_{n+1}) = 0.$$

Setting $\theta_i = \theta(x_i, x_{n+p})$ for each $i \in \mathbb{N}$, $p \geq 1$ and $d_\theta(x_0, x_1) = t$, we obtain

$$\begin{aligned}
d_\theta(x_n, x_{n+p}) &\leq \theta(x_n, x_{n+p})\big[d_\theta(x_n, x_{n+1}) + d_\theta(x_{n+1}, x_{n+p})\big] \\
&\leq \theta(x_n, x_{n+p})d_\theta(x_n, x_{n+1}) + \theta(x_n, x_{n+p}) \\
&\quad \cdot \theta(x_{n+1}, x_{n+p})[d_\theta(x_{n+1}, x_{n+2}) + d_\theta(x_{n+2}, x_{n+p})] \\
&\leq \cdots\cdots \\
&\leq \theta(x_n, x_{n+p})d_\theta(x_n, x_{n+1}) + \theta(x_n, x_{n+p})\theta(x_{n+1}, x_{n+p})d_\theta(x_{n+1}, x_{n+2}) \\
&\quad + \theta(x_n, x_{n+p})\theta(x_{n+1}, x_{n+p})\theta(x_{n+2}, x_{n+p})d_\theta(x_{n+2}, x_{n+3}) \\
&\quad + \cdots + \theta(x_n, x_{n+p})\theta(x_{n+1}, x_{n+p})\cdots\theta(x_{n+p-2}, x_{n+p})d_\theta(x_{n+p-2}, x_{n+p-1}) \\
&\quad + \cdots + \theta(x_n, x_{n+p})\theta(x_{n+1}, x_{n+p})\cdots\theta(x_{n+p-2}, x_{n+p})d_\theta(x_{n+p-1}, x_{n+p}) \\
&\leq \theta(x_n, x_{n+p})d_\theta(x_n, x_{n+1}) + \theta(x_n, x_{n+p})\theta(x_{n+1}, x_{n+p})d_\theta(x_{n+1}, x_{n+2}) \\
&\quad + \theta(x_n, x_{n+p})\theta(x_{n+1}, x_{n+p})\theta(x_{n+2}, x_{n+p})d_\theta(x_{n+2}, x_{n+3}) \\
&\quad + \cdots + \theta(x_n, x_{n+p})\theta(x_{n+1}, x_{n+p})\cdots\theta(x_{n+p-2}, x_{n+p})d_\theta(x_{n+p-2}, x_{n+p-1}) \\
&\quad + \cdots + \theta(x_n, x_{n+p})\theta(x_{n+1}, x_{n+p})\cdots\theta(x_{n+p-1}, x_{n+p})d_\theta(x_{n+p-1}, x_{n+p}) \\
&\leq \theta_n\psi^n\big(d_\theta(x_0, x_1)\big) + \theta_n\theta_{n+1}\psi^{n+1}\big(d_\theta(x_0, x_1)\big) \\
&\quad + \cdots + \theta_n\theta_{n+1}\cdots\theta_{n+p-1}\psi^{n+p-1}\big(d_\theta(x_0, x_1)\big) \\
&= \theta_n\psi^n(t) + \theta_n\theta_{n+1}\psi^{n+1}(t) + \cdots + \theta_n\theta_{n+1}\cdots\theta_{n+p-1}\psi^{n+p-1}(t) \\
&= \sum_{i=n}^{n+p-1} \psi^i(t)\prod_{j=n}^{i}\theta_j \leq \sum_{i=n}^{n+p-1} \psi^i(t)\prod_{j=1}^{i}\theta_j
\end{aligned}$$

$$= \sum_{i=1}^{n+p-1} \psi^i(t) \prod_{j=1}^{i} \theta_j - \sum_{i=1}^{n-1} \psi^i(t) \prod_{j=1}^{i} \theta_j.$$

Notice that

$$\lim_{n\to\infty} \frac{\theta(x_n, x_{n+p})\psi^n\left(d_\theta(x_0, x_1)\right)}{\psi^{n-1}\left(d_\theta(x_0, x_1)\right)} = \lim_{n\to\infty} \frac{\theta_n \psi^n(t)}{\psi^{n-1}(t)} < 1,$$

then, by the Ratio test the series, $\sum_{i=1}^{\infty}\psi^i(t)\prod_{j=1}^{i}\theta_j$ converges.

Let $S = \sum_{i=1}^{\infty}\psi^i(t)\prod_{j=1}^{i}\theta_j$ and $S_n = \sum_{i=1}^{n}\psi^i(t)\prod_{j=1}^{i}\theta_j$ be the sequence of partial sum. Consequently, for any $n \geq 1$ and $p \geq 1$, we obtain

$$d_\theta(x_n, x_{n+p}) \leq S_{n+p-1} - S_{n-1}.$$

Taking the limit as $n \to \infty$ from both side of the above inequality, we make a conclusion that $\{x_n\}$ is a Cauchy sequence in X. □

Lemma 3 ([29]). *Let $\{x_n\}$ be a sequence in an extended b-metric space (X, d_θ) such that*

$$\lim_{n,m\to\infty} \theta(x_n, x_m) < \frac{1}{k}$$

and

$$0 < d_\theta(x_n, x_{n+1}) \leq k d_\theta(x_{n-1}, x_n)$$

for any $m > n \geq 2$, $n, m \in \mathbb{N}$, where $k \in [0, 1)$, then $\{x_n\}$ is a Cauchy sequence in X.

