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Continuity and Analyticity for the Generalized Benjamin–Ono Equation

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Abstract: This work mainly focuses on the continuity and analyticity for the generalized Benjamin–Ono (g-BO) equation. From the local well-posedness results for g-BO equation, we know that its solutions depend continuously on their initial data. In the present paper, we further show that such dependence is not uniformly continuous in Sobolev spaces $H^s(\mathbb{R})$ with $s > 3/2$. We also provide more information about the stability of the data-solution map, i.e., the solution map for g-BO equation is Hölder continuous in H^r-topology for all $0 \leq r < s$ with exponent α depending on s and r. Finally, applying the generalized Ovsyannikov type theorem and the basic properties of Sobolev–Gevrey spaces, we prove the Gevrey regularity and analyticity for the g-BO equation. In addition, by the symmetry of the spatial variable, we obtain a lower bound of the lifespan and the continuity of the data-to-solution map.

Keywords: generalized Benjamin–Ono equation; non-uniform dependence; Hölder continuous; symmetry; analyticity; Gevrey regularity

MSC: 35G2; 35L05; 35Q50

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1. Introduction

In this paper, we study the Cauchy problem for the generalized Benjamin–Ono equation

$$\begin{cases} \partial_t u + \mathcal{H}\partial_x^2 u + u^k \partial_x u = 0, & t > 0, x \in \mathbb{R}, \\ u(x,0) = u_0(x), & t = 0, x \in \mathbb{R}, \end{cases} \tag{1}$$

where $\mathcal{H}$ is the spatial symmetrical Hilbert transform

$$\mathcal{H}(f)(x) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{f(y)}{x-y} dy.$$

The Benjamin–Ono equation ($k = 1$) was derived by Benjamin [1], and later Ono [2]. This equation can be see as a model to describe the wave motion at the interface of a two-layer fluid system of incompressible inviscid fluids, in which a heterogeneous layer is situated above or underneath an infinitely-deep layer of homogeneous fluid. The function $u(x,t)$ denotes the deviation of the interface from its resting position at the point x in the direction of propagation at time t. It is assumed that the deviation of the interface makes no significance in the direction orthogonal to x.

For the variables that are nondimensional, the Benjamin–Ono equation has been normalized to reach the tidy form (1). From the last century in 1960s, the Benjamin equation was of high concern, particularly because it is completely integrable, defines Hamiltonian systems, possesses infinite conserved quantities and has multi-soliton solutions, cf. [1–4].

The Cauchy problem for the Benjamin–Ono equation was studied extensively. The local well-posedness and global well-posedness for initial data in the classical Sobolev spaces $H^s(\mathbb{R})$ were investigated, cf. [5–11]. More precisely, the local well-posedness for initial data $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$ was shown in [7], and the globally well-posed in $H^s(\mathbb{R})$ for $s \geq \frac{3}{2}$ was also obtained in [10]. By the half Strichartz estimates for linear problems with variable coefficients, Koch and Tzvetkov [9] obtained the local well-posedness when $s > \frac{5}{4}$.

Subsequently, Kenig and Koenig [8] extended this result to $s > \frac{9}{8}$. Tao [11] obtained global well-posedness in $H^s(\mathbb{R})$ for $s \geq 1$ by a gauge transformation as for the derivative Schrödinger equation. Recently, the Benjamin–Ono equation was proved to be local well-posedness in $H^s(\mathbb{R})$ with $s > \frac{1}{4}$ in [5] and global well-posedness in $H^s(\mathbb{R})$ with $s \geq 0$ in [6].

More interestingly, based on the well-posedness results, Koch and Tzvetkov [12] showed that the solution mapping was not even locally uniformly continuous in $H^s(\mathbb{R})$ for $s > 0$. Fonseca and Ponce [13] established persistence properties and proved some unique continuation properties of the solution flow in the weighted Sobolev spaces $Z_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx)$.

For $k \geq 2$, the g-BO equation presents the interesting fact that the dispersive effect is described by a nonlocal operator and is weaker than that exhibited by the generalized KdV equation. In addition, it possesses three conservation laws,

$$I(u) = \int_{\mathbb{R}} u(t,x)dx, \quad M(u) = \int_{\mathbb{R}} u^2(t,x)dx$$

and

$$E(u) = \int_{\mathbb{R}} \left(\frac{1}{2}|D_x^{1/2}u(t,x)|^2 - c_k|u(t,x)|^{k+2} \right) dx,$$

where $D_x = (-\partial_x^2)^{1/2}$. The local well-posedness of the g-BO equation was also known in [14,15], and the global result was proven by Molinet and Ribaud [16]. In a sharp contrast with the case $k = 1$, the best known results about the g-BO equation with small initial data were obtained by using contraction methods [14].

More precisely, Kenig et al. [14] proved that the locally and globally well-posed for the solution of g-BO equation in different $H^s(\mathbb{R})$. Molinet and Ribaud [15] further studied these results for g-BO equation with small initial data. Recently, using the frequency-uniform decomposition method, the global well-posedness of solution for the Cauchy problem of the g-BO with the small rough data in certain modulation spaces $M^s_{2,1}(\mathbb{R})$ was investigated in [17].

Motivated by the results mentioned above, the goals of this paper are to study the continuity and analyticity for the generalized Benjamin–Ono Equation (1). From the local well-posedness results [14–16], we know that the solutions of g-BO Equation (1) continuously rely on their initial data in Sobolev spaces—that is, if, for a given $u_0 \in H^s(\mathbb{R})$ with $s > 3/2$, there exists a $T = T(\|u_0\|_{H^s})$ such that, for any sequence $u_0^n \in H^s$ and $\|u_0^n - u_0\|_{H^s} \to 0$ $(n \to \infty)$, the corresponding solutions $u^n(t)$ of g-BO satisfy $\|u^n(t) - u(t)\|_{H^s} \to 0$ $(n \to \infty)$ for $0 \leq t < T$.

In the present paper, we show that such dependence is not uniformly continuous in $H^s(\mathbb{R})$ with $s > \frac{3}{2}$. The uniformly continuous of the data-to-solution map means that: $\forall u_1^n(0), u_2^n(0) \in H^s$ be the sequences of initial data for the Equation (1), if $\lim_{n\to\infty} ||u_1^n(0) - u_2^n(0)||_{H^s} \to 0$, then the correspond sequences of solution $u_1^n(t)$, $u_2^n(t) \in \mathcal{C}([0,T]; H^s(\mathbb{R}))$ for the initial-value problem (1) satisfy $\lim_{n\to\infty} ||u_1^n(t) - u_2^n(t)||_{H^s} \to 0$ for $t \in [0,T)$. By the technique of approximate solutions [12], we find two suitable sequences of solutions $u_{1,\lambda}(t)$ and $u_{-1,\lambda}(t)$ to g-BO Equation (1) in $\mathcal{C}([0,T]; H^s(\mathbb{R}))$ such that

$$\|u_{1,\lambda}(t)\|_{H^s(\mathbb{R})} + \|u_{-1,\lambda}(t)\|_{H^s(\mathbb{R})} \lesssim 1, \quad \lim_{\lambda\to\infty} \|u_{1,\lambda}(0) - u_{-1,\lambda}(0)\|_{H^s(\mathbb{R})} = 0;$$

however, the following inequality holds

$$\liminf_{\lambda\to\infty} \|u_{1,\lambda}(t) - u_{-1,\lambda}(t)\|_{H^s(\mathbb{R})} \gtrsim \liminf_{\lambda\to\infty} \left|\sin \lambda^{1/k} t\right|, \quad 0 < t \leq T,$$

which implies $\liminf_{\lambda\to\infty} \|u_{1,\lambda}(t) - u_{-1,\lambda}(t)\|_{H^s(\mathbb{R})} \neq 0$ at any time.

In [12], Koch and Tzvetkov prove that the flow map of the Benjamin–Ono equation cannot be uniformly continuous on bounded sets of $H^s(\mathbb{R})$ for $s > 0$. We compare with the Benjamin–Ono equation, and the g-BO equation has a higher order nonlinear term $u^k\partial_x u$. If taking the similarly approximate solutions as Koch and Tzvetkov [12], we cannot successively estimate the error in suitable Sobolev norm, instead we must select a more complicated form of the low and high frequency parts for the approximate solutions (see (13) and (15)).

Motivated by the results obtained in [18–21], we use the interpolation properties of the Sobolev spaces and commutator estimates to present that the data-to-solution map as continuous but not uniformly continuous in Sobolev spaces $H^s(\mathbb{R})$ with $s > \frac{3}{2}$. Our results extend the work of Koch and Tzvetkov [12] to more general equations with higher-order nonlinearities. Our main result is stated as follows:

Theorem 1. *If the initial data $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$, then the data-to-solution map $u_0 \to u(t)$ for the g-BO Equation (1) is not uniformly continuous from any bounded subset of $H^s(\mathbb{R})$ into $\mathcal{C}([0,T]; H^s(\mathbb{R})) \times \mathcal{C}([0,T]; H^{s-1}(\mathbb{R}))$.*

Theorem 1 shows that the data-solution map depends on the initial data being continuous but not uniformly continuous. Our next result will provide information about the stability of the data-solution map. Our next result establishes the stability of the data-solution map, i.e., the solution map for g-BO equation is Hölder continuous in H^σ-topology.

