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Synchronous Steady State Solutions of a Symmetric Mixed Cubic-Superlinear Schrödinger System

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Abstract: Systems of coupled nonlinear PDEs are applied in many fields as suitable models for many natural and physical phenomena. This makes them active and attractive subjects for both theoretical and numerical investigations. In the present paper, a symmetric nonlinear Schrödinger (NLS) system is considered for the existence of the steady state solutions by applying a minimizing problem on some modified Nehari manifold. The nonlinear part is a mixture of cubic and superlinear nonlinearities and cubic correlations. Some numerical simulations are also illustrated graphically to confirm the theoretical results.

Keywords: steady states; variational methods; energy functional; existence of solutions; Nehari manifold; NLS systems

MSC: 35Q41; 35J50

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Citation: Chteoui, R.; Aljohani, A.F.; Ben Mabrouk, A. Synchronous Steady State Solutions of a Symmetric Mixed Cubic-Superlinear Schrödinger System. *Symmetry* **2021**, *13*, 190. https://doi.org/10.3390/sym13020190

Academic Editors: Carlo Cattani, Praveen Agarwal and Shilpi Jain
Received: 11 December 2020
Accepted: 13 January 2021
Published: 26 January 2021

1. Introduction

The purpose of the present study is to investigate a system of nonlinear Schrödinger (NLS) equations coupled by cubic and superlinear nonlinearities. We will focus mainly on a theoretical study of the existence of steady state solutions, especially positive ones. Consider for $p \in \mathbb{R}$, the model function

$$g(u,v) = |u|^{p-1} + v^2.$$

We aim to study the evolutive NLS system

$$\begin{cases} iu_t + \Delta u + g(u,v)u = 0, \\ iv_t + \Delta v + g(v,u)v = 0, \\ (u,v) = (u(x,t), v(x,t)) \in \mathbb{C}^2, \end{cases} \quad (1)$$

$(x,t) \in \mathbb{R}^N \times (t_0, +\infty)$, $N \geq 1$, $t_0 \geq 0$, u_t is the first order time-derivative of u, $\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator on $\mathbb{R}^N$.

Systems of coupled NLS equations have been tackled widely in the recent decades from both theoretical and applied aspects. However, the majority are relatively simple from the point of view of nonlinearities and/or coupling terms. It may be noticed that numerical studies have focused on standing and solitary waves. See for instance [1–3]. Many techniques have been applied to investigate such systems such as Palais-Smale notion. This later exhibits a bounded energy functional associated to the problem. Other methods have been applied such as Darboux transformation, least energy solutions, Fokas method, Morse index, etc. The readers may refer to [3–15]).

Coupled nonlinear Schrödinger equations are related to theoretical and applied fields. In hydrodynamics, NLS systems may be applied for waves propagation modeling. In [16] for example, directional effects have been confirmed such as the overlapping group velocity projection. In optics, short pulses propagation may be also governed by a system of NLS equations. Indeed, it is noticed in [17] that a single-mode optical fiber may be bimodal due to the birefringence. The original pulse may be subdivided into two possible pulses according to the directions of polarization. The polarizations are next coupled via an NLS system of the form (1). More backgrounds about these systems and their applications may be found in [18–22].

In the present paper, we are interested in an NLS system where the components of the solution are coupled by means of a mixture of nonlinear terms. Notice that the system in our case has a variational form as the nonlinear parts may be seen as derivatives of some functions of (u, v). Such systems which are special types of Hamiltonians are also related to biology, especially population dynamics modeling.

It is worth to notice that the main difficulty for these problems is generally the lack of Sobolev's embeddings' compactness. In the literature, to overcome this difficulty, researchers converted the problem under investigation to a radial form. The obtained function spaces, which are radially symmetric, possess compact embeddings. See [5,23–26].

In the next section, a full study of existence of steady state solutions to problem (1) will be developed. The main tool is by applying the notion of Nehari manifolds, Ekeland's variational principle and Palais-Smale condition. These concepts will be recalled with some details in Appendix A. Recall that a steady state solution of problem (1) is written as

$$W(x,t) = (e^{i\omega_1 t}u(x), e^{i\omega_2 t}v(x)),$$

where $\omega_1, \omega_2 \in \mathbb{R}$. Remark that in this case, the couple (u, v) satisfies the time-independent or stationary system

$$\begin{cases} \Delta u + (g(u,v) - \omega_1)u = 0, \\ \Delta v + (g(v,u) - \omega_2)v = 0, \end{cases} \tag{2}$$

$x \in \mathbb{R}^N$. This follows easily by substituting W in the system (1). Remark that a first advantage from the pure mathematical point of view is that steady state solutions permit to reduce the number of variables in the model. This permits in general to convert the original problem to a PDE easy to handle, for example when considering radially symmetric problems. In fact, symmetry notion has the principal role in the study of PDEs and coupled PDEs. Many types of symmetries have been applied by researchers to reduce the difficulties encountered. See for example [6,23,24,27–30].

In the following parts, we assume that the frequencies of the components u and v are equal and thus denote $\omega_1 = \omega_2 = \omega > 0$. We thus consider synchronous components u and v forming a pair solution of the system

$$\begin{cases} \Delta u + (g(u,v) - \omega)u = 0, \\ \Delta v + (g(v,u) - \omega)v = 0, \\ x \in \mathbb{R}^N. \end{cases} \tag{3}$$

A first question that may be asked is the following. Why ground state solutions are important? These are in fact confirmed in many applied fields. In general, atoms and/or molecules may occur in different situations such as excited and ground states. In atomic physics, excited states are quantum states of an atom or a molecule with more energy than the ground state. To understand these excited states we shall understand the fundamental state or the zero-point energy for the whole system known as ground state. Indeed, in quantum physics/mechanics, for a quantum mechanical system, the ground state posses in fact the lowest energy. It is sometimes called the vacuum state.

Steady state solutions may be also met in economic dynamics such as economies with capital and depreciation. Recall that for a given level of investment, there are always three logic possibilities. An investor may spend more than the depreciation of the capital stock, less that the value of the depreciation of the capital stock, or equal. In the last situation (equality), the investor will have the same amount of capital next period as it is done in the present. This means that if the level of investment remains the same for next period as in the present, the investor will just cover depreciation with no amount left over. The capital stock will remain the same in the third period also. Continuing with this process, the level of capital constitutes a steady state.

The main problem in studying physical systems is the existence and the multiplicity of ground states. The third thermodynamics law states that an absolute zero-temperature system is in a ground state. How many ground states exist? Mathematically speaking, the existence of one ground state may be explained by the so-called degeneracy.

Motivated by the numerous experimental and theoretical examinations of multi condensates, multi-component optical fibres, coupled dynamic economic models, the coupled systems of nonlinear PDEs such as (1) or other variants have constituted much motivations and have been attracting the attention of researchers. In the next section, more discussions on the importance of these systems will be exposed to reinforces the motivation of considering the problem studied here and its relation to applied fields.

The organization of the present paper is described as follows. In Section 2, a review on the physical framework of the problem investigated in the present paper is developed, provided with a mathematical overview. Section 3 is devoted to our main results on steady state solutions of problem (1). Nehari manifold concept, Ekeland's variational principle and Palais-Smale notion are applied to show the existence of positive solutions. Numerical simulations of some cases are also illustrated in order to confirm the theoretical results. Section 4 is a conclusion. An appendix is developed next to recall briefly some essential concepts applied in our work such as the Ekeland's variational principle, Palais-Smale sequences and Nehari manifold notion.

2. NLS Systems Mathematical Physics Overview

The purpose of this section is to provide mathematical and physical overviews on systems of coupled NLS equations such as the present problem as well as a general framework. We will discuss relative applications, the nature of solutions, the mathematical methods, and some other related concepts such as standing wave solutions, solitary wave solutions, and steady states.

NLS systems are indeed met in many situations in physics, such as nonlinear optics in modeling optical soliton propagation in fibres where the electric field propagates with at least two polarized components [31]. Already in nonlinear optics, the components of the solutions of the coupled NLS system are used to describe the components of a light beam in refractive materials. In [32], two-dimensional photo-refractive screening solutions with self-trapping have been experimentally observed.

NLS systems are also met in Bose-Einstein and spin Bose-Einstein condensates, waves propagation, etc. See for example [33]. In Bose-Einstein many condensates, possible mixtures of states occur. In [34], for example, triplet states in the magnetic insulator have been investigated. Coupled NLS systems of equations have been also applied to study matter waves in spinor Bose-Einstein condensate.

In some situations, NLS equations are coupled to other types to model more phenomenon. In [35] for example, a Schrödinger-Newton system has been discussed for understanding the concept of quantization of gravitational field.

In [36], the existence of solutions for a symmetric nonlinear multi-component Schrödinger system has been investigated. Using radial symmetry, the problem is converted into a generalized Ermakov-Pinney form. Positive periodic solutions have been next investigated. In [37], a system of coupled cubic-cubic NLS equations has been considered for homoclinic solutions. The authors discussed a specific physical application dealing with the possibility of coexistence of different condensates, which are spatially separated and weakly coupled.

