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New Results of the Fifth-Kind Orthogonal Chebyshev Polynomials

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Abstract: The principal objective of this article is to develop new formulas of the so-called Chebyshev polynomials of the fifth-kind. Some fundamental properties and relations concerned with these polynomials are proposed. New moments formulas of these polynomials are obtained. Linearization formulas for these polynomials are derived using the moments formulas. Connection problems between the fifth-kind Chebyshev polynomials and some other orthogonal polynomials are explicitly solved. The linking coefficients are given in forms involving certain generalized hypergeometric functions. As special cases, the connection formulas between Chebyshev polynomials of the fifth-kind and the well-known four kinds of Chebyshev polynomials are shown. The linking coefficients are all free of hypergeometric functions.

Keywords: Chebyshev polynomials; connection and linearization coefficients; generalized hypergeometric functions; moments formulas

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1. Introduction

The special functions in general and orthogonal polynomials, in particular, are related to a large number of problems in different disciplines such as approximation theory, theoretical physics, chemistry, and some other mathematical branches. Special functions have been investigated theoretically by a large number of authors, for example, the authors in [1–4] have investigated some special functions including Humbert and Genocchi polynomials.

There are four well-known kinds of Chebyshev polynomials. They are direct special kinds of Jacobi polynomials with certain parameters. They have their roles, especially in the scope of solving different types of differential equations (see, for example, [5–7]). Chebyshev polynomials of the fifth- and sixth-kinds are two special classes of the so-called generalized ultraspherical polynomials (see, [8,9]). They also can be considered as special classes of a general class of symmetric polynomials generated by the extended Sturm–Liouville problem which was investigated by Jamei [10]. In addition, they are categorized as Chebyshev polynomials due to their trigonometric representations. From a numerical point of view, these polynomials have been recently employed by some authors. The authors in [11–13] utilized them to obtain spectral solutions to some types of fractional differential equations. The authors in [14] have developed an approximate solution of a certain variable-order fractional integro-differential equations by using Chebyshev polynomials of the sixth-kind. In addition, recently, Abd-Elhameed in [15] utilized the Chebyshev polynomials of the sixth-kind to treat numerically the non-linear one-dimensional Burgers' differential equation.

In Jamei [10], the author extracted the Chebyshev polynomials of the fifth- and sixth-kinds, but to the best of our knowledge, The theoretical results concerned with the Chebyshev polynomials of the fifth-kind are not complete. For example, the connection problems of these polynomials with other orthogonal polynomials are not yet solved. In addition, the moments and linearization formulas of these polynomials are also not found. This gives us the motivation to develop such formulas. In addition, it is expected that these formulas will be useful in some applications. This gives us another motivation to investigate theoretically the fifth-kind Chebyshev polynomials.

Two important problems related to the orthogonal polynomials are the so-called connection and linearization problems. Regarding the connection problem, if we consider the two polynomial sets $\{\phi_i(x)\}_{i\geq 0}$ and $\{\psi_j(x)\}_{j\geq 0}$, then the two connection problems between them are to determine the connection coefficients $A_{i,j}$ and $\bar{A}_{i,j}$, such that

$$\phi_i(x) = \sum_{j=0}^{i} A_{i,j}\,\psi_j(x),$$

and

$$\psi_i(x) = \sum_{j=0}^{i} \bar{A}_{i,j}\,\phi_j(x).$$

Regarding the linearization problem, if we consider the three polynomial sets $\{\phi_i(x)\}_{i\geq 0}$, $\{\psi_j(x)\}_{j\geq 0}$, and $\{\theta_k(x)\}_{k\geq 0}$, then the problem

$$\phi_i(x)\,\psi_j(x) = \sum_{k=0}^{i+j} L_{k,i,j}\,\theta_k(x), \tag{1}$$

is called the general linearization problem. We comment here that if $\phi_i(x) \equiv \psi_i(x) \equiv \theta_i(x)$, then the linearization problem in such case is called Clebsch–Gordan type linearization problem. To solve the general linearization problem (1), we have to find the linearization coefficients $L_{k,i,j}$. Several articles were interested in investigating the connection and linearization formulas of different orthogonal polynomials using different approaches. For example, a symbolic approach is followed in [16] to find the connection coefficients between different polynomials. The linearization problem of Jacobi polynomials and the non-negativity of the linearization coefficients were investigated in [17–19]. Another study for the non-negativity of the linearization coefficients of orthogonal polynomials was considered in [20]. In [21], the author studied the linearization problems of some classes of Jacobi polynomials. Chaggara and Koepf in [22] succeeded in finding closed analytical linearization formulas of some classes of Jacobi polynomials. In the papers [23–25], the authors derived closed linearization formulas for some other classes of Jacobi polynomials. Recently, two different approaches based on connection and moments formulas are followed in [26] to obtain linearization formulas of certain Jacobi polynomials. In [27], the authors presented the expansions of a product of Laguerre and Legendre polynomials in series of such polynomials. Some approaches were followed in [28–30] to treat Clebsch–Gordan type linearization formulas. Some other studies can be found in [31–33]. We refer here that most connection and linearization coefficients of different orthogonal polynomials may be expressed in terms of certain generalized hypergeometric forms that can be reduced in specific cases of the involved parameters either by using some reduction formulas in the literature or via employing some symbolic algorithms, such as Zeilberger's algorithm [34]. Some other important problems related to linearization problems can be found in [35–38].

Among the important formulas concerned with a set of orthogonal polynomials is the moments formula. That is, if we consider a polynomial set $\{\phi_j(x)\}_{j\geq 0}$, then to find the moments formula

$$x^m\,\phi_j(x) = \sum_{\ell=0}^{m+j} M_{\ell,m,j}\,\phi_{j-\ell+m}(x), \quad m \geq 0, \tag{2}$$

we have to find the moments coefficients $M_{\ell,m,j}$ in (2).

In order to show our theoretical contribution, we summarize the aims of the current article in the following items:

- Introducing some elementary formulas of the fifth-kind Chebyshev polynomials;

- Obtaining explicit formulas of the moments of these polynomials;
- Solving the linearization problem of these polynomials with the aid of the derived moments formulas;
- Solving the connection problems that join Chebyshev polynomials of the fifth-kind with some other orthogonal polynomials.

It is worth pointing out here that another motivation for our interest in developing the presented formulas in this paper is that these formulas may be useful in some applications. Some of their expected uses are listed as follows:

- The moments formulas are useful in the numerical treatment of ordinary differential equations with polynomials coefficients;
- The linearization coefficients are useful in the numerical treatment of some non-linear differential equations as followed in [15];
- The connection coefficients are very useful in investigating the convergence analysis as followed in [11].

The following is a breakdown of the paper's structure. Section 2 covers some basic properties and elementary relations of fifth-kind Chebyshev polynomials. New formulas for the moments of these polynomials are obtained in Section 3. In Section 4, the linearization formulas of the fifth-kind Chebyshev polynomials are derived using these formulas. Connection problems between these polynomials and some other orthogonal polynomials are solved in Section 5. Finally, in Section 6, some findings are presented.

2. Some Properties and Essential Formulas

In this section, we give an account on Chebyshev polynomials of the fifth-kind $C_j(x)$, and we introduce some of their fundamental properties. These polynomials are orthogonal on $[-1,1]$ with respect to the weight function: $w(x) = \dfrac{x^2}{\sqrt{1-x^2}}$ in the sense that [11]

$$\int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}}\, C_m(x)\ C_n(x)\, dx = \begin{cases} h_n, & \text{if } m = n, \\ 0, & \text{if } m \neq n, \end{cases} \tag{3}$$

with

$$h_n = \frac{\pi}{2^{2\,n+1}} \begin{cases} 1, & n \text{ even}, \\ \dfrac{n+2}{n}, & n \text{ odd}. \end{cases}$$

The three-term recurrence relation satisfied by $C_n(x)$ may be written in the following form:

$$C_n(x) = x\, C_{n-1}(x) - \frac{(n-1)^2 + n + (-1)^n\,(2n-1)}{4\,n(n-1)}\, C_{n-2}(x), \quad n \geq 2,$$

with the initial values

$$C_0(x) = 1, \quad C_1(x) = x.$$

The power form representations of these polynomials can be written as [10]

$$C_{2j}(x) = (2j+1) \sum_{r=0}^{j} \frac{(-1)^r\,(2j-r)!}{r!\,2^{2r}\,(2j-2r+1)!}\, x^{2j-2r}, \tag{4}$$

$$C_{2j+1}(x) = \frac{\Gamma\left(j+\frac{5}{2}\right)}{(2j+1)!} \sum_{r=0}^{j} \frac{(-1)^r \binom{j}{j-r}\,(2j-r+1)!}{\Gamma\left(j-r+\frac{5}{2}\right)}\, x^{2j-2r+1}. \tag{5}$$

Of the important formulas concerned with any set of polynomials is its inversion formula. The following lemma gives the inversion formulas to Formulas (4) and (5).

