symmetry

MDPI

Article

Existence and Convergence Results for Generalized Mixed Quasi-Variational Hemivariational Inequality Problem

Shih-Sen Chang [1,*], Salahuddin [2], Lin Wang [3], Gang Wang [3] and Yunhe Zhao [3]

[1] Center for General Education, China Medical University, Taichung 40402, Taiwan
[2] Department of Mathematics, Jazan University, Jazan 45142, Saudi Arabia; dr_salah12@yahoo.com
[3] College of Statistics and Mathematics, Yunnan University of Finance and Economics, Kunming 650221, China; WL64mail@aliyun.com (L.W.); wg631208@sina.com (G.W.); zyh961019@163.com (Y.Z.)
* Correspondence: changss2013@163.com

Abstract: The main purpose of this paper is threefold. One is to study the existence and convergence problem of solutions for a class of generalized mixed quasi-variational hemivariational inequalities. The second one is to study the existence of optimal control for such kind of generalized mixed quasi-variational hemivariational inequalities under given control $u \in \mathcal{U}$. The third one is to study the relationship between the optimal control and the data for the underlying generalized mixed quasi-variational inequality problems and a class of minimization problem. As an application, we utilize our results to study the elastic frictional problem in a class of Hilbert spaces. The results presented in the paper extend and improve upon some recent results.

Keywords: generalized mixed quasi-variational hemivariational inequality problems; optimal control problems; convergence theory; contact problems; elastic frictional problems; Hausdorff-Lipschitz continuity

JEL Classification: 47J20; 47J22; 49J40; 49J53

Citation: Chang, S.-S.; Salahuddin; Wang, L.; Wang, G.; Zhao, Y. Existence and Convergence Results for Generalized Mixed Quasi-Variational Hemivariational Inequality Problem. *Symmetry* **2021**, *13*, 1882. https://doi.org/10.3390/sym13101882

Academic Editors: Sergei D. Odintsov and Alexander Zaslavski

Received: 12 August 2021
Accepted: 22 September 2021
Published: 5 October 2021

1. Introduction

Variational inequality theory is a very effective and powerful tool for studying a wide range of problems that arise in differential equations, mechanics, contact problems in elasticity, the optimization and control problem, as well as unilateral, obstacle and moving problems (see, for example, [1–8]).

Hemivariational inequalities, which were first initiated by Panagiotopoulos [9], deal with certain mechanical problems involving nonconvex and nonsmooth energy functions. If the energy function is convex, then the hemivariational inequalities reduce to the variational inequalities that have been previously considered by many authors. The hemivariational inequalities have emerged as one of the most promising branches of pure, applied, and industrial mathematics and have achieved a great achievement in the field of mathematical analysis (see, for example, [10–22]).

The main purpose of this article is:

(1) To study the existence and convergence problem of solutions of the following generalized mixed quasi-variational hemivariational inequality, i.e., to find $x \in \mathscr{A}(x)$ and $x^\star \in \mathscr{F}(x)$ such that

$$\langle x^\star - f, y - x\rangle + G^\circ(\hat{x}; \hat{y} - \hat{x}) + \varphi(y, x) - \varphi(x, x) \geq 0, \quad \forall\, y \in \mathscr{A}(x), \tag{1}$$

where $\mathbb{E}$ is a real Banach space, and $\mathbb{E}^\star$ is its dual space. Ω is a nonempty closed convex subset of $\mathbb{E}$, and $\mathscr{A} : \Omega \to 2^\Omega$ is a mapping such that for every $x \in \Omega$, the set $\mathscr{A}(x)$ is a nonempty closed convex subset of Ω. $\mathscr{F} : \mathbb{E} \to 2^{\mathbb{E}^\star}$ is a set-valued mapping, and its domain and the graph are defined by

$$\mathcal{D}(\mathscr{F}) = \{x \in \mathbb{E} : \mathscr{F}(x) \neq \emptyset\} \text{ and } \mathcal{G}(\mathscr{F}) = \{(x, x^\star) : x \in \mathcal{D}(\mathscr{F}), x^\star \in \mathscr{F}(x)\},$$

respectively. $\mathscr{T} : \mathbb{E} \to L^p(\triangle; \mathbb{R}^\ell)$ is a linear continuous operator, where $\ell \geq 1, 1 < p < \infty$. $G^\circ(x; y)$ is the Clarkes generalized directional derivative of the locally Lipschitz mapping $G : L^p(\triangle; \mathbb{R}^\ell) \to \mathbb{R}$ at the point $x \in L^p(\triangle; \mathbb{R}^\ell)$ with respect to direction $y \in L^p(\triangle; \mathbb{R}^\ell)$. $\varphi : \Omega \times \Omega \to \mathbb{R} \cup \{+\infty\}$ is a functional, $f \in \mathbb{E}^\star$, and $\hat{x} = \mathscr{T}x$.

(2) To study the optimal control of (1), for given control $u \in \mathcal{U}$, i.e., to find $x \in \mathscr{A}(x)$ and $x^\star \in \mathscr{F}(x)$ such that

$$\langle x^\star - f, y - x\rangle + G^\circ(\hat{x}; \hat{y} - \hat{x}) + \varphi(y, x) - \varphi(x, x) \geq \langle \mathscr{B}(u), y - x\rangle, \forall y \in \mathscr{A}(x), \quad (2)$$

where $\mathcal{V}$ is the control space, $\mathcal{U} \subset \mathcal{V}$ is the admissible controls (a nonempty closed convex set), and $\mathscr{B} : \mathcal{V} \to \mathbb{E}^\star$ is a compact mapping.

(3) To study the relationship between the solution $x(u)$ of (2), corresponding to the control u, and the optimal control problem to seek an optimal pair $(u, x(u)) \in U \times E$ to solve the following minimization problem:

$$\min_{u \in \mathcal{U}} \complement(u, x(u)), \quad (3)$$

where $\complement(u, x(u))$ is the cost function defined by

$$\complement(u, x(u)) = \|\urcorner(x(u)) - \Im\|^2 + \epsilon\|u\|^2,$$

As an application, we utilize our results to study the elastic frictional problem in a class of Hilbert spaces. The results presented in the paper extend and improve upon some recent results.

2. Preliminaries

In this section, we present some basic concepts that will be used in proving our main results.

In the sequel, we denote by $\to$ the strong convergence and by $\rightharpoonup$ the weak convergence.

Definition 1 ([23]). *The bifunction $\varphi : \Omega \times \Omega \to \mathbb{R} \cup \{+\infty\}$ is called skew-symmetric if and only if*

$$\varphi(x, x) - \varphi(x, y) - \varphi(y, x) - \varphi(y, y) \geq 0, \quad \forall\, x, y \in \Omega. \quad (4)$$

Clearly, if the skew-symmetric (bifunction $\varphi(\cdot, \cdot)$) is bilinear, then

$$\varphi(x, x) - \varphi(x, y) - \varphi(y, x) + \varphi(y, y) = \varphi(x - y, x - y) \geq 0, \quad \forall\, x, y \in \Omega. \quad (5)$$

Lemma 1 ([24,25]). *Let $\jmath : \Omega \to \mathbb{R}$ be locally Lipschitz of rank $\mathscr{L}_x > 0$ near x. Let $\jmath^\circ(x; y)$ be the Clarkes generalized directional derivative of $\jmath : \mathbb{E} \to \mathbb{R}$ at the point $x \in \mathbb{E}$ in the direction $y \in \mathbb{E}$, that is*

$$\jmath^\circ(x; y) = \limsup_{\lambda \to 0^+, \varsigma \to x} \frac{\jmath(\varsigma + \lambda y) - \jmath(\varsigma)}{\lambda}.$$

Let $\partial\jmath(x)$ be the Clarkes subdifferential or generalized gradient of $\jmath$ at $x \in \mathbb{E}$ defined by

$$\partial\jmath(x) = \{x^\star \in \mathbb{E}^\star : \jmath^\circ(x; y) \geq \langle x^\star, y\rangle_{\mathbb{E}}, \quad \forall\, y \in \mathbb{E}\}.$$

