

Article

Non-Linear Neutral Differential Equations with Damping: Oscillation of Solutions

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Abstract: The oscillation of non-linear neutral equations contributes to many applications, such as torsional oscillations, which have been observed during earthquakes. These oscillations are generally caused by the asymmetry of the structures. The objective of this work is to establish new oscillation criteria for a class of nonlinear even-order differential equations with damping. We employ different approach based on using Riccati technique to reduce the main equation into a second order equation and then comparing with a second order equation whose oscillatory behavior is known. The new conditions complement several results in the literature. Furthermore, examining the validity of the proposed criteria has been demonstrated via particular examples.

Keywords: even-order equations; non-linear equations; oscillation; Riccati substitution; comparison technique



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1. Introduction

In the present paper, an attempt is made to study the oscillation properties of the non-linear even-order neutral differential equations with damping

$$\left(a(t) \left(y^{(j-1)}(t) \right)^\alpha \right)' + p_1(t) \left(y^{(j-1)}(t) \right)^\alpha + p_2(t) x^\alpha(\gamma(t)) = 0, \quad (1)$$

where $t \geq t_0$ and

$$y(t) := x(t) + q(t)x(\beta(t)). \quad (2)$$

Our novel outcomes are obtained by considering the following suppositions:

- (L₁) $a \in C^1([t_0, \infty))$, $a(t) > 0$, $a'(t) \geq 0$, $q, p_1, p_2 \in C([t_0, \infty))$, $p_1(t) > 0$, $p_2(t) > 0$, $0 \leq q(t) < q_0 < 1$, p_2 is not identically zero for large t , $\beta \in C^1([t_0, \infty))$, $\gamma \in C([t_0, \infty))$, $\beta'(t) > 0$, $\beta(t) \leq t$ and $\lim_{t \rightarrow \infty} \beta(t) = \lim_{t \rightarrow \infty} \gamma(t) = \infty$.
 (L₂) The following relations are satisfied

$$\gamma(t) < \beta(t), \quad \gamma'(t) \geq 0, \quad (3)$$

and

$$\int_{t_0}^{\infty} \left(\frac{1}{a(s)} \exp \left(- \int_{t_0}^s \frac{p_1(y)}{a(y)} dy \right) \right)^{1/\alpha} ds = \infty, \quad (4)$$

where $j \geq 4$ is an even natural number and α is a quotient of odd natural numbers.

Non-linear neutral differential equations have been extensively utilized to mathematically model several interesting phenomena that are observed in many areas of science

and technology such as economics, biology, fluid dynamics, physics, differential geometry, engineering, control theory, materials science, and quantum mechanics. Asymptotic properties of solutions of non-linear neutral differential equations have been the objective of many researchers. Oscillation theory, however, has gained particular attention due to its widespread applications in mechanical oscillations, earthquake structures, clinical applications, frequency measurements and harmonic oscillators, which involve symmetrical properties; see the remarkable monograph of Hale [1].

Exploring the past few years, the asymptotic behavior of non-linear neutral differential equations has become a significant research area in different disciplines. In context of oscillation theory, it has been the object of research for many academics, who have investigated this notion for non-linear neutral differential and difference equations; the reader can refer to [2–11].

In [12], Liu et al. used the integral averaging technique to establish oscillation conditions for the solutions of the equation

$$L'_x + p_1(t) \left| x^{(j-1)}(t) \right|^{p_1-2} x^{(j-1)}(t) + p_2(t) |(x(\beta(t)))|^{\theta_1-2} x(\beta(t)) = 0, \quad (5)$$

where $L_x = a(t) \left| x^{(j-1)}(t) \right|^{p_1-2} x^{(j-1)}(t)$. On the other hand, in [13], the authors obtained oscillation criteria for equations with damping via comparing with first-order equations.

Continuing the investigation, the authors in [14,15] considered equation of the form

$$y^{(j)}(t) + p_2(t)x(\gamma(t)) = 0, \quad (6)$$

and used the Riccati method to ensure that the equation is oscillatory if

$$\liminf_{t \rightarrow \infty} \int_{\gamma(t)}^t \Pi(s) ds > \frac{(j-1)2^{(j-1)(j-2)}}{e}, \quad (7)$$

and

$$\liminf_{t \rightarrow \infty} \int_{\gamma(t)}^t \Pi(s) ds > \frac{(j-1)!}{e}, \quad (8)$$

where $\Pi(t) := \gamma^{j-1}(t)(1 - \theta_1(\gamma(t)))p_2(t)$

The purpose of this paper is to improve and extend the results in [14,15] and establish new oscillation criteria for Equation (1). Our approach is based on the use of Riccati substitution to reduce Equation (1) into a second order equation and then compare it with a second order equation whose oscillatory behavior is known. For examining the validity of the proposed criteria, two examples with particular values are constructed.