Lemma 1. *For any non-negative integer m, the following two formulas are valid:*

$$x^{2m} =(2m+1)!\, \sum_{r=0}^{m} \frac{1}{2^{2r}\, r!\, (2m-r+1)!}\, C_{2m-2r}(x), \tag{6}$$

$$x^{2m+1} =\Gamma\left(m+\tfrac{5}{2}\right) \sum_{r=0}^{m} \frac{\binom{m}{m-r}\, (2m-2r+2)!}{\Gamma\left(m-r+\frac{5}{2}\right)\, (2m-r+2)!}\, C_{2m-2r+1}(x). \tag{7}$$

Proof. First, we can assume that the inversion formula of (4) is of the form

$$x^{2m} = \sum_{r=0}^{m} F_{r,m}\, C_{2m-2r}(x), \tag{8}$$

and we have to determine explicitly the coefficients $F_{r,m}$. If we multiply both sides of (8) by $C_{2k}(x)$, $k \geq 0$, integrate over the interval $[-1,1]$, and use the orthogonality relation (3), then the coefficients $F_{r,m}$ can be computed using the following integral form

$$F_{r,m} = \frac{1}{2^{-1-4m+4r}\,\pi} \int_{-1}^{1} \frac{x^{2m+2}}{\sqrt{1-x^2}} C_{2m-2r}(x)\, dx. \tag{9}$$

The power form representation in (4) enables one to convert the right-hand side of (9) into the form

$$F_{r,m} = \frac{2m-2r+1}{2^{-1-4m+4r}\pi} \sum_{\ell=0}^{m-r} \frac{(-1)^{\ell}\,(2m-2r-\ell)!}{2^{2\ell}\, \ell!\, (2m-2\ell-2r+1)!} \int_{-1}^{1} \frac{x^{4m-2\ell-2r+2}}{\sqrt{1-x^2}}\, dx.$$

With the aid of the identity

$$\int_{-1}^{1} \frac{x^{4m-2\ell-2r+2}}{\sqrt{1-x^2}}\, dx = \frac{\sqrt{\pi}\, \Gamma\left(\frac{3}{2}-\ell+2m-r\right)}{(2m-r-\ell+1)!},$$

the coefficients $F_{r,m}$ can be written as

$$F_{r,m} = \frac{(2m-2r+1)}{2^{-1-4m+4r}\,\sqrt{\pi}} \sum_{\ell=0}^{m-r} \frac{(-1)^{\ell}\,(2m-2r-\ell)!\,\Gamma\left(\frac{3}{2}-\ell+2m-r\right)}{2^{2\ell}\,\ell!\,(2m-2\ell-2r+1)!\,(2m-r-\ell+1)!}.$$

Now, regarding the sum

$$S_j = \sum_{\ell=0}^{j} \frac{(-1)^{\ell}(2j-\ell)!\,\Gamma\left(\frac{3}{2}+2j-\ell+r\right)}{\ell!\,2^{2\ell}\,(2j-2\ell+1)!(2j-\ell+r+1)!},$$

it can be written in a closed form with the aid of Zeilberger's algorithm [34]. In fact, it can be demonstrated that S_j obeys the following recurrence relation of order one

$$S_{j+1} - \frac{(2j+1)(r+j+1)(2j+2r+3)}{8(2j+3)(r+2j+2)(r+2j+3)}\, S_j = 0, \quad S_0 = \frac{\Gamma\left(r+\frac{3}{2}\right)}{(r+1)!}. \tag{10}$$

The recurrence relation (10) is simple to solve to yield

$$S_j = \frac{(r+1)_j\, \Gamma\left(\frac{3}{2}+r+j\right)}{2^{2j}\,(2j+1)\,(2j+r+1)!},$$

and, hence, the coefficients $F_{r,m}$ take the form

$$F_{r,m} = \frac{(2m+1)!}{2^{2r}\, r!\,(2m-r+1)!},$$

therefore Formula (6) is proved. Formula (7) can be similarly proved. □

The following two lemmas are useful in the follow-up. In the first lemma, a trigonometric representation of the polynomials $C_j(x)$ is given, and in the second, an explicit expression for the polynomials $C_j(x)$ for $j<0$ is given in terms of their counterparts $C_k(x)$ for $k>0$. Thus $C_j(x)$ can be defined for all $j \in \mathbb{Z}$.

Lemma 2 ([11])**.** *For every integer j, we have*

$$C_j(\cos\theta) = \begin{cases} \dfrac{\cos((j+1)\theta)}{2^j\cos(\theta)}, & j \text{ even}, \\ \dfrac{\sec(\theta)\{(2+j)\,\cos((1+j)\theta) - \cos((2+j)\theta)\,\sec(\theta)\}}{j\,2^j}, & j \text{ odd}. \end{cases} \tag{11}$$

Lemma 3. *For $j<0$, we have*

$$C_j(x) = \begin{cases} \dfrac{1}{2^{2j+2}}\,C_{-j-2}(x), & j \text{ even}, \\ \dfrac{j+2}{j\,2^{2j+2}}\,C_{-j-2}(x), & j \text{ odd}. \end{cases} \tag{12}$$

Proof. Identity (12) is a straightforward result of the trigonometric identity (11). □

Regarding the first four kinds of Chebyshev polynomials, they are special polynomials of the Jacobi polynomials. More precisely, if the normalized orthogonal Jacobi polynomials (see [23]) are defined as:

$$R_j^{(\alpha,\beta)}(x) = {}_2F_1\left(\begin{matrix} -j, j+\alpha+\beta+1 \\ \alpha+1 \end{matrix}\,\middle|\,\frac{1-x}{2}\right),$$

with $\alpha > -1$ and $\beta > -1$, then six celebrated classes of orthogonal polynomials can be deduced from $R_j^{(\alpha,\beta)}(x)$ as follows:

$$\begin{aligned} T_j(x) &= R_j^{(-\frac{1}{2},-\frac{1}{2})}(x), & U_j(x) &= (j+1)\,R_j^{(\frac{1}{2},\frac{1}{2})}(x), \\ V_j(x) &= R_j^{(-\frac{1}{2},\frac{1}{2})}(x), & W_j(x) &= (2j+1)\,R_j^{(\frac{1}{2},-\frac{1}{2})}(x), \\ P_j(x) &= R_j^{(0,0)}(x), & U_j^{(\lambda)}(x) &= R_j^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x), \end{aligned}$$

where $T_j(x), U_j(x), V_j(x)$ and $W_j(x)$ are, respectively, the first-, second-, third-, and fourth-kinds of Chebyshev polynomials, while $P_j(x)$ and $U_j^{(\lambda)}(x)$ are, respectively, the Legendre and ultraspherical polynomials. A comprehensive survey on Jacobi polynomials and their special ones can be found in Andrews et al. [39] and Mason and Handscomb [40].

It is worthy to mention here that the first four kinds of Chebyshev polynomials have explicit trigonometric representations. In fact, if $\theta = \cos^{-1} x$, we have [40]

$$T_j(\cos\theta) = \cos(j\,\theta), \qquad U_j(\cos\theta) = \frac{\sin((j+1)\,\theta)}{\sin\theta}, \tag{13}$$

$$V_j(\cos\theta) = \frac{\cos\left(\left(j+\frac{1}{2}\right)\theta\right)}{\cos\left(\frac{\theta}{2}\right)}, \qquad W_j(\cos\theta) = \frac{\sin\left(\left(j+\frac{1}{2}\right)\theta\right)}{\sin\left(\frac{\theta}{2}\right)}. \tag{14}$$

We comment that the following relation:

$$W_j(x) = (-1)^j\,V_j(-x),$$

connects $Wj(x)$ with $Vj(x)$. The four kinds of Chebyshev polynomials can be generated by the unified recurrence relation:

$$\phi_j(x) = 2\,x\,\phi_{j-1}(x) - \phi_{j-2}(x), \quad j \geq 2,$$

but with the following different initial values:

$$T_0(x) = 1,\ T_1(x) = x, \quad U_0(x) = 1,\ U_1(x) = 2x,$$

$$V_0(x) = 1,\ V_1(x) = 2x - 1, \quad W_0(x) = 1,\ W_1(x) = 2x + 1.$$

It is worth mentioning here that the polynomials $\phi_j(x)$ can be defined for all $j \in \mathbb{Z}$. More precisely, we have the following identities for every non-negative integer j

$$T_{-j}(x) = T_j(x), \quad U_{-j}(x) = -U_{j-2}(x),$$

$$V_{-j}(x) = V_{j-1}(x), \quad W_{-j}(x) = -W_{j-1}(x).$$

The following lemma gives the unified moments formula of $\phi_j(x)$.

Lemma 4. *Let m and j be For any non-negative integers. We have*

$$x^m\,\phi_j(x) = \frac{1}{2^m}\sum_{s=0}^{m}\binom{m}{s}\,\phi_{j+m-2s}(x). \tag{15}$$

Proof. The proof can be completed using similar steps that followed in [41]. □

3. Moments Formulas of Chebyshev Polynomials of the Fifth-Kind

This section focuses on obtaining new moments formulas of Chebyshev polynomials of the fifth-kind. These formulas are the keys for deriving some other important formulas of these polynomials such as their linearization formulas. First, the following lemma is useful in the sequel.