Then,

(i) *$\jmath^\circ(x;y)$ is an upper semicontinuous function of (x,y), and $y \mapsto \jmath^\circ(x;y)$ is Lipschitz of rank $\mathscr{L}_x$ near x on $\mathbb{E}$ and satisfies*
$$|\jmath^\circ(x;y)| \leq \mathscr{L}_x \|y\|_{\mathbb{E}};$$
(ii) *The gradient $\partial\jmath(x)$ is nonempty, convex and a weakly* compact subset of $\mathbb{E}^\star$, which is bounded by Lipschitz constant $\mathscr{L}_x$ near x;*
(iii) *For every $y \in \mathbb{E}$, we have*
$$\jmath^\circ(x;y) = \max\{\langle \omega, y\rangle \mid \omega \in \partial\jmath(x)\}.$$

In the sequel, we assume that Θ is a bounded open set in $\mathbb{R}^{\mathcal{N}} (\mathcal{N} \geq 1)$, and $\partial\Theta$ is its boundary. Denote Θ or $\partial\Theta$ by $\triangle$. We assume that $\vartheta : \triangle \times \mathbb{R}^\ell \to \mathbb{R}$ is a function such that the function
$$\vartheta(\cdot,\xi) : \triangle \to \mathbb{R} \text{ is measurable for every } \xi \in \mathbb{R}^\ell. \tag{6}$$

We assume that at least one of the following conditions holds: either there exists $\kappa \in L^q(\triangle;\mathbb{R})$ such that
$$|\vartheta(\theta,\xi_1) - \vartheta(\theta,\xi_2)| \leq \kappa(\theta)\, |\xi_1 - \xi_2|, \ \forall\, \xi_1,\xi_2 \in \mathbb{R}^\ell,\ \theta \in \triangle, \tag{7}$$
or the mapping
$$\vartheta(\theta,\cdot), \ \ \forall\theta \in \triangle \tag{8}$$
is locally Lipschitz continuous and there exists $\mu > 0$ such that
$$|\eta| \leq \mu\Big(1 + |\xi|^{p-1}\Big),\ \forall\, \theta \in \triangle, \eta \in \partial\vartheta(\theta,\xi). \tag{9}$$

Under the above conditions, we have the following result:

Lemma 2 ([24], Theorem 2.7.5)**.** *If*
$$G(\phi) = \int_\triangle \vartheta(\theta,\phi(\theta))d\theta, \tag{10}$$
and ϑ satisfies the conditions (6) and (7) or (6), (8) and (9), then G is Lipschitz on bounded subsets of $L^p(\triangle;\mathbb{R}^\ell)$, and one has
$$\partial G(\phi) \subset \int_\triangle \partial\vartheta(\theta,\phi(\theta))d\theta.$$
Furthermore, if ϑ is regular at $(x,\phi(x))$, then G is regular at ϕ, and equality holds.

Definition 2. *Let $\mathscr{F} : \Omega \to 2^{\mathbb{E}^\star}$ be a mapping. Then $\mathscr{F}$ is said to be*
(i) *Monotone, if for each $x, y \in \Omega$,*
$$\langle y^\star - x^\star, y - x\rangle \geq 0, \forall x^\star \in \mathscr{F}(x), y^\star \in \mathscr{F}(y);$$
(ii) *Maximal monotone, if the graph of the monotone mapping $\mathscr{F}$ is not included in the graph of any other monotone map with the same domain;*
(iii) *Pseudomonotone, if*
  (a) *For each $x \in \Omega$, the set $\mathscr{F}(x)$ is nonempty, bounded, closed and convex;*
  (b) *The mapping $\mathscr{F}$ is u.s.c. from each finite-dimensional subspace of $\mathbb{E}$ to $\mathbb{E}^\star$ endowed with the weak topology;*
  (c) *If $\{x_n\} \subset \mathbb{E}$ with $x_n \rightharpoonup x \in \mathbb{E}$, and $x_n^\star \in \mathscr{F}(x_n)$ such that*
$$\limsup_{n\to\infty} \langle x_n^\star, x_n - x\rangle \leq 0,$$

then for every $y \in \mathbb{E}$, there exists $x^\star(y) \in \mathscr{F}(x)$ such that

$$\langle x^\star(y), x - y\rangle \le \liminf_{n\to\infty}\langle x_n^\star, x_n - y\rangle;$$

(iv) *Generalized pseudomonotone, if for any sequence $\{x_n\} \subset \mathbb{E}$ with $x_n \rightharpoonup x \in \mathbb{E}$ and $x_n^\star \in \mathscr{F}(x_n)$ with $x_n^\star \rightharpoonup x^\star$ such that*

$$\limsup_{n\to\infty}\langle x_n^\star, x_n - x\rangle \le 0,$$

we have $x^\star \in \mathscr{F}(x)$ and

$$\langle x_n^\star, x_n\rangle \to \langle x^\star, x\rangle.$$

Definition 3. *A mapping $\mathscr{A} : \Omega \to 2^\Omega$ is called $\mathfrak{M}$-continuous, if the following conditions hold:*

(**M1**) *For any sequence $\{x_n\}_{n\ge1} \subset \Omega$ with $x_n \rightharpoonup x$, and for each $y \in \mathscr{A}(x)$, there exists $\{y_n\}_{n\ge1}$ such that $y_n \in \mathscr{A}(x_n)$ and $y_n \to y$;*

(**M2**) *For $y_n \in \mathscr{A}(x_n)$ with $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$, we have $y \in \mathscr{A}(x)$.*

3. Existence Theorems

This section is devoted to the existence theory of the generalized mixed quasi-variational hemivariational inequality problems.

Lemma 3. *Let $\mathbb{E}$ be a Banach space, Ω be a nonempty compact subset of $\mathbb{E}$. Let $\mathscr{F} : \Omega \to 2^{\mathbb{E}^\star}$ be a sequentially bounded (i.e., if $x_n \to x$, then $\bigcup_{n\ge1} \mathscr{F}(x_n)$ is bounded in $\mathbb{E}^\star$), pseudomonotone mapping. Let $\mathscr{T} : \mathbb{E} \to L^p(\triangle;\mathbb{R}^\ell)$ be a linear continuous operator and $G : L^p(\triangle;\mathbb{R}^\ell) \to \mathbb{R}$ be the locally Lipschitz function defined by (10). Assume that $\varphi : \Omega\times\Omega \to \mathbb{R}\cup\{+\infty\}$ is skew-symmetric, then for any $f \in \mathbb{E}^\star$, the mapping $\mathfrak{L} : \Omega \to \Omega$ defined by*

$$\mathfrak{L}(x) = \left\{ y \in \Omega \inf_{\substack{x^\star\in\mathscr{F}(x),\\ \omega\in\partial G(\hat{x})}} \langle x^\star + \mathscr{T}^\star\omega - f, x - y\rangle + \varphi(x,x) - \varphi(y,x) \le 0 \right\}$$

has a closed graph in $\Omega\times\Omega$.

Proof. Let $\{(x_n, y_n)\} \in \mathcal{G}(\mathfrak{L})$, $x_n \to x$ and $y_n \to y$. We prove that $(x,y) \in \mathcal{G}(\mathfrak{L})$.

In fact, for each $n \in \mathcal{N}$, we have

$$\inf_{\substack{x_n^\star\in\mathscr{F}(x_n),\\ \omega_n\in\partial G(\hat{x_n})}} \langle x_n^\star + \mathscr{T}^\star\omega_n - f, x_n - y_n\rangle + \varphi(x_n,x_n) - \varphi(y_n,x_n) \le 0.$$

Hence, there exist $\tilde{x}_n^\star \in \mathscr{F}(x_n)$, $\tilde{\omega}_n \in \partial G(\hat{x}_n)$ such that

$$\langle \tilde{x}_n^\star + \mathscr{T}^\star\tilde{\omega}_n - f, x_n - y_n\rangle + \varphi(x_n,x_n) - \varphi(y_n,x_n) \le \frac{1}{n}. \tag{11}$$

Since $\mathscr{F}$ is sequentially bounded, this implies that $\{\tilde{x}_n^\star\}$ is bounded. Therefore, we have

$$\limsup_{n\to\infty}\langle \tilde{x}_n^\star, x_n - x\rangle = 0.$$

Again, since $\mathscr{F}$ is pseudomonotone, there exists $x^\star(y) \in \mathscr{F}(x)$ such that

$$\begin{aligned}\langle x^\star(y), x-y\rangle &\le \liminf_{n\to\infty}\langle \tilde{x}_n^\star, x_n - y\rangle \\ &= \liminf_{n\to\infty}(\langle \tilde{x}_n^\star, x_n-y_n\rangle + \langle \tilde{x}_n^\star, y_n - y\rangle) \\ &= \liminf_{n\to\infty}(\langle \tilde{x}_n^\star, x_n - y_n\rangle.\end{aligned}$$

