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The Solvability of Generalized Systems of Time-Dependent Hemivariational Inequalities Enjoying Symmetric Structure in Reflexive Banach Spaces

Lu-Chuan Ceng *, Yi-Xuan Fu, Jie Yin, Liang He, Long He and Hui-Ying Hu

Department of Mathematics, Shanghai Normal University, Shanghai 200234, China; yixuanfu@shnu.edu.cn (Y.-X.F.); jieyin@shnu.edu.cn (J.Y.); lianghe@shnu.edu.cn (L.H.); longhe@shnu.edu.cn (L.H.); huiying@shnu.edu.cn (H.-Y.H.)

* Correspondence: zenglc@shnu.edu.cn

Abstract: In real reflexive Banach spaces, let the GSTDHVI, SHVI, DVIP, VIT, and KKM represent a generalized system of time-dependent hemivariational inequalities, a system of hemivariational inequalities, a derived vector inclusion problem, Volterra integral term, and Knaster–Kuratowski–Mazurkiewicz, respectively, where the GSTDHVI consists of two parts which are of symmetric structure mutually. By virtue of the surjectivity theorem for pseudo-monotonicity mappings and the Banach contraction mapping principle, instead of the KKM theorems exploited by other authors in recent literature for a SHVI, we consider and study a GSTDHVI with VITs. Under quite mild assumptions, it is shown that there exists only a solution to the investigated problem via demonstrating that a DVIP with VIT is solvable.

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1. Introduction

It is well known that the hemivariational inequality (HVI, for short) involving nonsmooth and nonconvex energy functions was first considered in [1]. It is a significant extension of variational inequality (VI, for short). As a helpful technique, the HVIs and their systems have played a crucial role in the study of quite meaningful problems of mechanics and engineering sciences, e.g., obstacle problems, thermoviscoelastic frictional contact problems, unilateral contact problems in nonlinear elasticity, etc.; please refer to [2–6]. Via the Clarke's generalized directional derivative and the Clarke's generalized gradient (appearing in Section 2), various HVIs and systems of HVIs (SHVIs, for short), e.g., stationary HVIs, evolutionary HVIs, and their systems, etc., have been investigated by numerous authors in the past more than 30 years; please refer to, e.g., [2–12].

There are two important approaches to studying the solvability to various HVIs in the recent literature. One approach is closely related to the KKM theorems and fixed point theorems, which are exploited in [9,13–15] to investigate stationary HVIs and stationary SHVIs. The other is closely related to the surjectivity theorems involving pseudo-monotonicity of coercive mappings, which are applied in [5,6,16–18] for various stationary HVIs and evolutionary HVIs; please refer to [18–23]. Unfortunately, as meaningful models for problems in mechanics and engineering science, the SHVIs, e.g., the stationary SHVIs and the evolutionary SHVIs, are not investigated extensively via the surjectivity theorems involving pseudo-monotonicity of coercive mappings. As put forth in [20], the existence of solutions to the SHVIs in dynamic thermoviscoelasticity has been an open problem.

In [20], the authors considered a dynamic thermoviscoelastic frictional contact problem and modeled it via the evolutionary SHVIs. They showed that there exists only a weak solution to the problem via a surjectivity result for pseudo-monotonicity mappings. Previously, Panagiotopulos formulated two SHVIs for the behavior of von Karman plates and linear thermoelastic materials in ([4], Chapter 7.3). Unfortunately, the regular conditions on the multi-valued terms are very unnatural for the solvability of two SHVIs. In 2015, Xiao et al. [24] investigated a system of time-dependent hemivariational inequalities (STDHVI, for short) with Volterra integral terms (VITs, for short) according to the surjectivity theorem involving pseudo-monotonicity of coercive mappings, and the Banach contraction mapping principle. Here, the STDHVI consists of two parts which are of symmetric structure mutually. Under quite appropriate assumptions, they showed that there exists only a solution to the considered problem through demonstrating that a derived vector inclusion problem (DVIP, for short) with VIT is solvable. Subsequently, the authors [12] considered the well-posedness for a STDHVI without VITs in real reflexive Banach spaces in 2017.

Inspired by the above research works, via the surjectivity theorem involving pseudo-monotonicity of coercive mappings and the Banach contraction mapping principle, instead of the KKM theorems exploited by other authors in recent literature for a SHVI, we investigate a generalized STDHVI (GSTDHVI, for short) with VITs in real reflexive Banach spaces. Here, the GSTDHVI consists of two parts which are of symmetric structure mutually. Under quite mild assumptions, it is shown that there exists only a solution to the investigated problem via demonstrating that the DVIP with VIT is solvable. The article is organized below. We first formulate the considered problem and then present the basic results and tools in Section 2. The relation between the considered problem and the DVIP with VIT is established under quite mild conditions in Section 3. It is shown that there exists only a solution to the investigated problem via the surjectivity theorem involving pseudo-monotonicity of coercive mappings and the Banach contraction mapping principle in Section 4. Finally, the concluding remarks are presented in Section 5.

Finally, it is worth pointing out that there are the obvious disadvantages of the method based on the KKM approach for studying generalized parabolic or evolutionary SHVIs. In fact, if the operators in the method based on the KKM approach are not the KKM mappings, there are several possibilities which happen in the demonstrating process, e.g., in particular, whenever studying generalized parabolic or evolutionary SHVIs. This might leads to no successful continuation of demonstration. Actually, this is exactly the drawback of the KKM-based approach.

2. Basic Results and Tools

We first formulate the considered problem in this paper, and then present the useful concepts and basic tools involving monotonicity mappings, and nonlinear and nonsmooth analysis; please refer to [6,18,25]. Moreover, we also introduce some new concepts, notations and conditions, which will be used in the sequel.