Theorem 2. *Let $s > \frac{3}{2}$ and $0 \le r < s$. Then, the data-to-solution map for the g-BO Equation (1) is Hölder continuous in $H^s(\mathbb{R})$ equipped with $H^r(\mathbb{R})$-norm. In particular, the solutions $u(t), v(t)$ to the g-BO Equation (1) corresponding to the initial data u_0, v_0 in the ball $B(0,\rho) = \{\psi \in H^s(\mathbb{R}) : \|\psi\|_{H^s(\mathbb{R})} \le \rho\}$ of $H^s(\mathbb{R})$ satisfy the following inequality*

$$\|u(t) - v(t)\|_{H^r(\mathbb{R})} \le C\|u_0 - v_0\|^{\alpha}_{H^r(\mathbb{R})},$$

where the parameter α is given by

$$\alpha = \begin{cases} 1, & (s,r) \in A_1 \doteq \{(s,r) : s > \frac{3}{2}, 0 \le r \le s-1, r+s \ge 2\}; \\ \frac{2(s-1)}{s-r}, & (s,r) \in A_2 \doteq \{(s,r) : 2 > s > \frac{3}{2}, 0 \le r < 2-s\}; \\ s-r, & (s,r) \in A_3 \doteq \{(s,r) : s > \frac{3}{2}, s-1 < r < s\}. \end{cases} \tag{2}$$

The lifespan T and the constant C only depend on r, s and ρ (see Figure 1).

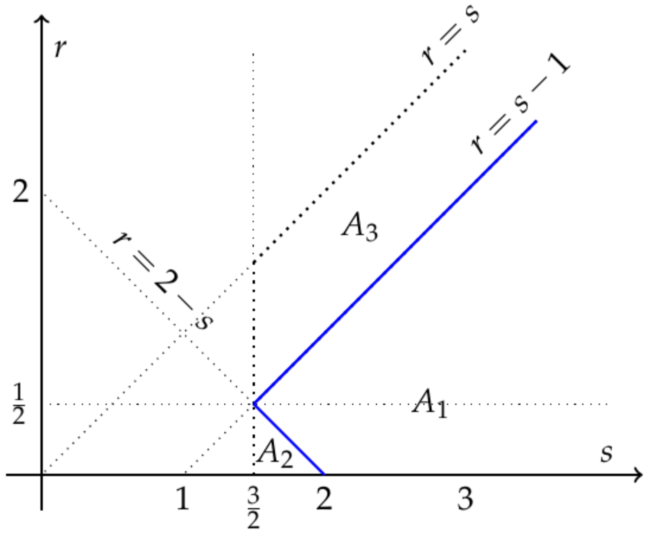

Figure 1. The relationship of r, s and α.

Many researchers have studied the analyticity of solutions to g-BO, cf. [12]. However, to our best acknowledge, the Gevrey regularity of solutions to the BO equation is still an open problem. The definition of Sobolev–Gevrey spaces is stated as follows.

Definition 1. *Let s be a real number and $\sigma, \delta > 0$. A function $f \in G^{\delta}_{\sigma,s}(\mathbb{R})$ if and only if $f \in C^{\infty}(\mathbb{R})$ and satisfies*

$$\|f\|_{G^{\delta}_{\sigma,s}(\mathbb{R})} = \left(\int_{\mathbb{R}} (1+|\xi|^2)^s e^{2\delta|\xi|^{1/\sigma}} |\widehat{f}(\xi)|^2 d\xi \right)^{1/2} < \infty.$$

Denoting the Fourier multiplier $e^{\delta(-\Delta)^{\frac{1}{2r}}}$ by $e^{\delta(-\Delta)^{\frac{1}{2r}}} f = \mathscr{F}^{-1}(e^{\delta|\xi|^{\frac{1}{r}}} \hat{f})$, we deduce that $||f||_{G^{\delta}_{r,s}(\mathbb{R})} = ||e^{\delta(-\Delta)^{\frac{1}{2r}}} f||_{H^s(\mathbb{R})}$. For $0 < r < 1$, it is called ultra-analytic function. If $r = 1$, it is a usual analytic (or holomorphic) function, and δ is called the radius of analyticity. If $r > 1$, it is the Gevrey class function.

By the generalized Ovsyannikov theorem [22] (see Theorem 6 in the Section 5), we can obtain the Gevrey regularity and analyticity of the g-BO equation.

Theorem 3. *Let $\sigma \geq 1$ and $s > \frac{3}{2}$. Assume that $u_0 \in G^{\delta}_{\sigma,s}(\mathbb{R})$. Then, for every $0 < \delta < 1$, there exists a $T_0 > 0$ such that the g-BO equation has a unique solution u, which is holomorphic in $|t| < \frac{T_0(1-\delta)^{2\sigma}}{2^{2\sigma}-1}$ with values in $G^{\delta}_{\sigma,s}(\mathbb{R})$. Moreover, there is a positive constant C such that $T_0 = \frac{C}{\|u_0\|_{G^1_{\sigma,s}(\mathbb{R})}}$.*

Theorem 3 tells us that solutions of g-BO equation are analytic in both space and time variables. Moreover, we give a lower bound of the analytic lifespan. Then, we continue to study the continuity of the data-to-solution.

Definition 2. *Let $\sigma \geq 1$ and $s > \frac{3}{2}$. We say that the data-to-solution map $u_0 \to u(t)$ of the g-BO is continuous, if for a given $u_0^{\infty} \in G^1_{\sigma,s}(\mathbb{R})$ there exists a $T = T(\|u_0^{\infty}\|_{G^1_{\kappa,s}})$ such that, for any sequence $u_0^n \in G^1_{\kappa,s}$ and $\|u_0^n - u_0^{\infty}\|_{G^1_{\kappa,s}} \to 0$ for $n \to \infty$, the corresponding solutions u^n of g-BO satisfy $\|u^n - u^{\infty}\|_{E_T} \to 0$ for $n \to \infty$, where*

$$\|f\|_{E_T} = \sup_{|t| < \frac{T(1-\delta)^{\kappa}}{2^{\kappa}-1}} \left(\|f\|_{G^{\delta}_{\kappa,s}} (1-\delta)^{\kappa} \sqrt{1 - \frac{|t|}{T(1-\delta)^{\kappa}}} \right).$$

Theorem 4. *Let $\sigma \geq 1$ and $s > \frac{3}{2}$. Assume that $u_0 \in G^1_{\sigma,s}(\mathbb{R})$. Then, the data-to-solution map $u_0 \mapsto u$ of the g-BO equation is continuous from $G^1_{\sigma,s}(\mathbb{R})$ into the solutions space.*

This paper is organized as follows. In Section 2, we recall some notation, give a prior well-posedness estimate for g-BO Equation (1), and determine a lower bound on the existence time of the solution in $H^s(\mathbb{R})$. In Section 3, adopting the method of approximate solutions and the well-posedness estimates, we show that the data-to-solution map fails to be locally uniformly continuous. In Section 4, we prove that the solution map for g-BO Equation (1) is Hölder continuous in H^r-topology for all $0 \leq r < s$. Finally, applying generalized Ovsyannikov type theorem and properties of Sobolev–Gevrey spaces, we establish the Gevrey regularity and analyticity of the g-BO equation and obtain the continuity of the data-to-solution map.

2. Priori Estimates and Lifespan of Solution

For any $s \in \mathbb{R}$, we take the operator $D^s = (1 - \partial_x^2)^{s/2}$ to be defined by

$$\widehat{D^s f}(\xi) = (1 + \xi^2)^{s/2} \hat{f}(\xi),$$

where $\hat{f}(\xi)$ is the Fourier transform,

$$\hat{f}(\xi) \doteq \int_{\mathbb{R}} e^{-ix\xi} f(x)dx, \ \ \xi \in \mathbb{R}.$$

Let H^s be the Sobolev space consisting of all tempered distributions f such that

$$\|f\|_{H^s} \doteq \|f\|_{H^s(\mathbb{R})} = \left(\int_{\mathbb{R}} (1+\xi^2)^s |\hat{f}(\xi)|^2 d\xi\right)^{1/2} < \infty.$$

Theorem 5. *Assume $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$. Let T be the maximal existence time of the solution u to g-BO Equation (1) with the initial data u_0. Then, T satisfies*

$$T \geq T_0 := \frac{2^k - 1}{2^k Ck\|u_0\|^k_{H^s}}, \tag{3}$$

where C_s is a constant depending only on s. We have

$$\|u(t)\|_{H^s} \leq 2\|u_0\|_{H^s}, \ \ 0 \leq t \leq T_0. \tag{4}$$

Proof. Applying the operator D^s to g-BO Equation (1), it can be rewritten as follows

$$\partial_t D^s u + D^s \mathcal{H}(\partial_x^2 u) + \left(D^s(u^k \partial_x u) - u^k D^s \partial_x u\right) + u^k \partial_x D^s u = 0. \tag{5}$$

Multiplying the g-BO Equation (5) by $D^s u$ and then integrating it with respect to $x \in \mathbb{R}$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{H^s} = &- \int_{\mathbb{R}} D^s \mathcal{H}(\partial_x^2 u) \cdot D^s u dx \\ &- \int_{\mathbb{R}} \left([D^s, u^k]\partial_x u\right) \cdot D^s u dx - \int_{\mathbb{R}} u^k \partial_x D^s u \cdot D^s u dx. \end{aligned} \tag{6}$$

Noting that $\int_{\mathbb{R}} D^s \mathcal{H}(\partial_x^2 u) \cdot D^s u dx = 0$. To estimate the second integral on the right-hand side of (6), we need the following lemma, which is derived from [23,24].

