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Refinements on Some Classes of Complex Function Spaces

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Abstract: Some weighted classes of hyperbolic function spaces are defined and studied in this paper. Finally, by using the chordal metric concept, some investigations for a class of general hyperbolic functions are also given.

Keywords: Bloch norm; hyperbolic norms; chordal metric

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1. Introduction

In the theory of holomorphic function spaces, the Besov and Bloch classes remain vital instruments which ensure the settings of Banach spaces. However, a good number of researchers generalized and extended these types of function spaces in certain numerous ways by choosing relevant weights, and by using auxiliary types of functions (hyperbolic, meromorphic, quaternion) as well as enlarging the classes of weighted function spaces for this kind of studied.

The major purpose of the present manuscript is to provide certain specific general concepts by concerned hyperbolic functions and discussing their properties in which the essentiality of the obtained results. What follows is a brief introduction to the concerned hyperbolic-type of function spaces.

To act the concerned aim, the symbol $\mathbb{D} = \{w : |w| < 1\}$ defines the open unit disk. In addition, the symbol $\mathcal{H}(\mathbb{D})$ stands for the space of all holomorphic functions in $\mathbb{D}$. Assuming also that $B(\mathbb{D})$ is a concerned subset of $\mathcal{H}(\mathbb{D})$ consisting of those $h \in \mathcal{H}(\mathbb{D})$, for which $|h(w)| < 1$, for all $w \in \mathbb{D}$. Furthermore, let $d\sigma_w = dx\,dy$ be the concerned normalized area measure on $\mathbb{D}$.

Moreover, the concerned Green's function of $\mathbb{D}$ is given by $g(w,b) = \log \frac{1}{|\varphi_b(w)|}$, with $\varphi_b(w) = \frac{b-w}{1-\overline{b}\,w}$, where the points $w, b \in \mathbb{D}$ may define a concerned Möbius transformation by the concerned singular point $b \in \mathbb{D}$.

Suppose that (M,d) defines a concerned metric space, then the concerned open and the concerned closed balls with center u and radius $\rho > 0$ can be defined by $B(u,\rho) := \{y_1 \in M : d(y_1,u) < \rho\}$ and $\overline{B}(u,\rho) := \{y_1 \in M : d(y_1,u) = \rho\}$, respectively.

Let $h^*(w) = \frac{|h'(w)|}{1-|h(w)|^2}$ define the concerned hyperbolic derivative of $h \in B(\mathbb{D})$. A specific function $h \in B(\mathbb{D})$ is said to belong to the hyperbolic α-Bloch class $\mathcal{B}^*_\alpha$ if

$$\|h\|_{\mathcal{B}^*_\alpha} = \sup_{w\in\mathbb{D}} h^*(w)(1-|w|^2)^\alpha < \infty, \quad \text{where } 0 < \alpha < 1.$$

The little concerned hyperbolic Bloch-type class $\mathcal{B}^*_{\alpha,0}$ consists of all $h \in \mathcal{B}^*_\alpha$, for which

$$\lim_{|w|\to 1} h^*(w)(1-|w|^2)^\alpha = 0.$$

The symbol $\simeq$ is used to express specific comparability.

We will use $C_1 \lesssim C_2$ to inform that there exists a positive constant k such that $C_1 \leq k\,C_2$. The symbol $\gtrsim$ can be also explained in a same way. During this manuscript, $\lambda, \lambda_1, \lambda_2$ and λ_3 are used to denote specific positive constants.

Let $b \in \mathbb{D}$ and $0 < |w| = r < 1$. The symbol $D(b,r)$ stands for pseudo-hyperbolic disk, with

$$|D(b,r)| = \frac{(1-|b|^2)^2}{(1-r^2|b|^2)^2} r^2.$$

Now, we introduce the following concerned general hyperbolic derivative:

$$h_n^*(w) = \frac{|h^{(n)}(w)|}{1-|h(w)|^{n+1}}; n \in \mathbb{N}.$$

Remarking that, when $n = 1$, then the usual hyperbolic derivative is obtained.

Throughout this manuscript both of the function ω, which maps from the interval $(0,1]$ into the interval $(0,\infty)$, the condition $\omega(0) \neq 0$ holds. Furthermore, the concerned weight function E which maps from the interval $[0,\infty)$ into itself is a right-continuous and nondecreasing concerned functions.

We are interested in the class of all hyperbolic functions which we define it by:

$$\mathcal{B}_{n,\omega}^* = \left\{ h \in B(\mathbb{D}) : \|h\|_{\mathcal{B}_{n,\omega}^\alpha} = \sup_{b\in\mathbb{D}} \frac{(1-|w|^2)^\alpha}{\omega(1-|\varphi_b(w)|)} h_n^*(w) < \infty, \text{ where } 0 < \alpha < \infty \right\}.$$

In addition, we define the class of general weighted hyperbolic functions by:

$$\mathcal{S}_{n,\omega}^* = \left\{ h \in B(\mathbb{D}) : \sup_{b\in\mathbb{D}} \int\int_{D(b,r)} \frac{(h_n^*(w))^2}{\omega^2(1-|\varphi_b(w)|)} d\sigma_w < \infty \quad for\ some\ r \in (0,1) \right\}. \tag{1}$$

Another interesting class of hyperbolic functions can be introduced as follows:

