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# Arithmetic Operations and Expected Values of Regular Interval Type-2 Fuzzy Variables

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**Abstract:** High computation complexity restricts the application prospects of the interval type-2 fuzzy variable (IT2-FV), despite its high degree of freedom in representing uncertainty. Thus, this paper studies the fuzzy operations for the regular symmetric triangular IT2-FVs (RSTIT2-FVs)—the simplest IT2-FVs having the greatest membership degrees of 1. Firstly, by defining the medium of an RSTIT2-FV, its membership function, credibility distribution, and inverse distribution are analytically and explicitly expressed. Secondly, an operational law for fuzzy arithmetic operations regarding mutually independent RSTIT2-FVs is proposed, which can simplify the calculations and directly output the inverse credibility of the functions. Afterwards, the operational law is applied to define the expected value operator of the IT2-FV and prove the linearity of the operator. Finally, some comparative examples are provided to verify the efficiency of the proposed operational law.

**Keywords:** interval type-2 fuzzy variable; membership function; expected value; operational law



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## 1. Introduction

There are various types of uncertainty in real life (e.g., fluctuations in the stock market, the costs and demands in the suppliers' market, etc.), forcing us to choose reasonable mathematical tools to describe the research objects. Therefore, the ordinary fuzzy set theory proposed by Zadeh [1] in 1965 has "deepened" over the past five decades. Among the vast amount of theories (e.g., the fuzzy random sets [2], the twofold fuzzy sets [3], the bifuzzy sets [4], the fuzzy soft sets [5], the interval-valued picture hesitant fuzzy set [6]), the type-2 fuzzy set (T2-FS) is an important branch attracting constant attention due to its superiority in representing the uncertainty of the membership function (MF).

The MF, which measures the degree to which an FV is likely to take a certain value, is usually provided by experts, based on their experiences or experiments under the circumstances where there are a lack of data, or where semantic ambiguity exists. In the ordinary fuzzy set theory, the MF of an FV should be uniquely determined. In reality, however, different people have their own understanding of the same fuzzy concept; that is, the determination process of the MF is subjective. Then, how could one guarantee that subjective judgments lead to objective results? There is no mature or effective method yet. Consequently, in order to reflect the inner- and inter-individual differences, Zadeh [7] proposed the concept of T2-FS in 1975, the MF (named primary MF) of which is also an FV.

For a general T2-FV, the fuzziness of its secondary MFs (the MFs of the primary MF) brings about both high computational complexity and visualization difficulty [8]. Although numerous scholars have made delicate research on type-reduction approaches, including the Karnik-Mendel (KM) algorithm-based iterative methods [9], the statistical sampling based geometric strategies [10], and the possibility measure based methods [11–13], the applicability of the T2-FVs is seriously restricted. Up until now, the studies on the application of T2-FVs have mainly focused on fuzzy logic systems, programmings, and game theory [14–18]. Readers may refer to [19–21] for related papers on T2-FS type-reduction and its application areas.

As a special case, the interval T2-FVs (IT2-FVs), whose secondary MFs are identically equal to 1 [22], the uncertainty of the secondary MFs disappears, and a single IT2-FV is uniquely determined by its upper MF (UMF) and lower MF (LMF). Therefore, it can be visualized in a plane. Compared to the generalized T2-FVs, the theories concerning IT2-FVs are, nowadays, relatively mature (see, e.g., [23,24]), and the application areas are extended to computer science, finance, commerce, etc. [25–28].

To the best of our knowledge, however, the existing studies focus on algorithm-based type-reductions and defuzzification methods, whereas the research for fuzzy arithmetic operations of the IT2-FVs are relatively limited. The first work is by Lee and Chen [29] who defined three arithmetic operations between two IT2-FVs, whose UMF and LMF are both r-polygonal shape, named addition, subtraction, and multiplication. Since then, some scholars have followed Lee and Chen's work while applying multi-attribute decision making approaches [30], such as the analytic hierarchy process [31,32] and TOPSIS [33], to make ranking and evaluation decisions [34–36]. Besides, for the trapezoidal interval type-2 fuzzy numbers, some other operators, such as the geometric Bonferroni mean operator [37] and Heronian mean operators [38] were proposed. Agüero and Vargas [39] is another work focusing on the operations between two IT2-FVs, where the vertex method, based on  $\alpha$ -cut, was suggested. Nevertheless, these operators involve complex operations themselves, and thus are inadequate at solving many realistic problems concerning more complex operations. Hence, there is an urgent need to put forward a brand-new and convenient operational law for fuzzy arithmetic operations of the IT2-FVs.

On the other hand, within the framework of the traditional fuzzy set theory, Zhou et al. [40] proposed an operational law for calculating the inverse credibility distribution of the strictly monotone functions between mutually independent type-1 FVs (T1-FVs) and verified that their method can greatly simplify the fuzzy arithmetical operations. Therefore, learning from their ideas, we attempt to extend the inverse credibility distribution-based operational law to the type-2 fuzzy set theory. To demonstrate the efficiency of the proposed method, the simplest and most special type of IT2-FVs, i.e., the regular symmetric triangular IT2-FV (RSTIT2-FV), is studied in this paper, as an extension of the most commonly used symmetric triangular FV in the type-1 fuzzy set system. The RSTIT2-FV this paper focuses on has two characteristics: (1) the shapes of the UMF and LMF are both symmetric triangles; (2) the vertexes of the UMF and LMF overlap at the membership degree of 1.

The main contributions of this paper are as follows. First, we give a clear mathematical definition of the RSTIT2-FV and define the medium of an RSTIT2-FV as a type-reduction operator, and then set forth the possibility, necessity, and credibility measures, as well as the credibility distribution and inverse credibility distribution. Second, the operational law based on the proposed inverse credibility distribution for mutually independent RSTIT2-FVs is given, which can help calculate for the strictly monotone functions of the RSTIT2-FVs in a relatively easy way. Thirdly, an expected value operator via inverse credibility distribution for the RSTIT2-FV is defined, and its linearity is proved. Finally, some numerical examples are illustrated to verify the efficiency of the proposed methodology.

The rest of this paper is organized as follows. In Section 2, the related concepts are briefly introduced. Then, the RSTIT2-FV is defined in Section 3, as well as the credibility distribution function and inverse credibility distribution. In Section 4, the operation law for the proposed RSTIT2-FV is provided. Finally, the expected value and its property of linearity are introduced and proved in Section 5.

## 2. Preliminaries

To begin, some fundamental concepts of the fuzzy set theory will be briefly introduced in advance.

**Definition 1.** (Zadeh [7]) A type-1 fuzzy set (T1-FS)  $B$  can be defined as

$$B = \int_{x \in X} \frac{\mu_B(x)}{x},$$

where  $\mu_B(x)$  is the real-valued membership function, and  $X$  is the universe of  $x$ .

**Definition 2.** (Liu [41]) Given that  $\Theta$  is a nonempty set,  $\Gamma(\Theta)$  is the power set of  $\Theta$ ,  $\text{Pos}$  is a possibility measure, and  $\mathbb{R}$  is a real number set. Let the triad  $(\Theta, \Gamma(\Theta), \text{Pos})$  be a possibility space, then the map  $\eta: (\Theta, \Gamma(\Theta), \text{Pos}) \rightarrow \mathbb{R}$  is called a type-1 fuzzy variable (T1-FV).

**Definition 3.** (Liu [41]) The credibility distribution for a T1-FV  $\eta$  can be calculated by

$$\Phi_\eta(x) = \text{Cr}\{\eta \leq x\} = \frac{1}{2} \left( \sup_{\eta \leq x} \mu_\eta(x) + 1 - \sup_{\eta > x} \mu_\eta(x) \right),$$

where  $\mu_\eta(x)$  is the MF of  $\eta$ .

**Definition 4.** (Doubois and Prade [42]) A T1-FV  $\eta$  is called an LR-type FV if its shape function and scalers  $\alpha > 0, c > 0$  satisfy

$$\mu_\eta(x) = \begin{cases} L\left(\frac{c-x}{\alpha}\right), & x \in (-\infty, c] \\ R\left(\frac{x-c}{\alpha}\right), & x \in [c, +\infty), \end{cases}$$

where the shape function  $L$  (for left) and  $R$  (for right) are decreasing functions from  $\mathbb{R}^+ \rightarrow [0, 1]$  and satisfy  $L(0) = R(0) = 1$  and  $L(1) = R(1) = 0$ .

**Definition 5.** (Zhou et al. [40]) An LR-type FV  $\eta$ , which has a continuous and strictly increasing credibility distribution  $\Phi_\eta$  is called a regular LR-FV.