For $k = 1,2$, let the real reflexive Banach spaces V_k and X_k enjoy dual spaces V_k^* and X_k^*, respectively, where V_k is separable. Given $T \in (0, +\infty)$, let $\mathcal{V}_k = L^2(0,T;V_k)$ for $k = 1,2$. Then, $\mathcal{V}_k^* = L^2(0,T;V_k^*)$ for $k = 1,2$. Unless otherwise specified, the $\langle \cdot, \cdot \rangle_{W^* \times W}$ indicates the duality pairing between W and W^*, and the $\|\cdot\|_W$ and $\|\cdot\|_{W^*}$ denote the norms in W and W^*, respectively, where $W \in \{V_k, X_k, \mathcal{V}_k, k = 1,2\}$. Inspired by the generalized mixed variational inequality in [26], we introduce and consider in this paper a GSTDHVI with VITs formulated below:

Find $(p_1, p_2) \in \mathcal{V}_1 \times \mathcal{V}_2$ s.t., for some selections $\Phi_l(t, p_l(t)) \in \mathbf{\Phi}_l(t, p_l(t))$ and $\Gamma_l(t) \in \mathbf{\Gamma}_l(t)$, $(l = 1,2)$,

$$\begin{cases} \langle \Phi_1(t,p_1(t)) + \Psi_1(t,p_2(t)), q_1 \rangle_{V_1^* \times V_1} + J_1^\circ(t, \Theta_1(p_1(t)), \Theta_2(p_2(t)); \Theta_1 q_1) \\ + \langle \int_0^t \Gamma_1(t-s) p_1(s) ds, q_1 \rangle_{V_1^* \times V_1} \geq \langle \varphi_1(t), q_1 \rangle_{V_1^* \times V_1}, \ \forall q_1 \in V_1, \text{ a.e. } t \in (0,T), \\ \langle \Phi_2(t,p_2(t)) + \Psi_2(t,p_1(t)), q_2 \rangle_{V_2^* \times V_2} + J_2^\circ(t, \Theta_1(p_1(t)), \Theta_2(p_2(t)); \Theta_2 q_2) \\ + \langle \int_0^t \Gamma_2(t-s) p_2(s) ds, q_2 \rangle_{V_2^* \times V_2} \geq \langle \varphi_2(t), q_2 \rangle_{V_2^* \times V_2}, \ \forall q_2 \in V_2, \text{ a.e. } t \in (0,T), \end{cases} \tag{1}$$

where, for $l, k = 1, 2$ and $k \neq l$, $\mathbf{\Phi}_l : (0, T) \times V_l \to 2^{V_l^*}$, $\Psi_l : (0, T) \times V_k \to V_l^*$, and $\varphi_l : (0, T) \to V_l^*$ are mappings with images in V_l^*. For $l \neq k = 1, 2$, $\Theta_l : V_l \to X_l$ is a linear bounded and compact mapping, $\mathbf{\Gamma}_l : (0, T) \to 2^{\mathcal{L}(V_l, V_l^*)}$ is an operator with images of linear continuous mappings from V_l to V_l^*, and $J_l^\circ(t, \omega_1, \omega_2; v)$ is the partial Clarke's generalized directional derivative of locally Lipschitz functional $J : (0, T) \times X_1 \times X_2 \to \mathbf{R}$ w.r.t. the lth argument at $\omega_l \in X_l$ in the direction $v \in X_l$ for given $\omega_k \in X_k$.

Recall some basic results and tools. Let the real Banach space E enjoy dual space E^*. Suppose that $\hbar : E \to \mathbf{R}$ is locally Lipschitz, $F : E \to E^*$ is single-valued, and $M : E \to 2^{E^*}$ is multi-valued.

Let $u, v \in E$. Clarke's generalized directional derivative of $\hbar$ at $u \in E$ in $v \in E$, denoted by $\hbar^\circ(u; v)$, is defined by

$$\hbar^\circ(u; v) := \limsup_{\omega \to u, \tau \to 0^+} \frac{\hbar(\omega + \tau v) - \hbar(\omega)}{\tau}.$$

Clarke's generalized gradient of $\hbar$ at $u \in E$, written by $\partial \hbar(u)$, is the set in E^* formulated by $\partial \hbar(u) := \{\xi \in E^* : \hbar^\circ(u; v) \geq \langle \xi, v \rangle, \ \forall v \in E\}$. As put forth in [5], in case $\hbar : E \to \mathbf{R}$ is convex and continuous, $\partial \hbar(u)$ is equal to the subdifferential of $\hbar$ at u in the sense of convex analysis. In case $\hbar$ is continuously differentiable, one has $\partial \hbar(u) = \{\hbar'(u)\} \ \forall u \in E$, with $\hbar'(u)$ being the Fréchet differential of $\hbar$ at u. For locally Lipschitz functional $\hbar$, it is well known that the conclusions hold below:

(a) $v \mapsto \hbar^\circ(u; v)$ is subadditive, positively homogeneous and finite;

(b) $(u, v) \mapsto \hbar^\circ(u; v)$ is u.s.c. (i.e., upper semicontinuous) on $E \times E$;

(c) $\partial \hbar(u)$ is nonempty, bounded, convex and weak*-compact in E^*;

(d) $\partial \hbar(u)$ enjoys the closed graph in $E \times (w^*\text{-}E^*)$.

Recall that $\hbar$ is referred to as being regular (in the sense of Clarke) at $u \in E$ iff

(a) directional derivative $\hbar'(u, v)$ exists for each $v \in E$;

(b) $\hbar'(u, v) = \hbar^\circ(u; v)$ for each $v \in E$, with $\hbar'(u, v)$ being the directional derivative of $\hbar$ at $u \in E$ in $v \in E$.