Lemma 1. *If $r > 0$, then*

$$\|[D^r, f]g\|_{L^2} \leq C_r(\|f_x\|_{L^\infty}\|D^{r-1}g\|_{L^2} + \|D^r f\|_{L^2}\|g\|_{L^\infty}),$$

where C_r is a positive constant depending only on r.

Using the Cauchy–Schwarz inequality and Lemma 1, we can estimate the second integral of (6)

$$\begin{aligned} |\int_{\mathbb{R}} \left([D^s, u^k]\partial_x u\right) \cdot D^s u dx| &\leq \left\|[D^s, u^k]\partial_x u\right\|_{L^2} \|D^s u\|_{L^2} \\ &\leq C_s(\|\partial_x u^k\|_{L^\infty}\|D^{s-1}\partial_x u\|_{L^2} + \|D^s u^k\|_{L^2}\|\partial_x u\|_{L^\infty})\|D^s u\|_{L^2} \\ &\leq C\|u\|^{k+2}_{H^s}, \end{aligned} \tag{7}$$

where we have used the equality $\|D^s u\|_{L^2} = \|u\|_{H^s}$ and the Sobolev embedding theorem $H^s \hookrightarrow L^\infty$ for $s > \frac{3}{2}$.

Estimating the third integral of the right-hand side of (6), integrating by parts, we deduce

$$\left|\int_{\mathbb{R}} u^k \partial_x D^s u \cdot D^s u dx\right| \leq \|\partial_x u^k\|_{L^\infty}\|D^s u\|^2_{L^2} \leq C\|u\|^{k+2}_{H^s}. \tag{8}$$

Combining (6)–(8) we can obtain the following inequality

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{H^s}^2 \leq C\|u\|_{H^s}^{k+2}. \tag{9}$$

Solving differential inequality (9) yields

$$\|u\|_{H^s} \leq \frac{\|u_0\|_{H^s}}{(1 - Ckt\|u_0\|_{H^s}^k)^{1/k}}. \tag{10}$$

Letting $T_0 := \min\left\{\frac{1}{Ct\|u_0\|_{H^s}^k}, \frac{2^k-1}{2^kCk\|u_0\|_{H^s}^k}\right\}$, then $(1 - CkT_0\|u_0\|_{H^s}^k)^{1/k} \geq \frac{1}{2}$. From (10), the solution u exists for $0 \leq t \leq T_0$ with the following bound

$$\|u\|_{H^s} \leq \frac{\|u_0\|_{H^s}}{(1 - CkT_0\|u_0\|_{H^s}^k)^{1/k}} \leq 2\|u_0\|_{H^s}, \;\; 0 \leq t \leq T_0. \tag{11}$$

This completes the proof of Theorem 5. □

3. Nonuniform Dependence for the Solution to g-BO

3.1. Approximate Solutions

In this section, we consider approximate solutions of the Equation (1) of the form

$$u^{\omega,\lambda} = u_l + u_h, \tag{12}$$

where $\omega = \pm 1$ and $\lambda > 0$. The high frequency part is given by

$$u_h = -\lambda^{-(1+\delta)/2k-s}\phi_\lambda \cos\Phi, \tag{13}$$

where $\phi_\lambda = \phi(\frac{x}{\lambda^{(1+\delta)/k}})$, $\Phi = -\omega^2\lambda^{1/k}t + \omega\lambda^{1/2k}x - \omega^{k+1}\lambda^{-1+1/2k}t$ and ϕ is $C^\infty(\mathbb{R})$ cutoff functions such that

$$\phi(x) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 2. \end{cases} \tag{14}$$

The low frequency part u_l is the solution to system (1) with initial data

$$\begin{cases} \partial_t u_l + \mathcal{H}\partial_x^2 u_l + u_l^k \partial_x u_l = 0, & t > 0, x \in \mathbb{R}, \\ u_l(x,0) = \omega\lambda^{-1/k}\widetilde{\phi}_\lambda, & x \in \mathbb{R}, \end{cases} \tag{15}$$

where $\widetilde{\phi}$ is $C^\infty(\mathbb{R})$ functions such that

$$\widetilde{\phi}(x) = 1 \;\text{ if } x \in supp\phi.$$

Lemma 2 (See [12])**.** *Let $\psi \in \mathcal{S}(\mathbb{R})$, $0 < \delta < 2$ and $\alpha \in \mathbb{R}$. Then, for any $s \geq 0$, we have that*

$$\lim_{\lambda\to\infty} \lambda^{-\frac{1}{2}\delta - s}\left\|\psi(\frac{x}{\lambda^\delta})\cos(\lambda x - \alpha)\right\|_{H^s(\mathbb{R})} = \frac{1}{\sqrt{2}}\|\psi\|_{L^2(\mathbb{R})}. \tag{16}$$

Relation (16) is also true if cos *is replaced by* sin.

Lemma 3. *Let $0 < \delta < 1$ and $\phi \in C_0^\infty(\mathbb{R})$. Then, for any $N > 0$, there exists a positive constant C_N such that for every $\alpha \in \mathbb{R}$*

$$\|[\mathcal{H}, \phi_\lambda]\cos(\lambda x + \alpha)\|_{H^\sigma(\mathbb{R})} \leq C_N\lambda^{-N}. \tag{17}$$

Since the proofs of Lemma 3 are quite similar to lemma 2.2 in [12], they are omitted to make the paper concise.

Lemma 4. *Let $\omega = \pm 1$, $0 < \delta < 1$ and $\lambda \gg 1$. Then, the initial-value problem (1) has a unique solution $u_l \in C([0,T); H^s(\mathbb{R}))$, $s > \frac{3}{2}$. For all $\sigma \geq 0$, this solutions satisfies the estimate*

$$\|u_l(t)\|_{H^\sigma(\mathbb{R})} \leq C_s \lambda^{(\delta-1)/2k}. \tag{18}$$

Proof. Clearly, for any function $\phi \in \mathcal{S}(\mathbb{R})$, we can easily check that

$$\left\|\phi\left(\frac{x}{\lambda^{k\delta}}\right)\right\|_{H^\sigma} \leq \lambda^{k\delta/2}\|\phi\|_{H^\sigma}. \tag{19}$$

As per the relation $\widehat{\phi\left(\frac{x}{\rho}\right)}(\xi) = \rho\widehat{\phi}(\rho\xi)$, making the change of variables $\eta = \lambda^{k\delta}\xi$ yields

$$\begin{aligned}
\left\|\phi\left(\frac{x}{\lambda^{k\delta}}\right)\right\|_{H^\sigma}^2 &= \frac{1}{2\pi}\int_{\mathbb{R}}(1+\xi^2)^\sigma \left|\lambda^{k\delta}\widehat{\phi}(\lambda^{k\delta})\right|^2 d\xi \\
&= \frac{\lambda^{k\delta}}{2\pi}\int_{\mathbb{R}}\left(1+\frac{\eta^2}{\lambda^{2k\delta}}\right)^\sigma |\widehat{\phi}(\eta)| d\xi \\
&\leq \frac{\lambda^{k\delta}}{2\pi}\int_{\mathbb{R}}\left(1+\eta^2\right)^\sigma |\widehat{\phi}(\eta)| d\xi \\
&= \lambda^{k\delta}\|\phi\|_{H^\sigma}.
\end{aligned} \tag{20}$$