**Definition 6.** (Zadeh [7], Mendel and John [43]) A type-2 fuzzy set (T2-FS), denoted as  $S$  and characterized by a type MF  $\mu_S(x, u)$ . And  $S$  can be expressed as

$$S = \int_{x \in X} \int_{u \in J_x} \frac{\mu_S(x, u)}{(x, u)}.$$

where  $u \in J_x \subseteq [0, 1], x \in X$  is called the primary MF of  $x$ ,  $\mu_S(x, u)$  is called the secondary MF of  $u$ .

It can be easily seen from Definition 6 that T2-FS is defined on a three-dimensional space. To better characterize its features, Mendel and John [43] defined the information it mapped to the two-dimensional plane as the footprint of uncertainty (FOU).

**Definition 7.** (Mendel and John [43]) For any  $x \in X$ , let the primary membership function of a T2-FS  $S$  be  $J_x \subseteq [0, 1]$ , then the FOU of  $S$  is the union of all the primary membership functions and, thus, can be expressed as

$$\text{FOU}(S) = \bigcup_{x \in X} J_x.$$

The upper and lower bounds of the FOU are called the upper membership function (UMF) and lower membership function (LMF) of  $S$ , respectively.

**Definition 8.** (Mendel et al. [22]) For a T2-FS  $S$  who has an  $N$ -element discrete universe of discourse  $X$  and primary membership functions  $J_{x_1}, J_{x_2}, \dots, J_{x_N}$ , an embedded T1-FS  $S_e$  has  $N$  elements, one each from  $J_{x_i}, i = 1, 2, \dots, N$ , namely  $u_1, u_2, \dots, u_N$ , i.e.,

$$S_e = \sum_{i=1}^N \frac{u_i}{x_i}, \quad u_i \in J_{x_i} \subseteq [0, 1].$$

**Remark 1.** The embedded T1-FS can be extended to the continuous situations and expressed as

$$S_e = \int \frac{u}{x}, \quad u \in J_x \subseteq [0, 1].$$

Obviously, there are uncountable numbers of embedded T1-FSs in infinite domains.

**Definition 9.** (Liu and Liu [11]) Let  $S$  be a T2-FV, then the secondary possibility distribution function,  $\tilde{\mu}_S(x)$ , is a map  $\mathbb{R} \rightarrow [0, 1]$  satisfying  $\tilde{\mu}_S(x) = \widehat{\text{Pos}}\{\gamma \in \Theta | S(\gamma) = x\}$ ,  $x \in \mathbb{R}$ , and the type-2 possibility distribution function,  $\mu_S(x, u)$ , is a map  $\mathbb{R} \times J_x \rightarrow [0, 1]$  satisfying  $\mu_S(x, u) = \text{Pos}\{\tilde{\mu}_S(x) = u\}$ ,  $(x, u) \in \mathbb{R} \times J_x$ .

**Remark 2.** In fact, for a given  $x = x'$ ,  $\mu_S(x')$  in Definition 9 is exactly the secondary MF,  $\mu_S(x', u)$ , in Definition 6. Therefore,  $J_x$  in Definition 9 can also be regarded as the primary MF in Definition 6.

**Definition 10.** (Men et al. [44]) For any  $x \in X$  and  $u \in J_x$ , if  $\mu_S(x, u)$  is identically equal to 1, then the T2-FS  $S$  is an interval T2-FV (IT2-FV).

According to Definition 10, the secondary membership of an IT2-FV is unique and equal everywhere, i.e., the uncertainty of the MF and the information of the third dimension for an IT2-FV can be ignored, and the IT2-FV can be uniquely determined by the region bounded by its UMF and LMF.

### 3. Symmetric Triangular Interval Type-2 Fuzzy Variable

In this section, a special kind of IT2-FV, named regular symmetric triangular IT2-FV, is introduced.

#### 3.1. RSTIT2-FV and Its Medium

The specific definition of the IT2-FV studied in this paper is as follows.

**Definition 11.** An IT2-FV  $S$  is called a regular symmetric triangular IT2-FV (RSTIT2-FV) if its UMF and LMF are as in the following forms,

$$\text{UMF} = \begin{cases} \frac{1}{l_U}x - \frac{c - l_U}{l_U}, & x \in [c - l_U, c) \\ -\frac{1}{l_U}x + \frac{c + l_U}{l_U}, & x \in [c, c + l_U] \\ 0, & \text{otherwise,} \end{cases} \quad (1)$$

and

$$\text{LMF} = \begin{cases} \frac{1}{l_L}x - \frac{c - l_L}{l_L}, & x \in [c - l_L, c) \\ -\frac{1}{l_L}x + \frac{c + l_L}{l_L}, & x \in [c, c + l_L] \\ 0, & \text{otherwise,} \end{cases} \quad (2)$$

and can be denoted as  $\begin{pmatrix} c - l_U & c & c + l_U \\ c - l_L & c & c + l_L \end{pmatrix}$ , where the spreads of the UMF and LMF,  $l_U$  and  $l_L$ , satisfy  $l_U > l_L$ , and the peak of them, 1, are reached when  $x$  is equal to  $c$ .

An RSTIT2-FV is visualized in Figures 1 and 2, respectively, in a three-dimensional space and two-dimensional plane, respectively, and Figure 2 can be understood as a top view of Figure 1.

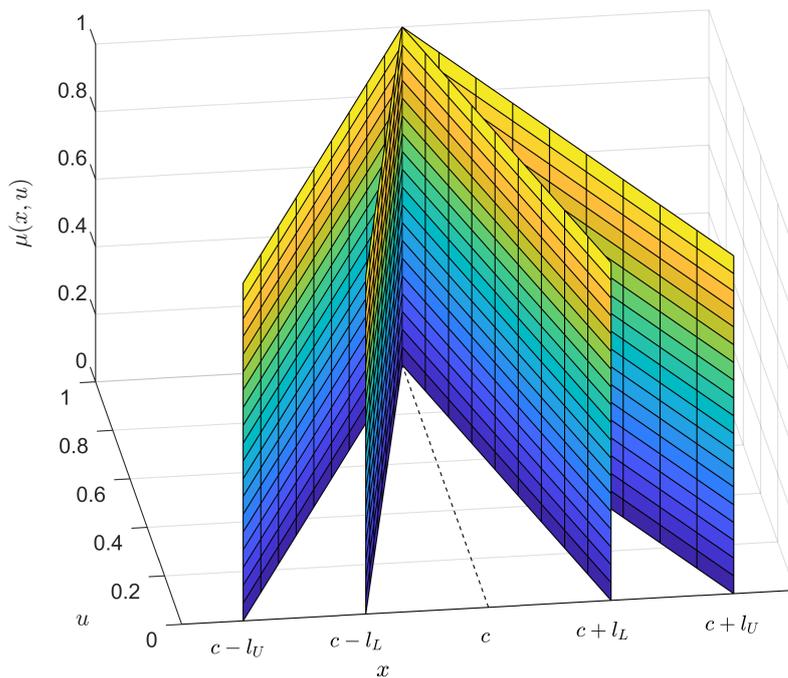


Figure 1. The solid visualization of an RSTIT2-FV.

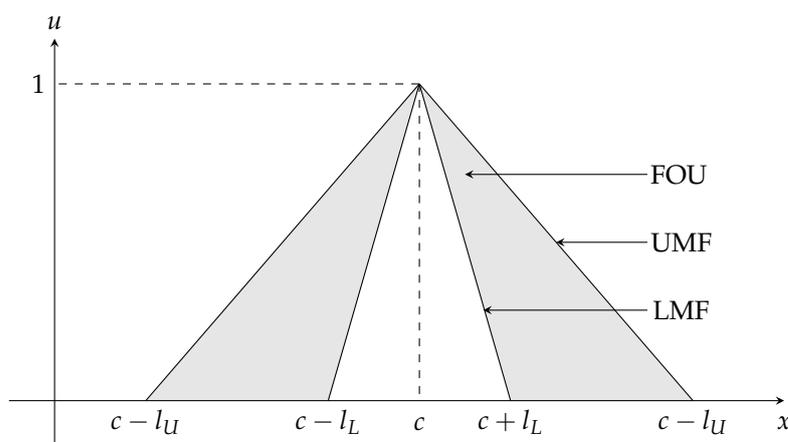


Figure 2. The plane visualization of an RSTIT2-FV.

In order to facilitate the subsequent definitions of credibility distribution and operational law for the RSTIT2-FV, the medium of an RSTIT2-FV is defined in advance.

**Definition 12.** Let  $S$  be an RSTIT2-FV,  $\eta$  be a T1-FV, and their MFs satisfy

$$\mu_{\eta}(x) = \frac{1}{2}\text{UMF} + \frac{1}{2}\text{LMF},$$

then  $\eta$  is called the medium of  $S$ .