Lemma 5. *Let j, ℓ, m be non-negative integers. The following reduction formula holds*

$$\begin{aligned}
&{}_4F_3\left(\begin{array}{c} -j, -\ell, -\frac{1}{2} - j, -1 - 2j + \ell - 2m \\ -2j, -\frac{1}{2} - j - m, -j - m \end{array}\middle|\, 1\right) \\
&= \begin{cases} \dfrac{(2m)!\,(2j - \ell + 2m + 1)!}{(2j + 2m + 1)!\,(2m - \ell)!}, & 0 \leq \ell \leq j, \\[2ex] \dfrac{(2m)!\left(\frac{\ell!}{(\ell-2j-1)!} + \frac{(2j-\ell+2m+1)!}{(2m-\ell)!}\right)}{(2j + 2m + 1)!}, & \ell \geq j + 1. \end{cases}
\end{aligned}$$

Proof. First, set

$$A_{\ell,m,j} = {}_4F_3\left(\begin{array}{c} -j, -\ell, -\frac{1}{2} - j, -1 - 2j + \ell - 2m \\ -2j, -\frac{1}{2} - j - m, -j - m \end{array}\middle|\, 1\right). \tag{16}$$

Due to the appearance of the two non-negative integers j and ℓ in the numerator parameters of the hypergeometric function (16), the following two cases should be taken into consideration:

(i) The case corresponds to $\ell \leq j$. Making use of Zeilberger's algorithm, we conclude that the following recurrence relation is satisfied by $A_{\ell,m,j}$

$$\begin{aligned}
&(-1+\ell)(-2+\ell-2m)\,A_{\ell-2,m,j} - 2\Big(2 + 2\,j - 3\,\ell - 2\,j\,\ell + \ell^2 + 2\,m + 2\,j\,m - 2\,\ell m\Big)\,A_{\ell-1,m,j} \\
&+ (-1 - 2\,j + \ell)(-2 - 2\,j + \ell - 2m)\,A_{\ell,m,j}, \quad A_{0,m,j} = 1,\ A_{1,m,j} = \frac{2m}{2j + 2m + 1}.
\end{aligned}$$

The exact solution of the last recurrence relation is

$$A_{\ell,m,j} = \frac{(2m)!\,(2j-\ell+2m+1)!}{(2j+2m+1)!\,(2m-\ell)!}.$$

(ii) The case corresponds to $\ell > j$. Making use of Zeilberger's algorithm again yields the following recurrence relation for $A_{\ell,m,j}$

$$\begin{aligned}
&(2-2j+\ell)(3-2j+\ell)(1-2j+\ell-2m)(2-2j+\ell-2m)(-2j+\ell-m)A_{\ell,m,j-2}\\
&-4(1-2j+\ell-m)(-1+j+m)(-1+2j+2m)\times\\
&\left(-4j+4j^2+2\ell-4j\ell+\ell^2-m+4jm-2\ell m+2m^2\right)A_{\ell,m,j-1}\\
&+4(2-2j+\ell-m)(-1+j+m)(j+m)(-1+2j+2m)(1+2j+2m)\,A_{\ell,m,j}=0,
\end{aligned}$$

with the initial values

$$A_{\ell,m,0}=1,\ A_{\ell,m,1}=\frac{3\,\ell\,(\ell-2m-3)}{2(m+1)\,(2m+3)}+1,$$

whose exact solution is explicitly given as:

$$A_{\ell,m,j} = \frac{(2m)!\left(\frac{\ell!}{(\ell-2j-1)!}+\frac{(2j-\ell+2m+1)!}{(2m-\ell)!}\right)}{(2j+2m+1)!}.$$

From the two cases (i) and (ii), the proof of Lemma 5 is now complete. □

Now, in the following theorem, we give the formula that expresses explicitly the moments of the polynomials $C_j(x)$ in terms of their original polynomials.

Theorem 1. *Let m and j be any non-negative integers. The following moments formula holds*

$$x^m\,C_j(x) = \sum_{\ell=0}^{m} S_{\ell,m,j}\,C_{j-2\,\ell+m}(x), \tag{17}$$

with the moments coefficients $A_{\ell,m}$ given by

$$S_{\ell,m,j} = \frac{m!}{2^{2\,\ell}\ell!\,(m-\ell)!}$$
$$\times\begin{cases}1, & m \text{ and } j \text{ even},\\[4pt] \dfrac{j^2(-1+m)+2\ell(1-2\ell+m)+j\left(-2-2\ell(-1+m)+m+m^2\right)}{j(2+j-2\ell+m)(m-1)}, & m \text{ even, } j \text{ odd},\\[4pt] \dfrac{-2\ell(1+m)+m(2+j+m)}{m(2+j-2\ell+m)}, & m \text{ odd, } j \text{ even},\\[4pt] \dfrac{2\ell+j\,m}{j\,m}, & m \text{ and } j \text{ odd}.\end{cases}$$

Proof. The four formulas that make up (17) are as follows:

$$
\begin{aligned}
x^{2m}\, C_{2j}(x) =&(2m)! \sum_{\ell=0}^{2m} \frac{1}{2^{2\,\ell}\, \ell!\, (2m-\ell)!}\, C_{2j+2m-2\ell}(x),\\
x^{2m}\, C_{2j+1}(x) =&\frac{m}{2j+1} \sum_{\ell=0}^{2m+1} \frac{(1-\ell+2m)_{\ell-2}}{\ell!\, 2^{2\,\ell-1}\, (3+2j-2\ell+2m)} \{ -3-4(-1+\ell)\ell+4m(1+m)\\
&-4\,j(-2+\ell-m)(-1+2m)-4\,j^2(1-2\,m)\}\, C_{2\,j+2\,m-2\,\ell+1}(x),\\
x^{2m+1}\, C_{2j}(x) =&m \sum_{\ell=0}^{2m+1} \frac{(2-\ell+2m)_{\ell-2}}{\ell!\, 2^{2\ell-1}\, (3+2j-2\ell+2m)} \times\\
&\{3-4\ell(1+m)+4m(2+m)+2\,j(1+2m)\}\, C_{2j+2m-2\,\ell+1}(x),\\
x^{2m+1}\, C_{2j+1}(x) =&\frac{(2m)!}{2j+1} \sum_{\ell=0}^{2m+1} \frac{(1+2\ell+2m+2\,j(1+2m))}{\ell!\, 2^{2\,\ell}\, (2m-\ell+1)!}\, C_{2j+2m-2\,\ell+2}(x).
\end{aligned} \tag{18}
$$

The proofs of the four formulas are similar, we prove only Formula (18). The power form expression of $C_{2j}(x)$ in (4) yields the following formula

$$
x^{2m}\, C_{2j}(x) = (2j+1) \sum_{r=0}^{j+m} \frac{(-1)^r\, (2j-r)!}{2^{2r}\, (2j-2r+1)!\, r!}\, x^{2j+2m-2r},
$$

which can be turned with the aid of relation (6) into the form

$$
\begin{aligned}
x^{2m}\, C_{2j}(x) =&(2j+1) \sum_{r=0}^{j+m} \frac{(-1)^r\, (2j-r)!\, (2j+2m-2r+1)!}{2^{2r}\, (2j-2r+1)!\, r!} \times\\
&\sum_{\ell=0}^{j+m-r} \frac{1}{2^{2\ell}\, \ell!\, (2j+2m-2r-\ell+1)!}\, C_{2j+2m-2\ell-2r}(x).
\end{aligned}
$$

Performing some manipulations on the last formula converts it into the following hypergeometric expression

$$
\begin{aligned}
x^{2m}\, C_{2j}(x) =&(2j+2m+1)! \sum_{\ell=0}^{j+m} \frac{1}{2^{2\ell}\, \ell!\, (2j-\ell+2m+1)!} \times\\
&{}_4F_3\left(\left. \begin{array}{c} -j, -\ell, -\frac{1}{2}-j, -1-2j+\ell-2m \\ -2j, -\frac{1}{2}-j-m, -j-m \end{array} \right| 1 \right) C_{2j+2m-2\ell}(x).
\end{aligned} \tag{19}
$$