2. Oscillation Conditions

The following lemmas are essential in the sequel.

Lemma 1 ([16]). *Let $x \in C^j([t_0, \infty), (0, \infty))$. Assume that $x^{(j)}(t)$ is of fixed sign and not identically zero on $[t_0, \infty)$ and that there exists a $t_1 \geq t_0$ such that $x^{(j-1)}(t)x^{(j)}(t) \leq 0$ for all $t \geq t_1$. If $\lim_{t \rightarrow \infty} x(t) \neq 0$, then for every $\mu \in (0, 1)$, there exists $t_\mu \geq t_1$ such that*

$$x(t) \geq \frac{\mu}{(j-1)!} t^{j-1} \left| x^{(j-1)}(t) \right| \text{ for } t \geq t_\mu,$$

for every $\mu \in (0, 1)$.

Lemma 2 ([17]). *If $x^{(i)}(t) > 0$, $i = 0, 1, \dots, j$, and $x^{(j+1)}(t) < 0$, then*

$$\frac{x(t)}{t^j/j!} \geq \frac{x'(t)}{t^{j-1}/(j-1)!}.$$

Lemma 3 ([18]). *Let*

$$x \text{ is an eventually positive solution of (1).} \quad (9)$$

Then, we have these cases:

$$\begin{aligned} (\mathbf{I}_1) : & \quad y(t) > 0, y'(t) > 0, y''(t) > 0, y^{(j-1)}(t) > 0 \text{ and } y^{(j)}(t) < 0, \\ (\mathbf{I}_2) : & \quad y(t) > 0, y^{(m)}(t) > 0, y^{(m+1)}(t) < 0 \text{ for some odd integer} \\ & \quad m \in \{1, 2, \dots, j-3\}, y^{(j-1)}(t) > 0 \text{ and } y^{(j)}(t) < 0, \end{aligned}$$

for $t \geq t_1$, *where* $t_1 \geq t_0$ *is sufficiently large.*

For the sake of simplification, we use some notations.

$$\begin{aligned} \mu_{t_0}(t) & : = \exp\left(\int_{t_0}^t \frac{\theta_1(y)}{a(y)} dy\right), \\ \tilde{\mu}_0(t) & : = \left(\frac{1}{\mu_{t_1}(t)a(t)} \int_t^\infty p_2(s)\mu_{t_1}(s)q_2^\alpha(\gamma(s))ds\right)^{1/\alpha} \\ \tilde{\mu}_k(t) & : = \int_t^\infty \tilde{\mu}_{k-1}(s)ds, \quad k = 1, 2, \dots, j-2 \end{aligned}$$

and

$$q_m(t) := \frac{1}{q(\beta^{-1}(t))} \left(1 - \frac{(\beta^{-1}(\beta^{-1}(t)))^{m-1}}{(\beta^{-1}(t))^{m-1}q(\beta^{-1}(\beta^{-1}(t)))}\right), \quad m = 2, j.$$

Lemma 4. *Let (9) hold. Then*

$$\left(\mu_{t_0}(t)a(t)\left(y^{(j-1)}(t)\right)^\alpha\right)' + \mu_{t_0}(t)p_2(t)(1-q(\gamma(t)))^\alpha y^\alpha(\gamma(t)) \leq 0, \text{ for } q_0 < 1 \quad (10)$$

and

$$\begin{aligned} & \left(\mu_{t_0}(t)a(t)\left(y^{(j-1)}(t)\right)^\alpha\right)' \\ & + \frac{\mu_{t_0}p_2(t)}{q^\alpha(\beta^{-1}(\gamma(t)))} \left(y(\beta^{-1}(\gamma(t))) - \frac{y(\beta^{-1}(\beta^{-1}(\gamma(t))))}{q(\beta^{-1}(\beta^{-1}(\gamma(t))))}\right)^\alpha \leq 0, \end{aligned} \quad (11)$$