It follows from Lemma 1 (ii) that $\{\tilde{\omega}_n\}$ is bounded. By Proposition 2.1.5 of [24], without loss of generality, we may assume the sequence $\{\tilde{\omega}_n\}$ converges weakly to some $\omega \in \partial G(\tilde{x})$. Hence, we obtain

$$\langle \mathscr{T}^\star\omega - f, x-y\rangle + \varphi(y,x) - \varphi(x,x) = \lim_{n\to\infty}\langle \mathscr{T}^\star\tilde{\omega}_n - f, x_n - y\rangle + \varphi(y,x) - \varphi(x_n,x).$$

Consequently, from (11), we have

$$\begin{aligned}&\langle x^\star(y) + \mathscr{T}^\star\omega - f, x-y\rangle + \varphi(x,x) - \varphi(y,x) \\ &\qquad \le \liminf_{n\to\infty}\langle \tilde{x}_n^\star + \mathscr{T}^\star\tilde{\omega}_n - f, x_n - y_n\rangle + \varphi(x_n,x_n) - \varphi(y_n,x_n) \\ &\qquad \le 0,\end{aligned}$$

which shows that $(x,y) \in \mathcal{G}(\mathfrak{L})$. The proof is completed. □

Theorem 1. *Let $\mathbb{E}$ be a separable Banach space and Ω be a nonempty compact convex subset of $\mathbb{E}$. Suppose that $\mathscr{F} : \Omega \to 2^{\mathbb{E}^\star}$ is a sequentially bounded, pseudomonotone mapping and for any $x \in \Omega$, $\mathscr{F}(x)$ is weakly* compact and convex. Let $\mathscr{T} : \mathbb{E} \to L^p(\triangle;\mathbb{R}^\ell)$ be a linear continuous operator and $G : L^p(\triangle;\mathbb{R}^\ell) \to \mathbb{R}$ be the locally Lipschitz function defined by (10). Let $\mathscr{A} : \Omega \to 2^\Omega$ be an l.s.c. mapping with a closed graph and nonempty convex values. Assume that $\varphi : \Omega\times\Omega \to \mathbb{R}\cup\{+\infty\}$ is skew-symmetric, then for any $f \in \mathbb{E}^\star$, (1) has at least one solution.*

Proof. It follows from Lemma 1 (ii) that for every $y \in \Omega$, $\partial G(\hat{y})$ is weakly* compact and convex. Again, by the assumption that $\mathscr{F}(y)$ is weakly* compact and convex, this implies that

$$\mathscr{F}(y) + \partial G(\hat{y})$$

is also compact and convex. By virtue of Lemma 2 of [26] and Lemma 3, there exist $x \in \mathscr{A}(x), x^\star \in \mathscr{F}(x)$ and $\omega \in \partial G(\hat{x})$ such that

$$\langle x^\star + \mathscr{T}^\star\omega - f, y-x\rangle + \varphi(y,x) - \varphi(x,x) \ge 0 \ \ \forall\, y \in \mathscr{A}(x),$$

i.e.,

$$\langle x^\star - f, y-x\rangle + \langle \mathscr{T}^\star\omega, y-x\rangle + \varphi(y,x) - \varphi(x,x) \ge 0,\ \forall\, y \in \mathscr{A}(x).$$

Therefore, we have

$$\langle x^\star - f, y-x\rangle + G^\circ(\hat{x};\hat{y}-\hat{x}) + \varphi(y,x) - \varphi(x,x) \ge 0,\ \forall\, y \in \mathscr{A}(x).$$

This completes the proof of Theorem 1. □

Theorem 2. *Let $\mathbb{E}$ and $\mathcal{V}$ be two real reflexive Banach space, and $\mathcal{U}$ be a nonempty closed convex subset of $\mathcal{V}$. Let $\mathcal{W}$ be a Banach space, $\Im \in \mathcal{W}$ and $f \in \mathbb{E}^\star$. Assume further that*

(i) $\mathscr{A} : \mathbb{E} \to 2^{\mathbb{E}}$ *is $\mathfrak{M}$-continuous;*

(ii) $\mathscr{F} : \mathbb{E} \to 2^{\mathbb{E}^\star}$ *is bounded, pseudomonotone, and there is a bounded, closed and convex set $\mathbb{S} \subset \mathbb{E}$ such that*

$$\mathscr{A}(x) \cap \mathbb{S} = \emptyset,\ \ \textit{for every}\ \ x \in \mathbb{E},$$

$$\inf_{x^\star \in \mathscr{F}(x)} \frac{\langle x^\star, x-y\rangle}{\|x\|} \to \infty \ \textit{as}\ \|x\| \to \infty \ \textit{uniformly in}\ y \in \mathbb{S}, \tag{12}$$

(iii) $G : L^p(\triangle;\mathbb{R}^\ell) \to \mathbb{R}$ *is the Lipschitz function defined by* (10);
(iv) $\mathscr{T} : \mathbb{E} \to L^p(\triangle;\mathbb{R}^\ell)$, $\daleth : \mathbb{E} \to \mathcal{W}$, $\mathscr{B} : \mathcal{V} \to \mathbb{E}^\star$ *are compact;*
(v) $\varphi : \Omega \times \Omega \to \mathbb{R} \cup \{+\infty\}$ *is a functional.*

If for every control $u \in \mathcal{U}$, (2) *has a solution, then the optimal control problem* (3) *has a solution pair* $(u, x(u)) \in \mathcal{U} \times \mathbb{E}$.

Proof. Let $\{(u_n, x_n)\} \subset \mathcal{U} \times \mathbb{E}$ be a minimizing sequence such that

$$\lim_{n\to\infty} \complement(u_n, x_n) = \min\{\complement(v, x(v)) : v \in \mathcal{U}\},$$

where $u_n \in \mathcal{U}$, and x_n is a solution of (2) that corresponds to the control u_n, that is,

$$x_n = x(u_n).$$

Consequently,

$$x_n \in \mathscr{A}(x_n) \text{ for some } x_n^\star \in \mathscr{F}(x_n);$$

hence, we have

$$\langle x_n^\star - f, y - x_n\rangle + G^\circ(\hat{x}_n; \hat{y} - \hat{x}_n) + \varphi(y, x_n) - \varphi(x_n, x_n) \geq \langle \mathscr{B}(u_n), y - x_n\rangle, \quad \forall\, y \in \mathscr{A}(x_n). \tag{13}$$

When n is large enough, we have

$$\begin{aligned} \epsilon\|u_n\|^2 &\leq \|\daleth(x_n) - \Im\|^2 + \epsilon\|u_n\|^2 \\ &\leq \lim_{n\to\infty} \complement(u_n, x_n) + 1. \end{aligned}$$

Hence, $\{u_n\}$ is a bounded sequence in $\mathcal{V}$. Since $\mathcal{V}$ is a reflexive space, there is a subsequence of $\{u_n\}$, denoted by $\{u_n\}$ again, such that

$$u_n \rightharpoonup \bar{u} \text{ for some } \bar{u} \in \mathcal{V}.$$

Since $\mathcal{U}$ is closed convex from Theorem 1.33 of [27], we deduce that $\mathcal{U}$ is weakly closed, and, hence, $\bar{u} \in \mathcal{U}$.

Next, we choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ corresponding to the subsequence of controls $\{u_{n_j}\}$, which is a solutions of (2). Now, we prove that $\{x_{n_j}\}$ remains bounded.

Suppose that, on the contrary, we assume that $\{x_{n_j}\}$ is unbounded. Let $\{x_{n_i}\}$ be a subsequence of$\{x_{n_j}\}$ (for simplicity, we denote it by $\{x_m\}$) such that

$$\|x_m\| \to \infty \text{ as } m \to \infty.$$

We choose an arbitrary $s_m \in \mathscr{A}(x_m) \cap \mathbb{S}$. Since the set $\mathbb{S}$ is bounded, the sequence $\{s_m\}$ remains bounded. By substituting $y = s_m$ in (13), we obtain

$$\langle x_m^\star - f, s_m - x_m\rangle + G^\circ(\hat{x}_m; \hat{s_m} - \hat{x_m}) + \varphi(s_m, x_m) - \varphi(x_m, x_m) \geq \langle \mathscr{B}(u_m), s_m - x_m\rangle.$$

Since

$$|G^\circ(\hat{x_m}; \hat{s_m} - \hat{x_m})| \leq \mathscr{L}\|\mathscr{T}\|\|s_m - x_m\|, \text{ for } \mathscr{L} > 0,$$

we have

$$\frac{\langle x_m^\star, s_m - x_m\rangle}{\|x_m\|} \leq (\|f\| + \|\mathscr{B}(u_m)\| + \mathscr{L}\|\mathscr{T}\|)\left(1 + \frac{\|s_m\|}{\|x_m\|}\right).$$

As $m \to \infty$, the above inequality is bounded, which is a contradiction to (12). This shows the boundedness of $\{x_{n_j}\}$.