Definition 1. *The concerned function $h \in B(\mathbb{D})$, is belonging to the hyperbolic classes $H_{E,n,\omega}^*$ if*

$$\sup_{b\in\mathbb{D}} \int_{\mathbb{D}} (h_n^*(w))^2 \frac{E(g(w,b))}{\omega^2(1-|\varphi_b(w)|)}\, d\sigma_w < \infty. \tag{2}$$

Remark 1. *When $n = 1$ and $\omega \equiv 1$ and considering the case of analytic or meromorphic functions, then we obtain some weighted analytic or meromorphic classes from the concerned classes $\mathcal{B}_{n,\omega}^*, \mathcal{S}_{n,\omega}^*, H_{E,n,\omega}^*$ which are researched by a number of authors (see [1–5] and others). On the other hand there are some results in Clifford analysis (see [6–10] and others).*

Remark 2. *It is said that the hyperbolic space $H_{E,n,\omega}^*$ is trivial when $H_{E,n,\omega}^*$ consisting of constant functions only. Furthermore, the hyperbolic space $H_{E,n,\omega}^*$ can be also trivial or not, this may be determined after considering the behavior of the concerned integral (convergent or divergent)*

$$J = \int_0^{1/e} E(\ln(1/\rho))\rho\, d\rho = \int_1^\infty E(s)e^{-2s}\, ds. \tag{3}$$

Proposition 1.

(i) When the concerned integral $J = \infty$, then the hyperbolic space $H_{E,n,\omega}^*$ must be trivial.

(ii) When the concerned integral $J < \infty$, so $H_{E,n,\omega}^* \subset \mathcal{B}_{\alpha,n,\omega}^*$.

2. Hyperbolic Type Classes

We investigate some general hyperbolic classes and their connection with some others. Important concerned properties of the used weights are also discussed. It is demand to see for what specific functions belonging to the concerned hyperbolic classes in $H_{E,n,\omega}^*$ may be null (trivial).

Theorem 1. *If the integral (3) diverges, thus the concerned hyperbolic spaces $H_{E,n,\omega}^*$ will contain specific constant functions only.*

Proof. The proof can be obtained from the following concerned calculations,

$$\begin{aligned}
\int\int_{\mathbb{D}}(h_n^*(w))^2\frac{E(g(w,b))}{\omega(1-|\varphi_b(w)|)}\,d\sigma_w &\geq \int\int_{D(b,r)}(h_n^*(w))^2\frac{E(g(w,b))}{\omega^2(1-|\varphi_b(w)|)}\,d\sigma_w \\
&= \int\int_{D(b,r)}\left(\frac{|h^{(n)}(w)|}{1-|h(w)|^{n+1}}\right)^2\frac{E(g(w,b))}{\omega^2(1-|\varphi_b(w)|)}\,d\sigma_w \\
&= \int\int_{|\varphi_b(w)|<r}\left(\frac{|h^{(n)}\varphi_b(w)|}{1-|h(\varphi_b(w))|^{n+1}}\right)^2|\varphi_b'(w)|^2\frac{E(\ln(1/|w|))}{\omega^2(1-|w|)}\,d\sigma_w \\
&\geq \frac{\pi}{2}\left(\frac{(1-|b|^2)|h^{(n)}(b)|}{(1-|h(b)|^{n+1})\omega(1-|b|)}\right)^2\int_0^r R\,E(\ln(1/R))dR=\infty.
\end{aligned}$$

which implies a contradiction, therefore the proof is established. □

As proved in [11], we will state the next result.

Theorem 2. *Assuming that $h\in B(\mathbb{D})$ and let*

$$\int_0^1\frac{r\,dr}{\omega^2(1-r)}<\infty.$$

Thus,

$$h\ \in\mathcal{B}_{n,\omega}^*\iff\sup_{b\in\mathbb{D}}\int\int_{D(b,r)}\frac{(h_n^*(w))^2}{\omega^2(1-|\varphi_b(w)|)}d\sigma_w<\lambda\,\pi\,,$$

where $\lambda>0$ is a constant.

Remark 3. *Theorem 1 is introduced for hyperbolic functions but Yamashita's result in [11] is proved for meromorphic functions. When we put $n=1$, $\omega\equiv 1$ and replacing the $h^*(w)$ by $h^\#(w)=\frac{|h'(w)|}{1+|h(w)|^2}$, then Yamashita's result [11] can be obtained.*

The following interesting question is considered.

Question 1

Let $h\in B(\mathbb{D})$ and assume that $r\in(0,1)$, is the condition

$$\sup_{b\in\mathbb{D}}\int\int_{D(b,r)}(h_n^*(w))^2\frac{E(g(w,b))}{\omega^2(1-|\varphi_b(w)|)}\,d\sigma_w<\infty \tag{4}$$

can be an actual necessary and sufficient to hold that $h\in\mathcal{B}_{n,\omega}^*$?