for $t \geq t_1$, *where* $t_1 \geq t_0$ *is sufficiently large.*

Proof. Let (9) hold. It is not difficult to see that

$$\begin{aligned} & \frac{1}{\mu_{t_0}(t)} \frac{d}{dt} \left(\mu_{t_0}(t)a(t)\left(y^{(j-1)}(t)\right)^\alpha\right) \\ & = \frac{1}{\mu_{t_0}(t)} \left(\mu_{t_0}(t)\left(a(t)\left(y^{(j-1)}(t)\right)^\alpha\right)' + \mu_{t_0}'(t)a(t)\left(y^{(j-1)}(t)\right)^\alpha\right) \\ & = \left(a(t)\left(y^{(j-1)}(t)\right)^\alpha\right)' + \frac{\mu_{t_0}'(t)}{\mu_{t_0}(t)}a(t)\left(y^{(j-1)}(t)\right)^\alpha \\ & = \left(a(t)\left(y^{(j-1)}(t)\right)^\alpha\right)' + p_1(t)\left(y^{(j-1)}(t)\right)^\alpha. \end{aligned} \quad (12)$$

Since j is even, by Lemma 3, $\omega(\rho) > 0$ and from (2), we get that $x(t) \geq (1-q(t))y(t)$. Thus, from (1) and (12), we conclude that (10) holds.

On the other hand, from (2), we obtain

$$\begin{aligned} q(\beta^{-1}(t))x(t) & = y(\beta^{-1}(t)) - x(\beta^{-1}(t)) \\ & = y(\beta^{-1}(t)) - \left(\frac{y(\beta^{-1}(\beta^{-1}(t)))}{q(\beta^{-1}(\beta^{-1}(t)))} - \frac{x(\beta^{-1}(\beta^{-1}(t)))}{q(\beta^{-1}(\beta^{-1}(t)))}\right) \\ & \geq y(\beta^{-1}(t)) - \frac{1}{q(\beta^{-1}(\beta^{-1}(t)))}y(\beta^{-1}(\beta^{-1}(t))), \end{aligned} \quad (13)$$

which with (1), (12) and (13) give (11). The proof is complete. \square

It is well known (see [19]) that the equation

$$\left[r(t)(x'(t))^\alpha \right]' + p_2(t)x^\alpha(\beta(t)) = 0, \quad t \geq t_0, \tag{14}$$

where $\alpha > 0, r, \theta_2 \in C[t_0, \infty)$, is nonoscillatory if and only if the inequality

$$v'(t) + \alpha r^{\frac{-1}{\alpha}}(t)(v(t))^{(1+\alpha)/\alpha} + p_2(t) \leq 0, \quad \text{on } [t, \infty)$$

is satisfied.

Theorem 1. Assume that (3) and (4) hold. If

$$\left(\frac{((j-2)!)^\alpha \alpha \mu_{t_1}(\gamma(t)) a(\gamma(t))}{(\epsilon_1 \gamma'(t) \gamma^{j-2}(t))^\alpha} (x'(t))^\alpha \right)' + \mu_{t_1}(t) q_j^\alpha(\gamma(t)) p_2(t) x^\alpha(t) = 0 \tag{15}$$

and

$$\frac{1}{\gamma'(t)} x''(t) + \tilde{\mu}_{j-3}(t) x(t) = 0 \tag{16}$$

are oscillatory for some constant $\epsilon_1 \in (0, 1)$. Then (1) is oscillatory.

Proof. Let (9) hold. From Lemma 3, we have two possible cases (I₁) and (I₂).

Let (I₁) hold. From Lemma 2, we see $y(t) \geq \frac{1}{(j-1)} t y'(t)$ and then $(t^{1-j} y(t))' \leq 0$. Thus, we obtain

$$y(\beta^{-1}(\beta^{-1}(t))) \leq \frac{(\beta^{-1}(\beta^{-1}(t)))^{j-1}}{(\beta^{-1}(t))^{j-1}} y(\beta^{-1}(t)). \tag{17}$$

By Lemma 4, we have (11). Thus, (17) gives

$$\left(\mu_{t_1}(t) a(t) \left(y^{(j-1)}(t) \right)^\alpha \right)' + \mu_{t_1}(t) p_2(t) q_j^\alpha(\gamma(t)) y^\alpha(\beta^{-1}(\gamma(t))) \leq 0. \tag{18}$$