A similar study as our's has been developed in [38] for ground state solutions of three-dimensional coupled cubic nonlinear Schrödinger system. Existence of positive, radially symmetric, ground state solutions has been studied. Besides, a local well-posedness result is discussed by applying Strichartz estimates and the contraction mapping principle. Ma and Zhao considered in [39] a multi-component nonlinear Schrödinger system for existence and uniqueness of ground states in the whole space by applying Schwartz symmetrization and the radial symmetry.

Belmonte-Beitia in [40] applied Lie symmetries for the existence of many types of solutions such as bright-bright type, dark-dark type, and dark-bright solutions of multi-component NLS systems. In [41], existence and nonexistence of solutions to a doubly coupled multi-component system of NLS equations have been studied in symmetric and non symmetric cases. In particular, the authors investigated the multiplicity and bifurcation phenomena for positive solutions. See also [42,43].

In [44], analytical bright solutions of a coupled cubic-cubic NLS system have been investigated via the symbolic computation procedure and Hirota method. The system is applied for modeling collisions for simultaneous and parallel bright solitons in elastic and inelastic cases. The same system has been applied also to describe pulse propagation in optical fibres in [45] and soliton wavelength in [46,47]. Similar investigations have been conducted in [48–52] for analytical periodic solutions and higher dimensions.

In [53], solutions of an analogue problem to (1) have been discussed in the one-dimensional case with a nonlinear model function $g(u,v) = |u|^2 + |v|^2$. Asymptotic behavior at time infinity has been investigated. These solutions constitute a class of time-independent or steady state solutions corresponding to a pulsation $\omega = 0$. The results join in some sense those exposed in [54–56].

Solitary wave solutions have been also shown for other models such as [57] where a nonlinear Klein-Gordon system has been considered in the case of electromagnetic and electrostatic fields. substituting $u(x,t) = u(x)e^{iwt}$ in the evolutive equation, the function $u = u(x)$ has been shown to satisfy an equation of the form

$$\Delta u + |u|^{p-2}u + (\omega^2 - m_0^2)u = 0, \tag{4}$$

where p, m are some fixed parameters. By replacing p with $p+1$ and $\omega^2 - m_0^2$ with ω, and taking a single solution (u,u) of problem (2), we obtain a perturbation form of (4). This also joins the famous Brezis-Nirenberg problem [58]. Similar studies to (4) and the techniques applied may be found in [59–64].

Already related to NLS systems, existence and non-existence of solutions have been investigated in [65]. Numerical solutions of a coupled NLS system on unbounded domains have been investigated in [66] where the components of the solutions are physically due to the wave amplitude considered in two polarizations. The cubic coupling parameter is related to phase modulation. By applying the operator splitting technique, the authors decomposed the problem into two decoupled linear Schrödinger equations and nonlinear subproblems with a small time step.

In [67], the behavior of the solutions of a full-modulated nonlinearities' NLS system has been examined. The authors considered the system

$$\begin{cases} u_t + m_1 u_{xx} = (\alpha + i\beta)u + f_1|u|^2u + f_2|u|^4u + f_3|v|^2u + f_4|u|^2|v|^2u, \\ v_t + m_2 v_{xx} = (\gamma + i\delta)v + g_1|v|^2v + g_2|v|^4v + g_3|u|^2v + g_4|u|^2|v|^2v, \end{cases} \tag{5}$$

where the components u and v are the complex envelopes relative to a co-moving electric field. The quantities α and γ describe the trapping potentials. The parameters β and δ are due to the amplifications. The functions f_j and g_j, $j = 1,1,3,4$ describe the nonlinearities, and m_1 and m_2 are dispersion parameters

A symmetric system of NLS equations has been also considered in [68] to model atmospheric gravity waves. Solutions have been investigated for the coupled NLS system

$$\begin{cases} iu_t + \alpha_1 u_{xx} + (\sigma_1 |u|^2 + \Gamma_1 |v|^2)u = 0, \\ iv_t + iCv_x + \alpha_2 v_{xx} + (\Gamma_2 |u|^2 + \sigma_2 |v|^2)v = 0. \end{cases} \tag{6}$$

Problem (6) above is converted to Boussinesq equation in some cases. See also [69] for coupled Boussinesq systems.

In [70], a system of coupled NLS equations has been investigated such as

$$\begin{cases} iu_t + iu_x + u_{xx} + u + v + \sigma_1 f(u,v)u = 0, \\ iv_t - iv_x + v_{xx} + u - v + \sigma_2 g(u,v)v = 0. \end{cases} \tag{7}$$

In (7), the functions f and g are real valued, smooth, nonlinear, and depend only on $(|u|^2, |v|^2)$. The parameters σ_1, σ_2 are fixed real numbers.

In [71], Nehari manifold approach has been applied to study the nonlinear elliptic system

$$\begin{cases} -\Delta_{p_i} u_i = \lambda_i a_i(x)|u_i|^{p_i-2}u_i + (\alpha_i + 1)c(x)|u_i|^{\alpha_i - 1}u_i \widehat{|u_i|^{\alpha_i+1}}, & x \in \Omega, \\ \qquad i = 1, 2, \dots, n, & \\ u_1 = u_2 = \dots u_n = 0, & x \in \partial\Omega, \end{cases} \tag{8}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary. $\widehat{|u_i|^{\alpha_i+1}} = \prod_{j \neq i} |u_j|^{\alpha_j+1}$. For $s > 1$, Δ_s is the s-Laplace operator defined by $\Delta_s Z = div(|\nabla Z|^{s-2}\nabla Z)$. $p_1, p_2, \dots, p_n > 1$. The parameters λ_i and α_i are positive real numbers, for all $i = 1, 2, \dots, n$. By applying the Nehari manifold notion and the fibering maps concept, existence and multiplicity of the solutions have been studied in the case of sign-changing weights. Notice that such a system may be understood as a time-independent version of an NLS multi-component system.

In [72], ground state solutions of linearly and nonlinearly coupled NLS systems have been investigated under suitable conditions such as spatially dependence of coefficients, their mixed behavior, periodicity in some directions, and positive finite limits in other directions. Positive ground state solutions have been confirmed for the problem

$$\begin{cases} -iu_t = \Delta u - v_1(x)u + \mu_1 |v|^2 u + \beta |v|^2 u + \gamma v, \\ -iv_t = \Delta v - v_2(x)v + \mu_2 |u|^2 v + \beta |u|^2 v + \gamma u, \end{cases} \tag{9}$$

$x \in \mathbb{R}^N, t > 0$.

In [73], NLS and Heat equations have been combined into a coupled system. Exponential stability has been investigated in a torus region by transforming the problem into one-dimensional coupled system in polar coordinates.

Benrhouma in [74,75] applied a perturbation method, Nehari manifold concept, linking theorem, and Ekeland's variational principle for existence, uniqueness and positive solutions of a type system such as

$$\begin{cases} -\Delta u + u = \frac{\alpha}{\alpha+\beta} w(x)|v|^\beta |u|^{\alpha-2}u, \\ -\Delta v + v = \frac{\beta}{\alpha+\beta} w(x)|u|^\alpha |v|^{\beta-2}v \end{cases} \tag{10}$$

in $\mathbb{R}^N$. Notice that the solutions u and v may be seen as standing wave solutions of the form $(u,v)(x,t) = e^{-it}(u(x), v(x))$ for an associated NLS system to (10).

In [76], ground state solutions of a multi-component focusing NLS system have been studied. Besides, a potential method has been applied for global existence and finite-time blow-up.

3. The Steady State Problem

To study existence of the solutions of problem (1) or its stationary version (3), we propose to apply the Ekeland's variational principle, Palais-Smale notion and Nehari manifolds. We firstly introduce the functional framework. Consider the functional space

$$H = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$$

equipped with the inner product

$$< (u,v);(\varphi,\psi) >_H = \int_{\mathbb{R}^N} (\nabla u \nabla \varphi + \omega u \varphi + \nabla v \nabla \psi + \omega v \psi) dx$$

and the norm

$$\|(u,v)\|_H = \Big(< (u,v);(u,v) >_H \Big)^{1/2}.$$

In the rest of the paper, we assume that $1 \leq N \leq 4$, $1 < p < \infty$ for $N \leq 2$, and $1 < p \leq \dfrac{N+2}{N-2}$ for $N \geq 3$. The following lemma is easy to prove.

Lemma 1. $\Big(H, \|(u,v)\|_H\Big)$ *is a Hilbert space.*

Definition 1. *A pair $w = (u,v)$ is called a weak solution of problem (3) if $w \in H$, and for all $(\varphi,\psi) \in \mathcal{C}_c^\infty(\mathbb{R}^N) \times \mathcal{C}_c^\infty(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} (\nabla u \nabla \varphi + \omega u \varphi + \nabla v \nabla \psi + \omega v \psi - f(u,v)\varphi - f(v,u)\psi) dx = 0$$

where

$$f(u,v) = (|u|^{p-1} + v^2)u = g(u,v)u.$$

Generally, in PDEs, single and coupled, weak solutions are issued from variational methods if the concerned problem possess the variational structure. The problem of existence of weak solutions becomes a problem of existence of critical points of a corresponding energy functional.