Proof. Choose $\psi(t) = kt$, where $k \in [0, 1)$ in Lemma 2. Then, the proof is completed. □

2. Fixed Points of Rational Type Contractions

In this section, we assume that (X, d_θ) is an extended b-metric space with the continuous functional d_θ. Let $T : X \to X$ be a mapping. For $x, y \in X$, we always denote

$$\begin{aligned}\mathcal{N}(x,y) = \max\Big\{&d_\theta(x,y), \frac{d_\theta(y,Ty)d_\theta(x,Tx)}{d_\theta(x,y)}, \frac{d_\theta(x,Tx)[1+d_\theta(y,Ty)]}{1+d_\theta(x,y)},\\ &\frac{d_\theta(y,Ty)[1+d_\theta(x,Tx)]}{1+d_\theta(x,y)}\Big\},\\ \mathcal{K}(x,y) = \max\Big\{&d_\theta(x,y), \frac{d_\theta(x,Tx)d_\theta(x,Ty)+d_\theta(y,Ty)d_\theta(y,Tx)}{\max\{d_\theta(x,Ty), d_\theta(y,Tx)\}},\\ &\frac{d_\theta(x,Tx)d_\theta(y,Ty)+d_\theta(x,Ty)d_\theta(y,Tx)}{\max\{d_\theta(y,Ty), d_\theta(y,Tx)\}}\Big\}.\end{aligned}$$

Theorem 6. *Let T be a self mapping on a T-orbitally complete extended b-metric space (X, d_θ). Assume that there exist two functions $\alpha : X \times X \to [0, +\infty)$, $\psi \in \Psi$ such that*

$$\alpha(x,y)d_\theta(Tx,Ty) \leq \psi(\mathcal{N}(x,y)) \tag{2}$$

for all $x, y \in X$, $x \neq y$. That is, T is a rational type contraction. If

(i) T is α-orbitally admissible;
(ii) there exists $x_0 \in X$ satisfying $\alpha(x_0, Tx_0) \geq 1$;
(iii) (1) is satisfied for $x_n = T^n x_0$ $(n = 0, 1, 2, \cdots)$;

(iv) T is either continuous or, orbitally continuous on X.

Then, T possesses a fixed point $z \in X$. Moreover, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to $z \in X$.

Proof. By (ii), define a sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n = T^{n+1}x_0$, for all $n \in \mathbb{N} \cup \{0\}$.

If $x_n = x_{n+1}$, for, some $n \in \mathbb{N} \cup \{0\}$, then x_n is a fixed point of T. This completes the proof. Without loss of generality, we therefore assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N} \cup \{0\}$.

Based on (i), $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$ implies that $\alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geq 1$. Then, $\alpha(x_2, x_3) = \alpha(Tx_1, Tx_2) \geq 1$. Continuing this process, one has $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$.

Taking $x = x_{n-1}$ and $y = x_n$, for all $n \in \mathbb{N}$ in (2), we have

$$\begin{aligned} d_\theta(x_n, x_{n+1}) &= d_\theta(Tx_{n-1}, Tx_n) \\ &\leq \alpha(x_{n-1}, x_n) d_\theta(Tx_{n-1}, Tx_n) \\ &\leq \psi(\mathcal{N}(x_{n-1}, x_n)), \end{aligned} \tag{3}$$

where

$$\begin{aligned} &\mathcal{N}(x_{n-1}, x_n) \\ &= \max\Big\{ d_\theta(x_{n-1}, x_n), \frac{d_\theta(x_n, Tx_n) d_\theta(x_{n-1}, Tx_{n-1})}{d_\theta(x_{n-1}, x_n)}, \\ &\quad \frac{d_\theta(x_{n-1}, Tx_{n-1})[1 + d_\theta(x_n, Tx_n)]}{1 + d_\theta(x_{n-1}, x_n)}, \frac{d_\theta(x_n, Tx_n)[1 + d_\theta(x_{n-1}, Tx_{n-1})]}{1 + d_\theta(x_{n-1}, x_n)} \Big\} \\ &= \max\Big\{ d_\theta(x_{n-1}, x_n), \frac{d_\theta(x_n, x_{n+1}) d_\theta(x_{n-1}, x_n)}{d_\theta(x_{n-1}, x_n)}, \\ &\quad \frac{d_\theta(x_{n-1}, x_n)[1 + d_\theta(x_n, x_{n+1})]}{1 + d_\theta(x_{n-1}, x_n)}, \frac{d_\theta(x_n, x_{n+1})[1 + d_\theta(x_{n-1}, x_n)]}{1 + d_\theta(x_{n-1}, x_n)} \Big\} \\ &= \max\Big\{ d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1}), \frac{d_\theta(x_{n-1}, x_n)[1 + d_\theta(x_n, x_{n+1})]}{1 + d_\theta(x_{n-1}, x_n)} \Big\}. \end{aligned} \tag{4}$$

Similar to ([10], Theorem 2.1), we can prove

$$0 < d_\theta(x_n, x_{n+1}) \leq \psi(d_\theta(x_{n-1}, x_n)), \text{ for all } n \in \mathbb{N}. \tag{5}$$

In fact, we finish the proof via three cases.

(i) If $\mathcal{N}(x_{n-1}, x_n) = d_\theta(x_{n-1}, x_n)$, then by (3), it follows that

$$0 < d_\theta(x_n, x_{n+1}) \leq \psi(d_\theta(x_{n-1}, x_n)).$$

This is (5).