The result in Lemma 5 enables one to reduce the hypergeometric form ${}_4F_3(1)$ that appears in (19). Hence, Formula (19) is turned into

$$
x^{2m}\, C_{2j}(x) = \sum_{\ell=0}^{m+j} B_{\ell,m,j} C_{2m+2j-2\ell}(x),
$$

where

$$
B_{\ell,m,j} = \begin{cases} \dfrac{(2m)!}{\ell!\, 2^{2\,\ell} (2m-\ell)!}, & 0 \le \ell \le j,\\[2ex] 4^{-\ell} (2m)! \left(\dfrac{1}{\ell!\, (-\ell+2m)!} + \dfrac{1}{(\ell-2j-1)!\, (2j-\ell+2m+1)!} \right), & j+1 \le \ell \le j+m, \end{cases}
$$

and accordingly, we have

$$x^{2m}\,C_{2j}(x)=\sum_{\ell=0}^{m-j-1}\left\{\frac{4^{-j+\ell-m}(2m)!}{(j-\ell+m)!(-j+\ell+m)!}+\frac{4^{-j+\ell-m}(2m)!}{(-j-\ell+m-1)!(j+\ell+m+1)!}\right\}C_{2\ell}(x)$$
$$+\sum_{\ell=m-j}^{m+j}\frac{4^{-j+\ell-m}(2m)!}{(j-\ell+m)!(-j+\ell+m)!}\,C_{2\ell}(x),$$

which is equivalent to

$$x^{2m}\,C_{2j}(x)=\sum_{\ell=0}^{j+m}\frac{4^{-j+\ell-m}\,(2m)!}{(j-\ell+m)!\,(-j+\ell+m)!}\,C_{2\ell}(x)+\sum_{\ell=1}^{m-j}\frac{4^{-1-j+\ell-m}\,(2m)!}{(-j-\ell+m)!\,(j+\ell+m)!}\,C_{2\ell-2}(x). \tag{20}$$

Now, based on Lemma 3, the following identity is valid

$$C_{2\ell-2}(x)=2^{-4\ell+2}\,C_{-2\ell}(x),\quad \ell\geq 1,$$

and, therefore, Formula (20) can be written as

$$\begin{aligned}x^{2m}\,C_{2j}(x)&=\sum_{\ell=0}^{j+m}\frac{4^{-j+\ell-m}\,(2m)!}{(j-\ell+m)!\,(-j+\ell+m)!}\,C_{2\ell}(x)+\sum_{\ell=1}^{m-j}\frac{2^{-2(j+\ell+m)}\,(2m)!}{(-j-\ell+m)!\,(j+\ell+m)!}\,C_{-2\ell}(x)\\&=\sum_{\ell=0}^{j+m}\frac{(2m)!}{2^{2\ell}\,\ell!(2m-\ell)!}\,C_{2j+2m-2\ell}(x)+\sum_{\ell=j+m+1}^{2m}\frac{(2m)!}{2^{2\ell}\,\ell!\,(2m-\ell)!}\,C_{2j+2m-2\ell}(x)\\&=(2m)!\sum_{\ell=0}^{2m}\frac{1}{2^{2\ell}\,\ell!\,(2m-\ell)!}\,C_{2j+2m-2\ell}(x).\end{aligned}$$

This ends the proof. □

4. Linearization Formulas of Chebyshev Polynomials of the Fifth-Kind

In this section, and based on the moments formulas that obtained in Section 3, the linearization formulas of Chebyshev polynomials of the fifth-kind are given. Furthermore, the products of Chebyshev polynomials of the fifth-kind and the first four kinds of Chebyshev polynomials are linearized.

Theorem 2. *Let i and j be any non-negative integers. The following linearization formula holds*

$$C_i(x)\,C_j(x)=\sum_{p=0}^{\min(i,j)}L_{p,i,j}\,C_{i+j-2p}(x), \tag{21}$$

where the linearization coefficients $L_{p,i,j}$ are given explicitly by

$$L_{p,i,j}=\left(\frac{-1}{4}\right)^{p}\begin{cases}1, & i \text{ and } j \text{ even},\\ \dfrac{j(2+i+j)-2(1+i+j)p+2p^2}{j(2+i+j-2p)}, & i \text{ even, } j \text{ odd},\\ \dfrac{i(2+i+j)-2(1+i+j)p+2p^2}{i(2+i+j-2p)}, & i \text{ odd, } j \text{ even},\\ \dfrac{ij-2(1+i+j)p+2p^2}{ij}, & i \text{ and } j \text{ odd}.\end{cases}$$

Proof. First, we consider the case in which both i and j are even. That is, we have to show the following linearization formula:

$$C_{2i}(x)\, C_{2j}(x) = \sum_{p=0}^{2i} \left(\frac{-1}{4}\right)^p C_{2i+2j-2p}(x). \tag{22}$$

Based on relation (4), we can write

$$C_{2i}(x)\, C_{2j}(x) = (1+2i) \sum_{r=0}^{i} \frac{(-1)^r\,(2i-r)!}{r!\,2^{2r}\,(1+2i-2r)!}\, x^{2i-2r}\, C_{2j}(x).$$

Making use of the moments formula in (18), the last formula can be transformed into the form

$$C_{2i}(x)\, C_{2j}(x) = (1+2i) \sum_{r=0}^{i} \frac{(-1)^r\,(2i-r)!}{2^{2r}\,r!\,(1+2i-2r)} \sum_{\ell=0}^{2j-2r} \frac{1}{\ell!\,2^{2\ell}(2i-\ell-2r)!}\, C_{2i+2j-2r-2\ell}(x). \tag{23}$$

Some computations on the right-hand side of (23) enable one to convert it into the form

$$C_{2i}(x)\, C_{2j}(x) = (1+2i) \sum_{p=0}^{2i} \frac{(-1)^p}{2^{2p}} \left\{ \sum_{\ell=0}^{p} \frac{(-1)^\ell\,(-2i+\ell)_p}{(1+2i-2\ell)\,\ell!\,(p-\ell)!} \right\} C_{2i+2j-2p}(x),$$

but it can be shown that

$$\sum_{\ell=0}^{p} \frac{(-1)^\ell\,(-2i+\ell)_p}{(1+2i-2\ell)\,\ell!\,(p-\ell)!} = \frac{(-1)^p\,(2i)!}{(1+2i)(2i-p)!\,p!}\, {}_3F_2\left(\left. \begin{matrix} -p, -\frac{1}{2}-i, -2i+p \\ \frac{1}{2}-i, -2i \end{matrix} \right| 1 \right),$$

and, therefore, the following linearization formula can be obtained

$$C_{2i}(x)\, C_{2j}(x) = (2i)! \sum_{p=0}^{2i} \frac{1}{p!\,2^{2p}\,(2i-p)!}\, {}_3F_2\left(\left. \begin{matrix} -p, -\frac{1}{2}-i, -2i+p \\ \frac{1}{2}-i, -2i \end{matrix} \right| 1 \right) C_{2i+2j-2p}(x). \tag{24}$$

The balanced ${}_3F_2(1)$ in (24) can be summed by Pfaff-Saalschütz identity (see, Olver et al. [42]) to give

$${}_3F_2\left(\left. \begin{matrix} -p, -\frac{1}{2}-i, -2i+p \\ \frac{1}{2}-i, -2i \end{matrix} \right| 1 \right) = \frac{(-1)^p\,p!\,(2i-p)!}{(2i)!},$$

and, therefore, the linearization Formula (24) reduces into the following form

$$C_{2i}(x)\, C_{2j}(x) = \sum_{p=0}^{2i} \left(\frac{-1}{4}\right)^p C_{2i+2j-2p}(x).$$

Similarly, we can prove the following three linearization formulas

$$C_{2i}(x)\, C_{2j+1}(x) = \frac{1}{1+2j} \sum_{p=0}^{2i} \frac{(-1)^p\,\left((1+2j)(3+2i+2j) - 4(1+i+j)p + 2p^2\right)}{2^{2p}\,(3+2i+2j-2p)}\, C_{2i+2j-2p+1},$$

$$C_{2i+1}(x)\, C_{2j}(x) = \frac{1}{1+2i} \sum_{p=0}^{2i+1} \frac{(-1)^p\,\left((1+2i)\,(3+2i+2j) - 4\,(1+i+j)\,p + 2p^2\right)}{2^{2p}\,(3+2i+2j-2p)}\, C_{2i+2j-2p+1}(x),$$

$$C_{2i+1}(x)\, C_{2j+1}(x) = \frac{1}{(1+2i)\,(1+2j)} \times \sum_{p=0}^{2i+1} \left(\frac{-1}{4}\right)^p \left(1+2j-6p+2\Big(i+2ij-2(i+j)p+p^2\Big)\right) C_{2i+2j-2p+2}(x).$$

Merging the above three linearization formulas along with Formula (22), the linearization Formula (21) can be obtained. □

The next theorem gives a formula that linearizes the product of Chebyshev polynomials of the fifth-kind with any one of the first four kinds of Chebyshev polynomials.

Theorem 3. *Let i and j be any two non-negative integers and let $\phi_j(x)$ by any kind of the first four kinds of Chebyshev polynomials. The following linearization formulas hold*

$$
\begin{aligned}
C_{2\,i}(x)\,\phi_j(x) &= \frac{1}{2^{2i}}\sum_{p=0}^{2i}(-1)^p\,\phi_{j+2i-2p}(x),\\
C_{2\,i+1}\,\phi_j(x) &= \frac{1}{(2i+1)\,2^{2i+1}}\sum_{p=0}^{2i+1}(-1)^p\,(1+2i-2p)\,\phi_{j+2i-2p+1}(x).
\end{aligned}
$$

Proof. The proof is based on the application of the unified moments formula of $\phi_j(x)$ in (15) along with the power form representations (4) and (5). □

5. Connection Formulas with Some Orthogonal Polynomials

This section is devoted to solving the connection problems between Chebyshev polynomials of the fifth-kind and some other orthogonal polynomials. Among these connections are the connection formulas between Chebyshev polynomials of the fifth-kind and the first four kinds of Chebyshev polynomials. The following theorem exhibits the fifth-kind Chebyshev-ultraspherical connection formulas.