Definition 12 endows equal weights to every possible value of the MFs for a certain  $x$ , it is a relatively evenhanded way to eliminate the uncertainty of MFs in an uncertain and complex environment. Following from Definition 12, the analytical expression of  $\mu_{\eta}(x)$  can be calculated as

$$\mu_{\eta}(x) = \begin{cases} \frac{1}{2} \left( \frac{1}{l_U} x - \frac{c - l_U}{l_U} \right), & x \in [c - l_U, c - l_L) \\ \frac{1}{2} \left( \frac{1}{l_U} x - \frac{c - l_U}{l_U} \right) + \frac{1}{2} \left( \frac{1}{l_L} x - \frac{c - l_L}{l_L} \right), & x \in [c - l_L, c) \\ \frac{1}{2} \left( -\frac{1}{l_U} x + \frac{c + l_U}{l_U} \right) + \frac{1}{2} \left( -\frac{1}{l_L} x + \frac{c + l_L}{l_L} \right), & x \in [c, c + l_L) \\ \frac{1}{2} \left( -\frac{1}{l_U} x + \frac{c + l_U}{l_U} \right), & x \in [c + l_L, c + l_U] \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

according to Equations (1) and (2). Figure 3 shows the medium  $\eta$  with the red polyline representing its MF, from which we can readily infer that  $\eta$  is a regular LR-FV. Moreover,  $\mu_{\eta}$  can be regarded as one of the embedded type-1 sets of  $S$ , according to Definition 8.

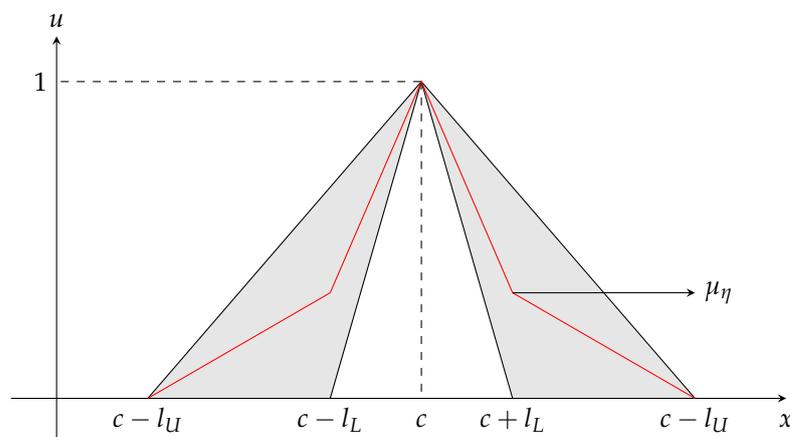


Figure 3. The visualization of the medium  $\eta$ .

### 3.2. The Credibility Distribution of the RSTIT2-FVs

By means of the medium  $\eta$ , we can define the credibility distribution and inverse credibility distribution in light of Liu [41]’s definition of credibility measure for the T1-FVs. Beforehand, the definition of the credibility measure for an RSTIT2-FV is given.

**Definition 13.** Let  $S$  be an RSTIT2-FV with the medium of  $\eta$ , and  $A$  be a fuzzy event from the universe, then the possibility, necessity, and credibility measures of  $A$  for  $S$  are, respectively,

$$\begin{aligned} \text{Pos}\{A\} &= \sup_{x \in A} \mu_{\eta}(x), \\ \text{Nec}\{A\} &= 1 - \sup_{x \in A^c} \mu_{\eta}(x), \\ \text{Cr}\{A\} &= \frac{1}{2}(\text{Pos}\{A\} + \text{Nec}\{A\}). \end{aligned}$$

It can be easily proven that the credibility measure of an RSTIT2-FV is self-dual and satisfies the monotonicity.

**Theorem 1.** For a fuzzy event  $A$  from the universe and the credibility measure of an RSTIT2-FV,  $\text{Cr}$ , we have

$$\text{Cr}\{A\} + \text{Cr}\{A^c\} = 1.$$

**Proof.** According to Definition 13, it can be deduced that

$$\begin{aligned} \text{Cr}\{A\} + \text{Cr}\{A^c\} &= \frac{1}{2}(\text{Pos}\{A\} + \text{Nec}\{A\}) + \frac{1}{2}(\text{Pos}\{A^c\} + \text{Nec}\{A^c\}) \\ &= \frac{1}{2}(\text{Pos}\{A\} + \text{Pos}\{A^c\} + 1 - \text{Pos}\{A^c\} + 1 - \text{Pos}\{A\}) \\ &= 1. \end{aligned}$$

□

**Theorem 2.** Suppose that  $C$  and  $D$  are two fuzzy events satisfying  $C \subseteq D$ , then for the credibility measure of the RSTIT2-FV, we have  $\text{Cr}\{C\} \leq \text{Cr}\{D\}$ .

**Proof.** In light of Definition 13, it can be obtained that

$$\begin{aligned} \text{Cr}\{C\} &= \frac{1}{2}(\text{Pos}\{C\} + 1 - \text{Pos}\{C^c\}), \\ \text{Cr}\{D\} &= \frac{1}{2}(\text{Pos}\{D\} + 1 - \text{Pos}\{D^c\}). \end{aligned} \tag{4}$$

On the other hand, we have  $D^c \subseteq C^c$  since  $C$  is a subset of  $D$ . Thus, we can further obtain

$$\begin{aligned} \sup_{x \in D^c} \mu_\eta(x) &\leq \sup_{x \in C^c} \mu_\eta(x), \\ \sup_{x \in C} \mu_\eta(x) &\leq \sup_{x \in D} \mu_\eta(x). \end{aligned}$$

Again, on the basis of Definition 13, we have

$$\begin{aligned} \text{Pos}\{C\} &= \sup_{x \in C} \mu_\eta(x) \leq \text{Pos}\{D\} = \sup_{x \in D} \mu_\eta(x), \\ \text{Pos}\{D^c\} &= \sup_{x \in D^c} \mu_\eta(x) \leq \text{Pos}\{C^c\} = \sup_{x \in C^c} \mu_\eta(x). \end{aligned} \tag{5}$$

Accordingly, it can be derived that

$$\text{Cr}\{C\} \leq \text{Cr}\{D\}$$

in view of Equations (4) and (5). □

Based on Definition 13 and Theorems 1 and measure-mono, the possibility, necessity, and credibility measures of a specific fuzzy event  $\{S \leq r\}$  for an RSTIT2-FV  $S$  can be calculated by

$$\begin{aligned} \text{Pos}\{S \leq r\} &= \sup_{x \leq r} \mu_\eta(x), \\ \text{Nec}\{S \leq r\} &= 1 - \sup_{x > r} \mu_\eta(x), \\ \text{Cr}\{S \leq r\} &= \frac{1}{2} \left( \sup_{x \leq r} \mu_\eta(x) + 1 - \sup_{x > r} \mu_\eta(x) \right). \end{aligned}$$

Next, we can define the credibility distribution of  $S$  as follows.

**Definition 14.** Suppose that  $S$  is an RSTIT2-FV with a medium of  $\eta$ , then the credibility distribution  $\Phi_S$  of  $S$  is defined as  $\text{Cr}\{S \leq x\}$ .

On the basis of Definition 13 and Equation (3), we can easily obtain that

$$\text{Pos}\{S \leq x\} = \begin{cases} 0, & x \in (-\infty, c - l_U] \\ \frac{1}{2} \left( \frac{1}{l_U} x - \frac{c - l_U}{l_U} \right), & x \in (c - l_U, c - l_L] \\ \frac{1}{2} \left( \frac{1}{l_U} x - \frac{c - l_U}{l_U} \right) + \frac{1}{2} \left( \frac{1}{l_L} x - \frac{c - l_L}{l_L} \right), & x \in (c - l_L, c] \\ 1, & x \in (c, +\infty), \end{cases} \tag{6}$$

and

$$\text{Nec}\{S \leq x\} = \begin{cases} 0, & x \in (-\infty, c] \\ 1 + \frac{1}{2} \left( \frac{1}{l_U} x - \frac{c+l_U}{l_U} \right) + \frac{1}{2} \left( \frac{1}{l_L} x - \frac{c+l_L}{l_L} \right), & x \in (c, c+l_L] \\ 1 + \frac{1}{2} \left( \frac{1}{l_U} x - \frac{c+l_U}{l_U} \right), & x \in (c+l_L, c+l_U] \\ 1, & x \in (c+l_U, +\infty). \end{cases} \tag{7}$$

Then by using Definition 14 and Equations (1) and (6), the credibility distribution  $\Phi_S$  can be easily calculated as

$$\Phi_S(x) = \text{Cr}\{S \leq x\} = \begin{cases} 0, & x \in (-\infty, c-l_U] \\ \frac{1}{4} \left( \frac{1}{l_U} x - \frac{c-l_U}{l_U} \right), & x \in (c-l_U, c-l_L] \\ \frac{1}{4} \left( \frac{1}{l_U} x - \frac{c-l_U}{l_U} \right) + \frac{1}{4} \left( \frac{1}{l_L} x - \frac{c-l_L}{l_L} \right), & x \in (c-l_L, c+l_L] \\ 1 + \frac{1}{4} \left( \frac{1}{l_U} x - \frac{c+l_U}{l_U} \right), & x \in (c+l_L, c+l_U] \\ 1, & x \in (c+l_U, +\infty). \end{cases} \tag{8}$$

Figure 4 gives the visualization of  $\Phi_S$ . Obviously,  $\Phi_S$  is a continuous and strictly increasing function.

