

Article

Second Hankel Determinant for a Certain Subclass of Bi-Close to Convex Functions Defined by Kaplan

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Abstract: In this paper, we consider the class of strongly bi-close-to-convex functions of order α and bi-close-to-convex functions of order β . We obtain an upper bound estimate for the second Hankel determinant for functions belonging to these classes. The results in this article improve some earlier result obtained for the class of bi-convex functions.

Keywords: bi-univalent functions; bi-convex functions; bi-close-to-convex functions; Hankel determinant



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1. Introduction

1.1. Bi-Univalence

If f is in the class $\mathcal{S}$, then f is one-to-one in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots, \quad (1)$$

then the inverse f^{-1} of f has Maclaurin expansion in a disk of radius at least $1/4$, say

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots \quad (2)$$

An analytic function f of the form (1) is said to be *bi-univalent* in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$, in the sense that f^{-1} has an univalent analytic continuation to $\mathbb{D}$. Let Σ denote the class of all bi-univalent functions in $\mathbb{D}$, given by the Taylor–Maclaurin series expansion (1). Family Σ has been the focus of attention for more than fifty years. In [1], Lewin established that for $f \in \Sigma$, $|a_2| \leq 1.51$. Later on, Brannan and Clunie [2] hypothesized that $|a_2| \leq \sqrt{2}$; however, their hypothesis has not been proved. One of the results which deserves more attention but somehow unnoticed is that of Netanyahu [3] who obtained a sharp upper bound $|a_2| \leq \frac{4}{3}$ for a class $\Sigma_1 \subset \Sigma$, consisting of the functions that are bi-univalent and its range contain $\mathbb{D}$. However, the sharp lower bound of the second coefficient $|a_2|$ in the class Σ is not known, as well as bounds for successive coefficients $|a_n|$ ($n > 2$). Some examples of bi-univalent functions are $\frac{z}{1-z}$, $\frac{1}{2} \log\left(\frac{1+z}{1-z}\right)$ or $-\log(1-z)$; however, the familiar Koebe function, or $\frac{z}{1-z^2}$, which are the members of $\mathcal{S}$, are not the elements of the class Σ .

1.2. Subfamilies of $\mathcal{S}$ and Related Bi-Univalent Functions

Let $0 \leq \beta < 1$. The subclasses of $\mathcal{S}$ consisting of *starlike functions of order β* (and *convex functions of order β* , respectively) are denoted by $\mathcal{ST}(\beta)$ ($\mathcal{CV}(\beta)$, resp.), and are defined analytically

$$\mathcal{ST}(\beta) = \{f \in \mathcal{S} : \Re(zf'(z)/f(z)) > \beta, \quad z \in \mathbb{D}\}, \quad (3)$$

$$\mathcal{CV}(\beta) = \{f \in \mathcal{S} : \Re(1 + zf''(z)/f'(z)) > \beta, \quad z \in \mathbb{D}\}. \quad (4)$$

A function f of the form (1) is said *close-to-convex* in $\mathbb{D}$, if and only if there exists a function $\Phi \in \mathcal{CV} = \mathcal{CV}(0)$ such that $\Re\left(\frac{f'(z)}{\Phi'(z)}\right) > 0, z \in \mathbb{D}$. The family of normalized close-to-convex functions was first introduced by Kaplan [4] and denoted $\mathcal{K}$.

Brannan and Taha [5] introduced the classes $\mathcal{ST}_{\Sigma}(\beta)$ (and $\mathcal{CV}_{\Sigma}(\beta)$) of *bi-starlike functions of order β* (and *bi-convex functions of order β* , resp.) corresponding to $\mathcal{ST}(\beta)$ and $\mathcal{CV}(\beta)$ defined by (3) and (4). They also found non-sharp estimates on $|a_2|$ and $|a_3|$ for its members of the form (1). Following Brannan and Taha [5], many researchers (see, for example, in [6,7]) have recently introduced and investigated several interesting subclasses of Σ and found non-sharp estimates on the first two Taylor–Maclaurin coefficients. Furthermore, in [5] the class of *strongly bi-starlike functions of order α* , where $0 < \alpha \leq 1$ has been defined and denoted $\mathcal{ST}_{\Sigma}[\alpha]$. A function f is in the class $\mathcal{ST}_{\Sigma}[\alpha]$, if

$$\left| \arg\left(\frac{zf'(z)}{f(z)}\right) \right| < \frac{\alpha\pi}{2} \quad \text{and} \quad \left| \arg\left(\frac{wg'(w)}{g(w)}\right) \right| < \frac{\alpha\pi}{2} \quad (z, w \in \mathbb{D}),$$

where g is the analytic continuation of f^{-1} to $\mathbb{D}$.

For $0 \leq \alpha \leq 1$, let $\mathcal{K}_{\alpha}$ denote the family of functions f of the form (1), analytic and locally univalent in $\mathbb{D}$, for which there exists a convex function ϕ such that

$$\left| \arg\left(\frac{f'(z)}{\phi'(z)}\right) \right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{D}). \quad (5)$$

The above class has been introduced by Kaplan [4] and later studied by Reade [8]. In particular, $\mathcal{K}_0$ is the family of convex univalent functions and $\mathcal{K}_1$ is the family of close-to-convex functions. Moreover, $\mathcal{K}_{\alpha_1}$ is a proper subclass of $\mathcal{K}_{\alpha_2}$ whenever $\alpha_1 < \alpha_2$. An extension of $\mathcal{K}_{\alpha}$ is a class $\mathcal{K}_{\alpha}(\beta)$ of close-to-convex functions of order β [8], given by

$$\Re\left(\frac{f'(z)}{\phi'(z)}\right) > \beta \quad (z \in \mathbb{D}). \quad (6)$$

Following Brannan and Taha, the related families of bi-univalent functions have been considered, for example, a class $\mathcal{K}_{\Sigma}$ of bi-close-to-convex functions [9]; a class of strongly bi-close-to-convex functions of order α , denoted by $\mathcal{K}_{\Sigma}[\alpha]$; and the class of bi-close-to-convex functions of order β , denoted by $\mathcal{K}_{\Sigma}(\beta)$.