Suppose that E_l is a real Banach space for $l = 1, 2$, $H : E_1 \times E_2 \to \mathbf{R}$ is locally Lipschitz on $E_1 \times E_2$ and H or $-H$ is regular at $(p_1, p_2) \in E_1 \times E_2$. Then, we recall that the following relation holds:

$$\partial H(p_1, p_2) \subset \partial_1 H(p_1, p_2) \times \partial_2 H(p_1, p_2),$$

or

$$H^\circ(p_1, p_2; q_1, q_2) \leq H_1^\circ(p_1, p_2; p_1) + H_2^\circ(p_1, p_2; p_2) \quad \forall (q_1, q_2) \in E_1 \times E_2.$$

It is worth pointing out that the converses of the above relationships are generally not valid.

On the other hand, suppose that E is a real reflexive Banach space. $F : E \to E^*$ is referred to as being pseudomonotone if F is bounded, s.t. $\liminf_{n\to\infty} \langle F p_n, p_n - q \rangle \geq \langle F p, p - q \rangle \ \forall q \in E$ provided $p_n \to p$ weakly in E and $\limsup_{n\to\infty} \langle F p_n, p_n - q \rangle \leq 0$. It is well known that F is pseudomonotone iff F is bounded s.t. $F p_n \to F p$ weakly in E^* and $\lim_{n\to\infty} \langle F p_n, p_n - p \rangle = 0$ whenever $p_n \to p$ weakly in E and $\limsup_{n\to\infty} \langle F p_n, p_n - p \rangle \leq 0$.

Definition 1. *$M : E \to 2^{E^*}$ is referred to as being pseudomonotone iff*

(a) Mp is a nonempty convex closed bounded set for every $p \in E$;

(b) M is u.s.c. from each finite-dimensional subspace of E to E^* equipped with the weak topology;

(c) if $\{p_n\} \subset E$ converging weakly to p and $p_n^* \in M p_n$ satisfying $\limsup_{n\to\infty} \langle p_n^*, p_n - p \rangle \leq 0$, then, $\forall q \in E$, $\exists p^*(q) \in Mp$ s.t. $\langle p^*(q), p - q \rangle \leq \liminf_{n\to\infty} \langle p_n^*, p_n - p \rangle$.

Definition 2. *$M : E \to 2^{E^*}$ is referred to as being generalized pseudomonotone if for $\{p_n\} \subset E$ and $\{p_n^*\} \subset E^*$ with $p_n^* \in M p_n$, the conditions that $p_n \to p$ weakly in E, $p_n^* \to p^*$ weakly in E^* and $\limsup_{n\to\infty} \langle p_n^*, p_n - p \rangle \leq 0$, imply that $p^* \in Mp$ and $\langle p_n^*, p_n \rangle \to \langle p^*, p \rangle$.*

Proposition 1. *Let $M : E \to 2^{E^*}$ be generalized pseudomonotone and bounded. Assume that, for every $p \in E$, Mp is nonempty, convex, and closed in E^*. Then, M is pseudomonotone.*

Definition 3. *$M : E \to 2^{E^*}$ is referred to as being*

(a) coercive if $\exists c : \mathbf{R}^+ \to \mathbf{R}$ with $\lim_{r\to\infty} c(r) = \infty$, s.t. $c(\|p\|)\|p\| \le \langle p^*, p\rangle\ \forall (p, p^*) \in G(M)$, with $G(M)$ being the graph of M;

(b) coercive with constant $\alpha > 0$ if $\alpha\|p\|^2 \le \langle p^*, p\rangle\ \forall (p, p^*) \in G(M)$.

Theorem 1. *(Surjectivity theorem). If $M : E \to 2^{E^*}$ is coercive and pseudomonotone, then the range of M is equal to E^*, i.e., $R(M) = E^*$.*

Suppose that $(E, \|\cdot\|)$ is a linear normed space and $\mathcal{H}$ is a Hausdorff metric on the family $CB(E)$ of all nonempty, bounded and closed sets in E, induced by the metric d according to $d(p, q) = \|p - q\|$, which is formulated below

$$\mathcal{H}(K, D) = \max\{\sup_{p\in K} \inf_{q\in D} \|p - q\|, \sup_{q\in D} \inf_{p\in K} \|p - q\|\} \quad \forall K, D \in CB(E).$$

Let $K, D \in CB(E)$. For any $\epsilon > 0$ and $p \in K$, according to Nadler's result [27], we know that $\exists q \in D$ s.t.

$$\|p - q\| \le (1 + \epsilon)\mathcal{H}(K, D).$$

In particular, whenever the sets $K, D \subset E$ are compact, for any $p \in K$, we know that $\exists q \in D$ s.t.

$$\|p - q\| \le \mathcal{H}(K, D).$$

It is remarkable that, as an important tool, the Nadler's result mentioned above exhibits a powerful role in the exploration of well-posedness of generalized mixed variational inequalities in [26].

Finally, we present the w-$\mathcal{H}$-continuity concept of $M : E \to 2^{E^*}$.

Definition 4. *$M : E \to 2^{E^*}$ is referred to as being w-$\mathcal{H}$-continuous if for any net $\{p_\alpha\} \subset E$ with $p_\alpha \to p$ weakly in E, one has $\mathcal{H}(M(p_\alpha), M(p)) \to 0$.*

Similarly, one can define the s-$\mathcal{H}$-continuous mapping M on E.

3. DVIP and Hypotheses

In this section, we first introduce a DVIP with VIT on $V_1 \times V_2$, and establish the relationship between the DVIP with VIT and GSTDHVI with VITs (i.e., problem (1)). Then, we impose the restrictions on the mappings for demonstrating that there exists only a solution to the GSTDHVI with VITs.