In the present case, it is obvious that $(0,0)$ is a solution of (3). Therefore, we will be interested for the rest of the paper in non-trivial solutions for problem (3) by applying critical point theory and the Nehari manifold. Consider the functional

$$I(u,v) = \frac{1}{2}\mathcal{L}(u,v) - \frac{1}{p+1}\mathcal{M}_p(u,v) - \frac{1}{2}\mathcal{N}(u,v), \tag{11}$$

where

$$\mathcal{L}(u,v) = \int_{\mathbb{R}^N} \Big(|\nabla u|^2 + |\nabla v|^2 + \omega(u^2 + v^2) \Big) dx,$$

$$\mathcal{M}_p(u,v) = \int_{\mathbb{R}^N} \Big(|u|^{p+1} + |v|^{p+1} \Big) dx$$

and

$$\mathcal{N}(u,v) = \int_{\mathbb{R}^N} u^2 v^2 dx.$$

The following lemma holds.

Lemma 2. **(i).** *The functional I is $\mathcal{C}^1(H,\mathbb{R})$.*

(ii). *Let (u,v) be a critical point of I on H. Then (u,v) is a weak solution of (3).*

(iii). *The functional I is not bounded neither from above nor from below on H.*

Proof. Straightforward calculus yield that

$$\mathcal{L}(u+\varphi, v+\psi) = \mathcal{L}(u,v) + 2\int_{\mathbb{R}^N} \Big(\nabla u \nabla \varphi + \nabla v \nabla \psi + \omega(u\varphi + v\psi)\Big) dx + O(\varphi, \psi). \quad (12)$$

Hence, $\mathcal{L}$ is differentiable and its differential is

$$\mathcal{L}'(u,v)(\varphi,\psi) = 2\int_{\mathbb{R}^N} \Big(\nabla u \nabla \varphi + \nabla v \nabla \psi + \omega(u\varphi + v\psi)\Big) dx.$$

Next, we have

$$\mathcal{M}_p(u+\varphi, v+\psi) = \mathcal{M}_p(u,v) + (p+1)\int_{\mathbb{R}^N} \Big(|u|^{p-1}u\varphi + |v|^{p-1}v\psi\Big) dx + O(\varphi,\psi). \quad (13)$$

Consequently, $\mathcal{M}_p$ is differentiable and its differential is

$$\mathcal{M}_p'(u,v)(\varphi,\psi) = (p+1)\int_{\mathbb{R}^N} \Big(|u|^{p-1}u\varphi + |v|^{p-1}v\psi\Big) dx.$$

Similarly, we get

$$\mathcal{N}(u+\varphi, v+\psi) = \mathcal{N}(u,v) + 2\int_{\mathbb{R}^N} (uv^2\varphi + vu^2\psi) dx + O(\varphi,\psi). \quad (14)$$

Hence, $\mathcal{N}$ is differentiable and its differential is

$$\mathcal{N}'(u,v)(\varphi,\psi) = 2\int_{\mathbb{R}^N} (uv^2\varphi + vu^2\psi) dx.$$

The quantity $O(\varphi,\psi)$ in Equations (12)–(14) above is a function of (φ,ψ) such that

$$\lim_{(\varphi,\psi)\to 0} O(\varphi,\psi) = 0.$$

It follows that I is differentiable and its differential is

$$\begin{aligned} I'(u,v)(\varphi,\psi) &= \mathcal{L}'(u,v)(\varphi,\psi) - \mathcal{M}_p'(u,v)(\varphi,\psi) - \mathcal{N}'(u,v)(\varphi,\psi) \\ &= \int_{\mathbb{R}^N} \Big(\nabla u \nabla \varphi + \nabla v \nabla \psi + \omega(u\varphi + v\psi)\Big) dx \\ &\quad - \int_{\mathbb{R}^N} \Big(|u|^{p-1}u\varphi + |v|^{p-1}v\psi\Big) dx \\ &\quad - \int_{\mathbb{R}^N} (uv^2\varphi + vu^2\psi) dx. \end{aligned}$$

□

To overcome the problem of the boundless of the functional I, we will apply the Nehari manifold notion for the existence of a non-trivial positive solution of problem (3). To do this, denote

$$\mathcal{NH} = \Big\{(u,v) \in H \setminus \{0\}\,,\ \Big\langle I'(u,v), (u,v)\Big\rangle = 0\Big\}.$$

The following proposition holds.

Proposition 1. *The Nehari manifold $\mathcal{NH}$ is not empty.*

Proof. Let $(u,v) \in H$ such that $u > 0$ and $v > 0$ on $\mathbb{R}^N$ and consider for $t \geq 0$,

$$\mathcal{H}_{u,v}(t) = I(tu,tv) = \frac{t^2}{2}\mathcal{L}(u,v) - \frac{t^{p+1}}{p+1}\mathcal{M}_p(u,v) - \frac{t^4}{2}\mathcal{N}(u,v).$$

It is straightforward that $\mathcal{H}_{u,v} \in \mathcal{C}^2(\mathbb{R}_+^*, \mathbb{R})$, and that

$$\mathcal{H}'_{u,v}(t) = t\Big(\mathcal{L}(u,v) - t^{p-1}\mathcal{M}_p(u,v) - 2t^2\mathcal{N}(u,v)\Big).$$

Denote next,

$$\mathcal{K}_{u,v}(t) = \mathcal{L}(u,v) - t^{p-1}\mathcal{M}_p(u,v) - 2t^2\mathcal{N}(u,v).$$

It is clear that $\mathcal{K}_{u,v}$ is strictly decreasing on $(0,\infty)$. Furthermore,

$$\mathcal{K}_{u,v}(0) = \mathcal{L}(u,v) > 0 \quad \text{and} \quad \lim_{t\to+\infty} \mathcal{K}_{u,v}(t) = -\infty.$$

Hence, there exists $t_0 > 0$ such that

$$\mathcal{K}_{u,v}(t_0) = 0.$$

At this point t_0, we get

$$\mathcal{H}'_{u,v}(t_0) = 0.$$

On the other hand,

$$\mathcal{H}'_{u,v}(t_0) = \Big\langle I'(t_0u, t_0v), (u,v)\Big\rangle.$$

Hence,

$$\Big\langle I'(t_0u, t_0v), (u,v)\Big\rangle = 0.$$

Denote next $\varphi = t_0u$ and $\psi = t_0v$. It is obvious that $(\varphi,\psi) \in H \setminus \{0\}$. Else, we have

$$\Big\langle I'(\varphi,\psi), (\varphi,\psi)\Big\rangle = 0.$$

Consequently, $(\varphi,\psi) \in \mathcal{NH}$. $\square$

Proposition 2. *For any sequence $(u_n, v_n)_n$ in $\mathcal{NH}$, the following assertion holds.*

$(I(u_n, v_n))_n$ is bounded (in $\mathbb{R}$) if and only if $(u_n, v_n)_n$ is bounded in H.

Proof. We claim that

$$I(u_n, v_n) \geq \max(\frac{p-1}{2(p+1)}, \frac{1}{4})\|(u_n, v_n)\|_H^2, \quad \forall (u_n, v_n)_m \subset \mathcal{NH}. \tag{15}$$

Indeed, recall that

$$I(u_n, v_n) = \frac{1}{2}\mathcal{L}(u_n, v_n) - \frac{1}{p+1}\mathcal{M}_p(u_n, v_n) - \frac{1}{2}\mathcal{N}(u_n, v_n), \tag{16}$$

On the other hand, $(u_n, v_n)_n \subset \mathcal{NH}$. Henceforth,

$$I'(u_n, v_n)(u_n, v_n) = \mathcal{L}(u_n, v_n) - \mathcal{M}_p(u_n, v_n) - 2\mathcal{N}(u_n, v_n) = 0.$$

Therefore,

$$\mathcal{M}_p(u_n, v_n) = \mathcal{L}(u_n, v_n) - 2\mathcal{N}(u_n, v_n). \tag{17}$$

Combining Equations (16) and (17) we get

$$I(u_n, v_n) = \frac{p-1}{2(p+1)}\mathcal{L}(u_n, v_n) + \frac{3-p}{2(p+1)}\mathcal{N}(u_n, v_n). \tag{18}$$

As a result, for $p \leq 3$, we obtain

$$I(u_n, v_n) \geq \frac{p-1}{2(p+1)}\mathcal{L}(u_n, v_n). \tag{19}$$

For $p > 3$, Equation (17) yields that

$$\mathcal{L}(u_n, v_n) = \mathcal{M}_p(u_n, v_n) + 2\mathcal{N}(u_n, v_n). \tag{20}$$

Consequently,

$$\mathcal{L}(u_n, v_n) \geq 2\mathcal{N}(u_n, v_n),$$

which yields that

$$\frac{3-p}{2(p+1)}\mathcal{N}(u_n, v_n) \geq \frac{3-p}{4(p+1)}\mathcal{L}(u_n, v_n). \tag{21}$$