Definition 1. ([9]) A function $f \in \Sigma$ of the form (1) belongs to the class of bi-close to convex functions $\mathcal{K}_{\Sigma}$, if there exist a function ϕ , convex and univalent for $z \in \mathbb{D}$, such that

$$\Re\left\{\frac{f'(z)}{\phi'(z)}\right\} \geq 0, \quad \text{and} \quad \Re\left\{\frac{g'(w)}{\phi'(w)}\right\} \geq 0 \quad (z, w \in \mathbb{D}),$$

where g is the analytic continuation of f^{-1} to $\mathbb{D}$ with a series expansion (2).

Definition 2. [9] Let $0 \leq \alpha \leq 1$. A function $f \in \Sigma$, given by (1), is said to be strongly bi-close-to-convex of order α if there exist bi-convex functions ϕ and ψ such that

$$\left| \arg\left(\frac{f'(z)}{\phi'(z)}\right) \right| < \alpha\pi/2 \quad \text{and} \quad \left| \arg\left(\frac{g'(w)}{\psi'(w)}\right) \right| < \alpha\pi/2 \quad (z, w \in \mathbb{D}). \quad (7)$$

Here, g is the analytic continuation of f^{-1} to $\mathbb{D}$. We denote the class of strongly bi-close-to-convex functions of order α by $\mathcal{K}_{\Sigma}[\alpha]$.

Remark 1. We note that $\mathcal{K}_\Sigma[1] \equiv \mathcal{K}_\Sigma$ and $\mathcal{K}_\Sigma[0] \equiv \mathcal{CV}_\Sigma$ [5].

Definition 3. [9] Let $0 \leq \beta < 1$. A function $f \in \Sigma$, given by (1), is said to be bi-close-to-convex of order β if there exist the bi-convex functions ϕ and $\psi \in \mathcal{CV}_\Sigma$ such that

$$\Re\left(\frac{f'(z)}{\phi'(z)}\right) > \beta \quad \text{and} \quad \Re\left(\frac{g'(w)}{\psi'(w)}\right) > \beta \quad (z, w \in \mathbb{D}), \quad (8)$$

where g is the analytic continuation of f^{-1} to $\mathbb{D}$. We denote the class of bi-close-to-convex functions of order β by $\mathcal{K}_\Sigma(\beta)$.

Remark 2. We note that $\mathcal{K}_\Sigma(0) \equiv \mathcal{K}_\Sigma$. Furthermore, for $\phi(z) = z$, the class $N_\Sigma(\alpha)$ ($0 \leq \alpha \leq 1$) reduces to the family of functions $f \in \Sigma$, satisfying the condition

$$|\arg f'(z)| < \alpha\pi/2 \quad \text{and} \quad |\arg g'(w)| < \alpha\pi/2 \quad (z, w \in \mathbb{D}),$$

and $\mathcal{K}_\Sigma(\beta)$ reduces to $N_\Sigma(\beta)$ defined by the conditions

$$\Re(f'(z)) > \beta \quad \text{and} \quad \Re(g'(w)) > \beta \quad (z, w \in \mathbb{D}),$$

where the function g is defined by (2). These classes were studied by Çağlar et al. [10]

Observe that if f is given by (1), then $g = f^{-1}$ is given by (2), and if

$$\phi(z) = z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \dots, \quad (9)$$

then

$$\psi(w) = \phi^{-1}(w) = w - c_2 w^2 + (2c_2^2 - c_3)w^3 - (5c_2^3 - 5c_2 c_3 + c_4)w^4 + \dots. \quad (10)$$

In the sequel, we assume that g, ϕ, ψ have Taylor expansions as in (2), (9), and (10).

1.3. Hankel Determinant

Towards the full understanding of a behavior of bi-univalence, it is necessary to extend our attention to the Hankel determinants, that is one of the most important tool in Geometric Function Theory, defined by Pommerenke [11,12]. Noonan and Thomas [13] defined the q^{th} Hankel determinant of f given by (1) for natural $n \geq 1$ and $q \geq 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

The importance of the Hankel determinants was recognized over half a century ago, and it has been studied in great details, see, for example, in [11,12]. The significance of the Hankel determinants follows from the study of singularities of analytic functions ([14], p. 329), see also in [15], and from the fact that it contains the Fekete-Szegő functional with its generalization [16]. Moreover, $H_2(2) = a_2 a_4 - a_3^2$ is the well-known second Hankel determinant. The Hankel determinant is useful for estimating the modulus of coefficients and the rate of growth of the coefficients. Both estimates determine the behavior of the studied function when the function itself and its properties are unknown. Extensive studies of the Hankel determinant in the theory of meromorphic functions are due to Wilson [17]; numerous applications in mathematical physics are given by Vein and Dale [18]. Recently, many authors have discussed upper bounds of the Hankel determinant and Fekete-Szegő functional for numerous subclasses of univalent functions [13,14,19–22] and references

therein. Very recently, the upper bounds of $H_2(2)$ for the classes $\mathcal{S}_\Sigma^*(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$ were investigated by Deniz et al. [23], and extended by Orhan et al. [24,25].

Sivasubramanian et al. [9] found the estimates of $|a_2|$ and $|a_3|$ in the classes $\mathcal{K}_\Sigma$, $\mathcal{K}_\Sigma[\alpha]$ and $\mathcal{K}_\Sigma(\beta)$. Further, they verified Brannan and Clunie's conjecture $|a_2| \leq \sqrt{2}$ for some of their subclasses.

Therefore, a naturally arising problem addressed in this paper is to investigate the behavior of the Hankel determinants in the newly defined families.

1.4. Some Useful Bounds

Let $\mathcal{P}$ denote the class of functions $p(z)$ of the form

$$p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \dots, \quad (11)$$

which are analytic in the open unit disk $\mathbb{D}$ and such that $\Re p(z) > 0$, $z \in \mathbb{D}$.

Lemma 1. [26] If the function $p \in \mathcal{P}$ is given by the series (11), then $|p_k| \leq 2$, $k = 1, 2, \dots$.