Let $\{x_n\}$ be a subsequence converging weakly to $\bar{x} \in \mathbb{E}$. We will prove that $\bar{x}$ is a solution of (2) that corresponds to $\bar{u}$, that is,

$$\bar{x} = x(\bar{u}).$$

Since $\mathscr{A}$ is $\mathfrak{M}$-continuous, we have

$$\bar{x} \in \mathscr{A}(\bar{x}).$$

Hence, for $\bar{x} \in \mathscr{A}(\bar{x})$, there exists $\{x'_n\}$ with $x'_n \in \mathscr{A}(x_n)$ and

$$x'_n \to \bar{x}.$$

Therefore, by substituting $y = x'_n$ in (13), utilizing the boundedness of $\mathscr{F}$, the compactness of $\mathscr{B}$, $\mathscr{T}$, Lemma 1(i) and rearranging the terms, we obtain

$$\begin{aligned}
\limsup_{n\to\infty}\langle x^\star_n, x_n - \bar{x}\rangle &\leq \limsup_{n\to\infty}\langle x^\star_n, x_n - x'_n\rangle + \limsup_{n\to\infty}\langle x^\star_n, x'_n - \bar{x}\rangle \\
&\leq \limsup_{n\to\infty}\Big[\langle f + \mathscr{B}(u_n), x_n - x'_n\rangle + G^\circ(\hat{x_n}; \hat{x'_n} - \hat{x_n}) + \varphi(x_n, \bar{x}) \\
&\qquad - \varphi(\bar{x}, \bar{x})\Big] \\
&\leq \limsup_{n\to\infty}\langle f + \mathscr{B}(u_n), x_n - \bar{x}\rangle + \limsup_{n\to\infty}\langle f + \mathscr{B}(u_n), \bar{x} - x'_n\rangle \\
&\qquad + \limsup_{n\to\infty} G^\circ(\hat{x_n}; \hat{x'_n} - \hat{x_n}) - \varphi(\bar{x}, \bar{x}) + \varphi(x_n, \bar{x}) \\
&\leq 0.
\end{aligned}$$

Since every pseudomonotone mapping is a generalized pseudomonotone, see [27], we deduce that $\mathscr{F}$ is generalized pseudomonotone. Thus, for a subsequence $\{x^\star_n\}$ such that

$$x^\star_n \rightharpoonup \bar{x}^\star,$$

we have

$$\bar{x}^\star \in \mathscr{F}(\bar{x})$$

and

$$\lim_{n\to\infty}\langle x^\star_n, x_n\rangle = \langle \bar{x}^\star, \bar{x}\rangle.$$

Let $\bar{y} \in \mathscr{A}(\bar{x})$ be arbitrary and $\{y_n\}$ be such that

$$y_n \in \mathscr{A}(x_n) \text{ and } y_n \to \bar{y}.$$

We have

$$\begin{aligned}
\langle \bar{x}^\star, \bar{x} - \bar{y}\rangle &= \limsup_{n\to\infty}\langle x^\star_n, x_n - y_n\rangle \\
&\leq \limsup_{n\to\infty}[\langle f + \mathscr{B}(u_n), x_n - y_n\rangle + G^\circ(\hat{x_n}; \hat{y_n} - \hat{x_n}) - \varphi(x_n, x_n) + \varphi(y_n, x_n)] \\
&\leq \langle f, \bar{x} - \bar{y}\rangle + \langle \mathscr{B}(\bar{u}), \bar{x} - \bar{y}\rangle + G^\circ(\hat{\bar{x}}; \hat{\bar{y}} - \hat{\bar{x}}) + \varphi(\bar{x}, \bar{x}) - \varphi(\bar{y}, \bar{x}).
\end{aligned}$$

Since $\bar{y} \in \mathscr{A}(\bar{x})$ is arbitrary, we have

$$\langle \bar{x}^\star - f, \bar{y} - \bar{x}\rangle + G^\circ(\hat{\bar{x}}; \hat{\bar{y}} - \hat{\bar{x}}) + \varphi(\bar{y}, \bar{x}) - \varphi(\bar{x}, \bar{x}) \geq \langle \mathscr{B}(\bar{u}), \bar{y} - \bar{x}\rangle, \ \forall\, \bar{y} \in \mathscr{A}(\bar{x}).$$

Hence, $\bar{x}$ is a solution of (2) that corresponds to the control $\bar{u}$, that is

$$\bar{x} = x(\bar{u}).$$

Finally, we have

$$\begin{aligned}\complement(\bar{u},\bar{x}) &= \|\urcorner(\bar{x}) - \Im\|^2 + \epsilon\|\bar{u}\|^2 \\ &\leq \liminf_{n\to\infty} \|\urcorner(x_n) - \Im\|^2 + \liminf_{n\to\infty} \epsilon\|u_n\|^2 \\ &\leq \liminf_{n\to\infty} \complement(u_n, x_n) \\ &= \lim_{n\to\infty} \complement(u_n, x_n) \\ &= \inf\{\complement(v, x(v)) : v \in \mathcal{U}\},\end{aligned}$$

which shows that $(\bar{u}, \bar{x})$ is an optimal pair and completes the proof. □

Theorem 3. *Let $\mathbb{E}$, $\mathcal{V}$, Ω, $\mathcal{U}$, and $\mathcal{W}$ be the same as in Theorem 2. Let $\Im \in \mathcal{W}$ and $f \in \mathbb{E}^\star$. Assume further that*

(i) *$\mathscr{A} : \Omega \to 2^{\Omega}$ is $\mathfrak{M}$-continuous;*
(ii) *$\mathscr{F} : \mathbb{E} \to 2^{\mathbb{E}^\star}$ is maximal monotone, $\Omega \subset int(\mathcal{D}(\mathscr{F}))$, and there exists $x_0 \in \bigcap_{y\in\Omega} \mathscr{A}(y)$ such that for every $x \in \Omega$, $x^\star \in \mathscr{F}(x)$,*

$$\frac{\langle x^\star, x - x_0\rangle}{\|x\|} \to \infty \ \text{as} \ \|x\| \to \infty; \tag{14}$$

(iii) *$G : L^p(\triangle; \mathbb{R}^\ell) \to \mathbb{R}$ is the uniformly Lipschitz function defined by (10);*
(iv) *$\mathscr{T} : \mathbb{E} \to L^p(\triangle; \mathbb{R}^\ell)$, $\urcorner : \mathbb{E} \to \mathcal{W}$, $\mathscr{B} : \mathcal{V} \to \mathbb{E}^\star$ are compact;*
(v) *$\varphi : \Omega \times \Omega \to \bar{\mathbb{R}}$ is a functional.*

If for every control $u \in \mathcal{U}$, (2) has a solution, then the optimal control problem (3) has a solution $(u, x(u))$.

Proof. The proof follows from the corollary of [28] and Theorem 2. □

4. Convergence Theory

Given an observation space $\mathcal{W}$, a compact mapping $\urcorner_n : \mathbb{E} \to \mathcal{W}$ $(n \in \mathcal{N})$, and a target $\Im \in \mathcal{W}$, we consider the following perturbed cost function:

$$\complement_n(u, x(u)) = \|\urcorner_n(x(u)) - \Im\|^2 + \epsilon\|u\|^2,$$

where $\epsilon > 0$, and $x(u)$ is a solution of (2), which corresponds to the control u through the following perturbed generalized mixed quasi-variational hemivariational inequality problem for finding $x \in \mathscr{A}_n(x)$, $x^\star \in \mathscr{F}_n(x)$. We have

$$\langle x^\star - f_n, y - x\rangle + G_n^\circ(\hat{x}; \hat{y} - \hat{x}) + \varphi(y, x) - \varphi(x, x) \geq \langle \mathscr{B}(u), y - x\rangle, \ \forall\, y \in \mathscr{A}_n(x), \tag{15}$$

where $f_n \in \mathbb{E}^\star$.