Let $V = V_1 \times V_2$. We equip V with the norm $\|\mathbf{p}\|_V := \sum_{k=1}^{2} \|p_k\|_{V_k}\ \forall \mathbf{p} = (p_1, p_2) \in V$. According to [25], we know that V is a reflexive Banach space and the duality pairing between V and V^* is formulated below:

$$\langle \mathbf{p}^*, \mathbf{p}\rangle_{V^*\times V} = \langle p_1^*, p_1\rangle_{V_1^*\times V_1} + \langle p_2^*, p_2\rangle_{V_2^*\times V_2}, \quad \forall \mathbf{p}^* = (p_1^*, p_2^*) \in V^*, \mathbf{p} = (p_1, p_2) \in V.$$

Similarly, we can also describe the product space $X = X_1 \times X_2$ with its dual X^*. For $t \in (0, T)$ and $\mathbf{p} = (p_1, p_2) \in V$, we construct the mappings below:

$$\begin{cases} \mathbf{\Phi} : (0, t) \times V \to 2^{V^*}, \quad \mathbf{\Phi}(t, \mathbf{p}) := (\mathbf{\Phi}_1(t, p_1) + \Psi_1(t, p_2)) \times (\mathbf{\Phi}_2(t, p_2) + \Psi_2(t, p_1)), \\ \mathbf{\Gamma} : (0, T) \to 2^{\mathcal{L}(V,V^*)}, \quad \mathbf{\Gamma}(t)\mathbf{p} := \mathbf{\Gamma}_1(t)p_1 \times \mathbf{\Gamma}_2(t)p_2, \\ \Theta : V \to X = X_1 \times X_2, \quad \Theta\mathbf{p} := (\Theta_1 p_1, \Theta_2 p_2), \\ \varphi : (0, T) \to V^*, \quad \varphi(t) := (\varphi_1(t), \varphi_2(t)). \end{cases} \tag{2}$$

In the rest of this paper, the range of variable t is always assumed to be the a.e. $t \in (0, T)$. For the convenience, we naturally omit the description of the a.e. $t \in (0, T)$. We formulate the DVIP with VIT below:

Find $\mathbf{p} \in \mathcal{V} = L^2(0,T;V)$ and $\Gamma(t) = (\Gamma_1(t), \Gamma_2(t)) \in \mathbf{\Gamma}(t)$ s.t.

$$\varphi(t) \in \mathbf{\Phi}(t, \mathbf{p}(t)) + \Theta^* \circ \partial J(t, \Theta(\mathbf{p}(t))) + \int_0^t \Gamma(t-s)\mathbf{p}(s)ds, \tag{3}$$

with Θ^* being the adjoint operator of Θ. The relations between the GSTDHVI with VITs and DVIP with VIT are established below.

Lemma 1. *If the locally Lipschitz $J(t,\cdot,\cdot)$ is regular on X, each solution $\mathbf{p} = (p_1, p_2) \in \mathcal{V}$ to the DVIP with VIT is a solution to the GSTDHVI with VITs.*

Proof. Suppose that $\mathbf{p} = (p_1, p_2) \in \mathcal{V}$ solves the problem (3). Then, $\exists \eta(t) \in \partial J(t, \Theta(\mathbf{p}(t))) \subset X^*$ and $\exists \Phi(t, \mathbf{p}(t)) \in \mathbf{\Phi}(t, \mathbf{p}(t))$ with

$$\begin{cases} \Phi(t, \mathbf{p}(t)) = (\Phi_1(t, p_1(t)) + \Psi_1(t, p_2(t)), \Phi_2(t, p_2(t)) + \Psi_2(t, p_1(t))), \\ \Phi_1(t, p_1(t)) \in \mathbf{\Phi}_1(t, p_1(t)) \ \text{ and } \ \Phi_2(t, p_2(t)) \in \mathbf{\Phi}_2(t, p_2(t)), \end{cases}$$

s.t.

$$\varphi(t) = \Phi(t, \mathbf{p}(t)) + \Theta^* \eta(t) + \int_0^t \Gamma(t-s)\mathbf{p}(s)ds, \quad \text{in } V^*. \tag{4}$$

For each $q_1 \in V_1$, by multiplying (4) with $\mathbf{q} = (q_1, \theta_2)$, where θ_2 is the zero vector of V_2, we have

$$\begin{aligned} \langle \varphi_1(t), q_1 \rangle_{V_1^* \times V_1} &= \langle \Phi_1(t, p_1(t)) + \Psi_1(t, p_2(t)), q_1 \rangle_{V_1^* \times V_1} + \langle \eta(t), \Theta \mathbf{q} \rangle_{X^* \times X} \\ &\quad + \langle \textstyle\int_0^t \Gamma_1(t-s) p_1(s)ds, q_1 \rangle_{V_1^* \times V_1}. \end{aligned}$$

Thus,

$$\begin{aligned} &\langle \varphi_1(t), q_1 \rangle_{V_1^* \times V_1} \\ &\le \langle \Phi_1(t, p_1(t)) + \Psi_1(t, p_2(t)), q_1 \rangle_{V_1^* \times V_1} + J_1^\circ(t, \Theta_1(p_1(t)), \Theta_2(p_2(t)); \Theta_1 q_1) \\ &\quad + \langle \textstyle\int_0^t \Gamma_1(t-s) p_1(s)ds, q_1 \rangle_{V_1^* \times V_1}. \end{aligned} \tag{5}$$

By a similar way, for each $q_2 \in V_2$, one can deduce by multiplying (4) with $\mathbf{q} = (\theta_1, q_2)$, where θ_1 is the zero vector of V_1 that

$$\begin{aligned} &\langle \varphi_2(t), q_2 \rangle_{V_2^* \times V_2} \\ &\le \langle \Phi_2(t, p_2(t)) + \Psi_2(t, p_1(t)), q_2 \rangle_{V_2^* \times V_2} + J_2^\circ(t, \Theta_1(p_1(t)), \Theta_2(p_2(t)); \Theta_2 q_2) \\ &\quad + \langle \textstyle\int_0^t \Gamma_2(t-s) p_2(s)ds, q_2 \rangle_{V_2^* \times V_2}, \end{aligned}$$

which, along with (5), ensures that $\mathbf{p} \in \mathcal{V}$ (i.e., $p_1 \in \mathcal{V}_1, p_2 \in \mathcal{V}_2$) is a solution to the GSTDHVI with VITs. □