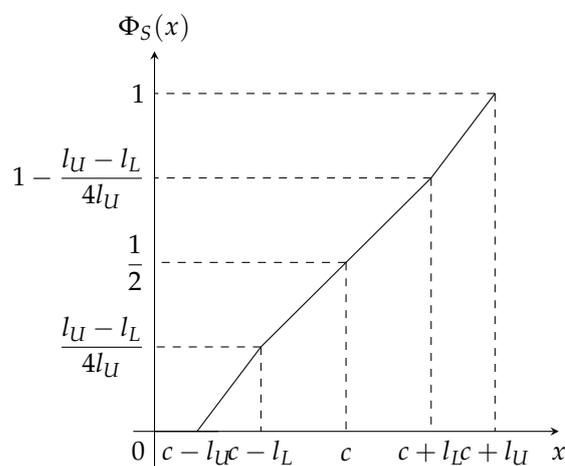


Figure 4. The credibility distribution of  $S$ ,  $\Phi_S$ .

**Definition 15.** Let  $S$  be an RSTIT2-FV, then its inverse credibility distribution is defined as the inverse function of  $\Phi_S$  (see Figure 5) and can be calculated by

$$\Phi_S^{-1}(\alpha) = \begin{cases} 4l_U\alpha + c - l_U, & \alpha \in \left[0, \frac{l_U - l_L}{4l_U}\right) \\ \frac{4l_U l_L \alpha - 2l_U l_L}{l_U + l_L} + c, & \alpha \in \left[\frac{l_U - l_L}{4l_U}, 1 - \frac{l_U - l_L}{4l_U}\right) \\ 4l_U\alpha + c - 3l_U, & \alpha \in \left[1 - \frac{l_U - l_L}{4l_U}, 1\right]. \end{cases} \tag{9}$$

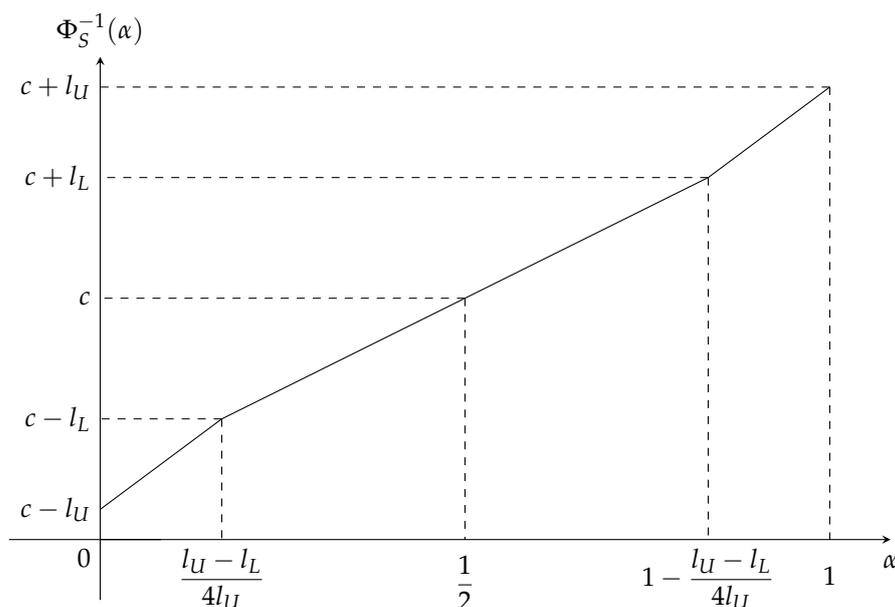


Figure 5. The inverse credibility distribution of  $S$ ,  $\Phi_S^{-1}$ .

**Remark 3.** According to the definition of the credibility distribution [45] and inverse credibility distribution [40] of the TI-FV, we have  $\Phi_\eta(x) = \Phi_S(x)$  and  $\Phi_\eta^{-1}(\alpha) = \Phi_S^{-1}(\alpha)$ , where  $\Phi_\eta$  and  $\Phi_\eta^{-1}$  are, respectively, the credibility distribution and inverse credibility distribution of the medium  $\eta$ .

**Example 1.** Denote an RSTIT2-FV as  $S_1 = \begin{pmatrix} 1 & 4 & 7 \\ 3 & 4 & 5 \end{pmatrix}$ , then according to Definition 11, we have  $l_U = 3$  and  $l_L = 1$ , and the UMF and LMF are, respectively,

$$UMF = \begin{cases} \frac{1}{3}x - \frac{1}{3}, & x \in [1, 4) \\ -\frac{1}{3}x + \frac{7}{3}, & x \in [4, 7) \\ 0, & \text{otherwise,} \end{cases} \quad LMF = \begin{cases} x - 3, & x \in [3, 4) \\ -x + 5, & x \in [4, 5) \\ 0, & \text{otherwise,} \end{cases}$$

Afterwards, based on Equation (3) in Definition 12, the MF of the medium  $\eta_1$  for  $S_1$  is

$$\mu_{\eta_1}(x) = \begin{cases} \frac{1}{6}x - \frac{1}{6}, & x \in [1, 3) \\ \frac{2}{3}x - \frac{5}{3}, & x \in [3, 4) \\ -\frac{2}{3}x + \frac{11}{3}, & x \in [4, 5) \\ -\frac{1}{6}x + \frac{7}{6}, & x \in [5, 7), \\ 0, & \text{otherwise,} \end{cases}$$

Consequently, on the basis of Definition 14, the possibility of the fuzzy event  $S \leq x$  is

$$Pos\{S \leq x\} = \begin{cases} 0, & x \in (-\infty, 1] \\ \frac{1}{6}x - \frac{1}{6}, & x \in (1, 3] \\ \frac{2}{3}x - \frac{5}{3}, & x \in (3, 4] \\ 1, & x \in (4, +\infty), \end{cases}$$

and the necessity of it is

$$\text{Nec}\{S \leq x\} = \begin{cases} 0, & x \in (-\infty, 4] \\ \frac{2}{3}x - \frac{8}{3}, & x \in (4, 5] \\ \frac{1}{6}x - \frac{1}{6}, & x \in (5, 7] \\ 1, & x \in (7, +\infty). \end{cases}$$

Thus, following from Definition 13 and Equation (14), the credibility (credibility distribution) is

$$\Phi_{S_1}(x) = \text{Cr}\{S \leq x\} = \begin{cases} 0, & x \in (-\infty, 1) \\ \frac{1}{12}x - \frac{1}{12}, & x \in [1, 3) \\ \frac{1}{3}x - \frac{5}{6}, & x \in [3, 5) \\ \frac{1}{12}x + \frac{5}{12}, & x \in [5, 7) \\ 1, & x \in (7, +\infty). \end{cases}$$

Finally, with a view to Equation (9) in Definition 15, we have

$$\Phi_{S_1}^{-1}(\alpha) = \begin{cases} 12\alpha + 1, & \alpha \in \left[0, \frac{1}{6}\right) \\ 3\alpha + \frac{5}{2}, & \alpha \in \left[\frac{1}{6}, \frac{5}{6}\right) \\ 12\alpha - 5, & \alpha \in \left[\frac{5}{6}, 1\right]. \end{cases} \quad (10)$$

**Example 2.** Given that  $S_2 = \begin{pmatrix} 0 & 4 & 8 \\ 2 & 4 & 6 \end{pmatrix}$  is an RSTIT2-FV, then the MF of its medium  $\eta_2$ , the credibility distribution and inverse credibility distribution are as follows,

$$\mu_{\eta_2}(x) = \begin{cases} \frac{1}{8}x, & x \in [0, 2) \\ \frac{3}{8}x - \frac{1}{2}, & x \in [2, 4) \\ -\frac{1}{4}x + \frac{5}{2}, & x \in [4, 6) \\ -\frac{1}{8}x + 1, & x \in [6, 8] \\ 0, & \text{otherwise,} \end{cases}$$

$$\Phi_{S_2}(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ \frac{1}{16}x, & x \in [0, 2) \\ \frac{3}{16}x - \frac{1}{4}, & x \in [2, 6) \\ \frac{1}{16}x + \frac{1}{2}, & x \in [6, 8] \\ 1, & x \in (8, +\infty), \end{cases}$$

$$\Phi_{S_2}^{-1}(\alpha) = \begin{cases} 16\alpha, & \alpha \in \left[0, \frac{1}{8}\right) \\ \frac{16}{3}\alpha + \frac{4}{3}, & \alpha \in \left[\frac{1}{8}, \frac{7}{8}\right) \\ 16\alpha - 8, & \alpha \in \left[\frac{7}{8}, 1\right]. \end{cases} \tag{11}$$

following from Equations (3), (8) and (9).