In this section, we are interested in the convergence behavior of the optimal control problem, which has an optimal pair $(u, x_n(u)) \in \mathcal{U} \times \mathbb{E}$ that solves the following minimization problem:

$$\min_{u\in\mathcal{U}} \complement_n(u, x(u)), \tag{16}$$

where $x_n(u)$ is a solution of (15), which corresponds to u. In order to obtain the result of this section, we need the following assumptions:

$(\mathcal{H}_{\mathscr{A}})$: For any $x_n, x \in \Omega$ with $x_n \rightharpoonup x$, there exists a continuous function $\tau_1 : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\mathcal{H}(\mathscr{A}(x), \mathscr{A}_n(x_n)) \leq \varrho_1^n \tau_1(\|x\|), \tag{17}$$

where

$$\mathcal{H}(\mathbb{Q}, \mathbb{S}) = \max\left\{ \sup_{y\in\mathbb{Q}} d(y, \mathbb{S}), \sup_{z\in\mathbb{S}} d(z, \mathbb{S}) \right\}$$

is the Hausdorff distance between the sets $\mathbb{Q}$ and $\mathbb{S}$, and $\{\varrho_1^n\}$ is a sequence of positive reals.

$(\mathcal{H}_{\mathscr{F}})$: For any $x \in \mathbb{E}$, there exists a continuous function $\tau_2 : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\mathcal{H}(\mathscr{F}(x), \mathscr{F}_n(x)) \leq \varrho_2^n \tau_2(\|x\|), \tag{18}$$

where $\{\varrho_2^n\}$ is a sequence of positive reals.

$(\mathcal{H}_G)$: For any $x, y \in L^p(\triangle; \mathbb{R}^\ell)$, there exists a continuous function $\tau_3 : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|G^\circ(x; y) - G_n^\circ(x; y)| \leq \varrho_3^n \tau_3(\|y\|), \tag{19}$$

where $\{\varrho_3^n\}$ is a sequence of positive reals.

$(\mathcal{H}_f)$: For every $n \in \mathcal{N}$,

$$\|f_n - f\| \leq \varrho_4^n, \quad \varrho_4^n \geq 0,$$

where $\{\varrho_4^n\}$ is a sequence of positive reals.

$(\mathcal{H}_0)$: For $n \to \infty$, the sequence is

$$\varrho_1^n \to 0, \quad \varrho_2^n \to 0, \quad \varrho_3^n \to 0, \quad \varrho_4^n \to 0. \tag{20}$$

We have the following theorem.

Theorem 4. *Let Ω, $\mathcal{U}$, $\mathbb{E}$, $\mathcal{V}$, and $\mathcal{W}$ be the same as in Theorem 3, $\Im \in \mathcal{W}$ and $f, f_n \in \mathbb{E}^\star$ $(n \in \mathcal{N})$. Assume that*

(i) $\mathscr{A}, \mathscr{A}_n : \Omega \to 2^\Omega$ are $\mathfrak{M}$-continuous;

(ii) $\mathscr{F}, \mathscr{F}_n : \mathbb{E} \to 2^{\mathbb{E}^\star}$ satisfies the assumptions of (ii) of Theorem 3;

(iii) $G : L^p(\triangle; \mathbb{R}^\ell) \to \mathbb{R}$ is the Lipschitz function defined by (10) and $G_n : L^p(\triangle; \mathbb{R}^\ell) \to \mathbb{R}$ $(n \in \mathcal{N})$ are uniformly Lipschitz defined by (10) corresponding to ϑ_n;

(iv) $\mathscr{T} : \mathbb{E} \to L^p(\triangle; \mathbb{R}^\ell)$, $\daleth : \mathbb{E} \to \mathcal{W}$, $\mathscr{B} : \mathcal{V} \to \mathbb{E}^\star$ are compact;

(v) $\varphi : \Omega \times \Omega \to \mathbb{R} \cup \{+\infty\}$ is a functional.

Suppose that for every control $u \in \mathcal{U}$, (15) has a solution and (2) has a unique solution. If $(\mathcal{H}_{\mathscr{A}}), (\mathcal{H}_G), (\mathcal{H}_{\mathscr{F}}), (\mathcal{H}_f), (\mathcal{H}_0)$ are satisfied, then for every $n \in \mathcal{N}$, (16) has a solution (u_n, x_n), and there exists a subsequence of $\{(u_n, x_n)\}$ that converges weakly to a solution of (3).

Proof. From Theorem 3, for every $n \in \mathcal{N}$, (16) has a solution (u_n, x_n). We first assume that $\{u_n\}$ is bounded. Therefore, we can extract a subsequence of $\{u_n\}$, denoted by $\{u_n\}$ again, that converges weakly to some $\bar{u} \in \mathcal{U}$. Let $\{x_n\}$ be a sequence of solutions of (15), which corresponds with the subsequence $\{u_n\}$. Therefore, $x_n \in \mathscr{A}_n(x_n)$, $x_n^\star \in \mathscr{F}_n(x_n)$, we have

$$\begin{aligned} \langle x_n^\star - f_n, y - x_n\rangle + G_n^\circ(\hat{x}_n; \hat{y} - \hat{x}_n) + \varphi(y, x_n) - \varphi(x_n, x_n) \geq \langle \mathscr{B}(u_n), y - x_n\rangle, \\ \forall\, y \in \mathscr{A}_n(x_n). \end{aligned} \tag{21}$$

We prove that $\{x_n\}$ is bounded. Suppose, to the contrary, we assume that $\{x_n\}$ is unbounded. Let $\{x_n\}$ be a subsequence $\|x_n\| \to \infty$ as $n \to \infty$. By substituting $y = x_0$ in (21), we obtain

$$\langle x_n^\star - f_n, x_0 - x_n\rangle + G_n^\circ(\hat{x}_n; \hat{x}_0 - \hat{x}_n) + \varphi(x_0, x_n) - \varphi(x_n, x_n) \geq \langle \mathscr{B}(u_n), x_0 - x_n\rangle.$$

After a rearrangement of terms, since

$$|G_n^\circ(\hat{x}_n; \hat{s}_n - \hat{x}_n)| \leq \mathscr{L}' \|\mathscr{T}\| \|s_n - x_n\|, \text{ for } \mathscr{L}' > 0,$$

we have

$$\frac{\langle x_n^\star, x_0 - x_n\rangle}{\|x_n\|} \le (\|f\| + \beta_n + \|\mathscr{B}(u_n)\| + \mathscr{L}'\|\mathscr{T}\|) + \left(1 + \frac{\|x_0\|}{\|x_n\|}\right).$$

Since the right-hand side of the above inequality is bounded as $n \to \infty$, we obtain a contradiction to (14). This implies the boundedness of $\{x_n\}$.

Therefore, from $(\mathcal{H}_{\mathscr{A}})$ and $(\mathcal{H}_0)$, there exists a subsequence $\{x_n\}$, which converges weakly to some $\bar{x} \in \mathscr{A}(\bar{x})$, and the corresponding sequence of controls $\{u_n\}$ such that for $\tilde{x}_n^\star \in \mathscr{F}_n(x_n)$, we have

$$\langle \tilde{x}_n^\star - f_n, y - x_n\rangle + G_n^\circ(\hat{x}_n; \hat{y} - \hat{x}_n) + \varphi(y, x_n) - \varphi(x_n, x_n) \ge \langle \mathscr{B}(u_n), y - x_n\rangle,\ \forall\, y \in \mathscr{A}_n(x_n). \tag{22}$$

Let $\bar{y} \in \mathscr{A}(\bar{x})$ be arbitrary and $\{y_n\}$ be such that $y_n \in \mathscr{A}_n(x_n)$,

$$\begin{aligned}
\|y_n - \bar{y}\| &\le d(\mathscr{A}_n(x_n), \bar{y}) + \varepsilon_n \\
&\le \sup_{y \in \mathscr{A}(\bar{x})} d(\mathscr{A}_n(x_n), y) + \varepsilon_n \\
&\le \mathcal{H}(\mathscr{A}_n(x_n), \mathscr{A}(\bar{x})) + \varepsilon_n \\
&\le \varrho_1^n \tau_1(\|\bar{x}\|) + \varepsilon_n,
\end{aligned} \tag{23}$$

where $\varepsilon_n \downarrow 0$, and we satisfy

$$\langle \tilde{x}_n^\star - f_n, y_n - x_n\rangle + G_n^\circ(\hat{x}_n; \hat{y}_n - \hat{x}_n) + \varphi(y_n, x_n) - \varphi(x_n, x_n) \ge \langle \mathscr{B}(u_n), y_n - x_n\rangle. \tag{24}$$