Concerned Answer

When (4) satisfies, so $h\in\mathcal{S}_{n,\omega}^*$. Hence, $H_{E,n,\omega}^*\subset\mathcal{S}_{n,\omega}^*$. On the other hand suppose that $h\in\mathcal{S}_{n,\omega}^*$. Further because E is bounded, it is very clear to find that the condition (4) can be satisfied. If E is unbounded and $h\in\mathcal{S}_{n,\omega}^*\setminus\mathcal{B}_{n,\omega}^*$, hence the specific supremum in (4) must be infinite $\forall\, r\in(0,1)$. To verify the assumption, we remark by Theorem 1 that when $h\in\mathcal{S}_{n,\omega}^*\setminus\mathcal{B}_{n,\omega}^*$, we deduce that

$$\sup_{b\in\mathbb{D}}\int\int_{D(b,r)}\frac{(h_n^*(w))^2d\sigma_w}{\omega^2(1-|\varphi_b(w)|)}\geq\pi\quad\forall\ r\in(0,1).$$

When $0<\rho<r$, then

$$\int\int_{D(b,r)}(h_n^*(w))^2\frac{E(g(w,b))}{\omega^2(1-|\varphi_b(w)|)}\,d\sigma_w\geq E(\ln(1/\rho))\int\int_{D(b,\rho)}\frac{(h_n^*(w))^2}{\omega^2(1-|\varphi_b(w)|)}\,d\sigma_w.$$

Hence,

$$\sup_{b\in\mathbb{D}} \int\int_{D(b,r)} (h_n^*(w))^2 \frac{E(g(w,b))}{\omega^2(1-|\varphi_b(w)|)}\, d\sigma_w \geq \pi\, E(\ln(1/\rho)),\ \ 0<\rho<r.$$

Letting $\rho \to 0$, we cannot obtain condition (4) with $r \in (0,1)$, therefore the concerned proof is finished.

Therefore, the concerned above condition (4) can be given as an actual sufficient condition for which the function $h \in \mathcal{B}^*_{n,\omega}$, can be used to be an actual necessary condition when the concerned weight function E is bounded too.
For the other case when we consider unbounded function E, suppose that $h \in \mathcal{B}^*_{n,\omega}$, it is not hard to show that (4) can be satisfied clearly.

The weighted functions E, ω playing essential roles in studying $H^*_{E,n,\omega}$. Some questions on the weights can be stated as follows:
Question 2

What further restrictions on the weighted functions E and ω may be added for $H^*_{E,n,\omega} \subset \mathcal{B}^*_{n,\omega}$?

When the concerned hyperbolic classes $H^*_{E_1,n}$ as well as $H^*_{E_2,n,\omega}$ can be congruent if $E_1 \neq E_2$?

In what follows we give answers for these important questions.

Proposition 2. *Suppose that $E(r) \to \infty$ when $r \to \infty$. Thus, $H^*_{E,n,\omega} \subset \mathcal{B}^*_{n,\omega}$.*

Now, the following elemental result shall be proved.

Theorem 3. *Suppose that $E(\infty) = 1$, thus we have*

$$h \in \mathcal{B}^*_{n,\omega} \Longleftrightarrow \sup_{b\in\mathbb{D}} \int\int_{D(b,r)} (h_n^*(w))^2 \frac{E(g(w,b))}{\omega^2(1-|\varphi_b(w)|)} d\sigma_w < \lambda\, \pi, \tag{5}$$

for some certain constant $r \in (0,1)$.

Proof. Suppose that $h \in B(\mathbb{D}$. Then for $0<r<1$, we have that

$$\begin{aligned}\int\int_{D(b,r)} (h_n^*(w))^2 \frac{E(g(w,b))}{\omega^2(1-|\varphi_b(w)|)}\, d\sigma_w &\leq \|h\|^2_{\mathcal{B}^*_{n,\omega}} \int\int_{D(b,r)} (1-|w|^2)^{-2} E(g(w,b))\, d\sigma_w \\ &\leq \lambda_1 \pi \|h\|^2_{\mathcal{B}^*_{n,\omega}} \int_0^r \frac{E(\ln(1/\rho))}{(1-\rho^2)^2} \rho\, d\rho. \end{aligned} \tag{6}$$

Thus, (6) holds by choosing r small enough.
On the other hand, assume that $C < \lambda\, \pi$ be the supremum in (6) which is given for some certain $r_0 \in (0,1)$. Assuming that $r \in (0, r_0)$. Because $D(b,r) = \{w \in \mathbb{D} : g(w,b) > \ln(1/r)\}$, we obtain that

$$\begin{aligned}\int\int_{D(b,r)} \frac{(h_n^*(w))^2}{\omega^2(1-|\varphi_b(w)|)} d\sigma_w &\leq \frac{1}{E(\ln(1/r))} \int\int_{D(b,r)} (h_n^*(w))^2 \frac{E(g(w,b))}{\omega^2(1-|\varphi_b(w)|)}\, d\sigma_w \\ &\leq \frac{C}{E(\ln(1/r))} < \lambda_2\, \pi.\end{aligned}$$

Thus, $h \in \mathcal{B}^*_{n,\omega}$ regarding to Theorem 1, then the concerned proof is completely finished. □

Corollary 1. *Letting $E(\infty) = 1$. When $h \in H^*_{E,n,\omega}$ as well as*

$$\sup_{b\in\mathbb{D}} \int\int_{\mathbb{D}} (h_n^*(w))^2 \frac{E(g(w,b)) d\sigma_w}{\omega^2(1-|\varphi_b(w)|} < \pi,$$

then $h \in \mathcal{B}^*_{n,\omega}$.

To act an interesting property on weights of hyperbolic-type spaces, the next emerging result shall be presented.