(ii) If $\mathcal{N}(x_{n-1}, x_n) = d_\theta(x_n, x_{n+1})$, then by (3), we have

$$0 < d_\theta(x_n, x_{n+1}) \leq \psi(d_\theta(x_n, x_{n+1})) < d_\theta(x_n, x_{n+1}),$$

which is a contradiction.

(iii) If $\mathcal{N}(x_{n-1}, x_n) = \frac{d_\theta(x_{n-1}, x_n)[1 + d_\theta(x_n, x_{n+1})]}{1 + d_\theta(x_{n-1}, x_n)}$, then by (4), it is easy to say that

$$\max\{d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1})\} \leq \frac{d_\theta(x_{n-1}, x_n)[1 + d_\theta(x_n, x_{n+1})]}{1 + d_\theta(x_{n-1}, x_n)}. \tag{6}$$

In this case, we discuss it with two subcases.

(i) If $\max\{d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1})\} = d_\theta(x_{n-1}, x_n)$, then

$$d_\theta(x_{n-1}, x_n) > d_\theta(x_n, x_{n+1}). \tag{7}$$

By (6), we get

$$d_\theta(x_{n-1},x_n) \leq \frac{d_\theta(x_{n-1},x_n)[1+d_\theta(x_n,x_{n+1})]}{1+d_\theta(x_{n-1},x_n)},$$

which means that

$$d_\theta(x_{n-1},x_n) \leq d_\theta(x_n,x_{n+1}).$$

This is in contradiction with (7).

(ii) If $\max\{d_\theta(x_{n-1},x_n), d_\theta(x_n,x_{n+1})\} = d_\theta(x_n,x_{n+1})$, then

$$d_\theta(x_n,x_{n+1}) > d_\theta(x_{n-1},x_n). \tag{8}$$

By (6), we get

$$d_\theta(x_n,x_{n+1}) \leq \frac{d_\theta(x_{n-1},x_n)[1+d_\theta(x_n,x_{n+1})]}{1+d_\theta(x_{n-1},x_n)},$$

which establishes that

$$d_\theta(x_n,x_{n+1}) \leq d_\theta(x_{n-1},x_n).$$

This is in contradiction with (8).

This is to say, (iii) does not occur.

Thus, (5) is satisfied. Accordingly, we speculate that

$$d_\theta(x_n,x_{n+1}) \leq \psi(d_\theta(x_{n-1},x_n)) \leq \cdots \leq \psi^n(d_\theta(x_0,x_1)).$$

Letting $n \to \infty$, we obtain that $\lim\limits_{n\to\infty} d_\theta(x_n,x_{n+1}) = 0$.

It follows from Lemma 2 that $\{T^n x_0\}$ is a Cauchy sequence in X. Since (X, d_θ) is T-orbitally complete, then there is $z \in X$ such that $\lim\limits_{n\to\infty} T^n x_0 = z$.

Assume that T is continuous, then

$$d_\theta(z,Tz) = \lim_{n\to\infty} d_\theta(x_n,Tx_n) = \lim_{n\to\infty} d_\theta(x_n,x_{n+1}) = 0.$$

Therefore, T possesses a fixed point z in X.

Assume that T is orbitally continuous on X, thus, $x_{n+1} = Tx_n = T(T^n x_0) \to Tz$ as $n \to \infty$. Since the limit of sequence in extended b-metric space is unique, then $z = Tz$. Thus, T possesses a fixed point z in X, i.e., $Fix(T) \neq \emptyset$. □

Example 8. *Under all the conditions of Example 3, let $T : X \to X$ be a continuous mapping defined by*

$$Tx = \begin{cases} \frac{2x}{3}, & 0 \leq x \leq 1, \\ 2x - \frac{4}{3}, & \textit{otherwise}. \end{cases}$$

In addition, we define a mapping $\alpha : X \times X \to [0, +\infty)$ as

$$\alpha(x,y) = \begin{cases} 1, & x,y \in [0,1], \\ 0, & \textit{otherwise}. \end{cases}$$

Let $x_0 \in X$ be a point with $\alpha(x_0, Tx_0) \geq 1$, then $x_0 \in [0,1] \subset X$ and $\alpha(Tx_0, T^2x_0) = \alpha\left(\frac{2x_0}{3}, \frac{4}{9}x_0\right) \geq 1$. Therefore, T is α-orbitally admissible.

Set $\psi(t) = kt$, for all $t > 0$, where $k = \frac{4}{9}$, then $\psi^n(t) = k^n t$.

For all distinct x, y in X, ones have

$$\alpha(x,y)d_\theta(Tx,Ty) \leq \frac{4}{9}(x^2+y^2) = kd_\theta(x,y) \leq k\mathcal{N}(x,y).$$

Moreover, there is $x_0 \in X$ with $\alpha(x_0, Tx_0) \geq 1$, then $\alpha(Tx_0, T^2x_0) \geq 1$. Now, we deduce inductively that $\alpha(x_n, x_{n+1}) \geq 1$, where $x_n = T^n x_0 = (\frac{2}{3})^n x_0$, for all $n \in \mathbb{N} \cup \{0\}$. Obviously, $x_n \to 0$ as $n \to \infty$. Thus, (X, d_θ) is T-orbitally complete.