In the previous subsection, the credibility distribution and the inverse credibility distribution for a single RSTIT2-FV had been given. Now, we consider a situation where more than one RSTIT2-FV are involved in an arithmetic operation.

**Definition 16.** Assume that  $S_i, i = 1, 2, \dots, n$  are RSTIT2-FVs with the mediums of  $\eta_i, i = 1, 2, \dots, n$ , respectively, and  $f$  is a function from  $\mathbb{R}^n$  to  $\mathbb{R}$ , then the credibility distribution of  $S = f(S_1, S_2, \dots, S_n), \Phi_S(x) = \text{Cr}\{S \leq x\}$  is defined as

$$\Phi_S(x) = \Phi_\eta(x),$$

where  $\eta = f(\eta_1, \eta_2, \dots, \eta_n)$  is the medium of  $S$ . The inverse credibility distribution of  $S$  is the inverse function of  $\Phi_S$ , i.e.,  $\Phi_S^{-1}$ .

**Remark 4.** According to Definition 16, it can be easily deduced that  $\Phi_S^{-1}(\alpha) = \Phi_\eta^{-1}(\alpha)$ .

#### 4. Operational Law

This section introduces the operational law for fuzzy arithmetic operations regarding RSTIT2-FVs, which can help efficiently calculate  $\Phi_S$  in Definition 16. As a prerequisite for its application, the definition of independence is introduced at first.

##### 4.1. Independence

Although Liu and Liu [11] has given the definition of independence for T2-FVs based on the possibility measure, it is somewhat more complex and is not that practicable. Therefore, a new definition of independence based on the credibility measure and the concept of medium in Definition 13 is proposed in this section.

**Definition 17.** The RSTIT2-FVs  $S_i, i = 1, 2, \dots, n$ .  $S_i, i = 1, 2, \dots, n$  are said to be mutually independent if

$$\text{Cr}\{S_i \in B_i, i = 1, 2, \dots, n\} = \min_{1 \leq i \leq n} \text{Cr}\{S_i \in B_i\}$$

for any subsets  $B_1, B_2, \dots, B_n$ .

**Definition 18.** (Liu and Gao [46]) Let  $\eta_i, i = 1, 2, \dots, n$  be  $n$  T1-FVs, they are said to be mutually independent if

$$\text{Cr}\{\eta_i \in B_i, i = 1, 2, \dots, n\} = \min_{1 \leq i \leq n} \text{Cr}\{\eta_i \in B_i\}$$

for any subsets  $B_1, B_2, \dots, B_n$ .

**Remark 5.** Since the independence of the RSTIT2-FVs are derived from their credibility measures, and the credibility measure of the RSTIT2-FVs are based on their mediums, T1-FVs  $\eta_i, i = 1, 2, \dots, n$ , once RSTIT2-FVs  $S_i, i = 1, 2, \dots, n$  are mutually independent, it means that

$$\text{Cr}\{\eta_i \in B_i, i = 1, 2, \dots, n\} = \min_{1 \leq i \leq n} \text{Cr}\{\eta_i \in B_i\}$$

holds. Thus, according to Definition 18, the mediums  $\eta_i, i = 1, 2, \dots, n$  are also mutually independent.

#### 4.2. Operational Law for Strictly Monotone Function of RSTIT2-FVs

Prior to the introduction of the operational law, the definition of the strictly monotone function is set forth.

**Definition 19.** (Zhou et al. [40]) A real-valued function  $f(x_1, x_2, \dots, x_n)$  is said to be strictly monotone if it is strictly increasing with respect to  $x_1, x_2, \dots, x_t$  and strictly decreasing with respect to  $x_{t+1}, x_{t+2}, \dots, x_n$ ; that is,

$$f(x_1, \dots, x_t, x_{t+1}, \dots, x_n) \leq f(y_1, \dots, y_t, y_{t+1}, \dots, y_n)$$

whenever  $x_i \leq y_i$  for  $i = 1, 2, \dots, t$  and  $x_i \geq y_i$  for  $i = t + 1, \dots, n$ , and

$$f(x_1, \dots, x_t, x_{t+1}, \dots, x_n) < f(y_1, \dots, y_t, y_{t+1}, \dots, y_n)$$

whenever  $x_i < y_i$  for  $i = 1, 2, \dots, t$  and  $x_i > y_i$  for  $i = t + 1, \dots, n$ .

**Theorem 3.** (Zhou et al. [40]) Let  $\eta_i, i = 1, 2, \dots, n$  be independent regular LR FVs with the credibility distributions of  $\Phi_{\eta_i}, i = 1, 2, \dots, n$ , respectively. If the function  $f(x_1, \dots, x_t, x_{t+1}, \dots, x_n)$  is strictly increasing with respect to  $x_i, i = 1, 2, \dots, t$  and strictly decreasing with respect to  $x_i, i = t + 1, t + 2, \dots, n$ , then the inverse credibility distribution of

$$\eta = f(\eta_1, \dots, \eta_t, \eta_{t+1}, \dots, \eta_n),$$

can be calculated by

$$\Phi_{\eta}^{-1}(\alpha) = f\left(\Phi_{\eta_1}^{-1}(\alpha), \dots, \Phi_{\eta_t}^{-1}(\alpha), \Phi_{\eta_{t+1}}^{-1}(1 - \alpha), \dots, \Phi_{\eta_n}^{-1}(1 - \alpha)\right).$$

**Theorem 4.** Suppose that  $S_i, i = 1, 2, \dots, n$  are mutually independent RSTIT2-FVs with the mediums of  $\eta_i, i = 1, 2, \dots, n$ , respectively. If the function  $f(x_1, \dots, x_t, x_{t+1}, \dots, x_n)$  is strictly increasing with respect to  $x_i, i = 1, 2, \dots, t$  and strictly decreasing with respect to  $x_i, i = t + 1, t + 2, \dots, n$ , then

$$S = f(S_1, \dots, S_t, S_{t+1}, \dots, S_n),$$

has the inverse credibility distribution of

$$\Phi_S^{-1}(\alpha) = f\left(\Phi_{S_1}^{-1}(\alpha), \dots, \Phi_{S_t}^{-1}(\alpha), \Phi_{S_{t+1}}^{-1}(1 - \alpha), \dots, \Phi_{S_n}^{-1}(1 - \alpha)\right).$$

**Proof.** Since  $\eta_i, i = 1, 2, \dots, n$  are the mediums of the RSTIT2-FVs  $S_i, i = 1, 2, \dots, n$ , according to Definition 16, we have

$$\Phi_{S_i}^{-1}(\alpha) = \Phi_{\eta_i}^{-1}(\alpha), \quad i = 1, 2, \dots, t,$$

$$\Phi_{S_i}^{-1}(1 - \alpha) = \Phi_{\eta_i}^{-1}(1 - \alpha), \quad i = t + 1, t + 2, \dots, n.$$

Let  $\eta = f(\eta_1, \dots, \eta_t, \eta_{t+1}, \dots, \eta_n)$ , then Definition 16, we have

$$\Phi_S^{-1}(\alpha) = \Phi_{\eta}^{-1}(\alpha),$$

By using Theorem 3, it can be deduced that

$$\Phi_{\eta}^{-1}(\alpha) = f\left(\Phi_{\eta_1}^{-1}(\alpha), \dots, \Phi_{\eta_t}^{-1}(\alpha), \Phi_{\eta_{t+1}}^{-1}(1 - \alpha), \dots, \Phi_{\eta_n}^{-1}(1 - \alpha)\right),$$

Accordingly, we can easily obtain that

$$\begin{aligned} \Phi_S^{-1}(\alpha) &= f\left(\Phi_{\eta_1}^{-1}(\alpha), \dots, \Phi_{\eta_t}^{-1}(\alpha), \Phi_{\eta_{t+1}}^{-1}(1-\alpha), \dots, \Phi_{\eta_n}^{-1}(1-\alpha)\right) \\ &= f\left(\Phi_{S_1}^{-1}(\alpha), \dots, \Phi_{S_t}^{-1}(\alpha), \Phi_{S_{t+1}}^{-1}(1-\alpha), \dots, \Phi_{S_n}^{-1}(1-\alpha)\right). \end{aligned} \tag{12}$$

□

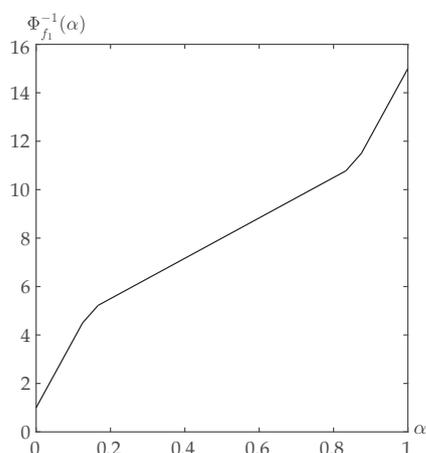
In order to show the efficiency of our proposed operational law in calculating the inverse credibility distribution of the fuzzy arithmetic operations, we present three comparative examples, as follows.