Finally, by combining Equations (18) and (21) we obtain

$$I(u_n, v_n) \geq \left(\frac{p-1}{2(p+1)} + \frac{3-p}{4(p+1)}\right)\mathcal{L}(u_n, v_n) = \frac{1}{4}\mathcal{L}(u_n, v_n). \tag{22}$$

The Equations (19) and (22) yield that

$$I(u_n, v_n) \geq \max(\frac{p-1}{2(p+1)}, \frac{1}{4})\mathcal{L}(u_n, v_n), \quad \forall (u_n, v_n)_m \subset \mathcal{NH}. \tag{23}$$

Observing that

$$\mathcal{L}(u_n, v_n) = \|(u_n, v_n)\|_H^2, \quad \forall (u_n, v_n)_m \subset \mathcal{NH},$$

the inequality (15) holds. Hence, the necessary condition follows. We next show the sufficient one. Assume that the sequence $(u_n, v_N)_n \subset \mathcal{NH}$ is bounded in H. Equation (20) written otherwise yields that

$$\|(u_n, v_n)\|_H^2 = \mathcal{M}_p(u_n, v_n) + 2\mathcal{N}(u_n, v_n). \tag{24}$$

Consequently, both $(\mathcal{M}_p(u_n, v_n))_n$ and $(\mathcal{N}(u_n, v_n))_n$ are bounded. Next, again Equation (16) written otherwise yields that

$$I(u_n, v_n) = \frac{1}{2}\|(u_n, v_n)\|_H^2 - \frac{1}{p+1}\mathcal{M}_p(u_n, v_n) - \frac{1}{2}\mathcal{N}(u_n, v_n). \tag{25}$$

So, $(I(u_n, v_n))_n$ is a linear combination of bounded sequences. Hence, it is also bounded. □

We now state our main result on the existence of positive solutions for problem (3).

Theorem 1. *Problem (3) has a non-trivial positive solution.*

Proof. We proceed by steps. The following claims are true.

Claim 1. $I_0 = \inf\{I(u, v), \ (u, v) \in \mathcal{NH}\} > 0.$

Claim 2. *There exists sequences $(u_n, v_n)_n \subset \mathcal{NH}$ and $(a_n)_n \subset \mathbb{R}$ such that*

$$\begin{cases} I(u_n, v_n) \longrightarrow I_0 \\ \text{and} \\ I'(u_n, v_n) - a_n F'(u_n, v_n) \longrightarrow 0 \text{ in } H', \end{cases} \tag{26}$$

as $n \to +\infty$, where H' is the dual of H and where

$$F(u, v) = \left\langle I'(u, v), (u, v) \right\rangle.$$

Claim 3. *Any sequence satisfying (26) is a Palais-Smale sequence of I in H.*

Claim 4. *There exists a sequence* $(u_n, v_n)_n \subset \mathcal{NH}$ *that is a Palais-Smale sequence of* I *in* H.

Claim 5. *There exists a pair* $(u, v) \in H$ *satisfying*

$$I(u, v) = I_0 \quad \textit{and} \quad I'(u, v) = 0. \tag{27}$$

Claim 6. *For* (u, v) *defined in Claim 5, the pair* $(|u|, |v|)$ *is a positive solution of (3).*

We now prove Claims 1–6.

Proof of Claim 1. Recall that from Equation (15) we obtain

$$I(u_n, v_n) \geq C_p \|(u_n, v_n)\|_H^2, \quad \forall (u_n, v_n)_n \subset \mathcal{NH}, \tag{28}$$

where

$$C_p = \max\left(\frac{p-1}{2(p+1)}, \frac{1}{4}\right).$$

Consequently, $I_0 \geq 0$. Assume next that $I_0 = 0$. There exists a sequence $(u_n, v_n)_n \subset \mathcal{NH}$ such that

$$I(u_n, v_n) \longrightarrow 0 \text{ as } n \to +\infty.$$

Using again Equation (28), it follows that

$$x_n = \|(u_n, v_n)\|_H \longrightarrow 0 \text{ as } n \to +\infty.$$

Next, from (24) it follows that

$$x_n^2 = \mathcal{M}_p(u_n, v_n) + 2\mathcal{N}(u_n, v_n). \tag{29}$$

From the Sobolev's embeddings' theorem, there exists a constant $C_{p,N} > 0$ such that

$$\mathcal{M}_p(u_n, v_n) \leq C_{p,N} \|(u_n, v_n)\|_H^{p+1}. \tag{30}$$

Similarly, there exists a constant $C_N > 0$ such that

$$\mathcal{N}(u_n, v_n) \leq C_N \|(u_n, v_n)\|_H^4. \tag{31}$$

As a result, by combining Equations (29)–(31), we get

$$x_n^2 \leq C_{p,N} x_n^{p+1} + C_N x_n^4. \tag{32}$$

Or equivalently,

$$1 \leq C_{p,N} x_n^{p-1} + C_N x_n^2. \tag{33}$$

We obtain a contradiction with the fact that $x_n \longrightarrow 0$ as $n \to +\infty$. □

Proof of Claim 2. By definition of I_0, we have

$$I_0 = \inf\{I(u, v) \in H \setminus \{0\}, \ F(u, v) = 0\}.$$

Hence, by applying Ekeland's variational principle, it follows that there exists $(u_n, v_n)_n \subset \mathcal{NH}$ satisfying (26). Hence, Claim 2 holds. □

Proof of Claim 3. Let $(u_n, v_n)_n$ be satisfying Claim 2. It suffices to show that

$$a_n F'(u_n, v_n) \longrightarrow 0 \text{ in } H' \text{ as } n \to +\infty. \tag{34}$$

Indeed, observing that

$$I(u_n, v_n) \longrightarrow I_0 \text{ as } n \to +\infty, \tag{35}$$

we deduce from Proposition 2 that $(u_n, v_n)_n$ is bounded in H. Hence,

$$\Big\langle I'(u_n, v_n), (u_n, v_n) \Big\rangle - a_n \Big\langle F'(u_n, v_n), (u_n, v_n) \Big\rangle \longrightarrow 0 \text{ as } n \to +\infty. \tag{36}$$

Observing that

$$\Big\langle I'(u_n, v_n), (u_n, v_n) \Big\rangle = 0, \tag{37}$$

it follows that

$$a_n \Big\langle F'(u_n, v_n), (u_n, v_n) \Big\rangle \longrightarrow 0 \text{ as } n \to +\infty. \tag{38}$$

On the other hand,

$$\Big\langle F'(u_n, v_n), (u_n, v_n) \Big\rangle = (1 - p)\mathcal{M}_p(u_n, v_n). \tag{39}$$

So, it is bounded due to (24). Consequently, Equation (38) yields that

$$a_n \longrightarrow 0 \text{ as } n \to +\infty. \tag{40}$$

Henceforth, (26) implies that

$$I'(u_n, v_n) \longrightarrow 0 \text{ in } H' \text{ as } n \to +\infty. \tag{41}$$

□

Proof of Claim 4. It follows from Claims 2 and 3 immediately. □

Proof of Claim 5. We will prove that the sequence $(u_n, v_n)_n$ of Claim 4 is convergent in H to a pair (u, v) (up to a subsequence) satisfying (27). Indeed, from Proposition 2, the sequence $(u_n, v_n)_n$ is bounded in H. Hence, there exists a pair $(u, v) \in H$ such that

$$u_n \rightharpoonup u \quad \text{and} \quad v_n \rightharpoonup v \text{ as } n \to +\infty \tag{42}$$

in $H^1(\mathbb{R}^N)$. We shall show that

$$I(u, v) = I_0 \quad \text{and} \quad I'(u, v) = 0. \tag{43}$$

Let $(\varphi, \psi) \in C_c^\infty(\mathbb{R}^N) \times C_c^\infty(\mathbb{R}^N)$ be a test pair. It follows from (42) that

$$\mathcal{L}'(u_n, v_n)(\varphi, \psi) \longrightarrow \mathcal{L}'(u, v)(\varphi, \psi) \text{ as } n \to +\infty. \tag{44}$$

Next, as $(\varphi, \psi) \in C_c^\infty(\mathbb{R}^N) \times C_c^\infty(\mathbb{R}^N)$, it results that for $n \to +\infty$,

$$\mathcal{M}_p'(u_n, v_n)(\varphi, \psi) \longrightarrow \mathcal{M}_p'(u, v)(\varphi, \psi). \tag{45}$$

Similarly, as $n \to \infty$, we get

$$\mathcal{N}'(u_n, v_n)(\varphi, \psi) \longrightarrow \mathcal{N}'(u, v)(\varphi, \psi). \tag{46}$$

As a result, as $n \to +\infty$, we obtain

$$0 = \Big\langle I'(u_n, v_n), (\varphi, \psi) \Big\rangle \longrightarrow \Big\langle I'(u, v), (\varphi, \psi) \Big\rangle. \tag{47}$$

Hence, the second part of (43) is proved. It remains to prove its first part. We already now, up to a subsequence, that

$$\|(u_n, v_n)\|_H \longrightarrow \|(u, v)\|_H \text{ as } n \to +\infty. \tag{48}$$

Since the functional I is continuous, we get the desired result. □

Proof of Claim 6. It follows from the parity of the functionals I and F that

$$I(|u|, |v|) = I_0 \quad \text{and} \quad F(|u|, |v|) = 0, \tag{49}$$

which means that $(|u|, |v|) \in \mathcal{NH}$. Therefore,

$$\exists\, a \in \mathbb{R}, \quad I'(|u|, |v|) = aF'(|u|, |v|). \tag{50}$$

If we succeed in proving that $a = 0$, we get immediately $I'(|u|, |v|) = 0$, which means that $(|u|, |v|)$ is a solution of problem (3). Of course, it is positive. Now, by applying similar techniques as for (17) and (39), we get

$$\left\langle F'(|u|, |v|), (|u|, |v|) \right\rangle = (1 - p)\mathcal{M}_p(|u|, |v|). \tag{51}$$

Consequently,

$$0 = \left\langle I'(|u|, |v|), (|u|, |v|) \right\rangle = a(1 - p)\mathcal{M}_p(|u|, |v|). \tag{52}$$

Hence, $a = 0$. □

As Claims (1–6) are satisfied, a non-trivial positive solution is guaranteed for problem (3). □

To achieve our work, we discuss here-after some special cases that may be deduced and/or related to it.