Note that $\lim_{n,m\to\infty} \theta(x_n, x_m) = 1 < \frac{9}{4} = \frac{1}{k}$, where $k = \frac{4}{9}$, that is to say,

$$\lim_{n,m\to\infty} k\,\theta(x_n,x_m) = \lim_{n,m\to\infty} \frac{\theta(x_n,x_m)\psi^n\big(d_\theta(x_0,x_1)\big)}{\psi^{n-1}\big(d_\theta(x_0,x_1)\big)} < 1.$$

Thus, all the conditions of Theorem 6 hold and hence T possesses a fixed point in X and $Fix(T) = \{0, \frac{4}{3}\}$.

Theorem 7. *In addition to all the conditions of Theorem 6, suppose that the T is α^*-orbitally admissible. Then, T possesses a unique fixed point $z \in X$.*

Proof. Following Theorem 6, T possesses a fixed point in X. Thus, $Fix(T) \neq \emptyset$. Assume that T is α^*-orbitally admissible. If possible, there exist $z, z^* \in Fix(T)$, $z \neq z^*$ such that $Tz = z$ and $Tz^* = z^*$, then $\alpha(z, z^*) = \alpha(Tz, Tz^*) \geq 1$.

Taking $x = z$, $y = z^*$ in (2), we obtain

$$\begin{aligned} d_\theta(z,z^*) = d_\theta(Tz,Tz^*) &\leq \alpha(z,z^*)d_\theta(Tz,Tz^*) \leq \psi\big(\mathcal{N}(z,z^*)\big) \\ &= \psi\Big(\max\Big\{d_\theta(z,z^*), \frac{d_\theta(z^*,Tz^*)d_\theta(z,Tz)}{d_\theta(z,z^*)}, \frac{d_\theta(z,Tz)[1+d_\theta(z^*,Tz^*)]}{1+d_\theta(z,z^*)}, \\ &\qquad \frac{d_\theta(z^*,Tz^*)[1+d_\theta(z,Tz)]}{1+d_\theta(z,z^*)}\Big\}\Big) \\ &= \psi(d_\theta(z,z^*)) \\ &< d_\theta(z,z^*), \end{aligned}$$

which is a contradiction. Therefore, T possesses a unique fixed point in X. □

Corollary 1. *([10], Theorem 2.1) Let T be a continuous self mapping on a complete extended b-metric space (X, d_θ) such that*

$$d_\theta(Tx,Ty) \leq k\mathcal{N}(x,y)$$

for all $x, y \in X$, $x \neq y$, where $k \in [0,1)$. That is, T is a rational type contraction. In addition, suppose that for all $x_0 \in X$,

$$\lim_{n,m\to\infty} \theta(x_n,x_m) < \frac{1}{k}, \tag{9}$$

where $x_n = T^n x_0$, $m > n \geq 1$. Then, T has a unique fixed point $z \in X$. Moreover, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to $z \in X$.

Proof. Setting $\alpha(x,y) = 1$, for all $x, y \in X$, then $\alpha(x, Tx) \geq 1$ implies that $\alpha(Tx, T^2x) \geq 1$. Therefore, T is α-orbitally admissible.

Let $\psi(t) = kt$, for all $t > 0$, where $0 \leq k < 1$, then $\psi^n(t) = k^n t$. Using (iii) of Theorem 6. In view of (9), then (iii) of Theorem 6 is satisfied. Thus, all the conditions of Theorem 6 hold. Therefore, T possesses a fixed point in X, i.e., $Fix(T) \neq \emptyset$. Because of $Fix(T) \subseteq X$, then T is α^*-orbitally admissible and hence, by Theorem 7, T has a unique fixed point in X. □

Remark 2. *(i) The uniqueness of fixed point is not guaranteed if T is not α^*-orbitally admissible. In Example 8, T is α-orbitally admissible and $Fix(T) = \{0, \frac{4}{3}\}$. However, $\alpha(\frac{4}{3}, T\frac{4}{3}) = 0$ so T is not α^*-orbitally admissible. Therefore, Theorem 7 is not applicable in this case.*

(ii) In Example 8, for $x = 1$ and $y = 2$, we obtain

$$d_\theta(T1, T2) = \frac{68}{9} > d_\theta(1, 2) = 5.$$

Therefore, ([3], Theorem 2) and ([10], Theorem 2.1) are not applicable in this case.

Motivated by Piri et al. [14], we extend a fixed point theorem for Khan type from metric spaces to extended b-metric spaces.

Theorem 8. *Let T be a self mapping on a T-orbitally complete extended b-metric space (X, d_θ). Suppose that $\alpha : X \times X \to [0, \infty)$, $\psi \in \Psi$ are two functions satisfying*

$$\alpha(x, y)d_\theta(Tx, Ty) \leq \begin{cases} \psi(\mathcal{K}(x, y)), & \text{whenever } \mathcal{A}(x, y) \neq 0 \text{ and } \mathcal{B}(x, y) \neq 0, \\ 0, & \text{otherwise}, \end{cases} \tag{10}$$

for all $x, y \in X$, $x \neq y$, where

$$\mathcal{A}(x, y) = \max\{d_\theta(x, Ty), d_\theta(y, Tx)\}, \qquad \mathcal{B}(x, y) = \max\{d_\theta(y, Ty), d_\theta(y, Tx)\}.$$

If
(i) T is α-orbitally admissible;
(ii) there exists $x_0 \in X$ and $\alpha(x_0, Tx_0) \geq 1$;
(iii) (1) is satisfied for $x_n = T^n x_0$ $(n = 0, 1, 2, \cdots)$.
Then, T possesses a fixed point $z \in X$. Moreover, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to $z \in X$.