**Example 3.** Suppose that  $S_1$  in Example 1 and  $S_2$  in Example 2 are mutually independent RSTIT2-FVs, and  $f_i(S_1, S_2), i = 1, 2, 3$  are three functions of them. Then the inverse credibility distributions of  $f_i(S_1, S_2)$  can be given directly by Equation (12).

**Case 1:** Set  $f_1(S_1, S_2) = S_1 + S_2$ . It is obviously seen that  $f_1$  is strictly increasing with respect to  $S_1$  and  $S_2$ . Hence, on the basis of Theorem 4 and Equations (10) and (11), we have

$$\Phi_{f_1}^{-1}(\alpha) = \begin{cases} (12\alpha + 1) + 16\alpha, & \alpha \in [0, 1/8) \\ (12\alpha + 1) + (16/3\alpha + 4/3), & \alpha \in [1/8, 1/6) \\ (3\alpha + 5/2) + (16/3\alpha + 4/3), & \alpha \in [1/6, 5/6) \\ (12\alpha - 5) + (16/3\alpha + 4/3), & \alpha \in [5/6, 7/8) \\ (12\alpha - 5) + (16\alpha - 8), & \alpha \in [7/8, 1]. \end{cases} \tag{13}$$

Figure 6 provides the graph of  $\Phi_{f_1}^{-1}(\alpha)$ .



**Figure 6.** The inverse credibility function of  $f_1(S_1, S_2) = S_1 + S_2$ .

**Case 2:** Set  $f_2(S_1, S_2) = S_1 - S_2$ . In this case,  $f_2$  is strictly increasing with respect to  $S_1$ , and strictly decreasing with respect to  $S_2$ . Following from Equation (11), we have

$$\Phi_{S_2}^{-1}(1-\alpha) = \begin{cases} 8 - 16\alpha, & \alpha \in [0, 1/8) \\ (20 - 16\alpha)/3, & \alpha \in [1/8, 7/8) \\ 16(1-\alpha), & \alpha \in [7/8, 1], \end{cases} \tag{14}$$

Then, with a view to Theorem 4 and Equations (10) and (14), we can readily obtain that

$$\Phi_{f_2}^{-1}(\alpha) = \begin{cases} (12\alpha + 1) - (8 - 16\alpha), & \alpha \in [0, 1/8) \\ (12\alpha + 1) - (20 - 16\alpha)/3, & \alpha \in [1/8, 1/6) \\ (3\alpha + 5/2) - (20 - 16\alpha)/3, & \alpha \in [1/6, 5/6) \\ (12\alpha - 5) - (20 - 16\alpha)/3, & \alpha \in [5/6, 7/8) \\ (12\alpha - 5) - 16(1 - \alpha), & \alpha \in [7/8, 1], \end{cases} \quad (15)$$

which can be visualized in Figure 7.

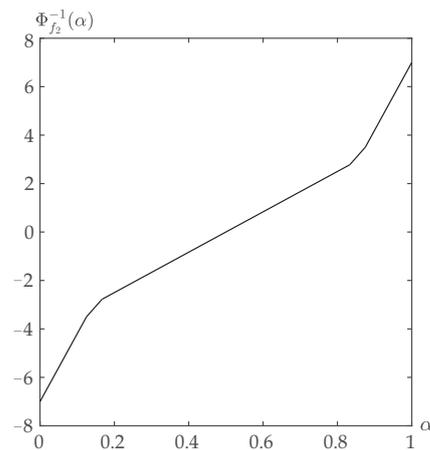


Figure 7. The inverse credibility function of  $f_2(S_1, S_2) = S_1 - S_2$ .

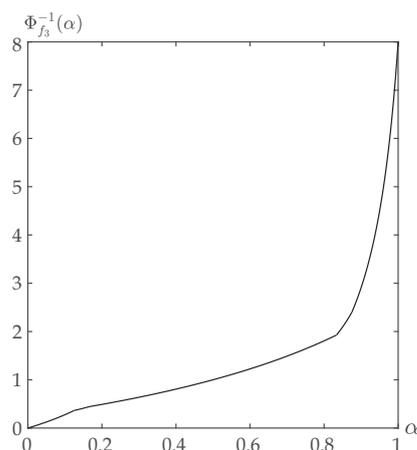
**Case 3:** Set  $f_3(S_1, S_2) = S_2/S_1$ . Similarly,  $f_3$  is strictly increasing with respect to  $S_2$ , and strictly decreasing with respect to  $S_1$ . According to Equation (10)

$$\Phi_{S_1}^{-1}(1 - \alpha) = \begin{cases} 7 - 12\alpha, & \alpha \in [0, 1/6) \\ 11/2 - 3\alpha, & \alpha \in [1/6, 5/6) \\ 13 - 12\alpha, & \alpha \in [5/6, 1], \end{cases} \quad (16)$$

Therefore, it can be obtained that

$$\Phi_{f_3}^{-1}(\alpha) = \begin{cases} \frac{16\alpha}{7 - 12\alpha}, & \alpha \in [0, 1/8) \\ \frac{16\alpha/3 + 4/3}{7 - 12\alpha}, & \alpha \in [1/8, 1/6) \\ \frac{16\alpha/3 + 4/3}{11/2 - 3\alpha}, & \alpha \in [1/6, 5/6) \\ \frac{16\alpha/3 + 4/3}{13 - 12\alpha}, & \alpha \in [5/6, 7/8) \\ \frac{16\alpha - 8}{13 - 12\alpha}, & \alpha \in [7/8, 1]. \end{cases} \quad (17)$$

following from Theorem 4 and Equations (11) and (16), as shown in Figure 8.



**Figure 8.** The inverse credibility function of  $f_3(S_1, S_2) = S_2/S_1$ .

Comparatively, if we calculate the inverse credibility function of  $S_1 + S_2$  by means of Zadeh’s extension principle [1], we should first determine the turning points of the MF of the medium  $f_i(\eta_1, \eta_2)$  for  $f_i(S_1, S_2)$  by identifying the key values of  $\mu_{\eta_1}(x)$  and  $\mu_{\eta_2}(x)$ , and then calculate the MF  $\mu_{f(\eta_1, \eta_2)}(x)$  and credibility distribution  $\Phi_{f_i(S_1, S_2)}(x)$ . Due to space limitations, the solution process is placed in Appendix A, from which we can see that the operational law proposed in this paper can simplify the fuzzy operations without computing the intermediate procedures and possess the correctness.

**5. Expected Value**

The expected value operator is a crucial numerical characteristic of fuzzy variables; however, to the best of our knowledge, there is not yet a clear expected value definition for the IT2-FVs in the existing literature. Therefore, we define the expected value of the IT2-FVs as well as their strictly monotone functions via the credibility distribution in this section.

**Definition 20.** Suppose that  $S$  is an RSTIT2-FV or an FV derived from multiple RSTIT2-FVs operations, then the expected value of  $S$  can be defined as

$$E[S] = \int_0^{+\infty} Cr\{S \geq x\}dx - \int_{-\infty}^0 Cr\{S \leq x\}dx. \tag{18}$$

Obviously, both integrals in Equation (18) are finite.