It is remarkable that the inverse of Lemma 1 might be false, i.e., the solution to the GSTDHVI with VITs might not solve the DVIP with VIT. In what follows, according to [14], we provide an example to support this assertion. For any $(v_1, v_2) \in X_1 \times X_2$, let $J : (0,T) \times X_1 \times X_2 \to \mathbf{R}$ be formulated by $J(t, v_1, v_2) := \phi(t, v_1) + \psi(t, v_2)$, where the functionals $\phi(t,\cdot)$ and $\psi(t,\cdot)$ are regular and locally Lipschitz on X_1 and X_2, respectively. Then, it is clear that $J(t,\cdot,\cdot)$ is regular and locally Lipschitz on X. Meantime, it is not hard to verify that the following relations hold:

$$\partial J(t, v_1, v_2) = \partial_1 J(t, v_1, v_2) \times \partial_2 J(t, v_1, v_2), \quad \forall (v_1, v_2) \in X, \tag{6}$$

or, equivalently, for any $(v_1, v_2), (\omega_1, \omega_2) \in X$,

$$J^\circ(t, v_1, v_2; \omega_1, \omega_2) = J_1^\circ(t, v_1, v_2; \omega_1) + J_2^\circ(t, v_1, v_2; \omega_2). \tag{7}$$

It is worth pointing out that, under the regularity assumption of $J(t,\cdot,\cdot)$, the above relationships (6) and (7) are not true in general.

Lemma 2. *If the locally Lipschitz $J(t,\cdot,\cdot)$ is regular on X and relationship (6) or (7) holds, $\mathbf{p} = (p_1, p_2) \in \mathcal{V}$ is a solution to the GSTDHVI with VITs if and only if it is a solution to the DVIP with VIT.*

Proof. By Lemma 1, we know that the sufficiency of this lemma is valid. In what follows, it is sufficient for us to show its necessity. To this aim, we suppose that $\mathbf{p} = (p_1, p_2) \in \mathcal{V}$ is a solution to the GSTDHVI with VITs. Adding the two inequalities in (1) and using the definitions of the operators in (2), one has

$$\begin{aligned}&\langle \Phi(t,\mathbf{p}(t)) + \textstyle\int_0^t \Gamma(t-s)\mathbf{p}(s)ds, \mathbf{q}\rangle_{V^*\times V} + J_1^\circ(t,\Theta(\mathbf{p}(t));\Theta_1 q_1) + J_2^\circ(t,\Theta(\mathbf{p}(t));\Theta_2 q_2)\\ &\geq \langle \varphi(t), \mathbf{q}\rangle_{V^*\times V}, \quad \forall \mathbf{q} = (q_1,q_2) \in V,\end{aligned}$$

with

$$\begin{aligned}\Phi(t,\mathbf{p}(t)) &= (\Phi_1(t,p_1(t)) + \Psi_1(t,p_2(t)), \Phi_2(t,p_2(t)) + \Psi_2(t,p_1(t)))\\ &\in (\mathbf{\Phi}_1(t,p_1(t)) + \Psi_1(t,p_2(t)) \times (\mathbf{\Phi}_2(t,p_2(t)) + \Psi_2(t,p_1(t))\\ &= \mathbf{\Phi}(t,\mathbf{p}(t)).\end{aligned}$$

Using the conditions (6) or (7), we infer that, for any $\mathbf{q} \in V$,

$$\langle \varphi(t) - \Phi(t,\mathbf{p}(t)) - \int_0^t \Gamma(t-s)\mathbf{p}(s)ds, \mathbf{q}\rangle_{V^*\times V} \leq J^\circ(t,\Theta(\mathbf{p}(t));\Theta\mathbf{q}). \tag{8}$$

Note that Θ is linear continuous and $J(t,\cdot,\cdot)$ is regular. Hence, according to ([6], Proposition 3.37), we obtain $J^\circ(t,\Theta(\mathbf{p}(t));\Theta\mathbf{q}) = (J(t,\cdot)\circ\Theta)^\circ(\mathbf{p}(t);\mathbf{q})$ and $\partial(J(t,\cdot)\circ\Theta)(\mathbf{p}(t)) = \Theta^*\circ\partial J(t,\Theta(\mathbf{p}(t))$ for all $\mathbf{q} \in V$. Consequently, from (8), we have

$$\begin{aligned}\varphi(t) &\in \Phi(t,\mathbf{p}(t)) + \Theta^*\circ\partial J(t,\Theta(\mathbf{p}(t)) + \textstyle\int_0^t \Gamma(t-s)\mathbf{p}(s)ds\\ &\subset \mathbf{\Phi}(t,\mathbf{p}(t)) + \Theta^*\circ\partial J(t,\Theta(\mathbf{p}(t)) + \textstyle\int_0^t \Gamma(t-s)\mathbf{p}(s)ds.\end{aligned}$$

Thus, $\mathbf{p}$ is a solution of the DVIP with VIT. □

Next, we impose the restrictions on the mappings $\mathbf{\Phi}_l, \Psi_l, \mathbf{\Gamma}_l$ $(l = 1,2)$ and J for demonstrating that there exists only a solution to the GSTDHVI with VITs.