Theorem 4. *Let j be any non-negative integer and let the $U_j^{(\lambda)}(x)$ be the ultraspherical polynomials. The fifth-kind Chebyshev and ultraspherical polynomials are connected with each other by the following two formulas:*

$$
\begin{aligned}
C_{2j}(x) =& \frac{(2j)!\,\Gamma(1+\lambda)}{2^{2j-1}\,\Gamma(1+2\lambda)}\sum_{\ell=0}^{j}\frac{(2j-2\ell+\lambda)\,\Gamma(2(j-\ell+\lambda))}{(2j-2\ell)!\,\ell!\,\Gamma(1+2j-\ell+\lambda)}\times\\
&{}_3F_2\left(\left.\begin{array}{c}-\ell,-j-\frac{1}{2},-2j-\lambda+\ell\\ -2j,\frac{1}{2}-j\end{array}\right|1\right)U^{(\lambda)}_{2j-2\ell}(x),
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
C_{2j+1}(x) =& \frac{4^{-j-\lambda}\sqrt{\pi}\,(1+2j)!}{\Gamma\left(\frac{1}{2}+\lambda\right)}\sum_{\ell=0}^{j}\frac{(1+2j-2\ell+\lambda)\Gamma(1+2j-2\ell+2\lambda)}{\ell!\,(2j-2\ell+1)!\,\Gamma(2+2j-\ell+\lambda)}\times\\
&{}_3F_2\left(\left.\begin{array}{c}-\ell,-j-\frac{3}{2},-2j-\lambda+\ell-1\\ -2j-1,-\frac{1}{2}-j\end{array}\right|1\right)U^{(\lambda)}_{2j-2\ell+1}(x).
\end{aligned}
\tag{26}
$$

Proof. First, assume the following connection formula

$$
C_{2j}(x)=\sum_{\ell=0}^{j}R_{\ell,j,\lambda}U^{(\lambda)}_{2j-2\ell}(x),
$$

where $R_{\ell,j,\lambda}$ are the connection coefficients that should be determined. If the orthogonality relation of ultraspherical polynomials is applied, then we can write

$$
R_{\ell,j,\lambda}=\frac{2^{1-2\lambda}\,(2j-2\ell+\lambda)\,\Gamma(2(j-\ell+\lambda))}{(2j-2\ell)!\,\Gamma\left(\frac{1}{2}+\lambda\right)^2}\int_{-1}^{1}(1-x^2)^{\lambda-\frac{1}{2}}\,C_{2j}(x)\,U^{(\lambda)}_{2j-2\ell}(x)\,dx,
$$

and in virtue of Rodrigues' formula for the ultraspherical polynomials [43], we obtain

$$R_{\ell,j,\lambda} = \frac{2^{1-2j+2\ell-2\lambda}\,(2j-2\ell+\lambda)\,\Gamma(2(j-\ell+\lambda))}{(2j-2\ell)!\,\Gamma\left(\frac{1}{2}+\lambda\right)\Gamma\left(\frac{1}{2}+2j-2\ell+\lambda\right)} \int_{-1}^{1}\left(1-x^2\right)^{2j-2\ell+\lambda-\frac{1}{2}} D^{2j-2\ell}\,C_{2j}(x)\,dx.$$

The application of the power form representation of $C_{2j}(x)$ in (4) turns the last relation into the form

$$R_{\ell,j,\lambda} = \frac{(1+2j)(2j-2\ell+\lambda)\,\Gamma(2(j-\ell+\lambda))}{(2j-2\ell)!\,\Gamma\left(\frac{1}{2}+\lambda\right)\Gamma\left(\frac{1}{2}+2j-2\ell+\lambda\right)} \sum_{s=0}^{j} \frac{(-1)^s\,2^{1-2j+2\ell-2s-2\lambda}\,(2j-s)!}{(1+2j-2s)\,(2\ell-2s)!s!}\times$$
$$\int_{-1}^{1}\left(1-x^2\right)^{2j-2\ell+\lambda-\frac{1}{2}} x^{2\ell-2s}\,dx.$$

It is not difficult to show that

$$\int_{-1}^{1}\left(1-x^2\right)^{2j-2\ell+\lambda-\frac{1}{2}} x^{2\ell-2s}\,dx = \frac{\Gamma\left(\frac{1}{2}+\ell-s\right)\Gamma\left(\frac{1}{2}+2j-2\ell+\lambda\right)}{\Gamma(1+2j-\ell-s+\lambda)},$$

and, therefore, the coefficients $R_{\ell,j}$ can be written in the form

$$\begin{aligned} R_{\ell,j,\lambda} =& \frac{2^{1-2j-2\lambda}\,(1+2j)\,\sqrt{\pi}\,(2j-2\ell+\lambda)\,\Gamma(2(j-\ell+\lambda))}{(2j-2\,\ell)!\,\Gamma\left(\frac{1}{2}+\lambda\right)}\times \\ & \sum_{s=0}^{\ell} \frac{(-1)^s\,(2j-s)!}{(1+2j-2s)\,(\ell-s)!\,s!\,\Gamma(1+2j-\ell-s+\lambda)}. \end{aligned} \tag{27}$$

The last summation in (27) can be written in terms of hypergeometric form as

$$\sum_{s=0}^{\ell} \frac{(-1)^s\,(2j-s)!}{(1+2j-2s)\,(\ell-s)!\,s!\,\Gamma(1+2j-\ell-s+\lambda)} =$$
$$\frac{(2j)!}{(1+2j)\,\ell!\,\Gamma(1+2j-\ell+\lambda)}\;{}_3F_2\left(\left.\begin{matrix}-\ell,-j-\frac{1}{2},-2j-\lambda+\ell\\ \frac{1}{2}-j,-2j\end{matrix}\right|1\right),$$

and, accordingly, the coefficients take the form

$$\begin{aligned} R_{\ell,j} =& \frac{2^{1-2j}\,(2j-2\ell+\lambda)\,(2j)!\,\Gamma(1+\lambda)\,\Gamma(2(j-\ell+\lambda))}{(2j-2\ell)!\,\ell!\,\Gamma(1+2j-\ell+\lambda)\,\Gamma(1+2\lambda)}\times \\ & {}_3F_2\left(\left.\begin{matrix}-\ell,-j-\frac{1}{2},-2j-\lambda+\ell\\ \frac{1}{2}-j,-2j\end{matrix}\right|1\right), \end{aligned}$$

and, accordingly, the connection Formula (25) can be obtained. Formula (26) can be proved in a similar way. □

Taking into consideration the three celebrated special polynomials of $U_j^{(\lambda)}(x)$, the following three connection formulas can be deduced.

Corollary 1. *The fifth-and first- kinds Chebyshev polynomials are connected with each other by the following formulas:*

$$\begin{aligned} C_{2j}(x) &= \frac{1}{2^{2j-1}} \sum_{\ell=0}^{j} (-1)^{\ell}\, \xi_{j-\ell}\, T_{2j-2\ell}(x), \\ C_{2j+1}(x) &= \frac{1}{2^{2j}\,(2j+1)} \sum_{\ell=0}^{j} (-1)^{\ell} (1+2j-2\ell)\, T_{2j-2\ell+1}(x), \end{aligned} \tag{28}$$

with

$$\xi_{\ell} = \begin{cases} \frac{1}{2}, & \ell = 0, \\ 1, & \ell > 0. \end{cases}$$

Proof. Substitution by $\lambda = 0$ into the connection Formula (25) yields

$$C_{2j}(x) = \frac{(2j)!}{2^{2j-1}} \sum_{\ell=0}^{j} \frac{\xi_{j-\ell}}{\ell!\,(2j-\ell)!}\ {}_3F_2\left(\left. \begin{array}{c} -\ell, -j-\frac{1}{2}, \ell-2j \\ \frac{1}{2}-j, -2j \end{array} \right| 1 \right). \tag{29}$$

The balanced ${}_3F_2(1)$ in (29) can be reduced making use of Pfaff-Saalschütz identity to give

$${}_3F_2\left(\left. \begin{array}{c} -\ell, -j-\frac{1}{2}, \ell-2j \\ \frac{1}{2}-j, -2j \end{array} \right| 1 \right) = \frac{(-1)^{\ell}\,\ell!}{(2j-\ell+1)_{\ell}},$$

and, consequently, the following connection formula can be obtained:

$$C_{2j}(x) = \frac{1}{2^{2j-1}} \sum_{\ell=0}^{j} (-1)^{\ell}\, \xi_{j-\ell}\, T_{2j-2\ell}(x).$$