**Theorem 5.** Given that  $S$  is an RSTIT2-FV or obtained from multiple RSTIT2-FVs operations, then its expected value can be simplified as the following form

$$E[S] = \int_0^1 \Phi_S^{-1}(\alpha)d\alpha.$$

**Proof.** In light of Definitions 5 and 20, we have

$$\begin{aligned} E[S] &= \int_0^{+\infty} Cr\{S \geq x\}dx - \int_{-\infty}^0 Cr\{S \leq x\}dx \\ &= \int_0^{+\infty} (1 - \Phi_S(x))dx - \int_{-\infty}^0 \Phi_S(x)dx \\ &= \int_{\Phi_S(0)}^1 \Phi_S^{-1}(\alpha)d\alpha + \int_0^{\Phi_S(0)} \Phi_S^{-1}(\alpha)d\alpha \\ &= \int_0^1 \Phi_S^{-1}(\alpha)d\alpha. \end{aligned}$$

□

**Example 4.** By means of Theorem 5, the expected value of the RSTIT2-FV  $S_1$  in Example 1 is

$$E[S_1] = \int_0^{1/6} (12\alpha + 1)d\alpha + \int_{1/6}^{5/6} \left(3\alpha + \frac{5}{2}\right)d\alpha + \int_{5/6}^1 (12\alpha - 5)d\alpha = 4.$$

Analogously, the expected value of  $S_2$  in Example 2 can be calculated as

$$E[S_2] = \int_0^{1/8} (16\alpha)d\alpha + \int_{1/8}^{7/8} \left(\frac{16}{3}\alpha + \frac{4}{3}\right)d\alpha + \int_{7/8}^1 (16\alpha - 8)d\alpha = 4.$$

From Example 4, it can be observed that  $E[S_1] = E[S_2] = 4$ . In fact, we can easily obtain that the expected value of an RSTIT2-FV  $S$  determined by the MF  $\begin{pmatrix} c - l_U & c & c + l_U \\ c - l_L & c & c + l_L \end{pmatrix}$  is constantly equal to the center value  $c$ , i.e.,

$$E[S] = \int_0^1 \Phi_S^{-1}(\alpha)d\alpha = c,$$

where  $\Phi_S^{-1}(\alpha)$  is the inverse credibility distribution of  $S$ , which can be calculated by Equation (9). The results show that the expected value operator we defined is an unbiased estimation.

**Example 5.** Following from Theorem 5 and Equation (13), the expected values of  $f_1 = S_1 + S_2$  in Example 3 can be calculated as

$$\begin{aligned} E[f_1] &= \int_0^1 \Phi_{f_1}^{-1}(\alpha)d\alpha \\ &= \int_0^{1/8} (28\alpha + 1)d\alpha + \int_{1/8}^{1/6} \frac{1}{3}(52\alpha + 7)d\alpha + \int_{1/6}^{5/6} \frac{1}{6}(50\alpha + 23)d\alpha \\ &\quad + \int_{5/6}^{7/8} \frac{1}{3}(52\alpha - 11)d\alpha + \int_{7/8}^1 (28\alpha - 13)d\alpha \\ &= 8. \end{aligned}$$

In a similar way, we can get  $E[f_2] = 0$  and  $E[f_3] = 1.3800$ .

**Theorem 6.** Let  $S$  be an RSTIT2-FV and  $k$  be a constant, then we have

$$E[kS] = kE[S].$$

**Proof.** When  $k > 0$ , it is easy to know from Definition 4 that the inverse credibility distribution of  $kS$  is  $k\Phi_S^{-1}(\alpha)$ . Then, by using Theorem 5, it can be deduced that

$$E[kS] = \int_0^1 k\Phi_S^{-1}(\alpha)d\alpha = k \int_0^1 \Phi_S^{-1}(\alpha)d\alpha = kE[S].$$

when  $k < 0$ , the inverse credibility distribution of  $kS$  is  $k\Phi_S^{-1}(1 - \alpha)$ , then we have

$$E[kS] = \int_0^1 k\Phi_S^{-1}(1 - \alpha)d\alpha = -k \int_1^0 \Phi_S^{-1}(t)dt = k \int_0^1 \Phi_S^{-1}(\alpha)d\alpha = kE[S].$$

□

**Theorem 7.** Let  $S_1$  and  $S_2$  be two mutually independent RSTIT2-FVs, then the linearity of the expected value operator concerning the two RSTIT2-FVs can be expressed in the following form,

$$E[k_1S_1 + k_2S_2] = k_1E[S_1] + k_2E[S_2].$$

**Proof.** When  $k_1 > 0$  and  $k_2 > 0$ ,  $(k_1S_1 + k_2S_2)$  is strictly increasing monotone with respect to  $S_1$  and  $S_2$  and, thus, the inverse credibility distribution of  $(k_1S_1 + k_2S_2)$  can be easily obtained as  $(k_1\Phi_{S_1}^{-1}(\alpha) + k_2\Phi_{S_2}^{-1}(\alpha))$  in the views of Definition 4. Following that, and based on Theorems 5 and 6, we have

$$\begin{aligned} E[k_1S_1 + k_2S_2] &= \int_0^1 (k_1\Phi_{S_1}^{-1}(\alpha) + k_2\Phi_{S_2}^{-1}(\alpha)) d\alpha \\ &= k_1E[S_1] + k_2E[S_2]. \end{aligned}$$

when  $k_1 > 0$  and  $k_2 < 0$ ,  $(k_1S_1 + k_2S_2)$  is strictly increasing with respect to  $S_1$  but strictly decreasing with respect to  $S_2$ . It follows immediately from Theorems 5 and 6 that

$$\begin{aligned} E[k_1S_1 + k_2S_2] &= \int_0^1 (k_1\Phi_{S_1}^{-1}(\alpha) + k_2\Phi_{S_2}^{-1}(1 - \alpha)) d\alpha \\ &= k_1E[S_1] + k_2E[S_2]. \end{aligned}$$

In the other two cases (i.e.,  $k_1 < 0$  and  $k_2 > 0$ ,  $k_1 < 0$  and  $k_2 < 0$ ), it is easy to verify that Theorem 7 still holds. Due to space limitations, the details of the proofs are not provided in this paper.  $\square$

**Remark 6.** Definition 7 can be extended to the general case as

$$E\left[\sum_{i=1}^n k_i S_i\right] = \sum_{i=1}^n k_i E[S_i],$$

where  $S_1, S_2, \dots, S_n$  are mutually independent RSTIT2-FVs and  $k_1, k_2, \dots, k_n$  are constants.

Theorem 7 and Remark 6 demonstrate that the introduced expected value operator is consistent with the property of linearity, which can help calculate the expected values of the functions of mutually independent RSTIT2-FVs without calculating the inverse credibility distribution of these RSTIT2-FVs.

**Example 6.** On the basis of Theorem 7 and Example 4, the expected value of the RSTIT2-FV  $f_1 = S_1 + S_2$  in Example 3 can be easily calculated by

$$E[f_1] = E[S_1] + E[S_2] = 4 + 4 = 8,$$

which is equal to the result figured out by the integral form in Example 5.

## 6. Conclusions

In this paper, we defined a special kind of TIT2-FV based on the most typical and simplest symmetric triangular type-1 FV, and proposed a novel operational law to calculate the arithmetic operations for the strictly monotone functions of the mutually independent RSTIT2-FVs as well as the expected value operator via the inverse credibility distribution. The comparative results of the numerical examples verified that the operational law is conducive to simplify the calculation processes, and the expected value operator is an unbiased estimator possessing the property of linearity. Thus, it can be applied to many areas involving complex fuzzy operations.

It is noteworthy that the methodology presented in this paper are not confined to the symmetric and triangular types, but could also be extended to the asymmetric cases

and many other types of type-2 fuzzy variables, such as trapezoidal and normal TIT2-FVs, and the regularity of the TIT2-FVs can be extended to more generalized cases, which would be analyzed in the near future. In other words, the proposed method is a universal methodology, providing new insight for future research of many scholars, in both theory and practice, regarding the type-2 fuzzy set theory.

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## Abbreviations

The following abbreviations are used in the manuscript:

IT2-FV	interval type-2 fuzzy variable
RSTIT2-FV	regular symmetric triangular Interval type-2 fuzzy variable
T2-FS	type-2 fuzzy set
MF	membership function
FV	fuzzy variable
T2-FV	type-2 fuzzy variable
UMF	upper membership function
LMF	lower membership function
T1-FS	type-1 fuzzy set
T1-FV	type-1 fuzzy variable
FOU	footprint of uncertainty

## Appendix A

**Example A1.** In views of Example 1, the key values of  $\mu_{\eta_1}(x)$  for  $S_1$  are, respectively, (1, 0), (3, 1/3), (4, 1), (5, 1/3) and (7, 0), as shown in Figure A1a. Similarly, those of  $\mu_{\eta_2}(x)$  for  $S_2$  are, respectively, (0, 0), (2, 1/4), (4, 1), (6, 1/4) and (8, 0), as Figure A1b shows.

According to Zadeh's extension principle [1], the fuzzy operations of  $\eta_1$  and  $\eta_2$  can only be conducted when the MFs are not piecewise functions. That is, the MFs of the medium for the functions  $f_i$  are piecewise functions which have turning points at  $\mu_{\eta_1+\eta_2}(x) = 0, 1/4, 1/3$  and 1, respectively. In order to obtain these turning points, the  $x$  values for  $\mu_{\eta_1}(x) = 1/4$  and for  $\mu_{\eta_2}(x) = 1/3$  should be calculated (see Table A1). Hence, the key values of  $\eta_1 + \eta_2$  can be deduced, as summarized in the last row of Table A1.