According to (19), we know that the initial data $u_l(0)$ satisfy the following estimate

$$\|u_l(0)\|_{H^\sigma} \leq \lambda^{(\delta-1)/2k},$$

which decays if $0 < \delta < 1$. Furthermore, the estimate (3) from Theorem 5 yields the lifespan $T = \frac{2^k - 1}{2^k Ck\|u_0\|_{H^s}^k} \geq 1$ for $\lambda \gg 1$ and $0 < \delta < 1$. If $\sigma \geq 0$, then the estimate (4) of Theorem 5 implies

$$\|u_l(t)\|_{H^\sigma} \leq \|u_l(t)\|_{H^{\sigma+2}} \leq C_s\|u_l(0)\|_{H^{\sigma+2}} \leq C_s\lambda^{(\delta-1)/2k},$$

which achieves Lemma 4. □

Now, we estimate the error in H^σ-norm of these approximate solutions. Substituting the approximate solution $u^{\omega,\lambda}(x,t)$ into Equation (1), we find the following error:

$$\begin{aligned}
F &= \partial_t u_h + \mathcal{H}\partial_x^2 u_h + u_l^k \partial_x u_h + \left(\sum_{i=1}^k c_i u_l^{k-i} u_h^i\right)\partial_x(u_l + u_h) \\
&= \omega\lambda^{-\delta/2k-s}\phi_\lambda \sin\Phi\left(u_l^k(x,t) - u_l^k(x,0)\right) - \lambda^{-3(1+\delta)/2k-s}u_l^k\phi_\lambda' \cos\Phi \\
&\quad + \lambda^{-(1+\delta)/2k-s}\left[2\omega\lambda^{-(1+2\delta)/2k}\mathcal{H}(\phi_\lambda' \sin\Phi) - \lambda^{-2(1+\delta)/k}\mathcal{H}(\phi_\lambda'' \cos\Phi)\right] \\
&\quad + \omega^2\lambda^{(1-\delta)/2k-s}[\mathcal{H},\phi_\lambda]\cos\Phi + \left(\sum_{i=1}^k c_i u_l^{k-i}u_h^i\right)\partial_x(u_l+u_h) \\
&:= F_1 + F_2 + \cdots + F_5.
\end{aligned} \tag{21}$$

Estimating the H^σ-norm of F_1. Apparently, applying the Cauchy–Schwarz inequality yields

$$\begin{aligned}
\|F_1\|_{H^\sigma} &\lesssim \lambda^{-\delta/2k-s}\left\|u_l^k(x,t) - u_l^k(x,0)\right\|_{H^\sigma}\|\phi_\lambda \sin\Phi\|_{H^\sigma} \\
&\lesssim \lambda^{1/2k+\sigma-s}\left\|u_l^k(x,t) - u_l^k(x,0)\right\|_{H^\sigma}.
\end{aligned} \tag{22}$$

To estimate the H^σ-norm of the difference $u^k(x,t) - u^k(x,0)$, we adopt the fundamental theorem of calculus in time variable to obtain

$$\left\| u_l^k(x,t) - u_l^k(x,0) \right\|_{H^\sigma} \le \int_0^t \|u_l^{k-1}(x,\tau)\|_{H^\sigma} \|\partial_t u_l(x,\tau)\|_{H^\sigma} d\tau. \tag{23}$$

Apply Lemma 4 and (15) to imply that

$$\|\partial_t u_l(x,\tau)\|_{H^\sigma} \lesssim \|\mathcal{H}\partial_x^2 u_l\|_{H^\sigma} + \|u_l^k \partial_x u_l\|_{H^\sigma} \lesssim \lambda^{(\delta-1)/2k}. \tag{24}$$

Substitute (17) and (24) into (23) to yield

$$\left\| u_l^k(x,t) - u_l^k(x,0) \right\|_{H^\sigma} \lesssim \lambda^{(\delta-1)/2}. \tag{25}$$

Finally, combining (22) and (25) gives

$$\|F_1\|_{H^\sigma} \lesssim \lambda^{(k\delta-k+1)/2k+\sigma-s}. \tag{26}$$

Estimating the H^σ-norm of F_2. Applying Lemma 2, we can easily estimate F_2

$$\|F_2\|_{H^\sigma} = \|\lambda^{-3(1+\delta)/2k-s} u_l^k \phi_\lambda' \cos\Phi\|_{H^\sigma} \lesssim \lambda^{-(1+\delta)/k+\sigma-s}. \tag{27}$$

Estimating the H^σ-norm of F_3. Similar to the type, we readily check

$$\begin{aligned} \|F_3\|_{H^\sigma} &= \left\| \lambda^{-(1+\delta)/2k-s} \left[2\omega\lambda^{-(1+2\delta)/2k} \mathcal{H}(\phi_\lambda' \sin\Phi) - \lambda^{-2(1+\delta)/k} \mathcal{H}(\phi_\lambda'' \cos\Phi) \right] \right\|_{H^\sigma} \\ &\lesssim \lambda^{-\delta/k+\sigma-s} + \lambda^{-2(1+\delta)/k+\sigma-s}. \end{aligned} \tag{28}$$

Estimating the H^σ-norm of F_4. Using Lemma 3, we achieve

$$\|F_4\|_{H^\sigma} = \left\| \omega^2 \lambda^{(1-\delta)/2k-s} [\mathcal{H}, \phi_\lambda] \cos\Phi \right\|_{H^\sigma} \lesssim \lambda^{-\delta/k+\sigma-s}. \tag{29}$$

Estimating the H^σ-norm of F_5. To estimate F_5, we need the following lemma.

Lemma 5 (see [23,24]). *If $\sigma > 0$, then $H^\sigma \cap L^\infty$ is an algebra. Moreover,*
(i) $\|fg\|_{H^\sigma} \le c_\sigma(\|f\|_{L^\infty}\|g\|_{H^\sigma} + \|g\|_{L^\infty}\|f\|_{H^\sigma})$, for $\sigma > 0$;
(ii) $\|fg\|_{H^\sigma} \le c_\sigma \|f\|_{H^\sigma}\|g\|_{H^\sigma}$, for $\sigma > \frac{1}{2}$.

Apply the Lemma 5 to obtain

$$\|F_5\|_{H^\sigma} \lesssim \lambda^{(\delta-1)/2k+\sigma-s} + \lambda^{1-(1+\delta)/2k+\sigma-2s}. \tag{30}$$

Collecting the estimates above, we can obtain the following proposition.

Proposition 1. *For $s > \frac{3}{2}$, $\frac{1}{2} < \sigma \le 1$ and $0 < \delta \le \frac{1}{2}$, we can find the following estimate*

$$\|F\|_{H^\sigma} \lesssim \lambda^{-\delta/k+\sigma-s}. \tag{31}$$

3.2. *Error Estimation between Approximate and Actual Solutions*

Let $u_{\omega,\lambda}(t,x)$ be the solution to the Cauchy problem (1)—that is, $u_{\omega,\lambda}(t,x)$ satisfies

$$\begin{cases} \partial_t u_{\omega,\lambda} + \mathcal{H}\partial_x^2 u_{\omega,\lambda} + u_{\omega,\lambda}^k \partial_x u_{\omega,\lambda} = 0, \\ u_{\omega,\lambda}(x,0) = u^{\omega,\lambda}(x,0) = \omega\lambda^{-2/k}\widetilde{\phi}_\lambda - \lambda^{-(1+\delta)/2k-s}\phi_\lambda \cos\omega\lambda^{1/k}x. \end{cases} \tag{32}$$

To estimate the error between approximate and actual solutions, let

$$v = u_{\omega,\lambda} - u^{\omega,\lambda}.$$

Clearly, v solves the following equation

$$\begin{cases} \partial_t v = -F - \mathcal{H}\partial_x^2 v - u_{\omega,\lambda}^k \partial_x v - v\partial_x u^{\omega,\lambda} \sum_{i+j=k-1} (u_{\omega,\lambda})^i (u^{\omega,\lambda})^j, \\ v(0,x) = 0, \end{cases} \tag{33}$$

where F is defined by (21) and satisfying the H^σ-estimate (31).

Proposition 2. *If $\lambda \gg 1$, $s > \frac{3}{2}$ and $\frac{1}{2} < \sigma \le s$, then*

$$\|v(t)\|_{H^\sigma} \doteq \|u^{\omega,\lambda}(t) - u_{\omega,\lambda}(t)\|_{H^\sigma} \lesssim \lambda^{-\delta/k+\sigma-s}, \text{ for } 0 \le t \le T. \tag{34}$$

Proof. Applying the operator D^σ to both sides of Equation (33), and multiplying the resulting equation by $D^\sigma v$, then integrating it with respect to $x \in \mathbb{R}$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v(t)\|_{H^\sigma}^2 &= \int_{\mathbb{R}} D^\sigma v \cdot D^\sigma F dx - \int_{\mathbb{R}} D^\sigma v \cdot \mathcal{H}\partial_x^2 D^\sigma v dx - \int_{\mathbb{R}} D^\sigma v \cdot D^\sigma (u_{\omega,\lambda}^k \partial_x v) dx \\ &\quad - \int_{\mathbb{R}} D^\sigma v \cdot D^\sigma \left(v\partial_x u^{\omega,\lambda} \sum_{i+j=k-1} (u_{\omega,\lambda})^i (u^{\omega,\lambda})^j \right) dx \\ &\doteq E_1 + E_2 + E_3 + E_4. \end{aligned} \tag{35}$$

Noting that $E_2 = \int_{\mathbb{R}} D^\sigma v \cdot \mathcal{H}\partial_x^2 D^\sigma v dx = 0$.