Define

$$w(t) = \mu_{t_1}(t) a(t) \frac{\left(y^{(j-1)}(t) \right)^\alpha}{y^\alpha(\gamma(t))}, \quad w(t) > 0.$$

Differentiating w and using (18), we get

$$\begin{aligned} w'(t) \leq & -\mu_{t_1}(t) \theta_2(t) q_j^\alpha(\gamma(t)) \frac{y^\alpha(\beta^{-1}(\gamma(t)))}{y^\alpha(\gamma(t))} \\ & - \frac{\mu_{t_1}(t) a(t) \left(y^{(j-1)}(t) \right)^\alpha}{y^{2\alpha}(\gamma(t))} \alpha y^{\alpha-1}(\gamma(t)) y'(\gamma(t)) \gamma'(t). \end{aligned} \tag{19}$$

From Lemma 1, we see

$$y'(\gamma(t)) \geq \frac{\epsilon}{(j-2)!} \gamma^{j-2}(t) y^{(j-1)}(\gamma(t)). \tag{20}$$

Since $\mu_{t_1}(t) a(t) \left(y^{(j-1)}(t) \right)^\alpha$ is decreasing, we obtain

$$\mu_{t_1}(t) a(t) \left(y^{(j-1)}(t) \right)^\alpha \leq \mu_{t_1}(\gamma(t)) a(\gamma(t)) \left(y^{(j-1)}(\gamma(t)) \right)^\alpha, \quad \text{for all } t \geq \gamma(t), \tag{21}$$

which implies

$$\frac{1}{\mu_{t_1}^{1/\alpha}(\gamma(t)) a^{1/\alpha}(\gamma(t))} \left(\mu_{t_1}(t) a(t) \right)^{1/\alpha} y^{(j-1)}(t) \leq y^{(j-1)}(\gamma(t)). \tag{22}$$

From (20) and (22), we have

$$y'(\gamma(t)) \geq \frac{\epsilon}{(j-2)!} \frac{\gamma^{j-2}(t)}{\mu_{t_1}^{1/\alpha}(\gamma(t))a^{1/\alpha}(\gamma(t))} (\mu_{t_1}(t)a(t))^{1/\alpha} y^{(j-1)}(t). \quad (23)$$

Since $\beta^{-1}(t) > t$ and $y'(t) > 0$, we have $y(\beta^{-1}(t)) > y(t)$ and so

$$\frac{y(\beta^{-1}(\gamma(t)))}{y(\gamma(t))} > 1. \quad (24)$$

By using (23) and (24) in (19), we have

$$w'(t) \leq -\mu_{t_1}(t)\theta_2(t)q_j^\alpha(\gamma(t)) - \frac{\mu_{t_1}(t)a(t)(y^{(j-1)}(t))^{\alpha+1}}{y^{\alpha+1}(\gamma(t))} \alpha \frac{\epsilon \gamma'(t) \gamma^{j-2}(t)}{(j-2)!} \left(\frac{\mu_{t_1}(t)a(t)}{\mu_{t_1}(\gamma(t))a(\gamma(t))} \right)^{1/\alpha}. \quad (25)$$

From the definition of w , we have

$$w'(t) \leq -\mu_{t_1}(t)\theta_2(t)q_j^\alpha(\gamma(t)) - \frac{\alpha \epsilon \gamma'(t) \gamma^{j-2}(t)}{(j-2)! (\mu_{t_1}(\gamma(t))a(\gamma(t)))^{1/\alpha}} w^{(\alpha+1)/\alpha}(t),$$

which yields,

$$w'(t) + \frac{\alpha \epsilon \gamma'(t) \gamma^{j-2}(t)}{(j-2)! (\mu_{t_1}(\gamma(t))a(\gamma(t)))^{1/\alpha}} w^{(\alpha+1)/\alpha}(t) + \mu_{t_1}(t)q_j^\alpha(\gamma(t))p_2(t) \leq 0. \quad (26)$$

Thus, we conclude that (26) is nonoscillatory for every constant $\epsilon \in (0, 1)$. From [19], we see that (15) is nonoscillatory, which is a contradiction.