Notice from the parity of the nonlinear parts $f_1(u, v) = g(u, v)u$ and $f_2(u, v) = f_1(v, u) = g(v, u)v$, that problem (1) presents symmetries according to the origin O and the axis $|u| = |v|$. Such symmetries show that if (u, v) is a solution, then, the pair $(\pm u, \pm v)$ is also a solution. This fact affects any study on classification, behavior, and uniqueness of the solutions of problem (3). Moreover, if we chose the two waves u and v with different pulsations $\omega_1 \neq \omega_2$, we get a non-symmetric problem and new difficulties may appear. A similar but more general variant of problem (3) has been already investigated in [77] for classification. The influence of the symmetry on the possibility of extending the obtained results on half-spaces and finite intervals has been discussed.

In the present paper, we focused on the existence of non-trivial solutions of problem (3) where both components u and v are simultaneously non-trivial (non identically zero). However, it is natural to investigate the case of semi-trivial solutions, where one of the pairs u or v is trivial. In this case, the system (3) becomes the single elliptic equation

$$\Delta u + (|u|^{p-1} - \omega)u = 0.$$

The stationary case $\omega = 0$ has been extensively studied, especially by Kajikyia [78–85]. The general case $\omega \neq 0$ may be deduced also from [59,60,63,77,86].

Now in a final stage of our work, we propose to develop few numerical examples due to problem (1). We recall, however, that a numerical study has been already developed in [77,87]. We although provide here some numerical simulations to illustrate the theoretical result obtained in the present paper. A nonstandard numerical method has been developed in [77] to approximate the solutions of problem (1) in higher dimensions. Besides, in [87], a classification of the steady state solutions of problem (3) has been developed with numerical experimentations. The readers may be referred to [77,87] for more details. This avoids to repeat the same developments and the redundancy. Besides, for convenience, we presented here few cases different slightly from those provided in [77,87].

To illustrate numerically and graphically the results of the present paper, we provided in Figures 1–3 the graphs of some cases of solutions of problem (3). In Figure 1, the graph illustrates the single non-trivial radial solution (u, u) for the cases $u(0) = 2$ and $u(0) = 0.75$ with $\omega = 2$ and $p = 1.5$. The figure shows easily the positivity of the solution. In Figure 2, a semi-trivial solution $(u, 0)$ is represented for $\omega = 2$ and $p = 1.5$ illustrating easily the positivity of u. Finally, in Figure 3, we plotted a non-trivial solution (u, v) for $\omega = 4$ and $p = 2.5$. We notice in this numerical example that the solution (u, v) has a limit (l, l) where l is the unique positive zero of $|x|^{p-1} + x^2 - \omega = 0$. In the present case, we have approximately $l = 1.4819$.

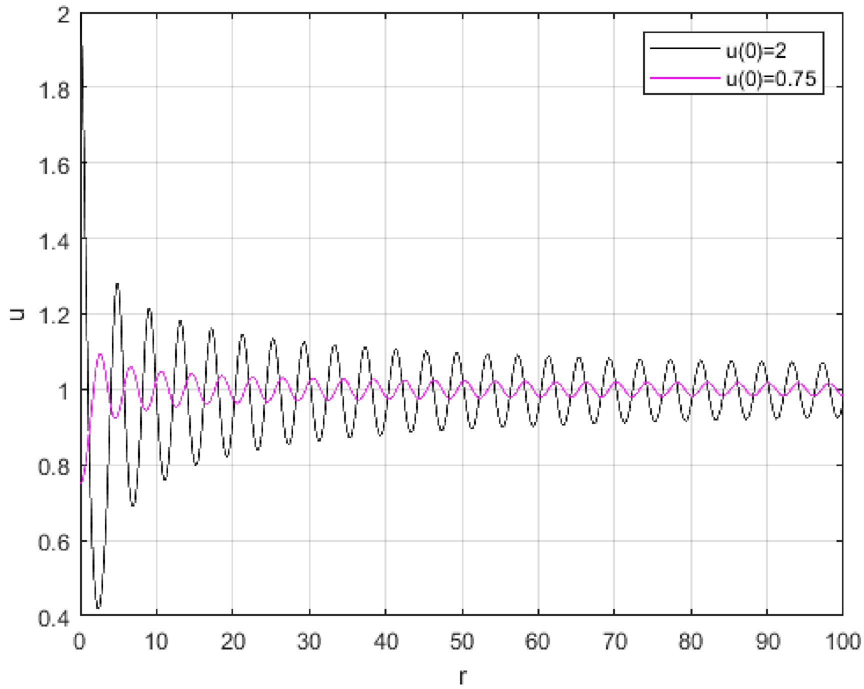

Figure 1. A positive single solution (u, u) for $\omega = 2$ and $p = 1.5$.

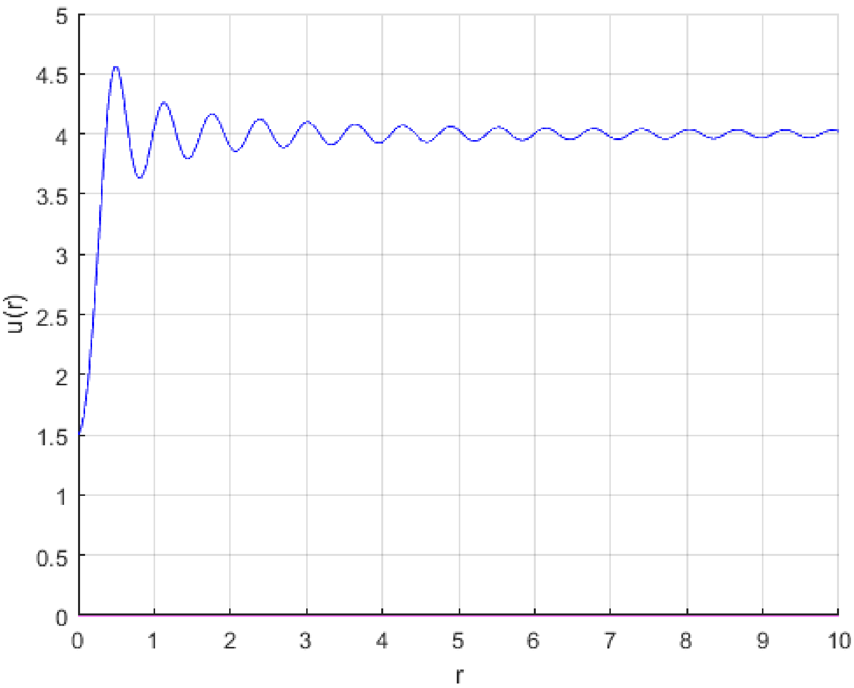

Figure 2. A semi-trivial positive solution $(u, 0)$ for $\omega = 2$ and $p = 1.5$.