Theorem 4. *Suppose that* $E(1) > 0$ *also let* $E_1(r) = \inf(E(r), E(1))$.

(i) *When the boundedness of the functions* E, ω *are hold, hence*

$$H^*_{E,n,\omega} = H^*_{E_1,n,\omega}.$$

(ii) *When the boundedness of the functions* E, ω *are not hold, so*

$$H^*_{E,n,\omega} = \mathcal{B}^*_{n,\omega} \cap H^*_{E_1,n,\omega}.$$

Proof. (i) Suppose that the weights E, ω are bounded, then

$$E_1(r) \leq E(r) \leq \frac{E(\infty)}{E(1)} E_1(r).$$

Hence, it is obvious to see that

$$H^*_{E,n,\omega} = H^*_{E_1,n,\omega}.$$

(ii) By using Proposition 2, the following inclusion can be obtained

$$H^*_{E,n,\omega} \subset \mathcal{B}^*_{n,\omega} \cap H^*_{E_1,n,\omega}.$$

Suppose that $h \in \mathcal{B}^*_{n,\omega}, \cap H^*_{E_1,n,\omega}$. Noting that $E(g(w,b)) = E_1(g(w,b))$ in $\mathbb{D}/D(b, 1/e)$. (For this pseudo-hyperbolic disk, we have $g(w,b) \leq 1$). By comparing the concerned integrals that defining the classes $H^*_{E,n,\omega}$ as well as $H^*_{E_1,n,\omega}$, it is enough to consider some certain integrals on $D(b, 1/e)$. In view of the concerned hypotheses $h \in \mathcal{B}^*_{n,\omega}$, we infer that

$$\begin{aligned}
\int\int_{D(b,1/e)} (h^*_n(w))^2 \frac{E(g(w,b))}{\omega^2(1-|\varphi_b(w)|)}\, d\sigma_w &\leq \|h\|^2_{\mathcal{B}^*_{n,\omega}} \int\int_{D(b,1/e)} (1-|w|^2)^{-2} E(g(w,b))\, d\sigma_w \\
&= \|h\|^2_{\mathcal{B}^*_{n,\omega}} \int\int_{D(0,1/e)<r} (1-|w_1|^2)^{-2} E(\ln(\frac{1}{r}))\, d\sigma_{w_1} \\
&= 2\pi \|h\|^2_{\mathcal{B}^*_{n,\omega}} \int_0^{1/e} r(1-|r|^2)^{-2}\, E(\ln(1/r)) dr.
\end{aligned}$$

Hence $h \in H^*_{E,n,\omega}$, therefore the concerned result is established completely. □

Some restrictions on the weighted function E_1 as well as on the function E_2, which make guarantee that

$$H^*_{E_1,n,\omega} = H^*_{E_2,n,\omega}$$

will be described in the next interesting result.

Theorem 5. *Let* E_1 *as well as* E_2 *be two weighted functions which both satisfying boundedness condition or both are not have boundedness condition. In addition assume that* $E_1(r) \approx E_2(r)$ *when* $r \to 0$. *Thus*

$$H^*_{E_1,n,\omega} = H^*_{E_2,n,\omega}.$$

Proof. Defining the weighted functions $E_{j,1}(r) = inf(E_j(r), E_j(1))$, $j = 1,2$. When the functions E_1 and E_2 are both bounded, then using the concerned hypothesis, we infer that

$$0 < c \leq E_1(r)/E_2(r) \leq c' < \infty, 0 < r < \infty,$$

where c and c' are two positive specific constants. Further, it is not hard to deduce that

$$H^*_{E_1,n,\omega} = H^*_{E_2,n,\omega}.$$

When, we have the unbounded case for the weighted functions E_1 and E_2, using Theorem 4, we thus obtain that

$$H^*_{E_1,n,\omega} = \mathcal{B}^*_{n,\omega} \cap H^*_{E_{1,1},n,\omega} = \mathcal{B}^*_{n,\omega} \cap H^*_{E_{2,1},n,\omega} = H^*_{E_2,n,\omega}.$$

Then, the proof is completely obtained. □

Theorem 6.

(i) *If E, ω are unbounded and (3) holds, then the equality $H^*_{E,n,\omega} = \mathcal{B}^*_{n,\omega}$ holds.*

(ii) *If E, ω are bounded and (3) holds, then the equality $H^*_{E,n,\omega} = \mathcal{S}^*_{n,\omega}$ holds too.*

(iii) *In the assertion (i) (respectively (ii)), the concerned condition (3) is an actual necessary condition for the equality $H^*_{E,n,\omega} = \mathcal{B}^*_{n,\omega}$ (respectively also the equality $H^*_{E,n,\omega} = \mathcal{S}^*_{n,\omega}$).*

Proof. For the proof of (i), the inclusion $H^*_{E,n,\omega} \subset \mathcal{B}^*_{n,\omega}$ can be obtained from the use of Proposition 2. On the other hand, when $h \in B^*_{n,\omega}$, we have that

$$\frac{(1-|w|^2)}{\omega^2(1-|\varphi_a(w)|)} h^*_n(w) \leq \lambda,$$

hence $h \in H^*_{E,n,\omega}$.