The connection formula in (28) can be similarly deduced. □

Corollary 2. *The fifth-and second- kinds Chebyshev polynomials are connected with each other by the following formulas:*

$$\begin{aligned} C_{2j}(x) &= \frac{1}{2^{2j-1}} \sum_{\ell=0}^{j} (-1)^{\ell}\, \xi_{\ell}\, U_{2j-2\ell}(x), \\ C_{2j+1}(x) &= \frac{1}{2^{2j+1}}\, U_{2j+1}(x) + \frac{1}{2^{2j-1}\,(2j+1)} \sum_{\ell=1}^{j} (-1)^{\ell} (1+j-\ell) U_{2j-2\ell+1}(x). \end{aligned} \tag{30}$$

Proof. Substitution by $\lambda = 1$ into the connection Formula (25) yields

$$C_{2j}(x) = \frac{(2j)!}{2^{2j}} \sum_{\ell=0}^{j} \frac{1+2j-2\ell}{\ell!(2j-\ell+1)!}\ {}_3F_2\left(\left. \begin{array}{c} -\ell, -\frac{1}{2}-j, -1-2j+\ell \\ \frac{1}{2}-j, -2j \end{array} \right| 1 \right) U_{2j-2\ell}(x).$$

Now, if we set

$$A_{\ell,j} = {}_3F_2\left(\left. \begin{array}{c} -\ell, -\frac{1}{2}-j, -1-2j+\ell \\ \frac{1}{2}-j, -2j \end{array} \right| 1 \right),$$

then it can be shown that the ${}_3F_2(1)$ in the last formula satisfies the following recurrence relation of order one:

$$(1+2j-2\ell)(1+\ell)\, A_{\ell,j} + \xi_{\ell}(-1+2j-2\ell)(1+2j-\ell)\, A_{\ell+1,j} = 0, \quad A_{0,j} = 1,$$

which can be solved quickly to provide

$$A_{\ell,j} = \frac{2\,\xi_\ell\,(-1)^\ell\,(1+2j)\,\ell!}{(2j-2\ell+1)\,(2j-\ell+2)_\ell},$$

and, therefore, the following connection formula holds

$$C_{2j}(x) = \frac{1}{2^{2j-1}}\sum_{\ell=0}^{j}(-1)^\ell\,\xi_\ell\,U_{2j-2\ell}(x).$$

The connection Formula (30) can be similarly proved. □

Corollary 3. *The fifth-kind Chebyshev and Legendre polynomials are connected with each other by the following formulas:*

$$C_{2j}(x) = \frac{\sqrt{\pi}\,(2j)!}{2^{2j+1}}\sum_{\ell=0}^{j}\frac{1+4j-4\ell}{\ell!\,\Gamma\left(\frac{3}{2}+2j-\ell\right)}\,{}_3F_2\left(\left.\begin{array}{c}-j-\frac{1}{2},-\ell,-2j+\ell-\frac{1}{2}\\ \frac{1}{2}-j,-2j\end{array}\right|1\right)P_{2j-2\ell}(x), \tag{31}$$

$$\begin{aligned}C_{2j+1}(x) = &\frac{\sqrt{\pi}\,(1+2j)!}{2^{2j+2}}\sum_{\ell=0}^{j}\frac{(3+4j-4\ell)}{\ell!\,\Gamma\left(\frac{5}{2}+2j-\ell\right)}\times\\ &{}_3F_2\left(\left.\begin{array}{c}-\ell,-j-\frac{3}{2},-2j+\ell-\frac{3}{2}\\ -2j-1,-j-\frac{1}{2}\end{array}\right|1\right)P_{2j-2\ell+1}(x).\end{aligned} \tag{32}$$

Proof. The substitution by $\lambda = \frac{1}{2}$ into the connection Formulas (25) and (26) yields, respectively, the two connection Formulas (31) and (32). □

In the following, we give the inversion connection formulas to the connection Formulas (25) and (26) and their special ones. First, the following lemma is needed.

Lemma 6. *For every non-negative integers ℓ and j, the following reduction holds:*

$${}_3F_2\left(\left.\begin{array}{c}-\ell,-\frac{1}{2}-j,-2-2j+\ell\\ -\frac{3}{2}-j,-2j-\lambda\end{array}\right|1\right) = \frac{(\lambda)_{\ell-1}\,(2j\lambda+4j\ell-2j+3\lambda-2\ell^2+4\ell-3)}{(2j+3)(2j+\lambda)(2j-\ell+\lambda+1)_{\ell-1}}.$$

Proof. If we set

$$S_{\ell,j} = {}_3F_2\left(\left.\begin{array}{c}-\ell,-\frac{1}{2}-j,-2-2j+\ell\\ -\frac{3}{2}-j,-2j-\lambda\end{array}\right|1\right),$$

after that, using Zeilberger's algorithm, the following recurrence relation for $S_{\ell,j}$ can be obtained

$$\begin{aligned}&(\lambda+\ell-2)\left(-2j\lambda-4j\ell+2j-3\lambda+2\ell^2-4\ell+3\right)S_{\ell-1,j}\\ &+(-2j-\lambda+\ell-1)\left(-2j\lambda-4j\ell+6j-3\lambda+2\ell^2-8\ell+9\right)S_{\ell,j}=0, \quad S_{0,j}=1.\end{aligned}$$

The above recurrence relation can be immediately solved to give

$$S_{\ell,j} = \frac{(\lambda)_{\ell-1}\,(2j\lambda+4j\ell-2j+3\lambda-2\ell^2+4\ell-3)}{(2j+3)(2j+\lambda)(2j-\ell+\lambda+1)_{\ell-1}}.$$

This proves Lemma 6. □

Theorem 5. *The ultraspherical-fifth-kind Chebyshev connection formulas are*

$$\begin{aligned} U_{2j}^{(\lambda)}(x) =& \frac{(2j)!\,\Gamma\left(\frac{1}{2}+\lambda\right)}{\sqrt{\pi}\,\Gamma(\lambda)\,\Gamma(2(j+\lambda))} \sum_{\ell=0}^{j} \frac{2^{-1+2j-2\ell+2\lambda}\,\Gamma(2j-\ell+\lambda)\,\Gamma(-1+\ell+\lambda)}{\ell!\,(2j-\ell+1)!} \times \\ & \Big\{-1-2(-1+\ell)\,\ell+\lambda+2j\,(-1+2\ell+\lambda)\Big\}\, C_{2j-2\ell}(x). \end{aligned} \tag{33}$$

$$\begin{aligned} U_{2j+1}^{(\lambda)}(x) =& \frac{(2j+1)!\,\Gamma\left(\frac{1}{2}+\lambda\right)}{\sqrt{\pi}\,\Gamma(\lambda)\,\Gamma(1+2j+2\lambda)} \sum_{\ell=0}^{j} \frac{2^{2(j-\ell+\lambda)}\,\Gamma(1+2j-\ell+\lambda)\,\Gamma(-1+\ell+\lambda)\,(1+j-\ell)}{\left(\frac{3}{2}+j-\ell\right)\ell!\,(2j-\ell+2)!} \times \\ & \times \Big\{-3+4\ell-2\ell^2+3\lambda+2j(-1+2\ell+\lambda)\Big\}\; C_{2j-2\ell+1}(x). \end{aligned} \tag{34}$$

Proof. We prove Formula (34). From the power form representation of the ultraspherical polynomials, we have

$$U_{2j+1}^{(\lambda)}(x) = \frac{(2j+1)!\,\Gamma(1+2\lambda)}{\Gamma(1+\lambda)\,\Gamma(1+2j+2\lambda)} \sum_{r=0}^{j} \frac{(-1)^r\,2^{2(j-r)}\,\Gamma(1+2j-r+\lambda)}{r!\,(2j-2r+1)!}\,x^{2j-2r+1}. \tag{35}$$

In virtue of relation (7), Formula (35) can be transformed into

$$\begin{aligned} U_{2j+1}^{(\lambda)}(x) =& \frac{(2j+1)!\,\Gamma(1+2\lambda)}{\Gamma(1+\lambda)\,\Gamma(1+2j+2\lambda)} \sum_{r=0}^{j} \frac{(-1)^r\,2^{1+2j-2r}\,(3+2j-2r)\,\Gamma(1+2j-r+\lambda)}{r!} \times \\ & \sum_{\ell=0}^{j-r} \frac{4^{-\ell}\,(-1-j+\ell+r)}{(-3-2j+2\ell+2r)\ell!(2j-\ell-2r+2)!}\; C_{2j-2\ell-2r+1}(x). \end{aligned}$$

The last formula can be transformed into the following one by certain algebraic computations.

$$\begin{aligned} U_{2j+1}^{(\lambda)}(x) =& \frac{(2j+1)!\,\Gamma\left(\frac{1}{2}+\lambda\right)}{\sqrt{\pi}\,\Gamma(1+2j+2\lambda)} \sum_{\ell=0}^{j} \frac{2^{1+2j-2\ell+2\lambda}\,(1+j-\ell)}{3+2j-2\ell} \times \\ & \left\{\sum_{p=0}^{\ell} \frac{(-1)^p\,(3+2j-2p)\,\Gamma(1+2j-p+\lambda)}{(2j-\ell-p+2)!\,p!\,(\ell-p)!}\right\} C_{2j-2\ell+1}(x). \end{aligned}$$