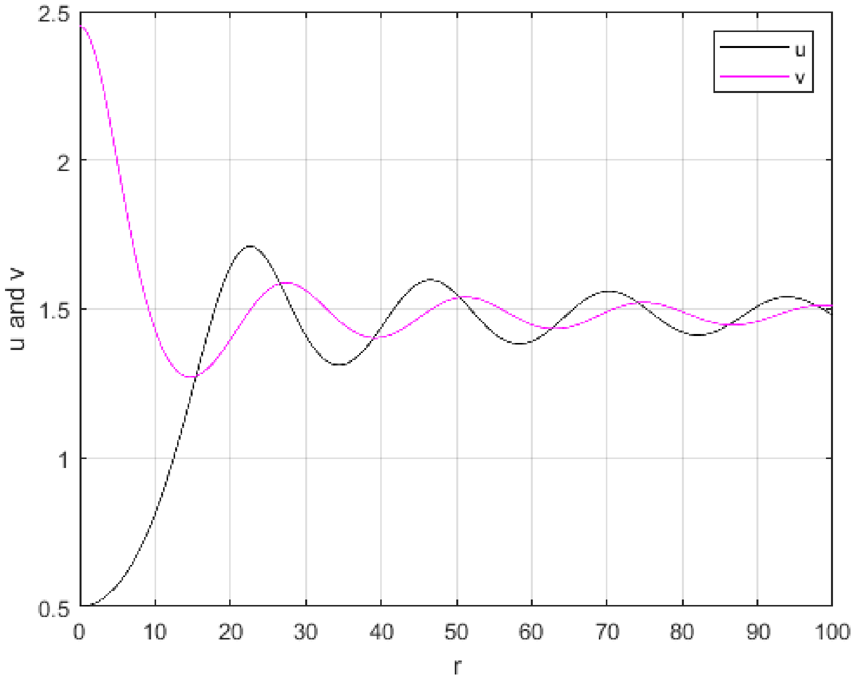

Figure 3. A positive non-trivial solution (u, v) for $\omega = 4$ and $p = 2.5$.

4. Conclusions

In the present work, existence of the solutions of a steady-state nonlinear Schrödinger system is investigated, especially for positive solutions. The problem is considered in the presence of mixed nonlinearities such as a cubic power law and a superlinear one with a cubic correlation. The main tools used in the analysis are the Nehari manifold notion, Ekeland's variational principle, and Palais-Smale sequences. Numerical simulations have been also provided in order to confirm the theoretical result. We showed a semi-trivial solution with a positive non-trivial component, a positive identical components solution, and a positive non-trivial and non-semi-trivial solution. We intend that the techniques applied in the present paper may be extended with necessary modifications to other systems with sublinear component $|u|^{p-1}u$ or general mixed nonlinear models on the form $(|u|^{p-1} \pm |v|^{q-1})u$ on bounded and unbounded domains. Existence, uniqueness, classification, nodal solutions, steady states, standing waves, radial, and non-radial solutions, numerical as a well as invariant solutions relatively to some structures such as groups are interesting extensions. For the moment, a full numerical study of problem (1) has been developed in [77]. Besides, in [87], the authors developed a full classification of the solutions of problem (1) for the one-dimensional case. We intend in the future to tackle the extensions raised here.

Author Contributions: Conceptualization, R.C., A.F.A. and A.B.M.; methodology, R.C., A.F.A. and A.B.M.; software, R.C., A.F.A. and A.B.M.; validation, R.C., A.F.A. and A.B.M.; formal analysis, R.C., A.F.A. and A.B.M.; investigation, R.C., A.F.A. and A.B.M.; writing—original draft preparation, R.C., A.F.A. and A.B.M.; writing—review and editing, R.C., A.F.A. and A.B.M.; visualization, A.B.M.; supervision, A.B.M.; project administration, A.B.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: The authors would like to extend their sincere appreciation to the editors and the reviewers for the interest they gave to our work, and for the time and efforts in evaluating it. Their comments have been of great role in improving the paper.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

In this appendix, we review the main tools applied in our study, and which are useful in similar problems. These tools may be resumed in Ekeland's variational principle, Palais-Smale sequences, and Nehari manifolds. Readers interested may refer to [4,88–94] and the references therein.

In a majority of problems such as the present one, researchers always come back to the notion of weak solutions. This necessitates to apply the so-called variational methods to pass from the PDE to critical point theory, where the attention goes to the energy functionals.

In the study of problem (1) for example, we call strong solution any pair (u,v) that satisfies it point-wise. This assumes for (u,v) to be at least $\mathcal{C}^2(\mathbb{R}^N)$. By considering the energy functional $I(u,v)$ associated to (1) and defined in (11), a weak solution to problem (1) will be any pair (u,v) satisfying $I'(u,v)(\varphi,\psi)=0$ for all $(\varphi,\psi)\in\mathcal{C}^\infty(\mathbb{R}^N)\times\mathcal{C}^\infty(\mathbb{R}^N)$. This apparently assumes for (u,v) to be at least $\mathcal{C}^1(\mathbb{R}^N)$ and that (u,v) is a critical point of I. The corresponding value $c=I(u,v)$ is known as a critical value of I. The critical points of I are the so-called weak solutions of (1). It is straightforward that strong solutions are weak. However, the converse, which is always the problem to resolve in PDEs theory, is not true. It needs extra assumptions on the nonlinear terms and the boundaries such as regularity conditions. One of the famous methods to find critical values of the energy functionals associated to PDEs is the Ekeland's variational principle. We will recall it in a general form. The authors may refer to the original works of Ekeland [89,90] for backgrounds

Ekeland's variational principle. Let X be a Banach space, X^* be its dual space and $I\in\mathcal{C}^1(X,\mathbb{R})$ be a functional on X. Denote finally $c=\inf\limits_{u\in X} I(u)>-\infty$. Then, there exists a minimizing sequence $\{u_n\}_1^\infty\subset X$ such that

$$I(u_n)\to c \ \text{ and } \ I'(u_n)\to 0 \text{ in } X^\star. \tag{A1}$$

It is easy to see that if X is an infinite dimensional space (which is the case for quasi all variational problems issued from PDEs), a minimizing sequence in X may not converge. In such situations, a compactness concept needs to be introduced such as the well known Palais-Smale (PS) condition.

Definition A1. *The functional I satisfies the Palais-Smale condition if for $\{u_n\}_1^\infty\subset X$ satisfying (A1), there exists a subsequence $\{u_{n_k}\}\subset\{u_n\}$ and $u_0\in X$ such that $u_{n_k}\to u_0$ as $k\to\infty$. In this case, the sequence $\{u_n\}$ is called a PS sequence.*

We notice from this definition and Ekeland's variational principle that the inf of I in X gives raise to a critical point u_0. However, other types of critical points may be also found such as local extrema and saddle points. In this case, other methods should be applied such as minimax theory. The main result in this direction is the Mountain-Pass theorem which will be recalled here-after.

Let X be a Banach space. Denote for $\rho>0$, $B_\rho=\{u\in X\,,\,\|u\|_X\le\rho\}$ and $S_\rho=\partial B_\rho=\{u\in X\,,\,\|u\|_X=\rho\}$. Finally, for $e\in X$, denote $\Gamma_e=\{\gamma\in\mathcal{C}([0,1],X)\,,\,\gamma(0)=0 \text{ and } \gamma(1)=e\}$. The Mountain-Pass theorem is stated as follows [88].

Mountain-Pass Theorem. Let $I\in\mathcal{C}^1(X,\mathbb{R})$ be a functional on X satisfying the PS condition and assume that

i. $I(0)=0$.

ii. there exists constants $\alpha,\rho>0$ such that $I(u)\ge\alpha$, $\forall u\in S_\rho$.

iii. there exists $e\in X\setminus B_p$ such that $I(e)\le 0$.

Then, I has a critical value c satisfying

$$c=\inf_{\gamma\in\Gamma_e}\max_{u\in\gamma([0,1])} I(u)\ge\alpha.$$

In many situations of PDEs and/or systems of coupled PDEs (such as the present problem), nonlinear parts and non-homogeneous terms necessitates to link critical values on the entire space to restrictions on the so-called Nehari manifold

$$\mathcal{NH} = \{u \in X|\{0\}\,; \ \ |I'(u)u = 0\}.$$

In some situations of symmetries, we may expect multiple solutions by using a symmetric version of variational methods.