(ii) Since, we have that

$$H^*_{E,n,\omega} \subset \mathcal{B}^*_{n,,\omega}.$$

Hence, we can prove that

$$\mathcal{B}^*_{n,,\omega} \subset H^*_{E,n,\omega}.$$

If $h \in \mathcal{S}^*_{n,\omega}$, then we may find $r \in (0,1)$, which

$$\int\int_{D(b,r)} \frac{(h^*_n(w))^2}{\omega^2(1-|\varphi_b(w))} \, d\sigma_w \leq \lambda_3 < \infty \quad \text{for all } \ b \in \mathbb{D}. \tag{7}$$

Now, we prove that we may get a concerned constant $k_1(r, E)$, for which

$$\int\int_{\mathbb{D}} (h^*_n(w))^2 \frac{E(\ln(1/|w|))}{\omega^2(1-|w|)} \, d\sigma_w \leq \lambda_3 \|E\|_\infty + k_1(r, E). \tag{8}$$

Hence, we estimate

$$\int\int_{|w|<r} (h^*_n(w))^2 \frac{E(\ln(1/|w|))}{\omega^2(1-|w|)} \, d\sigma_w \leq \lambda_3 \|E\|_\infty.$$

Let $U_m = \{z : z - (R)^m \leq |z| \leq 1 - (R)^{m+1}\}$, where $1 - r = R$ and $m = 1, 2$. In view of U_m with

$$|b| = 1 - (R)^{m+1},$$

we consider now the constant $M((1-R)(R)^{m+1})^{-1}$ such concerned disks, where M is a concerned an absolute positive constant. Then,

$$\begin{aligned}\int\int_{U_m}(h_n^*(w))^2\frac{E(\ln(1/|w|))}{\omega^2(1-|w|)}\,d\sigma_w &\le E\left(\ln\frac{1}{1-(R)^m}\right)^{-1}\lambda_3\,M((1-R)(R)^{m+1})^{-1}\\ &\le E((R)^m\gamma(1-R))\lambda_3\,M((1-R)(R)^{m+1})^{-1}.\end{aligned}$$

Setting $m(R,\omega)=(R)^{-1}\frac{\log(\frac{1}{1-R})}{\omega(R)}$. Then, we have

$$\begin{aligned}\int\int_{R<|w|<1}(h_n^*(w))^2\frac{E(\ln(1/|w|))}{\omega^2(1-|w|)}\,d\sigma_w &\le \lambda\,(1-R)^{-1}\sum_1^\infty(R)^{-k-1}E((R)^m m(R,\omega))\\ &\le \lambda\,(1-R)^{-2}(R)^{-2}\int_0^1 s^{-2}E(t\;m(R,\omega))\,dt.\\ &= \lambda\;m(R,\omega)(1-R)^{-2}(R)^{-2}\int_0^{m(R,\omega)}\ell^{-2}E(\ell)\,d\ell=k_1<\infty.\end{aligned}$$

From (3). We have deduced that (7) can be satisfied, $\forall\;\;h\in\mathcal{S}^*_{n,\omega}$.
Since, $\forall\;b\in\mathbb{D}$, we can infer that

$$\sup_{a\in\mathbb{D}}\int\int_{D(a,r)}\frac{((h\circ\varphi_b)^*_n(w))^2}{\omega^2(1-|\varphi_a(w)|)}d\sigma_w=\sup_{a\in\mathbb{D}}\int\int_{D(a,r)}\frac{(f_n^*(w))^2}{\omega^2(1-|\varphi_a(w)|)}\,d\sigma_w=\lambda.$$

Hence, using (7) and (8) with f_n^* replaced by $(f\circ\varphi_z)^*_n$ that

$$\sup_{b\in\mathbb{D}}\int\int_{\mathbb{D}}(h_n^*(w))^2\frac{E(\ln\frac{1}{|\varphi_z(w)|})}{\omega^2(1-|\varphi_z(w)|)}d\sigma_w=\sup_{z\in\mathbb{D}}\int\int_{\mathbb{D}}((h\circ\varphi_b)^*_n(w))^2\frac{E(\ln(1/|w|))}{\omega^2(1-|w|)}d\sigma_w\le C_1+\lambda\|E\|_\infty \tag{9}$$

this proves (ii) in Theorem 6.
(iii) For functions h_1 and h_2 in $\mathcal{B}^*_{n,\omega}$, for which

$$c_0=\inf_{w\in\mathbb{D}}\frac{(1-|w|^2)}{\omega^2(1-|\varphi_a(w)|)}(h^*_{n,1}(w)+h^*_{n,2}(w))>0 \tag{10}$$

If $H^*_{E,n,\omega}=\mathcal{B}^*_{n,\omega}$ or $H^*_{E,n,\omega}=\mathcal{S}^*_{n,\omega}\supset\mathcal{B}^*_{n,\omega}$, we have

$$\begin{aligned}\infty &> \sup_{a\in\mathbb{D}}\int\int_{\mathbb{D}}(h^*_{n,1}(w))^2+(h^*_{n,2}(w))^2\frac{E(g(w,a))}{\omega^2(1-|\varphi_a(w)|)}\,d\sigma_w.\\ &\ge \frac{1}{2}\int\int_{\mathbb{D}}(h^*_{n,1}(w)+h^*_{n,2}(w))^2\frac{E(g(w,0))}{\omega^2(1-|\varphi_0(w)|)}\,d\sigma_w.\\ &\ge (c_0^2/2)\int\int_{\mathbb{D}}(1-|w|^2)^{-2}E(g(w,0))\,d\sigma_w.\\ &= \pi c_0^2\int_0^1(1-r^2)^{-2}E(log(1/r))r\,dr.\end{aligned}$$