(HΦ): For $l = 1,2$, operator $\mathbf{\Phi}_l$ satisfies
(a) $\mathbf{\Phi}_l(t,\cdot) : V_l \to 2^{V_l^*}$ is w-$\mathcal{H}$-continuous;
(b) $\mathbf{\Phi}_l(t,\cdot) : V_l \to 2^{V_l^*}$ is bounded and pseudomonotone on V_l;
(c) $\mathbf{\Phi}_l(t,\cdot) : V_l \to 2^{V_l^*}$ is coercive with constant α_l;
(d) $\mathbf{\Phi}_l(t,\cdot) : V_l \to 2^{V_l^*}$ is of strong monotonicity with coefficient $\beta_l > 0$.

(HΨ): For $l,k = 1,2$ with $k \neq l$, operator $\Psi_l : (0,T) \times V_k \to V_l^*$ satisfies
(a) $\Psi_l(\cdot, v_k) : (0,T) \to V_l^*$ is measurable for any fixed $v_k \in V_k$;
(b) $\Psi_l(t,\cdot) : V_k \to V_l^*$ is weakly continuous;
(c) $\Psi_l(t,\cdot) : V_k \to V_l^*$ is bounded;
(d) $\langle \Psi_1(t,v_2), v_1\rangle_{V_1^*\times V_1} + \langle \Psi_2(t,v_1), v_2\rangle_{V_2^*\times V_2} = 0$ $\forall (v_1,v_2) \in V_1 \times V_2$.

Lemma 3. *If the conditions* **(HΦ)** *and* **(HΨ)** *hold, then $\mathbf{\Phi}(t,\cdot) : V \to 2^{V^*}$ is the bounded mapping with pseudo-monotonicity.*

Proof. Using the hypothesis **(HΦ)** (b), we deduce that, for $l = 1,2$, $\mathbf{\Phi}_l(t,\cdot) : V_l \to 2^{V_l^*}$ is pseudomonotone on V_l. By Definition 1 (a), $\mathbf{\Phi}_l(t,v_l)$ is nonempty, convex, closed, and bounded in V_l^* for any $v_l \in V_l$. This ensures that $\mathbf{\Phi}(t,\mathbf{v}) = (\mathbf{\Phi}_1(t,v_1) + \Psi_1(t,v_2)) \times (\mathbf{\Phi}_2(t,v_2) + \Psi_2(t,v_1))$ is nonempty, convex, closed, and bounded in V^* for any $\mathbf{v} \in V$

with $\mathbf{v} = (v_1, v_2)$. We now claim that $\mathbf{\Phi}(t,\cdot) : V \to 2^{V^*}$ is bounded. Indeed, we take $\Phi(t,\mathbf{v}) \in \mathbf{\Phi}(t,\mathbf{v})$, arbitrarily, with

$$\begin{cases} \Phi(t,\mathbf{v}) = (\Phi_1(t,v_1) + \Psi_1(t,v_2), \Phi_2(t,v_2) + \Psi_2(t,v_1)), \\ \Phi_1(t,v_1) \in \mathbf{\Phi}_1(t,v_1) \text{ and } \Phi_2(t,v_2) \in \mathbf{\Phi}_2(t,v_2). \end{cases}$$

Then, it is obvious that

$$\|\Phi(t,\mathbf{v})\|_{V^*} \le \|\Phi_1(t,v_1)\|_{V_1^*} + \|\Psi_1(t,v_2)\|_{V_1^*} + \|\Phi_2(t,v_2)\|_{V_2^*} + \|\Psi_2(t,v_1)\|_{V_2^*}.$$

Using the conditions **(HΦ)** (b) and **(HΨ)** (c), we deduce that $\mathbf{\Phi}(t,\cdot) : V \to 2^{V^*}$ is bounded. In what follows, according to Proposition 1, in order to attain the pseudo-monotonicity of the mapping $\mathbf{\Phi}(t,\cdot)$, it is sufficient for us to show the generalized pseudo-monotonicity of $\mathbf{\Phi}(t,\cdot)$. In fact, we assume that $\mathbf{v}^n \to \mathbf{v}$ weakly in V, $\Phi(t,\mathbf{v}^n) \in \mathbf{\Phi}(t,\mathbf{v}^n)$ with $\Phi(t,\mathbf{v}^n) \to \xi$ weakly in V^* and

$$0 \ge \limsup_{n\to\infty} \langle \Phi(t,\mathbf{v}^n), \mathbf{v}^n - \mathbf{v}\rangle_{V^*\times V}, \tag{9}$$

where $\mathbf{v}^n = (v_1^n, v_2^n)$, $\mathbf{v} = (v_1, v_2)$ and

$$\begin{cases} \Phi(t,\mathbf{v}^n) = (\Phi_1(t,v_1^n) + \Psi_1(t,v_2^n), \Phi_2(t,v_2^n) + \Psi_2(t,v_1^n)), \\ \Phi_1(t,v_1^n) \in \mathbf{\Phi}_1(t,v_1^n) \text{ and } \Phi_2(t,v_2^n) \in \mathbf{\Phi}_2(t,v_2^n). \end{cases}$$

Using the condition **(HΨ)** (b), one infers that $\Psi_1(t,v_2^n) \to \Psi_1(t,v_2)$ weakly in V_1^* and $\Psi_2(t,v_1^n) \to \Psi_2(t,v_1)$ weakly in V_2^*. This, along with the reflexivity of V_l $(l = 1,2)$, ensures that

$$\langle \Psi_1(t,v_2^n), v_1\rangle_{V_1^*\times V_1} \to \langle \Psi_1(t,v_2), v_1\rangle_{V_1^*\times V_1}, \tag{10}$$

and

$$\langle \Psi_2(t,v_1^n), v_2\rangle_{V_2^*\times V_2} \to \langle \Psi_2(t,v_1), v_2\rangle_{V_2^*\times V_2}. \tag{11}$$