Estimating the H^σ-norm of E_1. Referring to E_1 from (35) and using the Cauchy–Schwarz inequality gives us

$$|E_1| = \left| \int_{\mathbb{R}} D^\sigma v D^\sigma F dx \right| \le \|v\|_{H^\sigma} \|F\|_{H^\sigma}. \tag{36}$$

Estimating the H^σ-norm of E_3. Clearly, E_3 can be rewritten as

$$E_3 = -\int_{\mathbb{R}} D^\sigma v \cdot [D^\sigma, u_{\omega,\lambda}^k] \partial_x v dx - \int_{\mathbb{R}} D^\sigma v \cdot u_{\omega,\lambda}^k D^\sigma \partial_x v dx. \tag{37}$$

Using Lemma 1, we can estimate the first integral of the right-hand side of (37)

$$\begin{aligned} \int_{\mathbb{R}} D^\sigma v \cdot [D^\sigma, u_{\omega,\lambda}^k] \partial_x v dx &\le \|v\|_{H^\sigma} \left\| [D^\sigma, u_{\omega,\lambda}^k] \partial_x v \right\|_{L^2} \\ &\lesssim \|v\|_{H^\sigma} \left(\|\partial_x u_{\omega,\lambda}^k\|_{L^\infty} \|D^{\sigma-1}\partial_x v\|_{L^2} + \|D^\sigma u_{\omega,\lambda}^k\|_{L^2} \|\partial_x v\|_{L^\infty} \right) \\ &\lesssim \|u_{\omega,\lambda}\|_{H^\sigma}^k \|v\|_{H^\sigma}^2. \end{aligned} \tag{38}$$

Integrating by parts, we can estimate the second integral of the right-hand side of (37)

$$\begin{aligned} \left| \int_{\mathbb{R}} D^\sigma v \cdot u_{\omega,\lambda}^k D^\sigma \partial_x v dx \right| &= \left| \frac{1}{2} \int_{\mathbb{R}} u_{\omega,\lambda}^k \partial_x (D^\sigma v)^2 dx \right| \\ &= \left| \frac{1}{2} \int_{\mathbb{R}} \partial_x u_{\omega,\lambda}^k (D^\sigma v)^2 dx \right| \lesssim \|u_{\omega,\lambda}\|_{H^\sigma}^k \|v\|_{H^\sigma}^2. \end{aligned} \tag{39}$$

Now, (37)–(39) imply

$$|E_3| \lesssim \|u_{\omega,\lambda}\|_{H^\sigma}^k \|v\|_{H^\sigma}^2. \tag{40}$$

Estimating the H^σ-norm of E_4. Apply the Cauchy–Schwarz inequality and Lemma 5 (ii) to obtain

$$\begin{aligned}|E_4| &\lesssim \|v\|_{H^\sigma}\left\|v\partial_x u^{\omega,\lambda}\sum_{i+j=k-1}(u_{\omega,\lambda})^i(u^{\omega,\lambda})^j\right\|_{H^\sigma}\\ &\lesssim \|v\|_{H^\sigma}^2\|\partial_x u^{\omega,\lambda}\|_{H^\sigma}\left\|\sum_{i+j=k-1}(u_{\omega,\lambda})^i(u^{\omega,\lambda})^j\right\|_{H^\sigma}\\ &\lesssim \|v\|_{H^\sigma}^2.\end{aligned} \tag{41}$$

Combining the estimates of (35)–(41) yields the ODE

$$\frac{1}{2}\frac{d}{dt}\|v\|_{H^\sigma}^2 \lesssim \|v\|_{H^\sigma}^2 + \lambda^{-\delta/k+\sigma-s}\|v\|_{H^\sigma},$$

and thus

$$\frac{d}{dt}\|v\|_{H^\sigma} \lesssim \|v\|_{H^\sigma} + \lambda^{-\delta/k+\sigma-s},$$

which gives rise to the following estimate

$$\|v(t)\|_{H^\sigma} \lesssim \lambda^{-\delta/k+\sigma-s}, \text{ for } \lambda \gg 1,\ 0\le t\le T. \tag{42}$$

This proves the Proposition 2. □

3.3. *Proof of Theorem 1*

In this subsection, with the error estimation between approximate and actual solutions in hand, and using the interpolation properties of the Sobolev spaces, we can prove Theorem 1.

Proof. Let $s > \frac{3}{2}$ and define $u_{1,\lambda}(x,t)$ and $u_{-1,\lambda}(x,t)$ as the unique solutions to Equation (15) with the initial data $u_{1,\lambda}(x,0)$ and $u_{-1,\lambda}(x,0)$, respectively. From Lemma 2 and (4), we can obtain the following inequality

$$\|u_{1,\lambda}(t)\|_{H^s} + \|u_{-1,\lambda}(t)\|_{H^s} \le 2(\|u^{1,\lambda}(0)\|_{H^s} + \|u^{-1,\lambda}(0)\|_{H^s}) \lesssim 1.$$

At time $t=0$, we deduce

$$\lim_{\lambda\to\infty}\|u_{1,\lambda}(0) - u_{-1,\lambda}(0)\|_{H^s} = \lim_{\lambda\to\infty}\|2\omega\lambda^{-1/k}\widetilde{\phi_\lambda}\|_{H^s} = 0.$$

Next, we examine the H^s-norm of the difference when $t>0$. Using the triangle inequality, we find

$$\begin{aligned}\|u_{1,\lambda}(t) - u_{-1,\lambda}(t)\|_{H^s} \ge &\|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\|_{H^s}\\ &- \|u^{1,\lambda}(t) - u_{1,\lambda}(t)\|_{H^s} - \|u^{-1,\lambda}(t) - u_{-1,\lambda}(t)\|_{H^s}.\end{aligned} \tag{43}$$

With k-even, using the identity $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$, we find that

$$\begin{aligned}&\|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\|_{H^s}\\ &= \|u_{l,1,\lambda} - u_{l,-1,\lambda} - 2\lambda^{-(1+\delta)/2k-s}\phi_\lambda \sin\lambda^{1/2k}x\sin(\lambda^{1/k}t + \lambda^{-1+1/2k}t)\|_{H^s}.\end{aligned} \tag{44}$$

With k-odd, we deduce

$$\begin{aligned}&\|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\|_{H^s}\\ &= \|u_{l,1,\lambda} - u_{l,-1,\lambda} - 2\lambda^{-(1+\delta)/2k-s}\phi_\lambda \sin(\lambda^{1/2k}x - \lambda^{-1+1/2k}t)\sin\lambda^{1/k}t\|_{H^s}.\end{aligned} \tag{45}$$

Letting $m = [s] + 2 > 2$, apply Lemma 2 and (11) to find

$$\|u^{\omega,\lambda}(t) - u_{\omega,\lambda}(t)\|_{H^m} \lesssim \|u^{\omega,\lambda}(t)\|_{H^m} + \|u^{\omega,\lambda}(0)\|_{H^m} \lesssim \lambda^{m-s}, \;\; 0 < t \leq T. \tag{46}$$

Lemma 6. *Suppose $s_1 < s < s_2$ and $f \in H^s(\mathbb{R})$. Then,*

$$\|f\|_{H^s} \leq \|f\|_{H^{s_1}}^{\frac{s_2-s}{s_2-s_1}} \|f\|_{H^{s_2}}^{\frac{s-s_1}{s_2-s_1}}. \tag{47}$$

Employing the interpolation inequality in Lemma 6 with $s_1 = \sigma$ and $s_2 = [s] + 2 = m$ and Equations (42) and (46), we obtain

$$\begin{aligned} \|u^{\omega,\lambda}(t) - u_{\omega,\lambda}(t)\|_{H^s} &\leq \|u^{\omega,\lambda}(t) - u_{\omega,\lambda}(t)\|_{H^\sigma}^{\frac{m-s}{m-\sigma}} \|u^{\omega,\lambda}(t) - u_{\omega,\lambda}(t)\|_{H^m}^{\frac{s-\sigma}{m-\sigma}} \\ &\lesssim \lambda^{\frac{(-\delta/k+\sigma-s)(m-s)}{m-\sigma}} n^{\frac{(m-s)(s-\sigma)}{m-\sigma}} \\ &\lesssim \lambda^{\frac{(-\delta/k)(m-s)}{m-\sigma}}. \end{aligned} \tag{48}$$

Taking the limit infimum to both sides of (43) gives us

$$\begin{aligned} &\liminf_{\lambda\to\infty} \|u_{1,n}(t) - u_{-1,n}(t)\|_{H^s} \\ &\geq \liminf_{\lambda\to\infty} (\|u^{1,n}(t) - u^{-1,n}(t)\|_{H^s} - \|u^{1,n}(t) - u_{1,n}(t)\|_{H^s} - \|u^{-1,n}(t) - u_{-1,n}(t)\|_{H^s}) \\ &\gtrsim \liminf_{\lambda\to\infty} \left|\sin \lambda^{1/k} t\right|, \end{aligned} \tag{49}$$

apparently $\liminf_{\lambda\to\infty} \|u_{1,n}(t) - u_{-1,n}(t)\|_{H^s} \neq 0$; thus, we complete the proof of Theorem 1. □

4. Hölder Continuous in H^r-Topology

In this section, we continue to study the continuity properties for the solution map in Hölder spaces H^r. More precisely, we consider two solutions of Equation (1), u and v, which emanate from the initial data u_0 and v_0, respectively. We expect that if the initial data u_0 and v_0 are assigned in a ball with radius ρ in H^s, i.e.,

$$\|u_0\|_{H^s} \leq \rho, \|v_0\|_{H^s} \leq \rho, s > \frac{3}{2}, \tag{50}$$

and then we obtain

$$\|u(t) - v(t)\|_{H^r} \lesssim \|u_0 - v_0\|_{H^r}^{\alpha}, 0 \leq r < s,$$

where the Hölder exponent α is to be determined.