Hence the concerned condition (3) is verified. Therefore, (iii) is completely established, hence the concerned proof of Theorem 6 is obtained completely. □

On the boundary of the unit disc, we have following emerging spaces:

$$\mathcal{S}^*_{n,\omega,0}=\left\{h\in B(\mathbb{D}):\lim_{|b|\to1}\int\int_{D(b,r)}\frac{(h_n^*(w))^2}{\omega^2(1-|\varphi_b(w)|)}\,d\sigma_w=0\;\;\text{for some certain}\;\;r\in(0,1)\right\},$$

$$H^*_{E,n,\omega,0}=\left\{h\in B(\mathbb{D}):\lim_{|b|\to1}\int\int_{D(b,r)}(h_n^*(w))^2\frac{E(g(w,b))}{\omega^2(1-|\varphi_b(w)|)}\,d\sigma_w=0\right\},$$

$$\mathcal{B}^*_{n,\omega,0} = \left\{h \in B(\mathbb{D}) : \frac{(1-|w|^2)}{\omega^2(1-|\varphi_b(w)|)} h^*_n(w) \to 0\,,\ |w| \to\ 1\right\}.$$

Furthermore, the weighted hyperbolic Dirichlet class that can be given by

$$\mathcal{D}^*_{n,\omega} = \left\{h \in B(\mathbb{D}) : \int\int_{\mathbb{D}} \frac{(h^*_n(w))^2}{\omega^2(1-|\varphi_b(w)|)}\, d\sigma_w < \infty\right\}$$

Theorem 7. $H^*_{E,n,\omega,0} \subset \mathcal{S}^*_{n,\omega,0} = \mathcal{B}^*_{n,\omega,0}$.

Theorem 8. *Suppose that the condition (3) is satisfying, thus the equality*

$$H^*_{E,n,\omega,0} = \mathcal{B}^*_{n,\omega,0}$$

can be obtained.

Remark 4. *It is enough and not hard to show that the inclusion*

$$\mathcal{B}^*_{n,\omega,0} \subset H^*_{E,n,\omega,0}$$

can be verified.

Theorem 9.

(a) *Suppose that $E(0) > 0$, hence $\mathcal{D}^*_{n,\omega} = H^*_{E,n,\omega}$.*

(b) *For the weighted hyperbolic Dirichlet-type space, we obtain that*

$$\mathcal{D}^*_{n,\omega} \subset H^*_{E,n,\omega,0} \Longleftrightarrow E(0) = 0.$$

(c) *Suppose that $H^*_{E,n,\omega} \neq H^*_{E,n,\omega,0}$. Let $\mathcal{D}^*_{n,\omega} = H^*_{E,n,\omega}$, thus $E(0) > 0$.*

(d) *When the equality*

$$\mathcal{D}^*_{n,\omega} = H^*_{E,n,\omega} = H^*_{E,n,\omega,0}$$

holds, the following concerned equality $E(0) = 0$ can be deduced.

Proof. To prove (a), we will suppose first that $E(0) > 0$ by noting the inclusion

$$\mathcal{D}^*_n \subset \mathcal{B}^*_{n,\omega,0} = S_{n,\omega,0},$$

also assuming the boundedness for the functions E and ω, it is obvious to see that

$$H^*_{E,n,\omega} = \mathcal{D}^*_{n,\omega}.$$

The second case is to suppose that E, ω are not bounded functions, then by Theorem 3 and since

$$H^*_{E_1,n,\omega} = \mathcal{D}^*_{n,\omega},$$

we infer that

$$H^*_{E,n,\omega} = \mathcal{B}^*_{n,\omega} \cap H^*_{E_1,n,\omega} = \mathcal{B}^*_{n,\omega} \cap \mathcal{D}^*_{n,\omega} = \mathcal{D}^*_{n,\omega},$$

then (a) is proved. Furthermore, the proof of (b) is not hard.

To act the proof of (c), we note from (b) and the concerned hypothesis that

$$\mathcal{D}^*_{n,\omega} \not\subseteq H^*_{E,n,\omega,0}.$$

If the hypothesis in (d) holds, we can obtain that $\mathcal{D}^*_{n,\omega} \subset H^*_{E,n,\omega,0}$ and the inclusion is obtained using (b). □

Corollary 2.

$$\mathcal{D}^*_{n,\omega} \subset H^*_{p,n,\omega,0}, \quad \forall\ 0 < p < \infty,$$

where,

$$H^*_{p,n,\omega,0} = \left\{h \in B(\mathbb{D}) : \lim_{|b|\to 1} \int\int_{D(b,r)} (h^*_n(w))^2 \frac{(g(w,b))^p}{\omega^2(1-|\varphi_b(w)|)}\, d\sigma_w = 0\right\}.$$

Corollary 3.

$$\mathcal{D}^*_{n,1} \subset H^*_{p,n,1,0}, \quad \forall\ 0 < p < \infty,$$

where,

$$H^*_{p,n,1,0} = \left\{h \in B(\mathbb{D}) : \lim_{|b|\to 1} \int\int_{D(b,r)} (h^*_n(w))^2 (g(w,b))^p\, d\sigma_w = 0\right\}.$$