Lemma 2. [27] If the function $p \in \mathcal{P}$ is given by the series (11), then

$$\begin{aligned} 2p_2 &= p_1^2 + x(4 - p_1^2), \\ 4p_3 &= p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z, \end{aligned}$$

for some x, z with $|x| \leq 1$ and $|z| \leq 1$.

Lemma 3. [28] If the function $\phi \in \mathcal{CV}$, then for $\lambda \in \mathbb{R}$,

$$|c_3 - \lambda c_2^2| \leq \begin{cases} 1 - \lambda & \text{for } \lambda < 2/3, \\ 1 & \text{for } 2/3 \leq \lambda \leq 4/3, \\ \lambda - 1 & \text{for } \lambda > 4/3. \end{cases} \quad (12)$$

Lemma 4. [29] If the function $\phi \in \mathcal{CV}$, then $|c_2c_4 - c_3^2| \leq \frac{1}{8}$.

Lemma 5. [30] If the function $\phi \in \mathcal{CV}$, then $|c_2c_3 - c_4| \leq \frac{1}{6}$.

2. Second Hankel Determinant in Class $\mathcal{K}_\Sigma[\alpha]$ and $\mathcal{K}_\Sigma(\beta)$

The first aim of this section is to find the best bound of the second Hankel determinant in the class $\mathcal{K}_\Sigma[\alpha]$. A successful method of finding such bound has been exploited in [9] and other related publications.

2.1. The Class $\mathcal{K}_\Sigma(\beta)$

In the family of strongly bi-close-to-convex of order α , we have the following non-sharp estimates of $H_2(2)$; however, this bound, for a particular selection of α , improves the earlier results in [23].

Theorem 1. Let $0 \leq \alpha \leq 1$, and let the function f , given by (1), be in the class $\mathcal{K}_\Sigma[\alpha]$. Then,

$$|a_2a_4 - a_3^2| \leq \frac{1}{8} + \frac{3}{2}\alpha + \frac{43}{9}\alpha^2 + \frac{1}{3}\alpha^3 + \frac{4}{3}\alpha^4. \quad (13)$$

Proof. From the condition (7) it follows that there exists $p, q \in \mathcal{P}$ such that

$$f'(z) = \phi'(z)[p(z)]^\alpha \quad \text{and} \quad g'(w) = \psi'(w)[q(w)]^\alpha. \quad (14)$$

Let p be given by (11) and q has a series representations

$$q(w) = 1 + q_1w + q_2w^2 + q_3w^3 + \dots. \quad (15)$$

Then, equating the coefficients of both sides of (14), when f, p, q, ϕ , and ψ have given power series, we obtain a number of equalities, below.

$$2a_2 = 2c_2 + \alpha p_1, \quad -2a_2 = -2c_2 + \alpha q_1, \quad (16)$$

$$3a_3 = 3c_3 + 2\alpha c_2 p_1 + \alpha p_2 + \frac{1}{2}\alpha(\alpha-1)p_1^2, \quad (17)$$

$$4a_4 = 4c_4 + 3c_3\alpha p_1 + 2\alpha c_2 p_2 + \alpha p_3 + \alpha(\alpha-1)c_2 p_1^2 + \alpha(\alpha-1)p_1 p_2 + \frac{\alpha(\alpha-1)(\alpha-2)}{6}p_1^3, \quad (18)$$

$$6a_2^2 - 3a_3 = 6c_2^2 - 3c_3 - 2\alpha c_2 q_1 + \alpha q_2 + \frac{1}{2}\alpha(\alpha-1)q_1^2, \quad (19)$$

$$\begin{aligned} -20a_2^3 + 20a_2 a_3 - 4a_4 = & -20c_2^3 + 20c_2 c_3 - 4c_4 + 6c_2^2 \alpha q_1 - 3c_3 \alpha q_1 - 2\alpha c_2 q_2 \\ & + \alpha q_3 - \alpha(\alpha-1)c_2 q_1^2 + \alpha(\alpha-1)q_1 q_2 + \frac{\alpha(\alpha-1)(\alpha-2)}{6}q_1^3. \end{aligned} \quad (20)$$

The equality (16) immediately gives $p_1 = -q_1$. Next, by (17) and (19), we obtain

$$a_3 = c_3 + \alpha c_2 p_1 + \frac{1}{4}\alpha^2 p_1^2 + \frac{\alpha}{6}(p_2 - q_2). \quad (21)$$

Similarly, making necessary calculations of (18) and (20), we get

$$\begin{aligned} a_4 = & c_4 + \frac{\alpha}{8}(p_3 - q_3) + \frac{1}{4}c_2 \alpha(p_2 + q_2) + \frac{5}{24}\alpha^2 p_1(p_2 - q_2) \\ & + \frac{5}{12}\alpha c_2(p_2 - q_2) + \frac{5}{4}c_3 \alpha p_1 + \frac{1}{4}\alpha(\alpha-1)c_2 p_1^2 \\ & + \frac{1}{8}\alpha(\alpha-1)p_1(p_2 + q_2) - \frac{1}{2}c_2^2 \alpha p_1 + \frac{1}{24}\alpha(\alpha-1)(\alpha-2)p_1^3. \end{aligned} \quad (22)$$