References

1. Abazari, R. Numerical simulation of coupled nonlinear Schrödinger equation by RDTM and comparison with DTM. *J. Appl. Sci.* **2011**, *11*, 3454–3463. [CrossRef]
2. Ambrosetti, A.; Colorado, E. Standing waves of some coupled nonlinear Schrödinger equations. *J. Lond. Math. Soc.* **2007**, *75*, 67–82. [CrossRef]
3. Bartsch, T.; Wang, Z.-Q. Note on ground states of nonlinear Schrödinger systems. *J. Part. Diff. Equ.* **2006**, *19*, 200–207.
4. Bahri, A.; Lions, P.L. Morse index of some min-max critical points. I. Application to multiplicity results. *Commun. Pure Appl. Math.* **1988**, *41*, 1027–1037. [CrossRef]
5. Bermudez, R.F. Coupled system of nonlinear Schrödinger and Korteweg-de Vries equations. Master's Thesis in Mathematical Engineering, Universidad Carlos III de Madrid. *arXiv* **2016**, arXiv:1609.06655v2.
6. Geng, X.G.; Liu, H.; Zhu, J. Initial-boundary value problems for the coupled nonlinear Schrödinger equation on the half-line. *Stud. Appl. Math.* **2015**, *135*, 310–346. [CrossRef]
7. He, Y.; Qin, D.; Chen, D. Ground state solutions for Hamiltonian elliptic systems with super or asymptotically quadratic nonlinearity. *Bound Value Probl.* **2019**, *2019*, 158. [CrossRef]
8. Ling, L.M.; Zhao, L.C.; Guo, B.L. Darboux transformation and classification of solution for mixed coupled nonlinear Schrödinger equations. *Commun. Nonlinear Sci. Num. Simul.* **2016**, *32*, 285–304. [CrossRef]
9. Liu, Z.L.; Wang, Z.-Q. Ground states and bound states of a nonlinear Schrödinger system. *Adv. Nonlinear Stud.* **2010**, *10*, 175–193. [CrossRef]
10. Pomponio, A. Coupled nonlinear Schrödinger systems with potentials. *J. Differ. Equ.* **2006**, *227*, 258–281. [CrossRef]
11. Sirakov, B. Least energy solitary waves for a system of nonlinear Schrödinger equations in R^n. *Comm. Math. Phys.* **2007**, *271*, 199–221. [CrossRef]
12. Tian, S.F. Initial-boundary value problems for the general coupled nonlinear Schrödinger equation on the interval via the Fokas method. *J. Differ. Equ.* **2017**, *262*, 506–558. [CrossRef]
13. Tian, R.-S.; Wang, Z.-Q. Multiple solitary wave solutions of nonlinear Schrödinger systems. *Topol. Methods Nonlinear Anal.* **2011**, *37*, 203–223.
14. Wan, Y.; Avila, A.I. Multiple solutions of a coupled nonlinear Schrödinger system. *J. Math. Anal. Appl.* **2007**, *334*, 1308–1325. [CrossRef]
15. Willem, M. *Minimax Theorems*; Birkhauser: Boston, MA, USA, 1996.
16. Dhar, A.K.; Das, K.P. Fourth-order nonlinear evolution equation for two Stokes wave trains in deep water. *Phys. Fluids A Fluid Dyn.* **1991**, *3*, 3021. [CrossRef]
17. Menyuk, C.R. Stability of solitons in birefringent optical fibers. II. Arbitrary amplitudes. *J. Opt. Soc. Am. B* **1988**, *5*, 392–402. [CrossRef]
18. Kanna, T.; Lakshmanan, M.; Dinda, P.T.; Akhmediev, N. Soliton collisions with shape change by intensity redistribution in mixed coupled nonlinear Schrödinger equations. *Phys. Rev. E* **2006**, *73*, 026604. [CrossRef]
19. Lin, T.C.; Wei, J.C. Ground state of N coupled nonlinear Schrödinger equations in R^n; $n \geq 3$. *Commun. Math. Phys.* **2005**, *255*, 629–653. [CrossRef]
20. Liu, W.-M.; Kengne, E. *Schrödinger Equations in Nonlinear Systems*; Springer Nature Singapore Pte Ltd.: Singapore, 2019.
21. Prytula, V.; Vekslerchik, V.; Perez-Garcia, V.M. Collapse in coupled nonlinear Schrödinger equations: Sufficient conditions and applications. *Physica D* **2009**, *238*, 1462–1467. [CrossRef]
22. Sirakov, B.; Soares, S. Soliton solutions to systems of coupled Schrödinger equations of Hamiltonian type. *Trans. Am. Math. Soc. Am. Math. Soc.* **2010**, *362*, 5729–5744. [CrossRef]
23. Bluman, G.; Kumei, S. *Symmetries and Differential Equations*; Springer: New York, NY, USA, 1989.
24. Ibragimov, N.H. *CRC Handbook of Lie Group Analysis of Differential Equations*; Academic Press: New York, NY, USA, 1982.
25. Mawhin, J.; Willem, M. *Critical Point Theory and Hamiltonian System*; Springer: New York, NY, USA, 1989.
26. Noris, B.; Ramos, M. Existence and bounds of positive solutions for a nonlinear Schrödinger system. *Proc. AMS* **2010**, *138*, 1681–1692. [CrossRef]
27. Olver, P.J. *Applications of Lie Groups to Differential Equations*; Springer: New York, NY, USA, 1986.
28. Ovsiannikov, L.V. *Group Analysis of Differential Equations*; Nauka: Moscow, Russia, 1978.
29. Xu, J. Initial-boundary value problem for the two-component nonlinear Schrödinger equation on the half-line. *J. Nonlinear Math. Phys.* **2016**, *23*, 167–189. [CrossRef]

30. Yan, F.; Hua, C.; Liu, H.; Liu, Z. Lie symmetry analysis and exact solutions for the coupled Kuramoto-Sivashinsky Equations. In Proceedings of the 2011 Fourth International Workshop on Chaos-Fractals Theories and Applications, Hangzhou, China, 19–22 October 2011; IEEE Computer Soc.: Piscataway, NJ, USA, 2011; pp. 61–65.
31. Porsezian, K.; Nakkeeran, K. Optical soliton propagation in a coupled system of the nonlinear Schrödinger equation and the Maxwell-Bloch equations. *J. Mod. Opt.* **1995**, *42*, 1953–1958. [CrossRef]
32. Mitchell, M.; Chen, Z.; Shih, M.; Segev, M. Self-trapping of partially spatially incoherent light. *Phys. Rev. Lett.* **1996**, *77*, 490493. [CrossRef] [PubMed]
33. Terracini, S.; Verzini, G. Multipulse phase in k-mixtures of Bose-Einstein condenstates. *Arch. Rat. Mech. Anal.* **2009**, *194*, 717–741. [CrossRef]
34. Ruegg, C.; Cavadini, N.; Furrer, A.; Gudel, H.-U.; Kramer, K.; Mutka, H.; Wildes, A.; Habicht, K.; Vorderwischu, P. Bose-Einstein condensation of the triplet states in the magnetic insulator TlCuCl3. *Nature* **2003**, *423*, 62–65. [CrossRef]
35. Bahrami, M.; Großardt, A.; Donadi, S.; Bassi, A. The Schrödinger-Newton equation and its foundations. *New J. Phys.* **2014**, *16*, 115007. [CrossRef]
36. Belmonte-Beitia, J. A note on radial nonlinear Schrödinger systems with nonlinearity spacially modulated. *Electron. J. Differ. Equ.* **2008**, *2008*, 1–6.
37. Belmonte-Beitia, J.; Perez-García, V.M.; Torres, P.J. Solitary waves for linearly coupled nonlinear Schrödinger equations with inhomogeneous coefficients. *J. Nonlinear Sci.* **2009**, *19*, 437–451. doi 10.1007/s00332-008-9037-7. [CrossRef]
38. Farah, L.G.; Pastor, A. Scattering for a 3D coupled nonlinear Schrödinger system. *J. Math. Phys.* **2017**, *58*, 071502. [CrossRef]
39. Ma, L.; Zhao, L. Uniqueness of ground states of some coupled nonlinear Schrödinger systems and their application. *J. Differ. Equ.* **2008**, *245*, 2551–2565. [CrossRef]
40. Belmonte-Beitia, J.; Perez-García, V.M.; Brazhnyi, V. Solitary waves in coupled nonlinear Schrödinger equations with spatially inhomogeneous nonlinearities. *Commun. Nonlinear Sci. Numer. Simul.* **2011**, *16*, 158–172. [CrossRef]
41. Su, J.; Tian, R.; Wang, Z.-Q. Positive solutions of doubly coupled multicomponent nonlinear Schrödinger systems. *Discret. Contin. Dyn. Syst.* **2019**, *15*, 2143–2161. [CrossRef]
42. Tian, R. Existence and Multiplicity Results on Standing Wave Solutions of Some Coupled Nonlinear Schrödinger Equations. Ph.D. Thesis, Utah State University, Logan, UT, USA, 2013; Volume 1484.
43. Vijayajayanthi, M.; Kanna, T.; Lakshmanan, M. Bright-dark solitons and their collisions in mixed N-coupled nonlinear Schrödinger equations. *Phys. Rev. A* **2008**, *77*, 013820. [CrossRef]
44. Zhang, H.-Q.; Meng, X.-H.; Xu, T.; Li, L.-L.; Tian, B. Interactions of bright solitons for the (2+1)-dimensional coupled nonlinear Schrödinger equations from optical fibres with symbolic computation. *Phys. Scr.* **2007**, *75*, 537–542. [CrossRef]
45. Agrawal, G.P. *Nonlinear Fiber Optics*; Academic Press: New York, NY, USA, 1995.
46. Chakravarty, S.; Ablowitz, M.J.; Sauer, J.R.; Jenkins, R.B. Multisoliton interactions and wavelength-division multiplexing. *Opt. Lett.* **1995**, *20*, 136–138. [CrossRef]
47. Mollenauer, L.F.; Evangelides, S.G.; Gordon, J.P. Wavelength division multiplexing with solitons in ultra-long distance transmission using lumped amplifiers. *J. Lightwave Technol.* **1991**, *9*, 362 [CrossRef]
48. Aitchison, J.S.; Weiner, A.M.; Silberberg, Y.; Leaird, D.E.; Oliver, M.K.; Jackel, J.L.; Smith, P.W.E. Experimental observation of spatial soliton interactions. *Opt. Lett.* **1991**, *16*, 15–17. [CrossRef] [PubMed]
49. Hioe, F.T. Solitary waves for two and three coupled nonlinear Schrödinger equations. *Phys. Rev. E* **1998**, *58*, 6700. [CrossRef]
50. Hioe, F.T. Solitary waves for N coupled nonlinear Schrödinger equations. *Phys. Rev. Lett.* **1999**, *82*, 1152–1155. [CrossRef]
51. Hioe, F.T.; Salter, T.S. Special set and solutions of coupled nonlinear Schrödinger equations. *J. Phys. A Math. Gen.* **2002**, *35*, 8913–8928. [CrossRef]
52. Shalaby, M.; Reynaud, F.; Barthelemy, A. Experimental observation of spatial soliton interactions with a $\pi/2$ relative phase difference. *Opt. Lett.* **1992**, *17*, 778–780. [CrossRef] [PubMed]
53. Zhida, Y. Multi-soliton solutions of coupled nonlinear Schrödinger equations. *J. Chin. Phys. Lett.* **1987**, *4*, 185–187. [CrossRef]
54. Berkhoer, A.L.; Zakharov, V.E. Self excitation of Waves with different polarizations in nonlinear media. *JETP* **1970**, *31*, 486–490.
55. Bhakta, J.C.; Gupta, M.R. Stability of solitary wave solutions of simultaneous nonlinear Schrödinger equations. *J. Plas. Phys.* **1982**, *28*, 379–383. [CrossRef]
56. Gupta, M.R.; Som, B.K.; Dasgupta, B. Coupled nonlinear Schrödinger equations for Langmuir and elecromagnetic waves and extension of their modulational instability domain. *J. Plasma Phys.* **1981**, *25*, 499–507. [CrossRef]
57. Benci, V.; Fortunato, D.F. Solitary waves of the nonlinear Klein-Gordon equation coupled with Maxwell equations. *Rev. Math. Phys.* **2002**, *14*, 409–420. [CrossRef]
58. Brezis, H.; Nirenberg, L. Positive solutions of non linear elliptic equations involving critical Sobolev exponents. *Comm. Pure. Appl. Math.* **1983**, *36*, 437–477. [CrossRef]
59. Ben Mabrouk, A.; Ben Mohamed, M.L. On some critical and slightly super-critical sub-superlinear equations. *Far East J. Appl. Math.* **2006**, *23*, 73–90.
60. Mabrouk, A.B.; Ben Mohamed, M.L. Nodal solutions for some nonlinear elliptic equations. *Appl. Math. Comput.* **2007**, *186*, 589–597. [CrossRef]
61. Mabrouk, A.B.; Ben Mohamed, M.L.; Omrani, K. Finite difference approximate solutions for a mixed sub-superlinear equation. *Appl. Math. Comput.* **2007**, *187*, 1007–1016. [CrossRef]