3. Chordal Metric

The chordal metric between the two concerned points z and w in the known extended complex plane $\widehat{C} = C \cup \{\infty\}$ is given by (see [12]):

$$C(z,w) = \begin{cases} \frac{|z-w|}{(1+|z|^2)^{\frac{1}{2}}(1+|w|^2)^{\frac{1}{2}}}, & \text{when } z,w \neq \infty. \\ \frac{1}{(1+|z|^2)^{\frac{1}{2}}}, & \text{when } w = \infty. \end{cases}$$

For, $-1 < \alpha < \infty$, the concerned hyperbolic Bergman class M^p_α can be given as the set of those $f \in B(\mathbb{D})$, for which

$$\|f\|^p_{M^p_{\alpha,\omega}} = \int_{\mathbb{D}} \big(C(f(w),0)\big)^p \frac{(1-|w|^2)^\alpha}{\omega^p(1-|w|)}\, d\sigma_w < \infty.$$

In view of the concerned chordal metric, the following concerned result shall be proved.

Theorem 10. *Assuming $p \in (1,\infty)$, $\alpha \in (-1,\infty)$ and let $f \in B(\mathbb{D})$. Suppose that*

$$\int_{|w|}^1 \frac{\left(1-\frac{|w|}{t}\right)^\alpha}{\omega^p\left(1-\frac{|w|}{t}\right)} \frac{dt}{t^3} < \infty.$$

Then, we can find a constant $\eta > 0$, for which

$$\int_{\mathbb{D}} \big(C(f(w),f(0))\big)^p \frac{(1-|w|^2)^\alpha}{\omega(1-|w|)}\, d\sigma_w \leq \eta \int_{\mathbb{D}} \big(f^*_n(z)\big)^p \frac{(1-|w|^2)^{p+\alpha}}{\omega^p(1-|w|)} \frac{d\sigma_w}{|w|}.$$

Proof. Now, assume $p = 1$ and $t \in (0,1)$. Hence,

$$C(f(w),f(0)) \leq \int_0^1 f^*_n(tw)|w|\, dt.$$

Therefore, from integration method by parts and applying Fubini's theorem, we deduce

$$\begin{aligned}
\int_{\mathbb{D}} C(f(w),f(0))\, \frac{(1-|w|^2)^{\alpha}}{\omega^p(1-|w|)}\, d\sigma_w &\lesssim \int_{\mathbb{D}}\int_0^1 f_n^*(tz)\, dt|w| \frac{(1-|w|^2)^{\alpha}}{\omega^p(1-|w|)}\, d\sigma_w \\
&= \int_0^1 \int_{D(0,t)} f_n^*(w)\, |w|\, \frac{\left(1-\frac{|w|}{t}\right)^{\alpha}}{\omega^p\left(1-\frac{|w|}{t}\right)} \frac{dt}{t^3}\, d\sigma_w \\
&= \int_{\mathbb{D}} f_n^*(w)\, |w| \int_{|w|}^1 \frac{\left(1-\frac{|w|}{t}\right)^{\alpha}}{\omega^p\left(1-\frac{|w|}{t}\right)} \frac{dt}{t^3}\, d\sigma_w \\
&\lesssim \int_{\mathbb{D}} (f_n^*(w))|w| d\sigma_w,
\end{aligned}$$

this is the needed concerned asymptotic inequality when $p=1$. For the case of $p>1$, we can find

$$q > \frac{p-1}{p} \text{ with } \alpha - pq + p > 0.$$

Applying the inequality of Hölder, we deduce

$$\begin{aligned}
&C(f(w),f(0)) \le \int_0^1 f_n^*(t\,w)|w|\, dt = \int_0^1 \left(f_n^*(1-t|w|)\right)^q \frac{|w|\, dt}{(1-t|w|)^q} \\
&\le \left(\int_0^1 \left(f_n^*(t\,w)\right)^p \frac{(1-t|w|)^{pq}}{\omega^p(1-t|w|)}\, dt\right)^{1/p} \left(\int_0^1 \frac{|w|^{(p-1)/p}\, dt}{\omega^{\frac{-p}{p-1}}(1-t|w|)(1-t|w|)^{pq/(p-1)}}\right)^{(p-1)/p} \\
&\lesssim \left(\int_0^1 \left(f_n^*(tw)\right)^p (1-t|w|)^{pq}\, dt|w|(1-|w|)^{p-1-pq}\right)^{1/p}
\end{aligned}$$

therefore Fubini's theorem implies that

$$\begin{aligned}
\int_{\mathbb{D}} \left(C(f(w),f(0))\right)^p \frac{(1-|w|^2)^{\alpha}}{\omega^p(1-t|w|)}\, d\sigma_w &\lesssim \int_{\mathbb{D}}\int_0^1 \left(f_n^*(tw)\right)^p (1-t|w|)^{pq}\, dt|w| \frac{(1-|w|)^{\alpha+p-1-pq}}{\omega^p(1-t|w|)}\, d\sigma_w \\
&= \int_0^1 \int_{D(0,t)} \left(f_n^*(w)\right)^p (1-|w|)^{pq}|w| \frac{\left(1-\frac{|w|}{t}\right)^{\alpha-pq+p-1}}{\omega^p\left(1-\frac{|w|}{t}\right)} \frac{dt}{t^3} d\sigma_w \\
&= \int_{\mathbb{D}} \left(f_n^*(w)\right)^p |w| \int_{|w|}^1 \frac{\left(1-\frac{|w|}{t}\right)^{\alpha+p-1}}{\omega^p\left(1-\frac{|w|}{t}\right)} \frac{dt}{t^3} d\sigma_w \\
&\lesssim \int_{\mathbb{D}} \left(f_n^*(w)\right)^p |w| d\sigma_w.
\end{aligned}$$