We assume that $\{\tilde{x}_n^\star\}$ is bounded. Due to assumption $(\mathcal{H}_{\mathscr{F}})$ and similar to (23), there exists $\{x_n^\star\}$ with $x_n^\star \in \mathscr{F}(x_n)$ satisfying

$$\|\tilde{x}_n^\star - x_n^\star\| \le \varrho_2^n \tau_2(\|x_n\|) + \varepsilon_n. \tag{25}$$

For any $\bar{y}^\star \in \mathscr{F}(\bar{y})$, since $\mathscr{F}$ is monotone, we have

$$\begin{aligned}
\langle \bar{y}^\star, x_n - \bar{y}\rangle &\le \langle \bar{y}^\star, x_n - \bar{y}\rangle + \langle \tilde{x}_n^\star - f_n, y_n - x_n\rangle + G_n^\circ(\hat{x}_n; \hat{y}_n - \hat{x}_n) - \langle \mathscr{B}(u_n), y_n - x_n\rangle \\
&\quad - \varphi(y_n, x_n) + \varphi(x_n, x_n) \\
&= \langle \tilde{x}_n^\star - x_n^\star, y_n - x_n\rangle - \langle f_n + \mathscr{B}(u_n), y_n - x_n\rangle + \langle x_n^\star, y_n - \bar{y}\rangle \\
&\quad + \langle x_n^\star - \bar{y}^\star, \bar{y} - x_n\rangle + G_n^\circ(\hat{x}_n; \hat{y}_n - \hat{x}_n) - \varphi(y_n, x_n) + \varphi(x_n, x_n) \\
&\le \langle \tilde{x}_n^\star - x_n^\star, y_n - x_n\rangle - \langle f_n + \mathscr{B}(u_n), y_n - x_n\rangle + \langle x_n^\star, y_n - \bar{y}\rangle \\
&\quad + G_n^\circ(\hat{x}_n; \hat{y}_n - \hat{x}_n) - G^\circ(\hat{x}_n; \hat{y}_n - \hat{x}_n) + G^\circ(\hat{x}_n; \hat{y}_n - \hat{x}_n) - G^\circ(\hat{\bar{x}}; \hat{\bar{y}} - \hat{\bar{x}}) \\
&\quad + G^\circ(\hat{\bar{x}}; \hat{\bar{y}} - \hat{\bar{x}}) - \varphi(y_n, x_n) + \varphi(x_n, x_n) \\
&\le \langle \tilde{x}_n^\star - x_n^\star, y_n - x_n\rangle - \langle f, y_n - x_n\rangle + \langle f - f_n, y_n - x_n\rangle - \langle \mathscr{B}(u_n), y_n - x_n\rangle \\
&\quad + \langle x_n^\star, y_n - \bar{y}\rangle + G_n^\circ(\hat{x}_n; \hat{y}_n - \hat{x}_n) - G^\circ(\hat{x}_n; \hat{y}_n - \hat{x}_n) + G^\circ(\hat{x}_n; \hat{y}_n - \hat{x}_n) \\
&\quad - G^\circ(\hat{\bar{x}}; \hat{\bar{y}} - \hat{\bar{x}}) + G^\circ(\hat{\bar{x}}; \hat{\bar{y}} - \hat{\bar{x}}) - \varphi(y_n, x_n) + \varphi(x_n, x_n)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle \bar{y}^\star, \bar{x} - \bar{y}\rangle &= \limsup_{n\to\infty} \langle \bar{y}^\star, x_n - \bar{y}\rangle \\
&\le \limsup_{n\to\infty} \Big[(\varrho_2^n \tau_2(\|x_n\|) + \varrho_3^n \|\mathscr{T}\| \tau_3(\|\bar{x}\|) + \varrho_4^n) \|y_n - x_n\| + \|x_n^\star\| \|y_n - \bar{y}\| \\
&\quad + \langle f, x_n - y_n\rangle + \langle \mathscr{B}(u_n), x_n - y_n\rangle + G^\circ(\hat{x}_n; \hat{y}_n - \hat{x}_n) - G^\circ(\hat{\bar{x}}; \hat{\bar{y}} - \hat{\bar{x}}) \Big] \\
&\quad + G^\circ(\hat{\bar{x}}; \hat{\bar{y}} - \hat{\bar{x}}) - \varphi(y_n, x_n) + \varphi(x_n, x_n) \\
&\le \langle f, \bar{x} - \bar{y}\rangle + \langle \mathscr{B}(\bar{u}), \bar{x} - \bar{y}\rangle + G^\circ(\hat{\bar{x}}; \hat{\bar{y}} - \hat{\bar{x}}) - \varphi(\bar{y}, \bar{x}) + \varphi(\bar{x}, \bar{x}).
\end{aligned}$$

By virtue of the monotonicity of $\mathscr{F}$, the compactness of $\dot{\mathscr{T}}$, it follows from Lemma 2(i), (19), (22)–(25), and the boundedness of $\{\tilde{x}_n^\star\}$, we know that the sequence $\{x_n^\star\}$ is bounded. This shows that for any $\bar{y} \in \mathscr{A}(\bar{x})$ and $\bar{y}^\star \in \mathscr{F}(\bar{y})$, we have

$$\langle \bar{y}^\star - f, \bar{y} - \bar{x}\rangle + G^\circ(\hat{\bar{x}}; \hat{\bar{y}} - \hat{\bar{x}}) + \varphi(\bar{y}, \bar{x}) - \varphi(\bar{x}, \bar{x}) \geq \langle \mathscr{B}(\bar{u}), \bar{y} - \bar{x}\rangle.$$

By using a Minty lemma analog, see [28,29], for some $\bar{x}^\star \in \mathscr{F}(\bar{x})$, we have

$$\langle \bar{x}^\star - f, \bar{y} - \bar{x}\rangle + G^\circ(\hat{\bar{x}}; \hat{\bar{y}} - \hat{\bar{x}}) + \varphi(\bar{y}, \bar{x}) - \varphi(\bar{x}, \bar{x}) \geq \langle \mathscr{B}(\bar{u}), \bar{y} - \bar{x}\rangle, \ \forall \ \bar{y} \in \mathscr{A}(\bar{x}).$$

From above, $\bar{x} \in \mathscr{A}(\bar{x})$ and that the solution of (2) is unique shows that $\bar{x}$ is a solution of (2) that corresponds to the control $\bar{u}$.

Now, we have to show that $\{\tilde{x}_n^\star\}$ is bounded. Let $\bar{y}' \in \mathscr{A}(\bar{x})$ be arbitrary and $\{y_n'\}$ be such that

$$y_n' \in \mathscr{A}(x_n) \ \text{ and } \ y_n' \to \bar{y}'.$$

From Lemma 2, for $x_n^\star \in \mathscr{F}(x_n)$ satisfying (25), there are constants $\mu > 0$ and $r > 0$, such that

$$\begin{aligned} r\|x_n^\star - f\| &\leq \langle x_n^\star - f, x_n - \bar{y}'\rangle + \mu(r + \|x_n - \bar{y}'\|) + \varphi(x_n, \bar{y}') - \varphi(\bar{y}', \bar{y}') \\ &= \langle \tilde{x}_n^\star - f_n, x_n - y_n'\rangle + \langle x_n^\star - \tilde{x}_n^\star, x_n - y_n'\rangle + \langle f_n - f, x_n - y_n'\rangle \\ &\quad + \langle x_n^\star - f, y_n' - \bar{y}\rangle + \mu(r + \|x_n - \bar{y}'\|) + \varphi(x_n, \bar{y}') - \varphi(\bar{y}', \bar{y}') \\ &\leq \Big[\varrho_2^n \tau_2(\|x_n\|) + \varrho_4^n + \|\mathscr{B}(u_n)\|\Big]\|x_n - y_n'\| + |G^\circ(\hat{x}_n; \hat{y}_n' - \hat{x}_n)| \\ &\quad + \|x_n^\star - f\|\|y_n - \bar{y}\| + \mu(r + \|x_n - \bar{y}'\|) + \varphi(x_n, \bar{y}') - \varphi(\bar{y}', \bar{y}'). \end{aligned}$$

Since

$$\lim_{n\to\infty} \|y_n - \bar{y}\| = 0,$$

this implies that $\{\|x_n^\star - f\|\}$ is bounded. It further confirms the boundedness of $\{\tilde{x}_n^\star\}$.