Proof. By (ii), define a sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n = T^{n+1}x_0$, for all $n \in \mathbb{N} \cup \{0\}$. Since T is α-orbitally admissible, then $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1$ implies $\alpha(x_1, x_2) = \alpha(Tx_0, T^2x_0) \geq 1$. Thus, inductively, we obtain that $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$. In order to show that T possesses a fixed point in X, we assume that $x_{n-1} \neq x_n$, for all $n \in \mathbb{N}$. We divide the proof into the following two cases:

Case 1

Suppose that

$$\max\{d_\theta(x_{n-1}, Tx_n), d_\theta(x_n, Tx_{n-1})\} \neq 0$$

and

$$\max\{d_\theta(x_n, Tx_n), d_\theta(x_n, Tx_{n-1})\} \neq 0,$$

for all $n \in \mathbb{N}$. From (10), we obtain that

$$d_\theta(x_n, x_{n+1}) = d_\theta(Tx_{n-1}, Tx_n) \leq \alpha(x_{n-1}, x_n)d_\theta(Tx_{n-1}, Tx_n) \leq \psi\big(\mathcal{K}(x_{n-1}, x_n)\big),$$

where

$$\begin{aligned} \mathcal{K}(x_{n-1}, x_n) &= \max\Big\{d_\theta(x_{n-1}, x_n), \\ &\frac{d_\theta(x_{n-1}, Tx_{n-1})d_\theta(x_{n-1}, Tx_n) + d_\theta(x_n, Tx_n)d_\theta(x_n, Tx_{n-1})}{\max\{d_\theta(x_{n-1}, Tx_n),\, d_\theta(x_n, Tx_{n-1})\}}, \\ &\frac{d_\theta(x_{n-1}, T^2x_{n-1})d_\theta(x_n, Tx_n) + d_\theta(x_{n-1}, Tx_n)d_\theta(x_n, Tx_{n-1})}{\max\{d_\theta(x_n, Tx_n),\, d_\theta(x_n, Tx_{n-1})\}}\Big\} \\ &= \max\Big\{d_\theta(x_{n-1}, x_n), \frac{d_\theta(x_{n-1}, x_n)d_\theta(x_{n-1}, x_{n+1}) + d_\theta(x_n, x_{n+1})d_\theta(x_n, x_n)}{\max\{d_\theta(x_{n-1}, x_{n+1}),\, d_\theta(x_n, x_n)\}}, \end{aligned}$$

$$\left.\frac{d_\theta(x_{n-1},x_n)d_\theta(x_n,x_{n+1})+d_\theta(x_{n-1},x_{n+1})d_\theta(x_n,x_n)}{\max\{d_\theta(x_n,x_{n+1}),\, d_\theta(x_n,x_n)\}}\right\}$$
$$= d_\theta(x_{n-1},x_n).$$

Therefore,

$$0 < d_\theta(x_n,x_{n+1}) \le \psi\big(d_\theta(x_{n-1},x_n)\big).$$

Furthermore,

$$0 < d_\theta(x_n,x_{n+1}) \le \psi\big(d_\theta(x_{n-1},x_n)\big) \le \cdots \le \psi^n\big(d_\theta(x_0,x_1)\big).$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} d_\theta(x_n,x_{n+1}) = 0.$$

It follows from Condition (iii) and Lemma 2 that $\{T^n x_0\}$ is a Cauchy sequence in X. Notice that X is T-orbitally complete, thus, there is $z \in X$ with $x_n = T^n x_0 \to z$ as $n \to \infty$.

Assume, if possible, $Tz \neq z$. From (10) and the triangular inequality, we obtain

$$\begin{aligned} d_\theta(z,Tz) &\le \theta(z,Tz)[d_\theta(Tz,Tx_n)+d_\theta(Tx_n,z)] \\ &= \theta(z,Tz)d_\theta(Tz,Tx_n)+\theta(z,Tz)d_\theta(Tx_n,z) \\ &\le \theta(z,Tz)\alpha(z,x_n)d_\theta(Tz,Tx_n)+\theta(z,Tz)d_\theta(x_{n+1},z) \\ &\le \theta(z,Tz)\psi\Big(\mathcal{K}(z,x_n)\Big)+\theta(z,Tz)d_\theta(x_{n+1},z) \\ &< \theta(z,Tz)\mathcal{K}(z,x_n)+\theta(z,Tz)d_\theta(x_{n+1},z), \end{aligned} \tag{11}$$

where

$$\begin{aligned} \mathcal{K}(z,x_n) = \max\Big\{& d_\theta(z,x_n), \frac{d_\theta(z,Tz)d_\theta(z,Tx_n)+d_\theta(x_n,Tx_n)d_\theta(x_n,Tz)}{\max\{d_\theta(z,Tx_n),d_\theta(x_n,Tz)\}}, \\ & \frac{d_\theta(z,Tz)d_\theta(x_n,Tx_n)+d_\theta(z,Tx_n)d_\theta(x_n,Tz)}{\max\{d_\theta(x_n,Tx_n),d_\theta(x_n,Tz)\}}\Big\} \\ = \max\Big\{& d_\theta(z,x_n), \frac{d_\theta(z,Tz)d_\theta(z,x_{n+1})+d_\theta(x_n,x_{n+1})d_\theta(x_n,Tz)}{\max\{d_\theta(z,x_{n+1}),d_\theta(x_n,Tz)\}}, \\ & \frac{d_\theta(z,Tz)d_\theta(x_n,x_{n+1})+d_\theta(z,x_{n+1})d_\theta(x_n,Tz)}{\max\{d_\theta(x_n,x_{n+1}),d_\theta(x_n,Tz)\}}\Big\}. \end{aligned}$$

Taking $n \to \infty$ from both sides of (11), we have $d_\theta(z,Tz) \le 0$, which is in contradiction with $Tz \neq z$.