This completes the proof. □

Remark 5. *Studying some certain differential equations evolving some specific analytic function spaces were discussed by some authors (see [13,14]). For the defined general hyperbolic classes, the next question can be stated as follows:*
How one can apply some classes of hyperbolic-type to solve some specific differential equations?

4. Conclusions

The theory of complex function spaces has received interesting extensive attention due to its numerous applications in many branches of mathematics such as complex analysis, operator theory, measure theory, differential equations as well as functional analysis. The study of this theory depends on many tools.

The study of hyperbolic functions has been researched by many authors (see [15–19] and others).

Special attention of some generalizations of certain classes of weighted hyperbolic function spaces by using a general hyperbolic derivative is the object of this paper.

To this end, we propose interesting characterizations, a novel of general hyperbolic differentially private evolving hyperbolic-type of function classes releasing the general derivative which reduces the scales and tool up a clear utility. Moreover, a chordal metric characterization is presented for a class of hyperbolic functions.

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References

1. El-Sayed Ahmed, A.; Youssif, M. Classes of weighted tent function spaces and mixed norms with some applications. *Ital. J. Pure Appl. Math.* **2020**, *43*, 402–415.
2. Essén, M.; Wulan, H. On analytic and meromorphic functions and spaces of Q_K type. *Ill. J. Math.* **2002**, *46*, 1233–1258. [CrossRef]
3. Tang, S. Bers embedding of Q_K-Teichmüller space. *J. Math. Wuhan Univ.* **2020**, *40*, 210–218.
4. Tang, S.; Wu, P. Composition operator, boundedness, compactness, hyperbolic Bloch-type space $\mathcal{B}^*_\mu$, hyperbolic-type space. *J. Funct. Spaces* **2020**, *7*, 5390732.
5. Xiao, J. *Holomorphic Q Classes*; Lecture Notes in Mathematics, 1767; Springer: Berlin, Germany, 2001.
6. El-Sayed Ahmed, A. Hyperholomorphic weighted classes. *Math. Comput. Model.* **2012**, *55*, 1428–1435. [CrossRef]
7. El-Sayed Ahmed, A.; Omran, S. Weighted classes of quaternion-valued functions. *Banach J. Math.Anal.* **2012**, *6*, 180–191. [CrossRef]
8. El-Sayed Ahmed, A.; Omran, S. Extreme points and some quaternion functions in the unit ball of. *Adv. Appl. Clifford Algebr.* **2018**, *28*, 31. [CrossRef]
9. El-Sayed Ahmed, A.; Gürlebeck, K.; Reséndis, L.F.; Tovar, L.M. Characterizations for the Bloch space by $\mathbf{B}^{p,q}$ spaces in Clifford analysis. *Complex Var. EllipticEquations* **2006**, *51*, 119–136. [CrossRef]
10. Gürlebeck, K.; Kähler, U.; Shapiro, M.; Tovar, L.M. On Q_p spaces of quaternion-valued functions. *J. Complex Var.* **1999**, *39*, 115–135. [CrossRef]
11. Yamashita, S. Functions of uniformaly bounded characteristic. *Ann. Acad. Sci. Fenn. Ser. A. Math.* **1982**, *7*, 349–367. [CrossRef]
12. Wang, G. Metrics of Hyperbolic Type and Moduli Continuity of Maps. Ph.D. Thesis, Univrsity of Turku-Finland, Turku, Finland, 2013.
13. Gröhn, J.; Nicolau, A.; Rättyä, J. Mean growth and geometric zero distribution of solutions of linear differential equations. *J. Anal. Math.* **2018**, *134*, 747–768. [CrossRef]
14. Gröhn, J.; Huusko, J.; Rättyä, J. Linear differential equations with slowly growing solutions. *Trans. Am. Math. Soc.* **2018**, *370*, 7201–7227. [CrossRef]
15. El-Sayed Ahmed, A. Natural metrics and composition operators in generalized hyperbolic function spaces. *J. Inequalities Appl.* **2012**, *185*, 1–12. [CrossRef]
16. Kamal, A.; Ahmed, A.E.; Yassen, T.I. Lipschitz continuous and compact composition operator acting between some weighted general hyperbolic-type classes. *Korean J. Math.* **2016**, *24*, 647–662. [CrossRef]
17. Li, X. On hyperbolic Q classes, Dissertation, University of Joensuu, Joensuu. *Ann. Acad. Sci. Fenn. Math. Diss.* **2005**, *145*, 65.
18. Smith, W. Inner functions in the hyperbolic little Bloch class. *Mich. Math. J.* **1998**, *45*, 103–114. [CrossRef]
19. Yamashita, S. Hyperbolic Hardy classes and hyperbolically Dirichlet-finite functions. *Hokkaido Math. J.* **1981**, *10*, 709–722.