Meantime, using the condition **(HΨ)** (d), one has

$$\langle \Psi_1(t,v_2^n), v_1^n\rangle_{V_1^*\times V_1} + \langle \Psi_2(t,v_1^n), v_2^n\rangle_{V_2^*\times V_2} = 0. \tag{12}$$

Therefore, combining (9)–(12), we conclude that

$$\begin{aligned} 0 &\ge \limsup_{n\to\infty} \langle \Phi(t,\mathbf{v}^n), \mathbf{v}^n - \mathbf{v}\rangle_{V^*\times V} \\ &= \limsup_{n\to\infty} \{ \langle \Phi_1(t,v_1^n), v_1^n - v_1\rangle_{V_1^*\times V_1} + \langle \Phi_2(t,v_2^n), v_2^n - v_2\rangle_{V_2^*\times V_2} \\ &\quad - \langle \Psi_1(t,v_2^n), v_1\rangle_{V_1^*\times V_1} - \langle \Psi_2(t,v_1^n), v_2\rangle_{V_2^*\times V_2} \\ &\quad + \langle \Psi_1(t,v_2^n), v_1^n\rangle_{V_1^*\times V_1} + \langle \Psi_2(t,v_1^n), v_2^n\rangle_{V_2^*\times V_2} \} \\ &= \limsup_{n\to\infty} \{ \langle \Phi_1(t,v_1^n), v_1^n - v_1\rangle_{V_1^*\times V_1} + \langle \Phi_2(t,v_2^n), v_2^n - v_2\rangle_{V_2^*\times V_2} \}. \end{aligned} \tag{13}$$

Next, let us show that

$$\limsup_{n\to\infty} \langle \Phi_1(t,v_1^n), v_1^n - v_1\rangle_{V_1^*\times V_1} \le 0, \tag{14}$$

and

$$\limsup_{n\to\infty} \langle \Phi_2(t,v_2^n), v_2^n - v_2\rangle_{V_2^*\times V_2} \le 0. \tag{15}$$

Indeed, we conversely assume that the above claim is not valid. Then, there is at least one false inequality. We may suppose that (14) is false. Then, there exists $d > 0$ and a subsequence of $\{v_1^n\}$ that is still written by $\{v_1^n\}$, s.t.

$$\lim_{n\to\infty} \langle \Phi_1(t,v_1^n), v_1^n - v_1\rangle_{V_1^*\times V_1} = d > 0. \tag{16}$$

Thus, it follows from (13) and (16) that

$$\limsup_{n\to\infty}\langle \Phi_2(t,v_2^n), v_2^n - v_2\rangle_{V_2^*\times V_2} \le -d < 0. \tag{17}$$

Therefore, from (17) and the pseudomonotonicity of $\mathbf{\Phi}_2(t,\cdot)$, we have

$$\liminf_{n\to\infty}\langle \Phi_2(t,v_2^n), v_2^n - v_2\rangle_{V_2^*\times V_2} \ge 0,$$

which contradicts (17). Hence, (14) and (15) both are true.

Furthermore, using the reflexivity of V_1 and **(HΦ)** (b), we know that $\mathbf{\Phi}_1(t,\cdot) : V_1 \to 2^{V_1^*}$ is weakly compact-valued. In addition, from Nadler's result [27], it follows that $\exists \zeta_1^n \in \mathbf{\Phi}_1(t,v_1)$ s.t.

$$\|\Phi_1(t,v_1^n) - \zeta_1^n\| \le (1+\frac{1}{n})\mathcal{H}(\mathbf{\Phi}_1(t,v_1^n), \mathbf{\Phi}_1(t,v_1)).$$

Since $v_1^n \to v_1$ weakly in V_1, we deduce from the hypothesis **(HΦ)** (0) that as $n\to\infty$,

$$\|\Phi_1(t,v_1^n) - \zeta_1^n\|_{V_1^*} \le (1+\frac{1}{n})\mathcal{H}(\mathbf{\Phi}_1(t,v_1^n), \mathbf{\Phi}_1(t,v_1)) \to 0. \tag{18}$$

Since $\mathbf{\Phi}_1(t,v_1)$ is weakly compact, we may assume that $\exists \Phi_1(t,v_1) \in \mathbf{\Phi}_1(t,v_1)$ s.t. $\zeta_1^n \to \Phi_1(t,v_1)$ weakly in V_1^*. This, along with (18), yields

$$\Phi_1(t,v_1^n) \to \Phi_1(t,v_1) \text{ weakly in } V_1^*. \tag{19}$$

In addition, by the pseudo-monotonicity of $\mathbf{\Phi}_1(t,\cdot)$, we obtain from (14) that

$$\lim_{n\to\infty}\langle \Phi_1(t,v_1^n), v_1^n - v_1\rangle_{V_1^*\times V_1} = 0. \tag{20}$$

The similar conclusion that

$$\Phi_2(t,v_2^n) \to \Phi_2(t,v_2) \text{ weakly in } V_2^* \tag{21}$$

and

$$\lim_{n\to\infty}\langle \Phi_2(t,v_2^n), v_2^n - v_2\rangle_{V_2^*\times V_2} = 0 \tag{22}$$

holds due to the w-$\mathcal{H}$-continuity and pseudo-monotonicity of $\mathbf{\Phi}_2(t,\cdot)$. Therefore, we can conclude by (19), (21), $\Phi(t,\mathbf{v}^n) \rightharpoonup \xi$ and the weak continuity of $\Psi_l(t,\cdot)$ $(l=1,2)$ that

$$\begin{aligned}\Phi(t,\mathbf{v}^n) &= (\Phi_1(t,v_1^n) + \Psi_1(t,v_2^n), \Phi_2(t,v_2^n) + \Psi_2(t,v_1^n)) \\ &\rightharpoonup \xi = (\Phi_1(t,v_1) + \Psi_1(t,v_2), \Phi_2(t,v_2) + \Psi_2(t,v_1)) = \Phi(t,\mathbf{v}) \\ &\in (\mathbf{\Phi}_1(t,v_1) + \Psi_1(t,v_2)) \times (\mathbf{\Phi}_2(t,v_2) + \Psi_2(t,v_1)) = \mathbf{\Phi}(t,\mathbf{v}).\end{aligned} \tag{23}$$