Therefore,

$$\begin{aligned} |a_2 a_4 - a_3^2| = & \left| c_2 c_4 - c_3^2 + \frac{1}{2}\alpha c_4 p_1 - \frac{3}{4}c_2 c_3 \alpha p_1 + \frac{1}{8}c_3 \alpha^2 p_1^2 - \frac{5}{4}\alpha^2 p_1^2 c_2^2 \right. \\ & - \frac{1}{3}c_3 \alpha(p_2 - q_2) + \frac{5}{12}c_2^2 \alpha(p_2 - q_2) - \frac{1}{2}c_2^3 \alpha p_1 - \frac{1}{16}\alpha^4 p_1^4 \\ & + \frac{1}{12}c_2 \alpha^2 p_1(p_2 - q_2) + \frac{1}{48}\alpha^2(\alpha-1)(\alpha-2)p_1^4 \\ & + \frac{1}{8}\alpha^2(\alpha-1)c_2 p_1^3 + \frac{1}{24}c_2 \alpha(\alpha-1)(\alpha-2)p_1^3 \\ & + \frac{1}{8}\alpha c_2(p_3 - q_3) + \frac{1}{8}c_2 \alpha^2 p_1(p_2 + q_2) \\ & + \frac{1}{48}\alpha^3 p_1^2(p_2 - q_2) + \frac{1}{4}\alpha(\alpha-1)c_2^2 p_1^2 - \frac{1}{2}c_2 \alpha^3 p_1^3 \\ & + \frac{1}{16}\alpha^2 p_1(p_3 - q_3) + \frac{1}{16}\alpha^2(\alpha-1)p_1^2(p_2 + q_2) \\ & \left. + \frac{1}{4}c_2^2 \alpha(p_2 + q_2) + \frac{1}{8}c_2 \alpha(\alpha-1)p_1(p_2 + q_2) - \frac{1}{36}\alpha^2(p_2 - q_2)^2 \right|. \end{aligned} \quad (23)$$

Let us apply Lemma 2 to p_2 and q_2 . Then, for some x, y such that $|x| \leq 1$, $|y| \leq 1$, it holds

$$2p_2 = p_1^2 + x(4 - p_1)^2, \quad 2q_2 = q_1^2 + y(4 - q_1)^2,$$

from which we have

$$p_2 - q_2 = \frac{(4 - p_1^2)(x - y)}{2}; \quad p_2 + q_2 = p_1^2 + \frac{(4 - p_1^2)(x + y)}{2}. \quad (24)$$

Apply now Lemma 2 to p_3, q_3 and obtain

$$p_3 - q_3 = \frac{p_1^3}{2} + \frac{p_1(4 - p_1^2)(x + y)}{2} - \frac{p_1(4 - p_1^2)(x^2 + y^2)}{4} + \frac{(4 - p_1^2)}{2} [(1 - |x|^2)z - (1 - |y|^2)w], \quad (25)$$

for some x, y, z , and w with $|x| \leq 1$, $|y| \leq 1$, $|z| \leq 1$, and $|w| \leq 1$. Making use of (24) and (25) to (23) gives

$$\begin{aligned} |a_2 a_4 - a_3^2| &= \left| c_2 c_4 - c_3^2 + \frac{\alpha}{2} (c_4 - c_2 c_3) p_1 - \frac{1}{4} \alpha c_2 c_3 p_1 + \frac{1}{8} \alpha^2 (c_3 - 10 c_2^2) p_1^2 \right. \\ &\quad - \frac{1}{3} \alpha \left(c_3 - \frac{5}{4} c_2^2 \right) (p_2 - q_2) + \frac{1}{12} c_2 \alpha^2 p_1 (p_2 - q_2) - \frac{1}{2} c_2^3 \alpha p_1 - \frac{1}{2} c_2 \alpha^3 p_1^3 \\ &\quad - \frac{1}{16} \alpha^4 p_1^4 + \frac{1}{16} \alpha^2 (\alpha - 1) p_1^2 (p_2 - q_2) + \frac{1}{4} \alpha (\alpha - 1) c_2^2 p_1^2 + \frac{1}{48} \alpha^2 (\alpha - 1) (\alpha - 2) p_1^4 \\ &\quad + \frac{1}{8} \alpha^2 (\alpha - 1) c_2 p_1^3 + \frac{1}{24} c_2 \alpha (\alpha - 1) (\alpha - 2) p_1^3 + \frac{1}{8} \alpha c_2 (p_3 - q_3) \\ &\quad + \frac{1}{8} c_2 \alpha (\alpha - 1) p_1 (p_2 + q_2) + \frac{1}{8} c_2 \alpha^2 p_1 (p_2 + q_2) + \frac{1}{48} \alpha^3 p_1^2 (p_2 - q_2) \\ &\quad \left. + \frac{1}{16} \alpha^2 p_1 (p_3 - q_3) + \frac{1}{4} c_2^2 \alpha (p_2 + q_2) - \frac{1}{36} \alpha^2 (p_2 - q_2)^2 \right|. \end{aligned} \quad (26)$$

Without loss of generality, we can restrict our considerations to $p_1 := p \in [0, 2]$. Applying this and the triangle inequality to (26), we have

$$\begin{aligned} |a_2 a_4 - a_3^2| &\leq |c_2 c_4 - c_3^2| + \frac{\alpha}{2} |(c_4 - c_2 c_3)| p + \frac{1}{2} |c_2^3| \alpha p + \frac{\alpha}{4} |c_2 c_3| p \\ &\quad + \frac{1}{8} \alpha^2 |c_3 - 10 c_2^2| p^2 + \frac{1}{2} |c_2| \alpha^3 p^3 + \frac{1}{16} \alpha^4 p^4 + \frac{1}{4} |\alpha (\alpha - 1)| |c_2^2| p^2 \\ &\quad + \frac{1}{48} |\alpha^2 (\alpha - 1) (\alpha - 2)| p^4 + \frac{1}{8} |\alpha^2 (\alpha - 1)| |c_2| p^3 + \frac{1}{24} |c_2| |\alpha (\alpha - 1) (\alpha - 2)| p^3 \\ &\quad + \frac{1}{16} |\alpha^2 (\alpha - 1)| p^4 + \frac{1}{16} \alpha |c_2| p^3 + \frac{1}{8} |c_2| \alpha^2 p^3 + \frac{1}{8} |c_2| |\alpha (\alpha - 1)| p^3 + \frac{1}{32} \alpha^2 p^4 \\ &\quad + \frac{1}{4} |c_2^2| \alpha p^2 + (|x| + |y|) \left[\frac{\alpha}{6} \left| c_3 - \frac{5}{4} c_2^2 \right| + \frac{1}{8} |c_2^2| \alpha \right] (4 - p^2) \\ &\quad + \left[\frac{1}{16} |c_2| \alpha + \frac{5}{48} |c_2| \alpha^2 + \frac{1}{16} |c_2| |\alpha (\alpha - 1)| \right] p (4 - p^2) \\ &\quad + \left[\frac{1}{32} \alpha^2 + \frac{1}{96} \alpha^3 + \frac{1}{32} |\alpha^2 (\alpha - 1)| \right] p^2 (4 - p^2) \\ &\quad + (|x|^2 + |y|^2) \left[\frac{\alpha^2}{64} p^2 (4 - p^2) + \frac{\alpha}{32} |c_2| p (4 - p^2) \right] \\ &\quad + (1 - |x|^2) \left[\frac{\alpha}{16} |c_2| (4 - p^2) + \frac{\alpha^2}{32} p (4 - p^2) \right] \\ &\quad + (1 - |y|^2) \left[\frac{\alpha}{16} |c_2| (4 - p^2) + \frac{\alpha^2}{32} p (4 - p^2) \right] \\ &\quad + \frac{\alpha^2}{144} (4 - p^2)^2 (|x| + |y|)^2. \end{aligned} \quad (27)$$