However, it can be shown that

$$\begin{aligned} \sum_{p=0}^{\ell} \frac{(-1)^p\,(3+2j-2p)\,\Gamma(1+2j-p+\lambda)}{(2j-\ell-p+2)!\,p!\,(\ell-p)!} =& \frac{(3+2j)\Gamma(1+2j+\lambda)}{\ell!\,(2j-\ell+2)!} \\ \times\; {}_3F_2\left(\left.\begin{matrix} -\ell,-\frac{1}{2}-j,-2-2j+\ell \\ -\frac{3}{2}-j,-2j-\lambda \end{matrix}\right|1\right), \end{aligned}$$

and, therefore, the following connection formula is obtained:

$$\begin{aligned} U_{2j+1}^{(\lambda)}(x) =& \frac{(2j+3)\,(2j+1)!\,\Gamma\left(\frac{1}{2}+\lambda\right)\Gamma(1+2j+\lambda)}{\sqrt{\pi}\,\Gamma(1+2j+2\lambda)} \sum_{\ell=0}^{j} \frac{2^{1+2j-2\ell+2\lambda}(1+j-\ell)}{(3+2j-2\ell)\,\ell!\,(2j-\ell+2)!} \times \\ & \times\; {}_3F_2\left(\left.\begin{matrix} -\ell,-\frac{1}{2}-j,-2-2j+\ell \\ -\frac{3}{2}-j,-2j-\lambda \end{matrix}\right|1\right). \end{aligned}$$

Now, based on Lemma 6, and after performing some calculations, the following connection formula can be obtained:

$$U_{2j+1}^{(\lambda)}(x) = \frac{(2j+1)!\,\Gamma\left(\frac{1}{2}+\lambda\right)}{\sqrt{\pi}\,\Gamma(\lambda)\,\Gamma(1+2j+2\lambda)} \sum_{\ell=0}^{j} \frac{2^{2(j-\ell+\lambda)}\,\Gamma(1+2j-\ell+\lambda)\,\Gamma(-1+\ell+\lambda)\,(1+j-\ell)}{\left(\frac{3}{2}+j-\ell\right)\,\ell!\,(2j-\ell+2)!} \times$$
$$\times\left\{-3+4\ell-2\ell^2+3\lambda+2j(-1+2\ell+\lambda)\right\}\,C_{2j-2\ell+1}(x).$$

This proves Formula (34). Formula (33) can be similarly obtained. □

The following special connection formulas can be also deduced as special cases of Theorem 5.

Corollary 4. *The first-fifth kinds Chebyshev connection formulas are:*

$$T_{2j}(x) = 2^{-3+2j}\left(4\,C_{2j}(x)+C_{2j-2}(x)\right), \tag{36}$$

$$T_{2j+1}(x) = 4^{j-1}\left(4\,C_{2j+1}(x)+\frac{-1+2j}{1+2j}C_{2j-1}(x)\right). \tag{37}$$

Proof. If we substitute by $\lambda = 0$ into Formulas (33) and (34), then the connection Formulas (36) and (37) can be obtained. □

Corollary 5. *The second-fifth kinds Chebyshev connection formulas are:*

$$U_{2j}(x) = \sum_{\ell=0}^{j} 2^{1+2j-2\ell}\,\xi_\ell\,C_{2j-2\ell}(x), \tag{38}$$

$$U_{2j+1}(x) = 2^{2j+1}\,C_{2j+1}(x) + \sum_{\ell=1}^{j}\frac{2^{2j-2\ell+3}(1+j-\ell)}{3+2j-2\ell}C_{2j-2\ell+1}(x). \tag{39}$$

Proof. If we substitute by $\lambda = 1$ into Formulas (33) and (34), then the connection Formulas (38) and (39) can be obtained. □

Corollary 6. *The Legendre-fifth kind Chebyshev connection formulas are:*

$$P_{2j}(x) = \frac{1}{\pi}\sum_{\ell=0}^{j}\frac{2^{-1+2j-2\ell}\left(-(1-2\ell)^2-2\,j(1-4\,\ell)\right)\Gamma\left(\frac{1}{2}+2j-\ell\right)\Gamma\left(-\frac{1}{2}+\ell\right)}{\ell!\,(2j-\ell+1)!}C_{2j-2\ell}(x), \tag{40}$$

$$P_{2j+1}(x) = \frac{1}{\pi}\sum_{\ell=0}^{j}\frac{2^{1+2j-2\ell}\,(1+j-\ell)\,\Gamma\left(\frac{3}{2}+2j-\ell\right)\Gamma\left(-\frac{1}{2}+\ell\right)}{(3+2j-2\ell)\,\ell!\,(2j-\ell+2)!}\times$$
$$\left\{-3-4(-2+\ell)\ell-2\,j(1-4\,\ell)\right\}C_{2j-2\ell+1}(x). \tag{41}$$

Proof. If we substitute by $\lambda = \frac{1}{2}$ into Formulas (33) and (34), then the connection Formulas (40) and (41) can be obtained. □

Now, we give the connection formulas between the fifth-kind Chebyshev polynomials and the class of Jacobi polynomials $R_k^{(\alpha,\alpha+1)}(x)$ that generalize the third- and fourth- kinds Chebyshev polynomials.

Theorem 6. *Let j be any non-negative integer, we have*

$$C_{2j}(x) = \sum_{\ell=0}^{j} H_{\ell,j,\alpha}\, R_{2j-2\ell}^{(\alpha,\alpha+1)}(x) + 2\sum_{\ell=0}^{j-1} \frac{j-\ell}{1+2j-2\ell+2\alpha}\, H_{\ell,j,\alpha}\, R_{2j-2\ell-1}^{(\alpha,\alpha+1)}(x), \tag{42}$$

with

$$H_{\ell,j,\alpha} = \frac{2^{-1-2j-2\alpha}\sqrt{\pi}\,(2j)!\,\Gamma(2(1+j-\ell+\alpha))}{\ell!\,(2j-2\ell)!\,\Gamma(1+\alpha)\,\Gamma\left(\frac{3}{2}+2j-\ell+\alpha\right)}\, {}_3F_2\left(\left.\begin{array}{c} -\ell, -\frac{1}{2}-j, -\frac{1}{2}-2j+\ell-\alpha \\ \frac{1}{2}-j, -2j \end{array}\right| 1\right),$$

and

$$\begin{aligned} C_{2j+1}(x) = \sum_{r=0}^{j} &\frac{4^{-1-j-\alpha}\sqrt{\pi}\,(2j+1)!\,\Gamma(3+2j-2r+2\alpha)}{r!\,(2j-2r+1)!\,\Gamma(1+\alpha)\,\Gamma\left(\frac{5}{2}+2j-r+\alpha\right)}\, {}_3F_2\left(\left.\begin{array}{c} -r, -\frac{3}{2}-j, -\frac{3}{2}-2j+r-\alpha \\ -1-2j, -\frac{1}{2}-j \end{array}\right| 1\right) \times \\ &\left\{ R_{2j-2r+1}^{(\alpha,\alpha+1)}(x) + \frac{1+2j-2r}{2(1+j-r+\alpha)}\, R_{2j-2r}^{(\alpha,\alpha+1)}(x) \right\}. \end{aligned} \tag{43}$$

Proof. The proof of (42) and (43) is based on making use of the the power form representations of the polynomials $C_j(x)$ in (4) and (5), and the inversion formula of the polynomials $R_k^{(\alpha,\alpha+1)}(x)$ that introduced in [44] in form free of any hypergeometric functions. □

As special cases of Theorem 6, the following connection formulas can be deduced.

Corollary 7. *Let j be a non-negative integer. The fifth-third kinds of Chebyshev connection formulas are:*

$$C_{2j}(x) = \frac{1}{2^{2j}} \left\{ (-1)^j + \sum_{r=0}^{j-1} (-1)^r \left\{ V_{2j-2r}(x) + V_{2j-2r-1}(x) \right\} \right\}, \tag{44}$$

$$C_{2j+1}(x) = \frac{1}{2^{2j+1}\,(2j+1)} \sum_{r=0}^{j} (-1)^r\,(1+2j-2r) \left\{ V_{2j-2r}(x) + V_{2j-2r+1}(x) \right\}, \tag{45}$$

while the fifth-fourth kinds Chebyshev connection formulas are:

$$C_{2j}(x) = \frac{1}{2^{2j}} \left\{ (-1)^j + \sum_{r=0}^{j-1} (-1)^r \left\{ W_{2j-2r}(x) - W_{2j-2r-1}(x) \right\} \right\}, \tag{46}$$

$$C_{2j+1}(x) = \frac{1}{2^{2j+1}\,(2j+1)} \sum_{r=0}^{j} (-1)^r\,(1+2j-2r) \left\{ W_{2j-2r+1}(x) - W_{2j-2r}(x) \right\}. \tag{47}$$