Case 2

Assume that

$$\max\{d_\theta(x_{n-1},Tx_n),d_\theta(x_n,Tx_{n-1})\} = 0$$

or

$$\max\{d_\theta(x_n,Tx_n),d_\theta(x_n,Tx_{n-1})\} = 0,$$

for all $n \in \mathbb{N}$. Consider (10), it follows that

$$x_n = x_{n+1} = Tx_n.$$

Thus, T possesses a fixed point in X, i.e., $Fix(T) \neq \emptyset$. $\square$

Example 9. *Under all the conditions of Example 5, let $T : X \to X$ be a mapping defined by*

$$Tx = \begin{cases} 0, & 0 \leq x < \frac{3}{2}, \\ 2, & \frac{3}{2} \leq x < 500, \\ 100, & x \geq 500. \end{cases}$$

We also define a mapping $\alpha : X \times X \to [0, +\infty)$ as

$$\alpha(x, y) = \begin{cases} 1, & x, y \in [0, \frac{3}{2}), \\ 0, & otherwise. \end{cases}$$

Let $x \in X$ be a point such that $\alpha(x, Tx) \geq 1$, then $x \in [0, \frac{3}{2}) \subset X$ and $\alpha(Tx, T^2x) \geq 1$. Therefore, T is α-orbitally admissible.

Set $\psi(t) = kt$, for all $t > 0$, where $k = \frac{1}{2}$. For all $x, y \in X$, we obtain

$$\alpha(x, y)d_\theta(Tx, Ty) \leq k\mathcal{K}(x, y).$$

Clearly, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then $\alpha(Tx_0, T^2x_0) \geq 1$. Therefore, by the mathematical induction, we have $\alpha(x_n, x_{n+1}) = \alpha(T^n x_0, T^{n+1}x_0) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$. Consequently, $T^n x_0 \to 0$ as $n \to \infty$. This shows that (X, d_θ) is a T-orbitally complete extended b-metric space.

Moreover, it is easy to see that

$$\lim_{n,m\to\infty} \theta(T^n x_0, T^m x_0) = \frac{3}{2} < 2 = \frac{1}{k}.$$

Accordingly, all the conditions of Theorem 8 hold and, therefore, T possesses a fixed point and $Fix(T) = \{0, 2\}$.

Theorem 9. *In addition to Theorem 8, suppose that T is α^*-orbitally admissible. Then, T possesses a unique fixed point $z \in X$.*

Proof. By Theorem 8, T possesses a fixed point in X, i.e., $Fix(T) \neq \emptyset$. For the uniqueness, let $z, z^* \in Fix(T)$ such that $z \neq z^*$. Then, by the α^*-orbital admissibility of T, we have $\alpha^*(z, z^*) \geq 1$.

As in Theorem 8, we also divide the proof into two cases as follows:

Case 1

Suppose that

$$\max\{d_\theta(z, Tz^*), d_\theta(z^*, Tz)\} \neq 0$$

and

$$\max\{d_\theta(z^*, Tz^*), d_\theta(z^*, Tz)\} \neq 0.$$

From (10), we obtain

$$d_\theta(z, z^*) = d_\theta(Tz, Tz^*) \leq \alpha(z, z^*)d_\theta(Tz, Tz^*) \leq \psi(\mathcal{K}(z, z^*)),$$

where

$$\begin{aligned} \mathcal{K}(z, z^*) = \max\Big\{ & d_\theta(z, z^*), \frac{d_\theta(z, Tz)d_\theta(z, Tz^*) + d_\theta(z^*, Tz^*)d_\theta(z^*, Tz)}{\max\{d_\theta(z, Tz^*), d_\theta(z^*, Tz)\}}, \\ & \frac{d_\theta(z, Tz)d_\theta(z^*, Tz^*) + d_\theta(z, Tz^*)d(z^*, Tz)}{\max\{d_\theta(z^*, Tz^*), d_\theta(z^*, Tz)\}} \Big\} \\ = {} & d_\theta(z, z^*). \end{aligned}$$

Therefore,

$$d_\theta(z,z^*) \le \psi\Big(d_\theta(z,z^*)\Big) < d_\theta(z,z^*).$$

This is a contradiction.
Case 2
Assume that

$$\max\{d_\theta(z,Tz^*), d_\theta(z^*,Tz)\} = 0$$

or

$$\max\{d_\theta(z^*,Tz^*), d_\theta(z^*,Tz)\} = 0.$$

Consequently, $z = Tz^* = Tz = z^*$.
Thus, T possesses a unique fixed point in X. This completes the proof. □

Corollary 2. *Let T be a self mapping on a complete extended b-metric space (X,d_θ) such that*

$$d_\theta(Tx,Ty) \le k \begin{cases} \mathcal{K}(x,y), & \text{whenever } \mathcal{A}(x,y) \neq 0 \text{ and } \mathcal{B}(x,y) \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

for all $x,y \in X$, $x \neq y$, where $0 \le k < 1$, $\mathcal{A}(x,y)$ and $\mathcal{B}(x,y)$ are defined in Theorem 8. Furthermore, suppose, for all $x_0 \in X$, that (9) is satisfied. Then, T has a unique fixed point $z \in X$. Moreover, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to z.