Finally, we show that $(\bar{u}, \bar{x})$ is a solution of (3). From Theorem 3, we know that (3) has a solution. Let $(u', x(u'))$ be a solution of (3). We suggest a sequence $\{x_n'\}$ such that x_n' is a solution of the following generalized mixed quasi-variational hemivariational inequality problem that corresponds to the control u' for finding $x_n' \in \mathscr{A}_n(x_n')$ such that for some $x_n'^\star \in \mathscr{F}_n(x_n')$, we have

$$\langle x_n'^\star - f_n, y - x_n'\rangle + G_n^\circ(\hat{x}_n'; \hat{y} - \hat{x}_n') + \varphi(y, x_n') - \varphi(x_n', x_n') \geq \langle \mathscr{B}(u'), y - x_n'\rangle, \ \forall \, y \in \mathscr{A}_n(x_n').$$

By the same way as given above, we can also prove that $\{x_n'\}$ is bounded, and there exists a subsequence that converges weakly to some x' and that x' is a solution of (2) corresponds to u'.

Therefore, we have

$$\begin{aligned} \mathtt{C}(\bar{u}, \bar{u}) &= \|\daleth(\bar{x}) - \Im\|^2 + \epsilon\|\bar{u}\|^2 \\ &\leq \liminf_{n\to\infty} \|\daleth(x_n) - \Im\|^2 + \liminf_{n\to\infty} \epsilon\|u_n\|^2 \\ &\leq \liminf_{n\to\infty} \Big(\|\daleth(x_n) - \Im\|^2 + \epsilon\|u_n\|^2\Big) \\ &\leq \liminf_{n\to\infty} \Big(\|\daleth(x_n') - \Im\|^2 + \epsilon\|u'\|^2\Big) \\ &= \|\daleth(x') - \Im\|^2 + \epsilon\|u'\|^2, \end{aligned}$$

which shows that $\{u_n\}$ is bounded and $(\bar{u}, \bar{x})$ is an optimal pair. This completes the proof. $\square$

5. Applications

In this section, we will utilize our result presented in Section 4 to study the elastic frictional problem in a class of Hilbert spaces.

Let the elastic body be an open bounded connected set $\Theta \subset \mathbb{R}^d$ $(d = 1, 2, 3)$. Assume that the boundary $\daleth = \partial\Theta$ is Lipschitz continuous. Assume that $\daleth$ consists of three sets $\bar{\daleth}_{\mathcal{D}}, \bar{\daleth}_{\mathcal{N}}$ and $\bar{\daleth}_{\mathcal{C}}$, with mutually disjoint relatively open sets $\daleth_{\mathcal{D}}$, $\daleth_{\mathcal{N}}$ and $\daleth_{\mathcal{C}}$, such that $(\daleth_{\mathcal{D}}) > 0$. The classical model for the process is to find a displacement field $u : \Theta \to \mathbb{R}^d$ and a stress field $\sigma : \Theta \to \mathbb{S}^d$ such that

$$Div\, \sigma + f_0 = 0 \text{ in } \Theta, \tag{26}$$

is an equilibrium equation, where Div is the divergence operator, and f_0 is the density of applied forces;

$$\sigma = \mathcal{F}\epsilon(u) \text{ in } \Theta \tag{27}$$

is an elastic constitutive law, where $\mathcal{F}$ is the elasticity operator, and ϵ is the linearized deformation operator;

$$u = 0 \quad \text{on } \daleth_{\mathcal{D}} \tag{28}$$

and

$$\sigma\nu = f_{\mathcal{N}} \quad \text{on } \daleth_{\mathcal{N}} \tag{29}$$

denote the displacement and traction boundary conditions. Here $f_{\mathcal{N}}$ is the density of traction;

$$-\sigma_\nu \in \partial \jmath_\nu(u_\nu - \vartheta_0) \quad \text{on } \daleth_{\mathcal{C}} \tag{30}$$

is a contact condition;

$$-\sigma_\tau \in \complement_\tau(u_\nu - \vartheta_0)\partial \jmath_\tau(u_\tau) \quad \text{on } \daleth_{\mathcal{C}} \tag{31}$$

denotes the friction law, and ϑ_0 is the gap function, $\jmath_\nu, \jmath_\tau, \complement_\tau$ are given functions, ν is the outer normal, and

$$\begin{aligned} u_\nu \quad &= u \cdot \nu, \quad u_\tau = u - u_\nu\, \nu, \\ \sigma_\nu \quad &= (\sigma\, \nu) \cdot \nu, \quad \sigma_\tau = \sigma\, \nu - \sigma_\nu\, \nu. \end{aligned}$$

We use the spaces

$$\begin{aligned} &\mathbf{H} = L^2(\Theta; \mathbb{R}^d), \\ &\mathcal{H} = L^2(\Theta; \mathbb{S}^d), \\ &\mathcal{V} = \{v \in H^1(\Theta; \mathbb{R}^d) \big| v = 0 \text{ on } \daleth_{\mathcal{D}}\}, \end{aligned}$$

where $\mathbb{S}^d$ is the space of symmetric matrices of order d. Let $f_0 \in \mathbf{H}$, $f_{\mathcal{N}} \in L^2(\daleth_{\mathcal{N}}; \mathbb{R}^d)$, $\vartheta_0 \in L^\infty(\daleth_{\mathcal{C}})$, $\vartheta_0 \geq 0$ *a.e.* on $\daleth_{\mathcal{C}}$. Combining (26)–(31), the elastic frictional problem can be written as: to finding $u \in \mathcal{V}$ such that

$$\begin{aligned} &\langle \mathcal{F}\epsilon(u), \epsilon(v) - \epsilon(u)\rangle_{\mathbf{H}} \\ &\quad + \int_{\daleth_{\mathcal{C}}} \jmath_\nu^\circ(u_\nu - \vartheta_0; v_\nu - u_\nu) + \complement_\tau(u_\nu - \vartheta_0)\, \jmath_\tau^\circ(u_\tau - \vartheta_0; v_\tau - u_\tau) d\daleth \\ &\quad \geq \langle f, v - u\rangle_{\mathcal{V}} \quad \forall\, v \in \mathcal{V}, \end{aligned} \tag{32}$$

where $f \in \mathcal{V}^\star$ is given by

$$\langle f, v - u\rangle_{\mathcal{V}} = \langle f_0, v - u\rangle_{\mathcal{H}} + \langle f_{\mathcal{N}}, v - u\rangle_{L^2(\daleth_{\mathcal{N}}; \mathbb{R}^d)} \quad \forall\, v \in \mathcal{V}.$$

Now, we suggest the mapping $\mathscr{F} : \mathcal{V} \to \mathcal{V}^\star$ is given by

$$\langle \mathscr{F}(u), v - u\rangle_{\mathcal{V}} = \langle \mathcal{F}\epsilon(u), \epsilon(v) - \epsilon(u)\rangle_{\mathcal{H}}\ \forall\, u, v \in \mathcal{V}.$$

Then, the problem (32) turns to finding $u \in \mathscr{A}(u)$ such that

$$\begin{aligned}\langle \mathscr{F}(u), v-u\rangle_{\mathcal{V}} &+ \varphi(v,u) - \varphi(u,u) \\ &+ \int_{\daleth_{\mathcal{C}}} (\jmath_{\nu}^{\circ}(u_\nu - \vartheta_0; v_\nu - u_\nu) + \mathtt{C}_\tau(u_\nu - \vartheta_0)\jmath_{\tau}^{\circ}(u_\tau - \vartheta_0; v_\tau - u_\tau))d\daleth \\ &\geq \langle f, v-u\rangle_{\mathcal{V}}, \ \ \forall\, v \in \mathscr{A}(u). \end{aligned} \tag{33}$$