We now apply Lemma 3 and Lemma 4 with Lemma 5 to (27), and deduce that

$$\begin{aligned} |a_2 a_4 - a_3^2| &\leq \frac{1}{8} + \frac{5}{6} \alpha p + \left(\frac{9}{8} \alpha^2 + \frac{1}{4} \alpha (1 - \alpha) + \frac{1}{4} \alpha \right) p^2 \\ &\quad + \left(\frac{1}{16} \alpha + \frac{1}{8} \alpha^2 + \frac{1}{2} \alpha^3 + \frac{1}{8} \alpha (1 - \alpha) \frac{2\alpha + 5}{3} \right) p^3 \\ &\quad + \left[\frac{1}{16} \alpha^4 + \frac{1}{32} \alpha^2 + \frac{\alpha^2 (1 - \alpha) (5 - \alpha)}{48} \right] p^4 \\ &\quad + \frac{(\alpha)}{8} (4 - p^2) + \frac{(\alpha)^2}{16} p (4 - p^2) \end{aligned}$$

$$\begin{aligned}
& + (|x| + |y|) \left[\left(\frac{\alpha^3}{96} + \frac{\alpha^2}{32} + \frac{1}{32} \alpha^2 (1 - \alpha) \right) p^2 (4 - p^2) \right. \\
& + \left. \left(\frac{5\alpha^2}{48} + \frac{\alpha}{16} + \frac{\alpha(1 - \alpha)}{16} \right) p(4 - p^2) + \frac{\alpha}{6} (4 - p^2) \right] \\
& + (|x|^2 + |y|^2) \left[\frac{\alpha^2}{64} (p^2 - 4)(4 - p^2) \right] + \frac{\alpha^2}{144} (4 - p^2)^2 (|x| + |y|)^2.
\end{aligned}$$

Taking $\gamma_1 = |x| \leq 1, \gamma_2 = |y| \leq 1$, we rewrite the above as follows:

$$|a_2 a_4 - a_3^2| \leq S_1 + S_2(\gamma_1 + \gamma_2) + S_3(\gamma_1^2 + \gamma_2^2) + S_4(\gamma_1 + \gamma_2)^2 = F(\gamma_1, \gamma_2),$$

where

$$\begin{aligned}
S_1 = S_1(p) &= \frac{1}{8} + \frac{5}{6} \alpha p + \left(\frac{9}{8} \alpha^2 + \frac{1}{4} \alpha (1 - \alpha) + \frac{1}{4} \alpha \right) p^2 \\
&+ \left(\frac{1}{16} \alpha + \frac{1}{8} \alpha^2 + \frac{1}{2} \alpha^3 + \frac{1}{8} \alpha (1 - \alpha) \frac{2\alpha + 5}{3} \right) p^3 \\
&+ \left[\frac{1}{16} \alpha^4 + \frac{1}{32} \alpha^2 + \frac{\alpha^2 (1 - \alpha)(5 - \alpha)}{48} \right] p^4 \\
&+ \frac{(\alpha)}{8} (4 - p^2) + \frac{(\alpha)^2}{16} p(4 - p^2) \geq 0, \\
S_2 = S_2(p) &= \left(\frac{\alpha^3}{96} + \frac{\alpha^2}{32} + \frac{1}{32} \alpha^2 (1 - \alpha) \right) p^2 (4 - p^2) \\
&+ \left(\frac{5\alpha^2}{48} + \frac{\alpha}{16} + \frac{\alpha(1 - \alpha)}{16} \right) p(4 - p^2) + \frac{\alpha}{6} (4 - p^2) \geq 0, \\
S_3 = S_3(p) &= \frac{\alpha^2}{64} (p^2 - 4)(4 - p^2) \leq 0, \\
S_4 = S_4(p) &= \frac{\alpha^2}{144} (4 - p^2)^2 \geq 0.
\end{aligned}$$

In order to obtain an estimate of $|H_2(2)|$, we need to maximize $F(\gamma_1, \gamma_2)$ in the closed square

$$\Delta := \{(\gamma_1, \gamma_2) : 0 \leq \gamma_1 \leq 1, 0 \leq \gamma_2 \leq 1\}.$$

As $S_3 < 0$ and $S_3 + 2S_4 > 0$ and $p \in (0, 2)$, we conclude that $F_{\gamma_1 \gamma_1} F_{\gamma_2 \gamma_2} - (F_{\gamma_1 \gamma_2})^2 < 0$ for all $\gamma_1, \gamma_2 \in \text{int} \Delta$, and thus the function F can attain a maximum only on the boundary of Δ .