Proof. If we substitute by $\alpha = \frac{-1}{2}$, into the two connection Formulas (42) and (43), then the two connection Formulas (44) and (45) can be obtained. Moreover, if we take into consideration the two identities

$$C_j(-x) = (-1)^j\, C_j(x), \quad V_j(-x) = (-1)^j\, W_j(x), \tag{48}$$

then the two connection Formulas (46) and (47) are direct consequences of (44) and (45). □

Remark 1. *The connection formulas between the different Chebyshev polynomials can be translated into their corresponding trigonometric identities based on the trigonometric representations of Chebyshev polynomials. Some of these identities are presented in the corollary below.*

Corollary 8. *The following trigonometric identities are the translations to the connection Formulas (44)–(47), respectively.*

$$\sum_{r=0}^{j-1}(-1)^r\left(\cos\left(2j-2r+\tfrac{1}{2}\right)\theta+\cos\left(2j-2r-\tfrac{1}{2}\right)\theta\right)+(-1)^j\cos\left(\tfrac{\theta}{2}\right) = \cos\left(\tfrac{\theta}{2}\right)\sec(\theta)\cos((2j+1)\theta), \tag{49}$$

$$\sum_{r=0}^{j}(-1)^r(1+2j-2r)\left(\cos\left(2j-2r+\tfrac{1}{2}\right)\theta+\cos(2j-2r+\tfrac{3}{2})\theta\right) = ((3+2j)\cos(\theta)\cos(2(1+j)\theta)-\cos((3+2j)\theta))\,\sin\left(\tfrac{\theta}{2}\right)\sec^2(\theta), \tag{50}$$

$$\sum_{r=0}^{j-1}(-1)^r\left(\sin\left(2j-2r+\tfrac{1}{2}\right)\theta-\sin\left(2j-2r-\tfrac{1}{2}\right)\theta\right)+(-1)^j\sin\left(\tfrac{\theta}{2}\right) = \sin\left(\tfrac{\theta}{2}\right)\sec(\theta)\cos((2j+1)\theta), \tag{51}$$

$$\sum_{r=0}^{j}(-1)^r(1+2j-2r)\left(\sin(2j-2r+\tfrac{3}{2})\theta-\sin\left(2j-2r+\tfrac{1}{2}\right)\theta\right) = ((3+2j)\cos(\theta)\cos(2(1+j)\theta)-\cos((3+2j)\theta))\,\sin\left(\tfrac{\theta}{2}\right)\sec^2(\theta). \tag{52}$$

Proof. The trigonometric identities (49)–(52) are direct consequences of the connection Formulas (44)–(47). More precisely, If the trigonometric representations of the four kinds of Chebyshev polynomials in (13) and (14) along with the trigonometric representation of the fifth-kind Chebyshev polynomials in (11) are substituted into Formulas (44)–(47), then the four trigonometric identities (49),(50),(51) and (52) can be, respectively, obtained. □

Now, the following theorem gives the inversion formulas to Formulas (42) and (43).

Theorem 7. *Let j be any non-negative integer. We have*

$$R_{2j}^{(\alpha,\alpha+1)}(x)=\sum_{\ell=0}^{j}F_{\ell,j,\alpha}\,C_{2j-2\ell}(x)+\sum_{\ell=0}^{j-1}\bar{F}_{\ell,j,\alpha}\,C_{2j-2\ell-1}(x), \tag{53}$$

where

$$F_{\ell,j,\alpha}=\frac{4^{j-\ell+\alpha}\,\Gamma(1+\alpha)\,\Gamma(\tfrac{3}{2}+2j-\ell+\alpha)\,(2+2j-\ell)_{\ell-1}\,(\tfrac{3}{2}+\alpha)_{\ell-1}}{\sqrt{\pi}\,\ell!\,\Gamma(2(1+j+\alpha))}\times (1-4(\ell-1)\ell+2\alpha+j(2+8\ell+4\alpha)),$$

and

$$\bar{F}_{\ell,j,\alpha}=\frac{-2^{1+2j-2\ell+2\alpha}\,j\,(j-\ell)\,\Gamma(1+\alpha)\,\Gamma\left(\tfrac{1}{2}+2j-\ell+\alpha\right)\,(1+2j-\ell)_{\ell-1}\,(\tfrac{3}{2}+\alpha)_{\ell-1}}{(1+2j-2\ell)\sqrt{\pi}\,\ell!\,\Gamma(2(1+j+\alpha))}\times \left(1-4\ell^2+2\alpha+j(2+8\ell+4\alpha)\right).$$

Moreover,

$$R_{2j+1}^{(\alpha,\alpha+1)}(x)=\sum_{r=0}^{j}M_{r,j,\alpha}\left\{C_{2j-2r}(x)+\eta_{r,j,\alpha}\,C_{1+2j-2r}(x)\right\}, \tag{54}$$

with

$$M_{r,j,\alpha} = \frac{4^{j-r+\alpha}\,(1+2j)!\,\Gamma(1+\alpha)\,\Gamma\left(\frac{3}{2}+2j-r+\alpha\right)\Gamma\left(\frac{1}{2}+r+\alpha\right)}{\sqrt{\pi}\,(1+2j-r)!\,r!\,\Gamma\left(\frac{3}{2}+\alpha\right)\Gamma(3+2j+2\alpha)} \times (1-4(-1+r)r+2\alpha+2\,j\,(1+4\,r+2\,\alpha)),$$

and

$$\eta_{r,j,\alpha} = \frac{-2(1+j-r)(3+4j-2r+2\alpha)(3-4(-2+r)r+6\,\alpha+2\,j(1+4\,r+2\,\alpha))}{(3+2j-2r)(2+2j-r)(1-4(-1+r)r+2\alpha+2\,j\,(1+4\,r+2\,\alpha))}.$$

Proof. The proofs of (53) and (54) are based on making use of the the power form representation of the polynomials $R_k^{(\alpha,\alpha+1)}(x)$ that was introduced in [44] and the inversion formulas of $C_j(x)$ in (6) and (7). □

The following corollary displays the solutions of the third-fifth Chebyshev kinds and the fourth-fifth Chebyshev kinds connection problems. These connection formulas are spacial ones of those of Theorem 7.

Corollary 9. *Let j be a non-negative integer. The third-fifth kinds Chebyshev connection formulas are:*

$$\begin{aligned} V_{2j}(x) =& 2^{2j}C_{2j}(x) - 2^{2j-1}C_{2j-1}(x) + \sum_{\ell=1}^{j} 2^{2j-2\ell+1}\,C_{2j-2\ell}(x) \\ &- \sum_{\ell=1}^{j-1} \frac{2^{1+2j-2\ell}(j-\ell)}{1+2j-2\ell}\,C_{2j-2\ell-1}(x), \end{aligned} \tag{55}$$

$$\begin{aligned} V_{2j+1}(x) =& 2^{2j+1}C_{2j+1}(x) - 2^{2j}\,C_{2j}(x) - \sum_{r=1}^{j} 2^{2j-2r+1}\,C_{2j-2r}(x) \\ &+ \sum_{r=1}^{j} \frac{2^{2j-2r+3}(-j+r-1)}{-2j+2r-3}\,C_{2j-2r+1}(x), \end{aligned} \tag{56}$$

while, the fourth-fifth kinds Chebyshev connection formulas are:

$$\begin{aligned} W_{2j}(x) =& 2^{2j}C_{2j}(x) + 2^{2j-1}C_{2j-1}(x) + \sum_{\ell=1}^{j} 2^{2j-2\ell+1}\,C_{2j-2\ell}(x) \\ &+ \sum_{\ell=1}^{j-1} \frac{2^{1+2j-2\ell}(j-\ell)}{1+2j-2\ell}\,C_{2j-2\ell-1}(x), \end{aligned} \tag{57}$$

$$\begin{aligned} W_{2j+1}(x) =& 2^{2j+1}C_{2j+1}(x) + 2^{2j}\,C_{2j}(x) + \sum_{r=1}^{j} 2^{2j-2r+1}\,C_{2j-2r}(x) \\ &+ \sum_{r=1}^{j} \frac{2^{2j-2r+3}(-j+r-1)}{-2j+2r-3}\,C_{2j-2r+1}(x). \end{aligned} \tag{58}$$

Proof. If we substitute by $\alpha = \frac{-1}{2}$, into the two connection Formulas (53) and (54), then the two connection Formulas (55) and (56) can be obtained. Moreover, if we take into consideration the two identities in (48), the two Formulas (57) and (58) can be obtained. □

6. Conclusions

In this paper, a class of Chebyshev orthogonal polynomials was investigated from a theoretical point of view. Several interesting formulas concerned with this kind of orthogonal polynomials were proposed. Connections with the other well-known Chebyshev polynomials were presented. Many important problems that are useful in some applications were also proposed such as the moments, linearization, and connection formulas. Two different approaches are followed to obtain the connection coefficients between the fifth-kind Chebyshev polynomials with some other orthogonal polynomials. We utilized some standard reduction formulas as well as some symbolic algorithms such as Zeilberger's algorithm during the derivation of our formulas. As future work, and from a numerical point of view, we aim to employ some of the derived formulas in this paper along with suitable spectral methods to treat numerically the differential equations with polynomial coefficients, as well as some types of non-linear differential equations.

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