62. Berestycki, H.; Lions, P.L. Nonlinear scalar field equations I and II. *Arch. Rat. Anal.* **1983**, *82*, 333–345, 347–375.
63. Chteoui, R.; Ben Mabrouk, A.; Ounaies, H. Existence and properties of radial solutions of a sublinear elliptic equation. *J. Part. Diff. Eq.* **2015**, *28*, 1–9.
64. Strauss, W.A. Existence of solitary waves in higher dimensions. *Commun. Math. Phys.* **1977**, *55*, 149–162. [CrossRef]
65. Quaas, A.; Sirakov, B. Existence and non-existence results for fully nonlinear elliptic systems. *Indiana Univ. Math. J.* **2009**, *58*, 751–788. [CrossRef]
66. Zhou, S.; Cheng, X. Numerical solution to coupled nonlinear Schrödinger equations on unbounded domains. *Math. Comput. Simul.* **2010**, *80*, 2362–2373. [CrossRef]
67. Pinar, Z.; Deliktas, E. Solution behaviors in coupled Schrödinger equations with full-modulated nonlinearities. *AIP Conf. Proc.* **2017**, *1815*, 080019.
68. Ping, L.; Lou, S.Y. Coupled nonlinear Schrödinger equation: Symmetries and exact solutions. *Commun. Theor. Phys.* **2009**, *51*, 27–34. [CrossRef]
69. Bezia, A.; Ben Mabrouk, A.; Betina, K. Lyapunov-Sylvester operators for (2+1)-Boussinesq equation. *Electron. J. Differ. Equ.* **2016**, *286*, 1–19.
70. Aminikhah, H.; Pournasiri, F.; Mehrdoust, F. A novel effective approach for systems of coupled Schrödinger equation. *Pramana J. Phys. Indian Acad. Sci.* **2016**, *86*, 19–30. [CrossRef]
71. Rasouli, S.H.; Choubin, M. The Nehari manifold approach for a class of $n \times n$ nonlinear elliptic systems. *Monatshefte Math.* **2014**, *173*, 605–623. [CrossRef]
72. Liu, H. Ground states of linearly coupled Schrödinger systems. *Electron. J. Differ. Equ.* **2017**, *5*, 1–10.
73. Guo, Y.; Wang, J.M.; Zhao, D.X. Stability of an interconnected Schrödinger-heat system in a torus region. *Math. Methods Appl. Sci.* **2016**, *39*, 3735–3749. [CrossRef]
74. Benrhouma, M. Existence and uniqueness of solutions for a singular semilinear elliptic system. *Nonlinear Anal.* **2014**, *107*, 134–146. [CrossRef]
75. Benrhouma, M. Existence of solutions for a semilinear elliptic system. *ESAIM Control Optim. Calc. Var.* **2013**, *19*, 574–586. [CrossRef]
76. Saanouni, T. A note on coupled focusing nonlinear Schrödinger equations. *J. Appl. Anal. Int. J.* **2016**, *95*, 2063–2080. [CrossRef]
77. Chteoui, R.; Aljohani, A.F.; Ben Mabrouk, A. Classification and simulation of chaotic behaviour of the solutions of a mixed nonlinear Schrödinger system. *Electron. Res. Arch.* **2020**. [CrossRef]
78. Kajikyia, R. Necessary and sufficient condition for existence and uniqueness of nodal solutions to sublinear elliptic equations. *Adv. Diff. Equ.* **2001**, *11*, 1317–1346.
79. Kajikiya, R. Infinitely many radially symmetric solutions of certain semilinear elliptic equations. *Hiroshima Math. J.* **1991**, *21*, 557–585. [CrossRef]
80. Kajikiya, R. Existence and asymptotic behavior of nodal solutions for semilinear elliptic equations. *J. Differ. Equ.* **1993**, *106*, 238–256. [CrossRef]
81. Kajikiya, R. Nodal solutions of superlinear elliptic equations in symmetric domains. *Adv. Math. Sci. Appl.* **1993**, *3*, 219–266.
82. Kajikiya, R. A critical point theorem related to the symmetric mountain pass lemma and its applications to elliptic equations. *J. Funct. Anal.* **2005**, *225*, 352–370. [CrossRef]
83. Kajikiya, R. Radially symmetric solutions of semilinear elliptic equations, existence and Sobolev estimates. *Hiroshima Math. J.* **1991**, *21*, 111–161 . [CrossRef]
84. Kajikiya, R.; Tanaka, K. Existence of infinitely many solutions for some superlinear elliptic equations. *J. Math. Anal. Appl.* **1990**, *149*, 313–321 . [CrossRef]
85. Kajikiya, R. Multiple solutions of sublinear Lane-Emden elliptic equations. *Calc. Var.* **2006**, *26*, 29–48. [CrossRef]
86. Chteoui, R.; Ben Mabrouk, A.; Cattani, C. Mixed concave-convex sub-superlinear Schrödinger equation: Survey and development of some new cases. In *Nonlinear Analysis and Global Optimization*; Rassias, T.M., Pardalos, P.M., Eds.; Springer Optimization and Its Applications: Berlin/Heidelberg, Germany, 2021; Volume 167. [CrossRef]
87. Chteoui, R.; Aljohani, A.F.; Ben Mabrouk, A. Lyapunov-Sylvester computational method for numerical solutions of a mixed cubic-superlinear Schrödinger system. *Eng. Comput.* **2021**. [CrossRef]
88. Ambrosetti, A.; Rabinowitz, P.H. Dual variational methods in critical point theory and applications. *J. Funct. Anal.* **1973**, *14*, 349–381. [CrossRef]
89. Ekeland, I. On the variational principle. *J. Math. Anal. Appl.* **1974**, *47*, 324–353. [CrossRef]
90. Ekeland, I. Nonconvex minimization problems. *Bull. Am. Math. Soc. New Ser.* **1979**, *1*, 443–474. [CrossRef]
91. Nehari, Z. On a class of nonlinear second-order differential equations. *Trans. Am. Math. Soc.* **1960**, *95*, 101–123. [CrossRef]
92. Nehari, Z. Characteristic values associated with a class of nonlinear second-order differential equations. *Acta Math.* **1961**, *105*, 141–175. [CrossRef]
93. Nehari, Z. A nonlinear oscillation problem. *J. Differ. Equ.* **1969**, *5*, 452–460. [CrossRef]
94. Robinson, J. A survey of Ekeland's Variational Principle and Related Theorems and Applications. Master's Thesis, University of Nevada, Las Vegas, NV, USA, 2014; Volume 2291.