Let (I_2) hold. From Lemma 2, we obtain

$$y(t) \geq ty'(t) \quad (27)$$

and then $(t^{-1}y(t))' \leq 0$. Hence, since $\beta^{-1}(t) \leq \beta^{-1}(\beta^{-1}(t))$, we get

$$y(\beta^{-1}(\beta^{-1}(t))) \leq \frac{\beta^{-1}(\beta^{-1}(t))}{\beta^{-1}(t)} y(\beta^{-1}(t)), \quad (28)$$

which with (11) yield

$$\left(\mu_{t_1}(t)a(t) \left(y^{(j-1)}(t) \right)^\alpha \right)' + p_2(t)\mu_{t_1}(t)q_2^\alpha(\gamma(t))y^\alpha(\beta^{-1}(\gamma(t))) \leq 0. \quad (29)$$

Integrating (29) from t to ∞ , we obtain

$$\begin{aligned} -y^{(j-1)}(t) &\leq -\left(\frac{1}{\mu_{t_1}(t)a(t)} \int_t^\infty \theta_2(s)\mu_{t_1}(s)q_2^\alpha(\gamma(s))y^\alpha(\beta^{-1}(\gamma(s))) ds \right)^{1/\alpha} \\ &\leq -\tilde{\mu}_0(t)y(\beta^{-1}(\gamma(t))). \end{aligned}$$

Integrating this inequality $j-3$ times from t to ∞ , we get

$$y''(t) + \tilde{\mu}_{j-3}(t)y(\beta^{-1}(\gamma(t))) \leq 0. \quad (30)$$

Define

$$\varphi(t) = \frac{y'(t)}{y(\gamma(t))}, \quad w(t) > 0.$$

Differentiating w , we obtain

$$\varphi'(t) = \frac{y''(t)}{y(\gamma(t))} - \frac{y'(t)}{y^2(\gamma(t))} y'(\gamma(t)) \gamma'(t).$$

Since $y''(t) < 0$, we see $y'(\gamma(t)) > y'(t)$ for all $t \geq \gamma(t)$. Thus

$$\varphi'(t) \leq \frac{y''(t)}{y(\gamma(t))} - \left(\frac{y'(t)}{y(\gamma(t))} \right)^2 \gamma'(t). \tag{31}$$

From (30), we get

$$\varphi'(t) \leq -\frac{\tilde{\mu}_{j-3}(t)y(\beta^{-1}(\gamma(t)))}{y(\gamma(t))} - \left(\frac{y'(t)}{y(\gamma(t))} \right)^2 \gamma'(t).$$

Since $\beta^{-1}(t) > t$ and $y'(t) > 0$, we have $y(\beta^{-1}(t)) > y(t)$ and so

$$\varphi'(t) \leq -\tilde{\mu}_{j-3}(t) - \left(\frac{y'(t)}{y(\gamma(t))} \right)^2 \gamma'(t). \tag{32}$$

From the definition of φ , we have

$$\varphi'(t) \leq -\tilde{\mu}_{j-3}(t) - \gamma'(t)\varphi^2(t),$$

that is,

$$\varphi'(t) + \gamma'(t)\varphi^2(t) + \tilde{\mu}_{j-3}(t) \leq 0. \tag{33}$$

Thus, we conclude that (33) is nonoscillatory. From [19], we see that (16) is nonoscillatory, which is a contradiction. Thus, the proof is complete. \square

Corollary 1. *Let (4) hold. If*

$$\liminf_{t \rightarrow \infty} \int_{\gamma(t)}^t (1 - q(\gamma(s)))^\alpha p_2(s) \frac{\mu_{t_0}(s)}{\mu_{t_0}(\gamma(s))} \left(\frac{\epsilon \gamma^{j-1}(s)}{a^{1/\alpha}(\gamma(s))} \right)^\alpha ds > \frac{((j-1)!)^\alpha}{e}, \tag{34}$$

then (1) is oscillatory.

Corollary 2. *Let (3) and (4) hold. If*

$$\liminf_{t \rightarrow \infty} \int_{\beta^{-1}(\gamma(t))}^t p_2(s) \frac{\mu_{t_0}(s)}{\mu_{t_0}(\beta^{-1}(\gamma(s)))} \left(\frac{\epsilon(\beta^{-1}(\gamma(s)))^{j-1} q_j(\gamma(s))}{a^{1/\alpha}(\beta^{-1}(\gamma(s)))} \right)^\alpha ds > \frac{((j-1)!)^\alpha}{e}, \tag{35}$$

and

$$\liminf_{t \rightarrow \infty} \int_{\beta^{-1}(\gamma(t))}^t \beta^{-1}(\gamma(s)) \tilde{\mu}_{j-3}(s) ds > \frac{1}{e}, \tag{36}$$

then (1) is oscillatory.