Corollary 3. *Let T be a self mapping on a complete extended b-metric space (X,d_θ) such that*

$$d_\theta(Tx,Ty) \le k \begin{cases} \max\left\{d_\theta(x,y), \frac{d_\theta(x,Tx)d_\theta(x,Ty)+d_\theta(y,Ty)d_\theta(y,Tx)}{\mathcal{A}(x,y)}\right\}, & \text{if } \mathcal{A}(x,y) \neq 0, \\ 0, & \text{if } \mathcal{A}(x,y) = 0, \end{cases}$$

for all $x,y \in X$, $x \neq y$, where $0 \le k < 1$ and $\mathcal{A}(x,y) = \max\{d_\theta(x,Ty), d_\theta(y,Tx)\}$. Further suppose, for all $x_0 \in X$, that (9) is satisfied. Then, T has a unique fixed point $z \in X$. Moreover, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to z.

Corollary 4. *([10], Theorem 2.2) Let T be a self mapping on a complete extended b-metric space (X,d_θ) such that*

$$d_\theta(Tx,Ty) \le k \begin{cases} \max\left\{d_\theta(x,y), \frac{d_\theta(x,Tx)d_\theta(x,Ty)+d_\theta(y,Ty)d_\theta(y,Tx)}{\mathcal{C}(x,y)}\right\}, & \text{if } \mathcal{C}(x,y) \neq 0, \\ 0, & \text{if } \mathcal{C}(x,y) = 0, \end{cases}$$

for all $x,y \in X$, $x \neq y$, where $0 \le k < 1$ and $\mathcal{C}(x,y) = d_\theta(x,Ty) + d_\theta(y,Tx)$. Further assume, for all $x_0 \in X$, that (9) is satisfied. Then, T has a unique fixed point $z \in X$. Moreover, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to z.

Remark 3. *(i) In Example 9, T is α-orbitally admissible. Since $Fix(T) = \{0,2\}$, but $\alpha(2,T2) = \alpha(2,2) = 0$, T is not α^*-orbitally admissible. In this case, Theorem 9 is not applicable in Example 9.*
(ii) In Example 9, if $x = 2$ and $y = 500$, then

$$d_\theta(Tx,Ty) = d_\theta(T2,T500) = d_\theta(2,100) = 5 > \frac{1}{2}\max\left\{5,\frac{5}{2}\right\}.$$

This shows that Corollaries 2–4 are not applicable in Example 9.

3. Applications

In this section, by using fixed point theorems mentioned above, we cope with some problems for the unique solution to a class of Fredholm integral equations.

Let $X = \mathcal{C}[a,b]$ be a set of all real valued continuous functions on $[a,b]$. Define two mappings $d_\theta : X \times X \to [0,+\infty)$ by

$$d_\theta(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|^p,$$

and $\theta : X \times X \to [1,+\infty)$ by

$$\theta(x,y) = 2^{p-1} + |x(t)| + |y(t)|,$$

where $p > 1$ is a constant. Then, (X, d_θ) is a complete extended b-metric space.

Define a Fredholm integral equation by

$$x(t) = \eta(t) + \lambda \int_a^b \mathcal{I}(t,s,x(s))ds,$$

where $t \in [a,b]$, $|\lambda| > 0$ and $\mathcal{I} : [a,b] \times [a,b] \times X \to \mathbb{R}$ and $\eta : [a,b] \to \mathbb{R}$ are continuous functions. Let $T : X \to X$ be an integral operator defined by

$$Tx(t) = \eta(t) + \lambda \int_a^b \mathcal{I}(t,s,x(s))ds. \tag{12}$$

Theorem 10. *Let $T : X \to X$ be an integral operator defined in (12). Suppose that the following assumptions hold:*

(i) for any $x_0 \in X$, $\lim_{n,m\to\infty} \theta(T^n x_0, T^m x_0) < \frac{1}{k}$, where $k = \frac{1}{2^p}$,

(ii) for any $x, y \in X$, $x \neq y$, it satisfies

$$\big|\mathcal{I}(t,s,x(s)) - \mathcal{I}(t,s,y(s))\big| \le \xi(t,s)|x(s) - y(s))|, \tag{13}$$

where $(s,t) \in [a,b] \times [a,b]$ and $\xi : [a,b] \times [a,b] \to \mathbb{R}$ is a continuous function satisfying

$$\sup_{t\in[a,b]} \int_a^b \xi^p(t,s)ds < \frac{1}{2^p|\lambda|^p(b-a)^{p-1}}. \tag{14}$$

Then, the integral operator T has a unique solution in X.