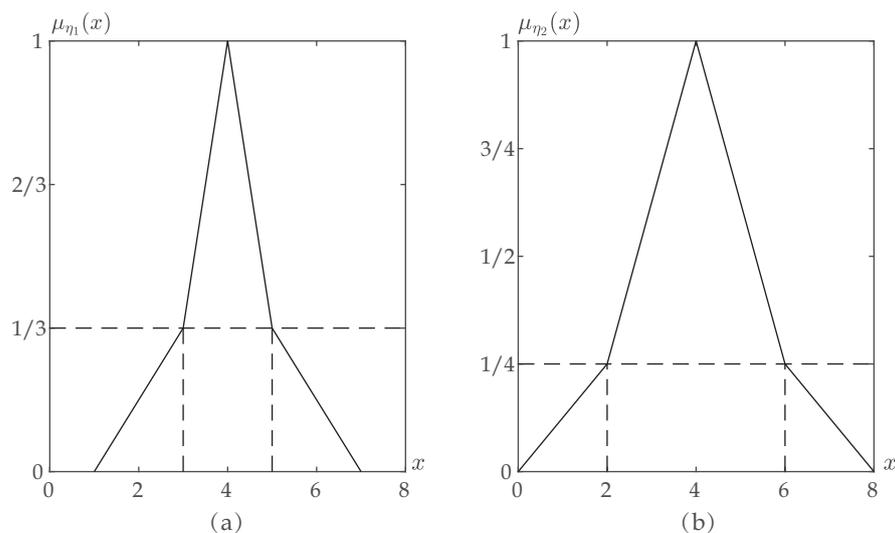


Figure A1. The MF of  $\eta_1$  for  $S_1$  (a) and  $\eta_2$  for  $S_2$  (b).

Table A1. The key values of the MFs of the mediums for  $S_1, S_2$  and  $S_1 + S_2$ .

$\mu$	0	1/4	1/3	1	1/3	1/4	0
$x$ for $\mu_{\eta_1}$	1	5/2	3	4	5	11/2	7
$x$ for $\mu_{\eta_2}$	0	2	20/9	4	52/9	6	8
$x$ for $\mu_{\eta_1+\eta_2}$	1	9/2	47/9	8	97/9	23/2	15

Subsequently, the MF of the medium  $\eta_1 + \eta_2$  for  $S_1 + S_2$  is (see Figure A2)

$$\mu_{\eta_1+\eta_2}(x) = \begin{cases} x/14 - 1/14, & x \in [1, 9/2) \\ 3x/26 - 7/26, & x \in [9/2, 47/9) \\ 6x/25 - 23/25, & x \in [47/9, 8) \\ -6x/25 + 73/25, & x \in [8, 97/9] \\ -3x/26 + 41/26, & x \in [97/9, 23/2] \\ -1x/14 + 15/14, & x \in [23/2, 15] \\ 0, & \text{otherwise,} \end{cases}$$

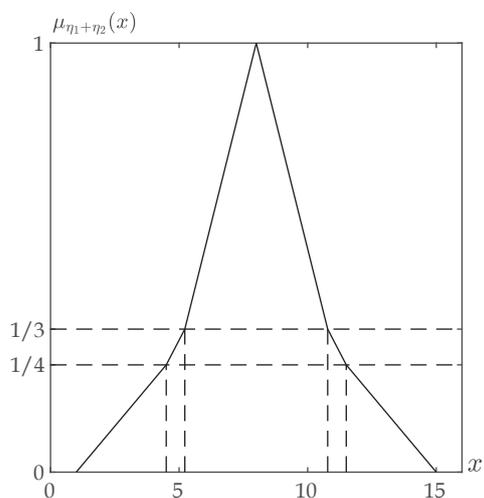


Figure A2. The MF of  $\eta_1 + \eta_2$ .

Accordingly, the credibility distribution can be obtained as

$$\Phi_{f_1}(x) = \begin{cases} x/28 - 1/28, & x \in [1, 9/2) \\ 3x/52 - 7/52, & x \in [9/2, 47/9) \\ 3x/25 - 23/50, & x \in [47/9, 97/9) \\ 3x/52 + 11/52, & x \in [97/9, 23/2) \\ x/28 + 13/28, & x \in [23/2, 15]. \end{cases}$$

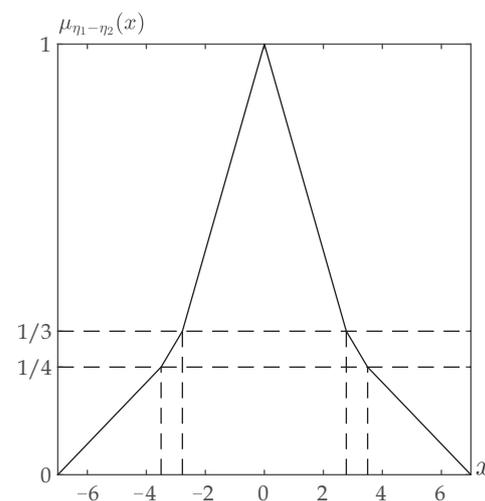
and the inverse credibility distribution is

$$\Phi_{f_1}^{-1}(\alpha) = \begin{cases} 28\alpha + 1, & \alpha \in [0, 1/8) \\ 52\alpha/3 + 7/3, & \alpha \in [1/8, 1/6) \\ 25\alpha/3 + 23/6, & \alpha \in [1/6, 5/6) \\ 52\alpha/3 - 11/3, & \alpha \in [5/6, 7/8) \\ 28\alpha - 13, & \alpha \in [7/8, 1]. \end{cases}$$

Analogously, on the basis of the  $x$  values for  $\mu_{\eta_1}$  and  $\mu_{\eta_2}$  at  $\mu = 0, 1/4, 1/3$  and  $1$ , the key values of the mediums for  $S_1 - S_2$  can be derived, as shown in Table A2 and Figure A3.

**Table A2.** The key values of the MFs of the mediums for  $S_1, -S_2$  and  $S_1 - S_2$ .

$\mu$	0	1/4	1/3	1	1/3	1/4	0
$x$ for $\mu_{\eta_1}$	1	5/2	3	4	5	11/2	7
$x$ for $\mu_{-\eta_2}$	-8	-6	-52/9	-4	-20/9	-2	0
$x$ for $\mu_{\eta_2-\eta_2}$	-7	-7/2	-25/9	0	25/9	7/2	7



**Figure A3.** The MF of  $\eta_1 - \eta_2$ .

Then, we have the MF of

$$\mu_{\eta_1 - \eta_2}(x) = \begin{cases} x/14 + 1/2, & x \in [-7, -7/2) \\ 3x/26 + 17/26, & x \in [-7/2, -25/9) \\ 6x/25 + 1, & x \in [-25/9, 0) \\ -6x/25 + 1, & x \in [0, 25/9] \\ -3x/26 + 17/26, & x \in [25/9, 7/2] \\ -x/14 + 1/2, & x \in [7/2, 7], \\ 0, & \text{otherwise,} \end{cases}$$

Then, the credibility distribution is

$$\Phi_{f_2}(x) = \begin{cases} x/28 + 1/4, & x \in [-7, -7/2) \\ 3x/52 + 17/52, & x \in [-7/2, -25/9) \\ 3x/25 + 1/2, & x \in [-25/9, 25/9) \\ 3x/52 + 35/52, & x \in [25/9, 7/2] \\ x/28 + 3/4, & x \in [7/2, 7] \end{cases}$$

and the inverse credibility distribution is

$$\Phi_{f_2}^{-1}(\alpha) = \begin{cases} 28\alpha - 7, & \alpha \in [0, 1/8) \\ 52\alpha/3 - 17/3, & \alpha \in [1/8, 1/6) \\ 25\alpha/3 - 25/6, & \alpha \in [1/6, 5/6) \\ 52\alpha/3 - 35/3, & \alpha \in [5/6, 7/8) \\ 28\alpha - 21, & \alpha \in [7/8, 1], \end{cases}$$

which is identical with Equation (15).

Following from Table A1, the key values of the medium for  $f_3 = S_2/S_1$  are summarized in Table A3, and we can get the identical inverse credibility of Equation (17) for  $f_3$ .

**Table A3.** The key values of the MF of the medium for  $S_2/S_1$ .

$\mu$	0	1/4	1/3	1	1/3	1/4	0
$x$ for $\mu_{\frac{1}{\eta_1}}$	1/7	2/11	1/5	1/4	1/3	2/5	1
$x$ for $\mu_{\eta_2}$	0	2	20/9	4	52/9	6	8
$x$ for $\mu_{\eta_2/\eta_1}$	0	4/11	4/9	1	52/27	12/5	8

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