It is well known (see [20]) that if

$$\int_{t_0}^\infty \frac{1}{r(t)} dt = \infty, \text{ and } \liminf_{t \rightarrow \infty} \left(\int_{t_0}^t \frac{1}{r(s)} ds \right) \int_t^\infty \theta_2(s) ds > \frac{1}{4},$$

then (14) is oscillatory.

Corollary 3. Let (4) hold and $\alpha = 1$. If

$$\int_{t_0}^{\infty} \frac{\epsilon_1 \gamma'(t) \gamma^{j-2}(t)}{(j-2)! \mu_{t_1}(\gamma(t)) a(\gamma(t))} dt = \infty,$$

$$\liminf_{t \rightarrow \infty} \left(\int_{t_0}^t \frac{\epsilon_1 \gamma'(s) \gamma^{j-2}(s)}{(j-2)! \mu_{t_1}(\gamma(s)) a(\gamma(s))} ds \right) \int_t^{\infty} p_2(s) \mu_{t_1}(s) q_j^\alpha(\gamma(s)) ds > \frac{1}{4},$$

$$\int_{t_0}^{\infty} \gamma'(t) dt = \infty,$$

and

$$\liminf_{t \rightarrow \infty} \left(\int_{t_0}^t \int_{t_0}^{\infty} \gamma'(s) ds \right) \int_t^{\infty} \tilde{\mu}_{j-3}(s) ds > \frac{1}{4},$$

then (1) is oscillatory.

3. Applications

For the sake of demonstrating the validity of the above hypotheses, this section presents some particular examples in correspondence with Equation (1).

Example 1. Consider the equation

$$\left(x(t) + \frac{1}{2} x\left(\frac{1}{2}t\right) \right)^{(4)} + \frac{\theta_0}{t^4} x\left(\frac{9}{10}t\right) = 0, \quad t \geq 1. \tag{37}$$

Note that $\alpha = 1, j = 4, a(t) = 1, p_2(t) = \theta_0/t^4, \gamma(t) = 9t/10$ and $\beta(t) = t/2$. Applying the conditions (7) and (8) to Equation (37), we get

Condition	(7)	(8)
Condition	$\theta_0 > 1839.2$	$\theta_0 > 59.5$

By Corollary 1, all solutions of (37) are oscillatory if $\theta_0 > 57.5$. From this, we conclude that our results are better than those of [14,15].

Example 2. For $t \geq 1$, consider the equation

$$\left(x(t) + \frac{1}{2} x\left(\frac{t}{3}\right) \right)^{(4)} + \frac{1}{t} y^{(3)}(t) + \frac{\theta_0}{t^4} x\left(\frac{t}{2}\right) = 0, \tag{38}$$

where $\theta_0 > 0$ is a constant. Let $\alpha = 1, j = 4, a(t) = 1, p_1(t) = 1/t, p_2(t) = \theta_0/t^4, \gamma(t) = t/2, \beta^{-1}(t) = (3/2)t$ and $\beta(t) = t/3$. Then

$$\mu_{t_0}(t) = t, \mu_{t_0}(\gamma(t)) = t/2.$$

Thus, we see that

$$\begin{aligned} & \liminf_{t \rightarrow \infty} \int_{\gamma(t)}^t (1 - q(\gamma(s)))^\alpha p_2(s) \frac{\mu_{t_0}(s)}{\mu_{t_0}(\gamma(s))} \left(\frac{\epsilon \gamma^{j-1}(s)}{a^{1/\alpha}(\gamma(s))} \right)^\alpha ds \\ &= \liminf_{t \rightarrow \infty} \int_{t/2}^t \frac{\theta_0}{t^4} \left(\frac{t^3}{8} \right) ds = \frac{\theta_0}{8} \ln 2. \end{aligned}$$

It follows that

$$\theta_0 > \frac{48}{e \ln 2}. \tag{39}$$

Using Corollary 1, we deduce that all solutions of (38) are oscillatory if $\theta_0 > 25.5$.

4. Conclusions

In this paper, we establish new oscillation criteria for a certain class of even-order non-linear differential equation with damping of the form (1). Our approach is different and obtained by using Riccati technique and comparing it with second-order equations. The new proposed conditions complement several results in the literature. Furthermore, some interesting examples are presented to examine the applicability of theoretical outcomes. Establishing oscillation criteria if $\int_{t_0}^{\infty} \left(\frac{1}{a(s)} \exp\left(-\int_{t_0}^s \frac{p_1(y)}{a(y)} dy\right) \right)^{1/\alpha} ds < \infty$ could be a promising topic for future work.