Proof. Let $x_0 \in X$ and define a sequence $\{x_n\}$ in X by $x_n = T^n x_0$, $n \ge 1$. From (12), we obtain

$$x_{n+1} = Tx_n(t) = \eta(t) + \lambda \int_a^b \mathcal{I}(t,s,x_n(s))ds.$$

Let $q > 1$ be a constant with $\frac{1}{p} + \frac{1}{q} = 1$. Making full use of (13) and the Hölder's inequality, we speculate that

$$\begin{aligned}
\big|Tx(t) - Ty(t)\big|^p &= \left|\lambda \int_a^b \mathcal{I}(t,s,x(s))ds - \lambda \int_a^b \mathcal{I}(t,s,y(s))ds\right|^p \\
&\le \Big(\int_a^b |\lambda|\big|\mathcal{I}(t,s,x(s)) - \mathcal{I}(t,s,y(s))\big|ds\Big)^p \\
&\le \Big(\int_a^b |\lambda|^q ds\Big)^{\frac{p}{q}} \left(\Big(\int_a^b \big|\mathcal{I}(t,s,x(s)) - \mathcal{I}(t,s,y(s))\big|^p ds\Big)^{\frac{1}{p}}\right)^p \\
&= |\lambda|^p(b-a)^{p-1}\Big(\int_a^b \big|\mathcal{I}(t,s,x(s)) - \mathcal{I}(t,s,y(s))\big|^p ds\Big) \\
&\le |\lambda|^p(b-a)^{p-1} \int_a^b \xi^p(t,s)|x(s) - y(s)|^p ds.
\end{aligned} \tag{15}$$

Making the most of (15) and (14), we deduce that

$$\begin{aligned} d_\theta(Tx,Ty) &= \sup_{t\in[a,b]}\Big|Tx(t)-Ty(t)\Big|^p \\ &\le |\lambda|^p(b-a)^{p-1}\sup_{t\in[a,b]}\Big[\int_a^b \xi^p(t,s)\big|x(s)-y(s)\big|^p ds\Big] \\ &\le |\lambda|^p(b-a)^{p-1}\sup_{s\in[a,b]}|x(s)-y(s)|^p\Big(\sup_{t\in[a,b]}\int_a^b \xi^p(t,s)ds\Big) \\ &\le \frac{1}{2^p}\mathcal{N}(x,y). \end{aligned}$$

Setting $k=\frac{1}{2^p}$, we obtain that

$$d_\theta(Tx,Ty)\le k\mathcal{N}(x,y).$$

Thus, all the conditions of Corollary 1 are satisfied and hence T possesses a unique fixed point in X. □

Theorem 11. *Let $T : X \to X$ be an integral operator defined by (12). Assume that the following assumptions hold:*
(i) $\lim_{n,m\to\infty}\theta(T^n x_0, T^m x_0) < \frac{1}{k}$, where $k=\frac{1}{2^p}$ for any $x_0\in X$;
(ii) for all distinct x, y in X, ones have

$$\big|\mathcal{I}\big(t,s,x(s)\big)-\mathcal{I}\big(t,s,y(s)\big)\big| \le \begin{cases}\xi(t,s)\mathcal{K}(x(s),y(s)), & \text{where } \mathcal{A}\neq 0 \text{ and } \mathcal{B}\neq 0,\\ 0, & \text{otherwise,}\end{cases}$$

where

$$\begin{aligned}\mathcal{A}=\mathcal{A}(x(s),y(s)) &= \sup\{|x(s)-Ty(s)|^p,\ |y(s)-Tx(s)|^p\},\\ \mathcal{B}=\mathcal{B}(x(s),y(s)) &= \sup\{|y(s)-Ty(s)|^p, |y(s)-Tx(s)|^p\},\end{aligned}$$

$(s,t)\in[a,b]\times[a,b]$ and $\xi:[a,b]\times[a,b]\to\mathbb{R}$ is a continuous function such that

$$\sup_{t\in[a,b]}\int_a^b \xi^p(t,s)ds < \frac{1}{2^p|\lambda|^p(b-a)^{p-1}}.$$

Then, the integral operator T has a unique solution in X.

Example 10. *Let $X=C[0,1]$ be a set of all real valued continuous functions defined on $[0,1]$. Then, (X,d_θ) is a complete extended b-metric space equipped with $d_\theta(x,y)=\sup_{t\in[0,1]}|x(t)-y(t)|^2$, where $\theta(x,y)=2+|x(t)|+|y(t)|$, for all $x,y\in X$. Let $T:X\to X$ be an operator defined by*

$$Tx(t)=\eta(t)+\int_0^1 \mathcal{I}\big(t,s,x(s)\big)ds,$$

where $\eta(t)=\frac{t}{4}$ and $\mathcal{I}\big(t,s,x(s)\big)=\frac{t(1+x^2(s))}{3}$, for all $(t,s)\in[0,1]\times[0,1]$.
We have

$$\begin{aligned}\big|Tx(t)-Ty(t)\big|^2 &= \Big|\int_0^1 \mathcal{I}\big(t,s,x(s)\big)ds-\int_0^1\mathcal{I}\big(t,s,y(s)\big)ds\Big|^2\\ &\le \int_0^1\Big|\frac{t}{3}\big(x^2(s)-y^2(s)\big)\Big|^2 ds. \end{aligned} \tag{16}$$

Taking the supremum on both sides of (16), *for all* $t \in [0,1]$, *we obtain*

$$d_\theta(Tx, Ty) = \sup_{t\in[0,1]} \left|Tx(t) - Ty(t)\right|^2 \le \frac{1}{9} d_\theta(x,y) < \frac{1}{6}\mathcal{N}(x,y).$$

In addition, $\lim_{m,n\to\infty} \theta(T^m x_0, T^n x_0) = 2 < \frac{1}{k}$, *where* $k = \frac{1}{6}$ *and* $x_0(t) = \frac{t}{4}$. *Thus, all the conditions of Theorem 10 are satisfied and hence the integral operator T has a unique solution.*

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