We first note that F is symmetric in γ_1 and γ_2 , therefore it is enough to consider $0 \leq \gamma_1 \leq 1$ and $0 \leq \gamma_2 \leq \gamma_1$. For $\gamma_2 = 0$ and $0 \leq \gamma_1 \leq 1$, we obtain

$$F(\gamma_1, 0) = G(\gamma_1) = S_1 + S_2 \gamma_1 + (S_3 + S_4) \gamma_1^2.$$

Fix $p \in [0, 2]$ and consider two separate cases:

(i) $S_3 + S_4 \geq 0$. In this case, $G'(\gamma_1) = 2(S_3 + S_4) \gamma_1 + S_2 > 0$, that is, $G(\gamma_1)$ is an increasing function. Therefore, for fixed $p \in [0, 2]$ the maximum of $G(\gamma_1)$ may occur only at $\gamma_1 = 1$, and

$$\max G(\gamma_1) = G(1) = S_1 + S_2 + S_3 + S_4.$$

(ii) $S_3 + S_4 < 0$. As $S_2 + 2(S_3 + S_4) \geq 0$ for $0 < \gamma_1 < 1$, it is clear that $S_2 + 2(S_3 + S_4) < 2(S_3 + S_4) \gamma_1 + S_2 < S_2$ so that $G'(\gamma_1) > 0$. Therefore, similarly as in the case (i) the maximum of $G(\gamma_1)$ is attained for $\gamma_1 = 1$.

For $\gamma_1 = 1$ and $0 \leq \gamma_2 \leq 1$, we obtain

$$F(1, \gamma_2) = H(\gamma_2) = (S_3 + S_4)\gamma_2^2 + (S_2 + 2S_4)\gamma_2 + S_1 + S_2 + S_3 + S_4.$$

Similarly, to the above cases of $S_3 + S_4$, we get that

$$\max H(\gamma_2) = H(1) = S_1 + 2S_2 + 2S_3 + 4S_4.$$

As $G(1) \leq H(1)$ for $p \in [0, 2]$, we have that $\max F(\gamma_1, \gamma_2) = F(1, 1)$ on the boundary of Δ and from this on the closed square Δ .

Next, let us define a function $K : [0, 2] \rightarrow \mathbb{R}$ as follows:

$$K(p) = \max F(\gamma_1, \gamma_2) = F(1, 1) = S_1 + 2S_2 + 2S_3 + 4S_4, \quad (28)$$

that is, in view of (28),

$$\begin{aligned} K(p) &= \frac{1}{8} + \frac{11}{6}\alpha - \frac{1}{18}\alpha^2 + \left(\frac{11}{6}\alpha + \frac{7}{12}\alpha^2\right)p \\ &+ \left(\frac{1}{24}\alpha + \frac{101}{72}\alpha^2 - \frac{1}{6}\alpha^3\right)p^2 \\ &+ \left[\frac{5}{12}\alpha^3 - \frac{19}{48}\alpha^2 + \frac{25}{48}\alpha\right]p^3 \\ &+ \left[\frac{1}{12}\alpha^4 - \frac{7}{48}\alpha^3 + \frac{11}{144}\alpha^2 + \frac{1}{16}\alpha\right]p^4. \end{aligned}$$

By an elementary calculation, we find that

$$\begin{aligned} K'(p) &= \left(\frac{11}{6}\alpha + \frac{7}{12}\alpha^2\right) + \left(\frac{1}{12}\alpha + \frac{101}{36}\alpha^2 - \frac{1}{3}\alpha^3\right)p \\ &+ \left[\frac{5}{4}\alpha^3 - \frac{19}{16}\alpha^2 + \frac{25}{16}\alpha\right]p^2 + \left[\frac{1}{3}\alpha^4 - \frac{7}{12}\alpha^3 + \frac{11}{36}\alpha^2 + \frac{1}{4}\alpha\right]p^3, \end{aligned}$$

that can be rewritten as

$$\begin{aligned} K'(p) &= \left(\frac{11}{6}\alpha + \frac{7}{12}\alpha^2\right) + \frac{\alpha p}{3} \left[\alpha(1 - \alpha) + \frac{89}{22}\alpha + \frac{1}{4}\right] \\ &+ \frac{\alpha p^2}{4} \left[5\alpha^2 + \frac{19}{4}(1 - \alpha) + \frac{3}{2}\right] + \frac{\alpha p^3}{3} \left[\alpha\left(\alpha^2 - \alpha + \frac{11}{12}\right) + \frac{3}{4}(1 - \alpha^2)\right], \end{aligned}$$

from which it is easily seen that $K'(p) > 0$ for $0 < \alpha \leq 1$. Therefore, $K(p)$ is an increasing function of p so that $K(p)$ attains its maximum at $p = 2$. Consequently, we have

$$\max_{0 \leq p \leq 2} K(p) = K(2) = \frac{1}{8} + \frac{3}{2}\alpha + \frac{43}{9}\alpha^2 + \frac{1}{3}\alpha^3 + \frac{4}{3}\alpha^4.$$

This completes the proof of the theorem. $\square$

Remark 3. For $\alpha = 1$, we have the following bound

$$|a_2 a_4 - a_3^2| \leq \frac{581}{72},$$

and when $\phi(z) = z$, Theorem 1 reduces to the Theorem 2 in [10]. Furthermore, when $\alpha = 0$, we get the estimate for the class of bi-convex functions, which significantly improves the bound due to Deniz et al. [23], below. Unfortunately, we do not know if that result is sharp.

Corollary 1. For $0 \leq \alpha < 1$, and $f \in \mathcal{K}_\Sigma \equiv \mathcal{K}_\Sigma[0]$ we have

$$|a_2a_4 - a_3^2| \leq \frac{1}{8}. \quad (29)$$

2.2. The Class $\mathcal{K}_\Sigma(\beta)$

In order to estimate the second Hankel determinate in $\mathcal{K}_\Sigma(\beta)$ we apply consideration similar to that used in the proof of 1.