Proof of Theorem 2. Lipschitz continuity in region A_1. Let v be another solution to the Cauchy problem for (1) corresponding to the initial data $v_0(x) \in H^s(\mathbb{R})$, i.e.

$$\begin{cases} \partial_t v + \mathcal{H}\partial_x^2 v + v^k \partial_x v = 0, & t > 0, x \in \mathbb{R}, \\ v(x,0) = v_0(x), & t = 0, x \in \mathbb{R}. \end{cases} \tag{51}$$

Subtracting (51) from (1) yields the Cauchy problem for w to

$$\partial_t w + \mathcal{H}\partial_x^2 w + \partial_x(v^k w) - w\partial_x v^k + w\partial_x u \sum_{i+j=k-1} (u^i v^j) = 0, \tag{52}$$

where $w = u - v$ and $i, j \in$. For a fixed $0 \leq r \leq s - 1$ with $r + s \geq 2$, estimating the H^r energy of w leads us to the following equation

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w(t)\|_{H^r}^2 = &-\int_{\mathbb{R}} D^r w \cdot \mathcal{H}\partial_x^2 D^r w dx - \int_{\mathbb{R}} D^r w \cdot D^r \partial_x (v^k w) dx \\ &+ \int_{\mathbb{R}} D^r w \cdot D^r (w\partial_x v^k) dx - \int_{\mathbb{R}} D^r w \cdot D^r \left(w\partial_x u \sum_{i+j=k-1} (u^i u^j) \right) dx.\end{aligned} \tag{53}$$

Noting that $\int_{\mathbb{R}} D^r w \cdot \mathcal{H}\partial_x^2 D^r w dx = 0$. Clearly, the second integral of the right-hand side of (53) can be rewritten as

$$-\int_{\mathbb{R}} D^r w \cdot D^r \partial_x (v^k w) dx = -\int_{\mathbb{R}} D^r w \cdot [D^r \partial_x, v^k] w dx - \int_{\mathbb{R}} D^r w \cdot v^k (D^r \partial_x w) dx. \tag{54}$$

To estimate the first term on the right-hand sides of (57), we need the following lemma (see [23,24]).

Lemma 7. *If $\sigma + 1 \geq 0, s > \frac{3}{2}, \sigma + 1 \leq s$, then*

$$\|[D^\sigma \partial_x, f]g\|_{L^2} \leq c_{\sigma,s} \|f\|_{H^s} \|g\|_{H^\sigma}.$$

Using Lemma 7 and (50), we find

$$\begin{aligned}\left| \int_{\mathbb{R}} D^r w \cdot [D^r \partial_x, v^k] w dx \right| &\leq C_{s,r} \|w\|_{H^r} \|[D^r \partial_x, v^k] w\|_{L^2} \leq C_{s,r} \|v\|_{H^s}^k \|w\|_{H^r}^2 \\ &\leq C_{s,r} \|v(0)\|_{H^s}^k \|w\|_{H^r}^2 \leq C_{s,r} \rho^k \|w\|_{H^r}^2,\end{aligned} \tag{55}$$

and we can easily yield the following estimates

$$\left| \int_{\mathbb{R}} D^r w \cdot v^k (D^r \partial_x w) dx \right| \leq C_{s,r} \|v\|_{H^s}^k \|w\|_{H^r}^2 \leq C_{s,r} \rho^k \|w\|_{H^r}^2. \tag{56}$$

Combining (55) and (56) yields the estimates

$$\left| \int_{\mathbb{R}} D^r w \cdot D^r \partial_x (v^k w) dx \right| \leq C_{s,r} \rho^k \|w\|_{H^r}^2. \tag{57}$$

For the third term on the right-hand sides of (53), we readily check

$$\left| \int_{\mathbb{R}} D^r w \cdot D^r (w\partial_x u^k) dx \right| \leq C_{s,r} \|w\|_{H^r} \|w\partial_x v^k\|_{H^r} \leq C_{s,r} \rho^k \|w\|_{H^r}^2, \tag{58}$$

where, in the second inequality, we used following lemma.

Lemma 8 (see [25,26]). *If $\sigma > -\frac{1}{2}$, then*

$$\|fg\|_{H^\sigma} \leq c_\sigma \|f\|_{H^{\sigma+1}} \|g\|_{H^\sigma}.$$

Similarly, we can estimate the last term of (53)

$$\left| \int_{\mathbb{R}} D^r w \cdot D^r \left(w\partial_x u \sum_{i+j=k-1} (u^i u^j) \right) dx \right| \leq C_{s,r} \rho^k \|w\|_{H^r}^2. \tag{59}$$

End of Lipschitz Continuity in A_1. Combining the above estimates generates the following energy inequality

$$\frac{d}{dt}\|w(t)\|_{H^r} \leq C_{r,s,\rho} \|w(t)\|_{H^r},$$

which implies

$$\|w(t)\|_{H^r} \leq e^{C_{r,s,\rho}T}\|w(0)\|_{H^r}.$$

Clearly, it is equivalent to

$$\|u(t)-v(t)\|_{H^r} \leq e^{C_{r,s,\rho}T}\|u(0)-v(0)\|_{H^r},$$

which is the desired Lipschitz continuity in A_1.

Hölder Continuity in A_2. As per the Lipschitz continuity in A_1 and the assumption $r<2-s$, we deduce

$$\|u(t)-v(t)\|_{H^r} \leq \|u(t)-v(t)\|_{H^{2-s}} \leq e^{C_{r,s,\rho}T}\|u(0)-v(0)\|_{H^{2-s}}.$$

Since $r<2-s<s$, by the interpolation between the H^r and the H^s norms described in Lemma 6, we find

$$\|u(0)-v(0)\|_{H^{2-s}} \leq \|u(0)-v(0)\|_{H^r}^{\frac{2(s-1)}{s-r}}\|u(0)-v(0)\|_{H^s}^{\frac{2-s-r}{s-r}} \leq C_{r,s,\rho}\|u(0)-v(0)\|_{H^r}^{\frac{2(s-1)}{s-r}},$$

which guarantees the Hölder continuity in A_2.

Hölder Continuity in A_3. For $s-1<r<s$, by the interpolation between H^{s-1} and H^s norms, we have

$$\|u(t)-v(t)\|_{H^r} \leq \|u(t)-v(t)\|_{H^{s-1}}^{s-r}\|u(t)-v(t)\|_{H^s}^{r-s+1}.$$

By the well-posedness size estimate (50), we find

$$\|u(t)-v(t)\|_{H^s} \lesssim \|u_0\|_{H^s}+\|v_0\|_{H^s} \lesssim \rho,$$

which, therefore, gives

$$\|u(t)-v(t)\|_{H^r} \leq C_{r,s,\rho}\|u(t)-v(t)\|_{H^{s-1}}^{s-r}.$$

The Lipschitz continuity in A_1 and the condition $s-1<r$ admit

$$\|u(t)-v(t)\|_{H^r} \leq C_{r,s,\rho}\|u(0)-v(0)\|_{H^{s-1}}^{s-r} \leq C_{r,s,\rho}\|u(0)-v(0)\|_{H^r}^{s-r},$$

which is the desired Hölder continuity in A_3. □

5. Gevrey Regularity and Analyticity for g-BO System

5.1. Analytic Solutions for g-BO in $G^{\delta}_{\sigma,s}$

In this section, By applying nonlinear Cauchy-Kowalevski theory, we will establish the Gevrey regularity and analyticity of solutions to g-BO system.