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References

- Hale, J.K. *Theory of Functional Differential Equations*; Springer: New York, NY, USA, 1977.
- Alzabut, J.; Bohner, M.; Grace, S.R. Oscillation of nonlinear third-order difference equations with mixed neutral terms. *Adv. Differ. Equ.* **2021**, *2021*, 3. [[CrossRef](#)]
- Baculikova, B.; Dzurina, J.; Graef, J.R. On the oscillation of higher-order delay differential equations. *Math. Slovaca* **2012**, *187*, 387–400. [[CrossRef](#)]
- Bazighifan, O.; Alotaibi, H.; Mousa, A.A.A. Neutral Delay Differential Equations: Oscillation Conditions for the Solutions. *Symmetry* **2021**, *13*, 101. [[CrossRef](#)]
- Grace, S.; Alzabut, J. Oscillation results for nonlinear second order difference equations with mixed neutral terms. *Adv. Differ. Equ.* **2020**, *2020*, 8. [[CrossRef](#)]
- Bazighifan, O.; Ramos, H. On the asymptotic and oscillatory behavior of the solutions of a class of higher-order differential equations with middle term. *Appl. Math. Lett.* **2020**, *107*, 106431. [[CrossRef](#)]
- Bazighifan, O.; Ahmad, H. Asymptotic Behavior of Solutions of Even-Order Advanced Differential Equations. *Math. Eng.* **2020**, *2020*, 8041857. [[CrossRef](#)]
- Bazighifan, O.; Grace, S.R.; Alzabut, J.; Özbekler, A. New results for oscillatory properties of neutral differential equations with a p-Laplacian like operator. *Miskolc Math. Notes* **2020**, *21*, 631–640. [[CrossRef](#)]
- Agarwal, R.P.; Bohner, M.; Li, T.; Zhang, C. A new approach in the study of oscillatory behavior of even-order neutral delay differential equations. *Appl. Math. Comput.* **2013**, *225*, 787–794. [[CrossRef](#)]
- Li, T.; Rogovchenko, Y.V. Oscillation criteria for even-order neutral differential equations. *Appl. Math. Lett.* **2016**, *61*, 35–41. [[CrossRef](#)]
- Chatzarakis, G.E.; Džurina, J.; Jadlovská, I. A remark on oscillatory results for neutral differential equations. *Appl. Math. Lett.* **2019**, *90*, 124–130. [[CrossRef](#)]
- Liu, S.; Zhang, Q.; Yu, Y. Oscillation of even-order half-linear functional differential equations with damping. *Comput. Math. Appl.* **2011**, *61*, 2191–2196. [[CrossRef](#)]
- Agarwal, R.P.; Bazighifan, O.; Ragusa, M.A. Nonlinear Neutral Delay Differential Equations of Fourth-Order: Oscillation of Solutions. *Entropy* **2021**, *23*, 129. [[CrossRef](#)] [[PubMed](#)]
- Zafer, A. Oscillation criteria for even order neutral differential equations. *Appl. Math. Lett.* **1998**, *11*, 21–25. [[CrossRef](#)]
- Zhang, Q.; Yan, J. Oscillation behavior of even order neutral differential equations with variable coefficients. *Appl. Math. Lett.* **2006**, *19*, 1202–1206. [[CrossRef](#)]
- Agarwal, R.; Grace, S.; O'Regan, D. *Oscillation Theory for Difference and Functional Differential Equations*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 2000.

17. Kiguradze, I.T.; Chanturiya, T.A. *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1993.
18. Moaaz, O.; Awrejcewicz, J.; Bazighifan, O. A new approach in the study of oscillation criteria of even-order neutral differential equations. *Mathematics* **2020**, *8*, 179. [[CrossRef](#)]
19. Agarwal, R.; Shieh, S.L.; Yeh, C.C. Oscillation criteria for second order retarded differential equations. *Math. Comput. Model.* **1997**, *26*, 1–11. [[CrossRef](#)]
20. Nehari, Z. Oscillation criteria for second order linear differential equations. *Trans. Am. Math. Soc.* **1957**, *85*, 428–445. [[CrossRef](#)]