Theorem 2. Let $0 \leq \beta < 1$, and let the function f given by (1) be in the class $\mathcal{K}_\Sigma(\beta)$. Then,

$$|a_2a_4 - a_3^2| \leq \frac{1}{8} + \frac{19}{6}(1-\beta) + 6(1-\beta)^2 + 4(1-\beta)^3 + (1-\beta)^4. \quad (30)$$

Proof. By (8) there exist $p, q \in \mathcal{P}$ such that

$$\frac{f'(z)}{\phi'(z)} = \beta + (1-\beta)[p(z)] \quad \text{and} \quad \frac{g'(w)}{\psi'(w)} = \beta + (1-\beta)[q(w)]. \quad (31)$$

Let p, q have series representations as in the previous section. Then, equating coefficients of z, z^2 and z^3 of both sides of (31), we obtain

$$2a_2 = 2c_2 + (1-\beta)p_1 \quad \text{and} \quad -2a_2 = -2c_2 + (1-\beta)q_1, \quad (32)$$

$$3a_3 = 3c_3 + 2(1-\beta)c_2p_1 + (1-\beta)p_2, \quad (33)$$

$$4a_4 = 4c_4 + 3c_3(1-\beta)p_1 + 2(1-\beta)c_2p_2 + (1-\beta)p_3. \quad (34)$$

$$6a_2^2 - 3a_3 = 6c_2^2 - 3c_3 - 2(1-\beta)c_2q_1 + (1-\beta)q_2, \quad (35)$$

$$\begin{aligned} -20a_2^3 + 20a_2a_3 - 4a_4 &= -20c_2^3 + 20c_2c_3 - 4c_4 + 6c_2^2(1-\beta)q_1 - 3c_3(1-\beta)q_1 \\ &\quad - 2(1-\beta)c_2q_2 + (1-\beta)q_3. \end{aligned} \quad (36)$$

From (32), we get $p_1 = -q_1$, and $a_2 = c_2 + \frac{(1-\beta)p_1}{2}$, and making use of (33), (35), and (36), we have

$$a_3 = c_3 + (1-\beta)c_2p_1 + \frac{(1-\beta)(p_2 - q_2)}{6} + \frac{(1-\beta)^2p_1^2}{4}, \quad (37)$$

$$\begin{aligned} a_4 &= c_4 + \frac{1}{8}(1-\beta)(p_3 - q_3) + \frac{1}{4}c_2(1-\beta)(p_2 + q_2) - \frac{1}{2}c_2^2(1-\beta)p_1 \\ &\quad + \frac{5}{4}c_3(1-\beta)p_1 + \frac{5}{12}c_2(1-\beta)(p_2 - q_2) + \frac{5}{24}(1-\beta)^2p_1(p_2 - q_2). \end{aligned} \quad (38)$$

Therefore,

$$\begin{aligned} |a_2a_4 - a_3^2| &= \left| c_2c_4 - c_3^2 + \frac{1}{2}(1-\beta)c_4p_1 - \frac{3}{4}c_2c_3(1-\beta)p_1 \right. \\ &\quad + \frac{1}{8}c_3(1-\beta)^2p_1^2 - \frac{5}{4}(1-\beta)^2p_1^2c_2^2 - \frac{1}{3}c_3(1-\beta)(p_2 - q_2) \\ &\quad + \frac{5}{12}c_2^2(1-\beta)(p_2 - q_2) - \frac{1}{2}c_2(1-\beta)^3p_1^3 + \frac{1}{12}c_2(1-\beta)^2p_1(p_2 - q_2) \\ &\quad - \frac{1}{2}c_2^3(1-\beta)p_1 - \frac{1}{16}(1-\beta)^4p_1^4 + \frac{1}{48}(1-\beta)^3p_1^2(p_2 - q_2) \\ &\quad + \frac{1}{16}(1-\beta)^2p_1(p_3 - q_3) + \frac{1}{8}(1-\beta)c_2(p_3 - q_3) \\ &\quad \left. + \frac{1}{4}c_2^2(1-\beta)(p_2 + q_2) + \frac{1}{8}c_2(1-\beta)^2p_1(p_2 + q_2) - \frac{1}{36}(1-\beta)^2(p_2 - q_2)^2 \right|. \end{aligned} \quad (39)$$

Now, we apply the relations (24) and (25) to (39) and obtain

$$\begin{aligned}
 |a_2a_4 - a_3^2| &= \left| c_2c_4 - c_3^2 + \frac{(1-\beta)}{2}(c_4 - c_2c_3)p_1 - \frac{1}{4}(1-\beta)c_2c_3p_1 \right. \\
 &\quad + \frac{1}{8}(1-\beta)^2(c_3 - 10c_2^2)p_1^2 - \frac{1}{3}(1-\beta)\left(c_3 - \frac{5}{4}c_2^2\right)\left(\frac{4-p_1^2}{2}(x-y)\right) \\
 &\quad - \frac{1}{2}c_2(1-\beta)^3p_1^3 + \frac{1}{12}c_2(1-\beta)^2p_1\left(\frac{4-p_1^2}{2}(x-y)\right) \\
 &\quad - \frac{1}{2}c_2^3(1-\beta)p_1 - \frac{1}{16}(1-\beta)^4p_1^4 + \frac{1}{48}(1-\beta)^3p_1^2\left(\frac{4-p_1^2}{2}(x-y)\right) \\
 &\quad + \frac{1}{16}(1-\beta)^2p_1\left[\frac{p_1^3}{2} + \frac{(4-p_1^2)p_1}{2}(x+y) \right. \\
 &\quad \left. - \frac{(4-p_1^2)p_1}{4}(x^2+y^2) + \frac{4-p_1^2}{2}[(1-|x|^2)z - (1-|y|^2)w] \right] \\
 &\quad + \frac{1}{8}(1-\beta)c_2\left[\frac{p_1^3}{2} + \frac{(4-p_1^2)p_1}{2}(x+y) \right. \\
 &\quad \left. - \frac{(4-p_1^2)p_1}{4}(x^2+y^2) + \frac{4-p_1^2}{2}[(1-|x|^2)z - (1-|y|^2)w] \right] \\
 &\quad + \frac{1}{4}c_2^2(1-\beta)\left(p_1^2 + \frac{4-p_1^2}{2}(x+y)\right) + \frac{1}{8}c_2(1-\beta)^2p_1\left(p_1^2 + \frac{4-p_1^2}{2}(x+y)\right) \\
 &\quad \left. - \frac{1}{36}(1-\beta)^2\frac{(4-p_1^2)^2(x-y)^2}{4} \right|, \tag{40}
 \end{aligned}$$

where x, y, z , and w are such that $|x| \leq 1$, $|y| \leq 1$, $|z| \leq 1$, and $|w| \leq 1$.