Theorem 6 (see [22]). *Let $(X_\delta, \|\cdot\|_{G^{\delta}_{\sigma,s}})_{0<\delta<1}$ be a scale of decreasing Banach spaces, namely, for any $\delta'<\delta$ we have $X_\delta \subset X_{\delta'}$ and $\|\cdot\|_{G^{\delta'}_{\sigma,s}} \leq \|\cdot\|_{G^{\delta}_{\sigma,s}}$. Consider the Cauchy problem*

$$\begin{cases} \frac{du}{dt} = F(t,u(t)), \\ u|_{t=0} = 0. \end{cases} \tag{60}$$

Let $T,R>0$ and $\sigma \geq 1$. For given $u_0 \in X_1$, assume that F satisfies the following conditions:
(1) If for any $0<\delta'<\delta<1$, the function $t \mapsto u(t)$ is holomorphic on $|t|<T$ and continuous on $|t|<T$ with values in X_δ and

$$\sup_{|t|<T}\|u(t)\|_{G^{\delta}_{\sigma,s}} < R,$$

and then $t \mapsto F(t, u(t))$ is a holomorphic function on $|t| < T$ with values in $X_{\delta'}$.
(2) For any $0 < \delta' < \delta < 1$ and $u, v \in \overline{B(u_0, R)} \subset X_\delta$—that is, $\|u\|_{G^\delta_{\sigma,s}} < R, \|v\|_{G^\delta_{\sigma,s}} < R$, there exits a positive constant L depending on u_0 and R such that

$$\sup_{|t|<T} \|F(t,u) - F(t,v)\|_{G^{\delta'}_{\sigma,s}} \le \frac{L}{(\delta - \delta')^{2\sigma}} \|u - v\|_{G^\delta_{\sigma,s}}.$$

(3) There exists a $M > 0$ depending on u_0 and R such that, for any $0 < \delta < 1$,

$$\sup_{|t|<T} \|F(t, u_0)\|_{G^\delta_{\sigma,s}} \le \frac{M}{(1-\delta)^{2\sigma}}.$$

Then, there exists a $T_0 \in (0, T)$ and a unique function $u(t)$ to the Cauchy problem (60) that is holomorphic in $|t| < \frac{(1-\delta)^{2\sigma} T_0}{2^{2\sigma}-1}$ with values in X_δ for every $\delta \in (0, 1)$.

Proposition 3 (see [22]). *Let $0 < \delta' < \delta, 0 < \sigma' < \sigma$ and $s' < s$. From Definition 1, one can check that $G^\delta_{\sigma,s} \hookrightarrow G^{\delta'}_{\sigma,s}, G^\delta_{\sigma',s} \hookrightarrow G^\delta_{\sigma,s}$ and $G^\delta_{\sigma,s} \hookrightarrow G^\delta_{\sigma,s'}$.*

Proposition 4. *Let s be a real number and $\sigma > 0$. Assume that $0 < \delta' < \delta$. Then, we have*

$$\|\partial_x f\|_{G^{\delta'}_{\sigma,s}(\mathbb{R})} \le \frac{e^{-\sigma}\sigma^\sigma}{(\delta-\delta')^\sigma} \|f\|_{G^\delta_{\sigma,s}(\mathbb{R})}, \|\partial_x^2 f\|_{G^{\delta'}_{\sigma,s}(\mathbb{R})} \le \frac{e^{-2\sigma}(2\sigma)^{2\sigma}}{(\delta-\delta')^{2\sigma}} \|f\|_{G^\delta_{\sigma,s}(\mathbb{R})}.$$

Proof. The first inequality can be found in [22], so we only prove the second inequality. Since $\widehat{\partial_x f} = i\xi \hat{f}$, it follows that

$$\begin{aligned}
\|\partial_x^2 f\|^2_{G^{\delta'}_{\sigma,s}} &= \int_{\mathbb{R}} (1+|\xi|^2)^s e^{2\delta'|\xi|^{1/\sigma}} |\xi|^4 |\hat{f}(\xi)|^2 d\xi \\
&= \frac{1}{(\delta-\delta')^{4\sigma}} \int_{\mathbb{R}} (1+|\xi|^2)^s e^{2\delta|\xi|^{1/\sigma}} e^{-2[((\delta-\delta')^\sigma)|\xi|]^{1/\sigma}} (\delta-\delta')^{4\sigma} |\xi|^4 |\hat{f}(\xi)|^2 d\xi \\
&\le \frac{\|f\|^2_{G^\delta_{\sigma,s}}}{(\delta-\delta')^{4\sigma}} \sup_{\xi\in\mathbb{R}} \{e^{-2[((\delta-\delta')^\sigma)|\xi|]^{1/\sigma}} (\delta-\delta')^{4\sigma} |\xi|^4\}.
\end{aligned} \tag{61}$$

Let $z = [((\delta-\delta')^\sigma)|\xi|]^{1/\sigma} \ge 0$ and consider the function $g(z) = e^{-2z} z^{4\sigma}$. By directly calculating, we have $\lim_{z\to0} g(z) = 0, \lim_{z\to\infty} g(z) = 0$ and $g'(z) = 2z^{4\sigma-1}(2\sigma - z)$. By solving $g'(z) = 0$, we obtain that $z = 2\sigma$, which implies that $g(z) \le g(2\sigma) = e^{-4\sigma}(2\sigma)^{4\sigma}$. Then, we deduce from (61) that

$$\|\partial_x^2 f\|^2_{G^{\delta'}_{\sigma,s}} \le \frac{e^{-2\sigma}(2\sigma)^{2\sigma}\|f\|_{G^\delta_{\sigma,s}}}{(\delta-\delta')^{2\sigma}}.$$

□

Proposition 5 (see [22]). *Let $s > \frac{1}{2}, \sigma \ge 1$ and $\delta > 0$. Then, $G^\delta_{\sigma,s}(\mathbb{R})$ is an algebra. Moreover, there exists a constant C_s such that*

$$\|fg\|_{G^\delta_{\sigma,s}(\mathbb{R})} \le C_s \|f\|_{G^\delta_{\sigma,s}(\mathbb{R})} \|g\|_{G^\delta_{\sigma,s}(\mathbb{R})}.$$

Proof of Theorem 3. We rewrite (g-BO) as follows:

$$\begin{cases} u_t = F(u) \doteq -\mathcal{H}\partial_x^2 u - u^k \partial_x u, \\ u|_{t=0} = 0. \end{cases} \tag{62}$$

For a fixed $\sigma \geq 1$ and $s > \frac{3}{2}$. By virtue of Propositions 3, 4 and 5, we deduce that, for any $0 < \delta' < \delta < 1$,

$$\begin{aligned}\|F(u)\|_{G^{\delta'}_{\sigma,s}} &\leq \|\partial_x^2 u\|_{G^{\delta'}_{\sigma,s}} + \|u^k \partial_x u\|_{G^{\delta'}_{\sigma,s}} \\ &\leq \frac{e^{-2\sigma}(2\sigma)^{2\sigma}}{(\delta-\delta')^{2\sigma}}\|u\|_{G^{\delta}_{\sigma,s}} + \frac{C_s e^{-\sigma}\sigma^{\sigma}}{(\delta-\delta')^{\sigma}}\|u\|^{k+1}_{G^{\delta}_{\sigma,s}} \\ &\leq \frac{C_s e^{-\sigma}(2\sigma)^{2\sigma}}{(\delta-\delta')^{2\sigma}}(\|u\|_{G^{\delta}_{\sigma,s}} + \|u\|^{k+1}_{G^{\delta}_{\sigma,s}}),\end{aligned} \tag{63}$$

which implies that F satisfies the condition (1) of Theorem 6. By the same token, we obtain that $\|F(u_0)\|_{G^{\delta}_{\sigma,s}} \leq \frac{C_s e^{-\sigma}(2\sigma)^{2\sigma}}{(1-\delta)^{2\sigma}}(\|u_0\|_{G^1_{\sigma,s}} + \|u_0\|^{k+1}_{G^1_{\sigma,s}})$. Thus, F satisfies the condition (3) of Theorem 6 with $M = C_s e^{-\sigma}(2\sigma)^{2\sigma}(\|u_0\|_{G^1_{\sigma,s}} + \|u_0\|^{k+1}_{G^1_{\sigma,s}})$. Finally, we will show that F satisfies the condition (2) of Theorem 6. Assume that $\|u-u_0\|_{G^{\delta}_{\sigma,s}} \leq R$, $\|v-u_0\|_{G^{\delta}_{\sigma,s}} \leq R$ and $w = u - v$. Applying Proposition 4, we find

$$\begin{aligned}\|F(u)-F(v)\|_{G^{\delta'}_{\sigma,s}} &= \left\|\mathcal{H}\partial_x^2 w + \partial_x(v^k w) - w\partial_x v^k + w\partial_x u \sum_{i+j=k-1}(u^i v^j)\right\|_{G^{\delta'}_{\sigma,s}} \\ &\leq \frac{e^{-2\sigma}(2\sigma)^{2\sigma}}{(\delta-\delta')^{2\sigma}}\|w\|_{G^{\delta}_{\sigma,s}} + \frac{C_s e^{-\sigma}\sigma^{\sigma}}{(\delta-\delta')^{\sigma}}\|w\|_{G^{\delta}_{\sigma,s}}\|v\|^k_{G^{\delta}_{\sigma,s}} + \frac{C_s e^{-\sigma}\sigma^{\sigma}}{(\delta-\delta')^{\sigma}}\|w\|_{G^{\delta}_{\sigma,s}}\|v\|^k_{G^{\delta}_{\sigma,s}} \\ &\quad + \frac{e^{-\sigma}\sigma^{\sigma}}{(\delta-\delta')^{\sigma}}\|w\|_{G^{\delta}_{\sigma,s}}\|u\|_{G^{\delta}_{\sigma,s}}\left\|\sum_{i+j=k-1}(u^i v^j)\right\|_{G^{\delta}_{\sigma,s}} \\ &\leq \frac{C_s e^{-2\sigma}(2\sigma)^{2\sigma} + C_s e^{-\sigma}\sigma^{\sigma}(\|u_0\|_{G^1_{\sigma,s}} + R)^k}{(\delta-\delta')^{2\sigma}}\|w\|_{G^{\delta}_{\sigma,s}} \\ &\leq \frac{C_s e^{-\sigma}(2\sigma)^{2\sigma}\left(1 + (\|u_0\|_{G^1_{\sigma,s}} + R)^k\right)}{(\delta-\delta')^{2\sigma}}\|w\|_{G^{\delta}_{\sigma,s}}.\end{aligned}$$