Let $\hat{u} = \mathscr{T}u$, where $\mathscr{T} : \mathbf{H}^{\delta}(\Theta;\mathbb{R}^d) \to L^2(\daleth_{\mathcal{C}};\mathbb{R}^d)$ $(\delta \in (\frac{1}{2},1))$ is the trace operator. Define the operators $\jmath : \daleth_{\mathcal{C}} \times \mathbb{R}^d \to \mathbb{R}$ by

$$\jmath(x,\hat{u}(x)) = \jmath_\nu(x, u_\nu(x) - \vartheta_0(x)) + \mathtt{C}_\tau(x, u_\nu(x) - \vartheta_0(x))\jmath_\tau(x, u_\tau(x) - \vartheta_0(x))$$

for *a.e.* $x \in \daleth_{\mathcal{C}}$, all $u \in \mathbf{H}^{\delta}(\Theta;\mathbb{R}^d)$, and $J : L^2(\daleth_{\mathcal{C}};\mathbb{R}^d) \to \mathbb{R}$ by

$$J(\hat{u}) = \int_{\daleth_{\mathcal{C}}} \jmath(x,\hat{u}(x))d\daleth \ \text{ for } \ u \in \mathbf{H}^{\delta}(\Theta;\mathbb{R}^d).$$

We can prove that J satisfies the conditions of Lemma 2. Since for all $z \in \partial J(\hat{u})$,

$$\langle z, \hat{v} - \hat{u}\rangle \leq J^{\circ}(\hat{u};\hat{v}-\hat{u}) \leq \int_{\daleth_{\mathcal{C}}} \jmath^{\circ}(\hat{u}(x);\hat{v}(x) - \hat{u}(x))d\daleth, \ \forall\, v \in \mathcal{V};$$

this implies that any solution of the following problem: to find $u \in \mathscr{A}(u)$ such that

$$\langle \mathscr{F}u, v-u\rangle_{\mathcal{V}} + \varphi(v,u) - \varphi(u,u) + J^{\circ}(\hat{u};\hat{v}-\hat{u}) \geq \langle f, v-u\rangle_{\mathcal{V}}, \ \forall\, v \in \mathscr{A}(u)$$

is a solution of (33). Therefore, the existence of a solution for the elastic frictional problem can be obtained from Theorem 4.

6. Conclusions

This paper is to initiate the optimal control of the generalized mixed variational-hemivariational inequality problem involving set-valued mapping under the assumption of monotonicity in real Banach space. As a simple innovative model problem, we have discussed the existence results of the optimal control and convergence of the optimal control under suitable conditions for generalized mixed variational-hemivariational inequality problems.

Author Contributions: All authors contributed equally to the writing of this paper. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Natural Science Foundation of China Medical University, Taiwan, and the Natural Science Foundation of China (No. 11361070).

Data Availability Statement: The data sets used and/or analyzed during the current study are available from the corresponding author upon reasonable request.

Acknowledgments: The authors are extremely grateful to the editor and the reviewers, whose valuable comments and suggestions have led to considerable improvement of this paper.

Conflicts of Interest: The authors declare that they have no competing interest.

References

1. Chang, S.S.; Salahuddin; Wang, L.; Wang, G.; Ma, Z.L. Error bounds for mixed set valued vector inverse quasi-variational inequalities. *J. Inequal. Appl.* **2020**, *160*, 1–16.
2. Fang, Y.P.; Huang, N.J. *H*-monotone operator and resolvent operator technique for variational inclusions. *Appl. Math. Computat.* **2003**, *145*, 795–803. [CrossRef]
3. Hassouni, A.; Moudafi, A. A perturbed algorithm for variational inclusions. *J. Math. Anal. Appl.* **1994**, *185*, 706–721. [CrossRef]
4. Salahuddin. Approximation solvability for a system of nonlinear variational type inclusions in Banach spaces. *Kyungpook Math. J.* **2019**, *59*, 101–123.

5. Salahuddin. Iteration method for non-stationary mixed variational inequalities. *Discont. Nonlinearity Complex*. **2020**, *9*, 647–655. [CrossRef]
6. Lee, B.S.; Salahuddin. Solutions for general class of hemivariational like inequality systems. *J. Nonlinear Convex Anal*. **2015**, *16*, 141–150.
7. Kim, J.K.; Dar, A.H.; Salahuddin. Existence solution for the generalized relaxed pseudomonotone variational inequalities. *Nonlinear Funct. Anal. Appl.* **2020**, *25*, 25–34.
8. Salahuddin; Alesemi, M. Iteration complexity of generalized complementarity problems. *J. Inequal. Appl*. **2019**, *79*, 113.
9. Motreanu, D.; Panagiotopoulos, P.D. *Minimax Theorem and Qualitative Properties of the Solutions of Hemivariational Inequalities and Applications*; Kluwer Academic Publishers: Dordrecht, The Netherland; Boston, MA, USA; London, UK, 1999.
10. Adly, S.; Bergounioux, M.; Mansour, M.A. Optimal control of a quasi-variational obstacle problem. *J. Glob. Optim.* **2010**, 47, 421–435. [CrossRef]
11. Denkowski, Z.; Migorski, S. Sensitivity of optimal solutions to control problems for systems described by hemivariational inequalities. *Control Cybern.* 2004, 33, 211–236.
12. Denkowski, Z.; Migorski, S. On sensitivity of optimal solutions to control problems for hyperbolic hemivariational inequalities. *Lect. Notes Pure Appl. Math.* **2005**, *240*, 145–156.
13. Dietrich, H. Optimal control problems for certain quasivariational inequalities. *Optimization* **2001**, *49*, 67–83. [CrossRef]
14. Kristly, A.; Radulescu, V.; Varga, C. *Variational Principles in Mathematical Physics, Geometry, and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems, Encylopedia of Mathematics*; Cambridge University Press: Cambridg, UK, 2010; Volume 136.
15. Micu, S.; Roventa, I.; Tucsnak, M. Time optimal boundary controls for the heat equation. *J. Funct. Anal.* **2012**, *263*, 25–49. [CrossRef]
16. Migorski, S.; Ochal, A. Optimal control of parabolic hemivariational inequalities. *J. Glob. Optim.* **2000**, *17*, 285–300. [CrossRef]
17. Migorski, S. Evolution hemivariational inequalities in infinite dimension and their control. *Nonlinear Anal. Theory Methods Appl.* **2001**, *47*, 101–112. [CrossRef]
18. Migorski, S.; Ochal, A.; Sofonea, M. History-dependent subdifferential inclusions and hemivariational inequalities in contact mechanics, Nonlinear Anal. *Real World Appl*. **2011**, *12*, 3384–3396. [CrossRef]
19. Migorski, S.; Szafraniec, P. A class of dynamic frictional contact problems governed by a system of hemivariational inequalities in thermoviscoelasticity. *Nonlinear Anal. Real World Appl*. **2014**, *15*, 158–171. [CrossRef]
20. Naniewicz, Z.; Panagiotopoulos, P.D. *Mathematical Theory of Hemivariational Inequalities and Applications*; Marcel Dekker: New York, NY, USA, 1995.
21. Liu, Z.; Zeng, B. Optimal control of generalized quasi-variational-hemivariational inequalities and its applications. *Appl. Math. Optim*. **2015**, *72*, 305–323. [CrossRef]
22. Zeng, B.; Migorski, S. Variational-hemivariational inverse problems for unilateral frictional contact. *Appl. Anal.* **2020**, *99*, 293–312. [CrossRef]
23. Salahuddin. The existence of solution for equilibrium problems in Hadamard manifolds. *Trans. A Razmad. Math. Inst.* **2017**, *171*, 381–388. [CrossRef]
24. Clarke, F.H. *Optimization and Nonsmooth Analysis*; Wiley: New York, NY, USA, 1983.
25. Denkowski, Z.; Migorski, S.; Papageorgiou, N.S. *An Introduction to Nonlinear Analysis: Theory*; Kluwer Academic/Plenum Publishers: Boston, MA, USA, 2003.
26. Alber, Y.I.; Notic, A.I. Perturbed unstable variational inequalities with unbounded operators on approximately given sets. *Set-Valued Anal*. **1993**, *1*, 393–402. [CrossRef]
27. Migorski, S.; Ochal, A.; Sofonea, M. *Nonlinear Inclusions and Hemivariational Inequalities: Models and Analysis of Contact Problems*; Springer: New York, NY, USA, 2013.
28. Khan, A.A.; Tammer, C.; Zalinescu, C. Regularization of quasi variational inequalities. *Optimization* **2015**, *64*, 1703–1724. [CrossRef]
29. Giannessi, F.; Khan, A.A. Regularization of non-coercive quasi variational inequalities. *Control Cybern*. **2000**, *29*, 91–110.