According to Lemma 4, we may assume without any restriction that $p_1 \in [0, 2]$. Thus, applying the triangle inequality and taking $p_1 = p$, we find that

$$\begin{aligned}
 |a_2a_4 - a_3^2| &\leq \left| c_2c_4 - c_3^2 \right| + \frac{(1-\beta)}{2}|(c_4 - c_2c_3)|p \\
 &\quad + \frac{1}{4}(1-\beta)|c_2c_3|p + \frac{1}{8}(1-\beta)^2|(c_3 - 10c_2^2)|p^2 \\
 &\quad + \frac{1}{2}|c_2|(1-\beta)^3p^3 + \frac{1}{2}|c_2^3|(1-\beta)p + \frac{(1-\beta)^4}{16}p^4 \\
 &\quad + \frac{(1-\beta)^2}{32}p^4 + \frac{1-\beta}{16}|c_2|p^3 + \frac{1}{4}|c_2^2|(1-\beta)p^2 + \frac{1}{8}|c_2|(1-\beta)^2p^3 \\
 &\quad + (|x| + |y|)\left[\frac{1-\beta}{24}|c_3 - \frac{5}{4}c_2^2|(4-p^2) + \frac{1}{24}|c_2|(1-\beta)^2p(4-p^2) \right. \\
 &\quad + \frac{1}{96}(1-\beta)^3p^2(4-p^2) + \frac{(1-\beta)^2}{32}p^2(4-p^2) + \frac{(1-\beta)}{16}|c_2|p(4-p^2) \\
 &\quad \left. + \frac{1}{8}|c_2^2|(1-\beta)(4-p^2) + \frac{(1-\beta)}{16}c_2p(4-p^2) \right] \\
 &\quad + (|x|^2 + |y|^2)\left[\frac{(1-\beta)^2}{64}p^2(4-p^2) + \frac{(1-\beta)}{32}|c_2|p(4-p^2) \right. \\
 &\quad + (1-|x|^2)\left[\frac{1-\beta}{16}|c_2|(4-p^2) + \frac{(1-\beta)^2}{32}p(4-p^2) \right] \\
 &\quad + (1-|y|^2)\left[\frac{1-\beta}{16}|c_2|(4-p^2) + \frac{(1-\beta)^2}{32}p(4-p^2) \right] \\
 &\quad \left. + \frac{(1-\beta)^2}{144}(4-p^2)^2(|x| + |y|)^2 \right]. \tag{41}
 \end{aligned}$$

We now apply the Lemmas 3–5, and set $\gamma_1 = |x| \leq 1$, $\gamma_2 = |y| \leq 1$. Then, (41) can be rewritten in the form

$$|a_2a_4 - a_3^2| \leq S_1 + S_2(\gamma_1 + \gamma_2) + S_3(\gamma_1^2 + \gamma_2^2) + S_4(\gamma_1 + \gamma_2)^2 = F(\gamma_1, \gamma_2),$$

where

$$\begin{aligned}
 S_1 = S_1(p) &= \frac{1}{8} + \frac{5}{6}(1-\beta)p + \left[\frac{9}{8}(1-\beta)^2 + \frac{(1-\beta)}{4} \right] p^2 \\
 &+ \left(\frac{1-\beta}{16} + \frac{(1-\beta)^3}{2} + \frac{(1-\beta)^2}{8} \right) p^3 \\
 &+ \left[\frac{(1-\beta)^4}{16} + \frac{(1-\beta)^2}{32} \right] p^4 \\
 &+ \frac{(1-\beta)}{8}(4-p^2) + \frac{(1-\beta)^2}{16}p(4-p^2) \geq 0, \\
 S_2 = S_2(p) &= \left(\frac{(1-\beta)^2}{32} + \frac{(1-\beta)^3}{96} \right) p^2(4-p^2) \\
 &+ \left(\frac{1-\beta}{16} + \frac{5(1-\beta)^2}{48} \right) p(4-p^2) + \frac{(1-\beta)}{6}(4-p^2) \geq 0, \\
 S_3 = S_3(p) &= \frac{(1-\beta)^2}{64}(p^2-4)(4-p^2) \leq 0, \\
 S_4 = S_4(p) &= \frac{(1-\beta)^2}{144}(4-p^2)^2 \geq 0.
 \end{aligned}$$

Maximizing $F(\gamma_1, \gamma_2)$ in a square $\Delta := \{(\gamma_1, \gamma_2) : 0 \leq \gamma_1 \leq 1, 0 \leq \gamma_2 \leq 1\}$ we conclude that $\max F(\gamma_1, \gamma_2) = F(1, 1)$. Defining now a function $K : [0, 2] \rightarrow \mathbb{R}$ as in Theorem 1 defined by

$$K(p) = \max F(\gamma_1, \gamma_2) = F(1, 1) = S_1 + 2S_2 + 2S_3 + 4S_4, \quad (42)$$

and analyzing its behavior, we infer that $K(p)$ is an increasing function of p and attains its maximum at $p = 2$. Consequently, we have

$$\max_{0 \leq p \leq 2} K(p) = K(2) = \frac{1}{8} + \frac{19}{6}(1-\beta) + 6(1-\beta)^2 + 4(1-\beta)^3 + (1-\beta)^4,$$

that completes the proof of the theorem. $\square$

Remark 4. For $\phi(z) = z$, Theorem 2 reduces to the Theorem 1 [10].

3. Conclusions

In the present paper, we have estimated a smaller upper bound and more accurate estimation for the functional $|a_2a_4 - a_3^2|$ for functions in the class of strongly bi-close-to-convex functions of order α , ($0 \leq \alpha \leq 1$) and the class of bi-close-to-convex functions of order β , ($0 \leq \beta < 1$). Obtaining a sharp estimate for $|a_2a_4 - a_3^2|$ in these classes are still open and keeps the researcher interested.

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