From the above inequality, we verify that F satisfies the condition (2) of Theorem 6 with $L = C_s e^{-\sigma}(2\sigma)^{2\sigma}\left(1 + (\|u_0\|_{G^1_{\sigma,s}} + R)^k\right)$. Moreover, $T_0 = \min\{\frac{1}{2^{4\sigma+4}L}, \frac{(2^{2\sigma}-1)R}{(2^{2\sigma}-1)2^{4\sigma+3}LR+M}\}$, by setting $R = \|u_0\|_{G^1_{\sigma,s}}$, we see that $L = C_s e^{-\sigma}(2\sigma)^{2\sigma}\left(1 + 2^k\|u_0\|^k_{G^1_{\sigma,s}}\right)$ and $M \leq 2^{4\sigma+3}LR$. Then, we have $T_0 = \frac{1}{C_s 2^{4\sigma+4}e^{-\sigma}(2\sigma)^{2\sigma}\left(1+2^k\|u_0\|^k_{G^1_{\sigma,s}}\right)}$. □

5.2. Continuity of the Data-to-Solution Map in $G^1_{\sigma,s}$

Proof of Theorem 4. Without loss of generality, we may assume that $t \geq 0$. Define that

$$T^n = \frac{1}{C_s 2^{4\sigma+4}e^{-\sigma}(2\sigma)^{2\sigma}\left(1+2^k\|u_0^n\|^k_{G^1_{\sigma,s}}\right)}, T^{\infty} = \frac{1}{C_s 2^{4\sigma+4}e^{-\sigma}(2\sigma)^{2\sigma}\left(1+2^k\|u_0^{\infty}\|^k_{G^1_{\sigma,s}}\right)}, \tag{64}$$

where C_s is given in Proposition 5. Since $\|u_0^n - u_0^{\infty}\|_{G^1_{\sigma,s}} \mapsto 0$; therefore, there exists a constant N such that

$$\|u_0^n\|_{G^1_{\sigma,s}} \leq \|u_0^{\infty}\|_{G^1_{\sigma,s}} + 1, \quad for\ n \geq N. \tag{65}$$

Then,

$$T \doteq \frac{1}{C_s 2^{4\sigma+4}e^{-\sigma}(2\sigma)^{2\sigma}\left(1+2^k(\|u_0^{\infty}\|_{G^1_{\sigma,s}}+1)^k\right)} < \min\{T^n, T^{\infty}\}, \quad for\ n \geq N. \tag{66}$$

Furthermore, as in the proof of Theorem 3, we see that T^n and T^∞ are the existence time corresponding to $\|u_0^n\|_{G^1_{\sigma,s}}$ and $\|u_0^\infty\|_{G^1_{\sigma,s}}$, respectively, which implies that, for any $n \geq N$

$$\begin{aligned} u^n(t,x) &= u_0^n(x) + \int_0^t F(u^n(t,\tau))d\tau, \quad 0 \leq t < \frac{T(1-\delta)^{2\sigma}}{2^{2\sigma}-1}, \\ u^\infty(t,x) &= u_0^\infty(x) + \int_0^t F(u^\infty(t,\tau))d\tau, \quad 0 \leq t < \frac{T(1-\delta)^{2\sigma}}{2^{2\sigma}-1}, \end{aligned} \tag{67}$$

where F is given in (62). Therefore, we verify that, for any $0 \leq t < \frac{T(1-\delta)^{2\sigma}}{2^{2\sigma}-1}$ and $0 < \delta < 1$

$$\|u^n - u^\infty\|_{G^\delta_{\sigma,s}} \leq \|u_0^n - u_0^\infty\|_{G^\delta_{\sigma,s}} + \int_0^t \|F(u^n(t,\tau)) - F(u^\infty(t,\tau))\|_{G^\delta_{\sigma,s}} d\tau. \tag{68}$$

Choosing $\delta(\tau) = \frac{1+\delta}{2} + \frac{1}{2}^{2+1/2\sigma}\Big\{\big[(1-\delta)^{2\sigma} - \frac{t}{T}\big]^{1/2\sigma} - \big[(1-\delta)^{2\sigma} + (2^{2\sigma+1}-1)\frac{t}{T}\big]^{1/2\sigma}\Big\}$. we find $\|F(u^n(t,\tau)) - F(u^\infty(t,\tau))\|_{G^\delta_{\sigma,s}} \leq \frac{L\|u^n-u^\infty\|_{G^{\delta(\tau)}_{\sigma,s}}}{(\delta(\tau)-\delta)^{2\sigma}}$ with $L = C_s e^{-\sigma}(2\sigma)^{2\sigma}\Big(1 + 2^k\|u_0\|^k_{G^1_{\sigma,s}}\Big)$ and $0 < \delta < \delta(\tau) < 1$. Using this in (68) yields

$$\begin{aligned} \|u^n - u^\infty\|_{G^\delta_{\sigma,s}} &\leq \|u_0^n - u_0^\infty\|_{G^\delta_{\sigma,s}} + L\int_0^t \frac{\|u^n - u^\infty\|_{G^{\delta(\tau)}_{\sigma,s}}}{(\delta(\tau)-\delta)^{2\sigma}} d\tau \\ &\leq \|u_0^n - u_0^\infty\|_{G^\delta_{\sigma,s}} + \frac{2^{4\sigma+3}LT\|u^n - u^\infty\|_{E_T}}{(1-\delta)^{2\sigma}}\sqrt{\frac{T(1-\delta)^{2\sigma}}{T(1-\delta)^{2\sigma}-t}}, \end{aligned} \tag{69}$$

where, in the last inequality, we used lemma 3.7 in [22]. Since $T = \frac{1}{C_s 2^{4\sigma+4}e^{-\sigma}(2\sigma)^{2\sigma}\Big(1+2^k(\|u_0\|_{G^1_{\sigma,s}}+1)^k\Big)}$ and $L = C_s e^{-\sigma}(2\sigma)^{2\sigma}\Big(1 + 2^k\|u_0\|^k_{G^1_{\sigma,s}}\Big)$, this yields that $2^{4\sigma+3}LT < \frac{1}{2}$. Then, we have

$$\|u^n - u^\infty\|_{G^\delta_{\sigma,s}} \leq \|u_0^n - u_0^\infty\|_{G^\delta_{\sigma,s}} + \frac{\|u^n - u^\infty\|_{E_T}}{2(1-\delta)^{2\sigma}}\sqrt{\frac{T(1-\delta)^{2\sigma}}{T(1-\delta)^{2\sigma}-t}}. \tag{70}$$

This leads to

$$\begin{aligned} &\|u^n - u^\infty\|_{G^\delta_{\sigma,s}}(1-\delta)^{2\sigma}\sqrt{1 - \frac{t}{T(1-\delta)^{2\sigma}}} \\ &\leq \|u_0^n - u_0^\infty\|_{G^\delta_{\sigma,s}}(1-\delta)^{2\sigma}\sqrt{1 - \frac{t}{T(1-\delta)^{2\sigma}}} + \frac{1}{2}\|u^n - u^\infty\|_{E_T} \\ &\leq \|u_0^n - u_0^\infty\|_{G^1_{\sigma,s}} + \frac{1}{2}\|u^n - u^\infty\|_{E_T}. \end{aligned} \tag{71}$$

Note that the right hand side of the above inequality is independent of t and δ. By the definition of E_T, we have

$$\|u^n - u^\infty\|_{E_T} \leq 2\|u_0^n - u_0^\infty\|_{G^1_{\sigma,s}}. \tag{72}$$

The above inequality holds true for any $n \geq N$, which leads to our desired result. □

6. Conclusions

The local well-posedness results in [14–16] imply that the existence, uniqueness and continuously dependence on their initial data of the solutions to the g-BO Equation (1) in in $H^s(\mathbb{R})$ with $s > \frac{3}{2}$. We showed that such a data-to-solution map is not uniformly continuous in Theorem 1 but Hölder continuous in H^σ-topology. On the other hand, in

Sobolev–Gevrey spaces, we proved that the solutions of g-BO equation are analytic in both space and time variables in Theorem 3. In addition, the continuity of the data-to-solution in Sobolev–Gevrey spaces was also obtained.

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