The above notation $\rightharpoonup$ represents the weak convergence in V^*. In addition, combining (10)–(12), (20), and (22) guarantees that $\lim_{n\to\infty}\langle \Phi(t,\mathbf{v}^n), \mathbf{v}^n - \mathbf{v}\rangle_{V^*\times V} = 0$. Hence, from (23), we get

$$\lim_{n\to\infty}\langle \Phi(t,\mathbf{v}^n), \mathbf{v}^n\rangle_{V^*\times V} = \langle \xi, \mathbf{v}\rangle_{V^*\times V}.$$

Consequently, in terms of Definition 2, $\mathbf{\Phi}(t,\cdot)$ is of generalized pseudomonotonicity. □

Lemma 4. *If the conditions* **(HΦ)** *(c) and* **(HΨ)** *(d) hold, then the operator* $\mathbf{\Phi}(t,\cdot)$ *is of coercivity with constant* $\alpha = \min\{\alpha_1,\alpha_2\}/2$. *In addition, if the conditions* **(HΦ)** *(d) and* **(HΨ)** *(d) hold, then the operator* $\mathbf{\Phi}(t,\cdot)$ *is of strong monotonicity with coefficient* $\beta = \min\{\beta_1,\beta_2\}/2$.

Proof. For any $\mathbf{v}, \mathbf{w} \in V$ with $\mathbf{v} = (v_1, v_2), \mathbf{w} = (\omega_1, \omega_2)$, we take $\Phi(t, \mathbf{v}) \in \mathbf{\Phi}(t, \mathbf{v})$ and $\Phi(t, \mathbf{w}) \in \mathbf{\Phi}(t, \mathbf{w})$ arbitrarily, with

$$\begin{cases} \Phi(t, \mathbf{v}) = (\Phi_1(t, v_1) + \Psi_1(t, v_2), \Phi_2(t, v_2) + \Psi_2(t, v_1)), \\ \Phi_1(t, v_1) \in \mathbf{\Phi}_1(t, v_1) \text{ and } \Phi_2(t, v_2) \in \mathbf{\Phi}_2(t, v_2), \end{cases}$$

and

$$\begin{cases} \Phi(t, \mathbf{w}) = (\Phi_1(t, \omega_1) + \Psi_1(t, \omega_2), \Phi_2(t, \omega_2) + \Psi\Psi_2(t, \omega_1)), \\ \Phi_1(t, \omega_1) \in \mathbf{\Phi}_1(t, \omega_1) \text{ and } \Phi_2(t, \omega_2) \in \mathbf{\Phi}_2(t, \omega_2). \end{cases}$$

We first show that $\mathbf{\Phi}(t, \cdot)$ is coercive. Indeed, it is clear that, under the conditions **(HΦ)** (c) and **(HΨ)** (d), the operator $\mathbf{\Phi}(t, \cdot) : V \to 2^{V^*}$ is coercive with constant α owing to the inequality below:

$$\begin{aligned} \langle \Phi(t, \mathbf{v}), \mathbf{v} \rangle_{V^* \times V} &= \langle \Phi_1(t, v_1) + \Psi_1(t, v_2), v_1 \rangle_{V_1^* \times V_1} + \langle \Phi_2(t, v_2) + \Psi_2(t, v_1), v_2 \rangle_{V_2^* \times V_2} \\ &\geq \alpha_1 \|v_1\|_{V_1}^2 + \alpha_2 \|v_2\|_{V_2}^2 \geq \alpha \|\mathbf{v}\|_V^2. \end{aligned}$$

Secondly, we show the strong monotonicity of $\mathbf{\Phi}(t, \cdot)$. In fact, from the strong monotonicity of $\mathbf{\Phi}_l(t, \cdot)$ $(l = 1, 2)$ and the condition **(HΨ)** (d), we have

$$\begin{aligned} &\langle \Phi(t, \mathbf{v}) - \Phi(t, \mathbf{w}), \mathbf{v} - \mathbf{w} \rangle_{V^* \times V} \\ &= \langle \Phi_1(t, v_1) - \Phi_1(t, \omega_1), v_1 - \omega_1 \rangle_{V_1^* \times V_1} + \langle \Phi_2(t, v_2) - \Phi_2(t, \omega_2), v_2 - \omega_2 \rangle_{V_2^* \times V_2} \\ &\quad + \langle \Psi_1(t, v_2), v_1 \rangle_{V_1^* \times V_1} + \langle \Psi_2(t, v_1), v_2 \rangle_{V_2^* \times V_2} - \langle \Psi_1(t, v_2), \omega_1 \rangle_{V_1^* \times V_1} \\ &\quad - \langle \Psi_2(t, \omega_1), v_2 \rangle_{V_2^* \times V_2} - \langle \Psi_1(t, \omega_2), v_1 \rangle_{V_1^* \times V_1} - \langle \Psi_2(t, v_1), \omega_2 \rangle_{V_2^* \times V_2} \\ &\quad + \langle \Psi_1(t, \omega_2), \omega_1 \rangle_{V_1^* \times V_1} + \langle \Psi_2(t, \omega_1), \omega_2 \rangle_{V_2^* \times V_2} \\ &\geq \beta_1 \|v_1 - \omega_1\|_{V_1}^2 + \beta_2 \|v_2 - \omega_2\|_{V_2}^2 \\ &\geq \beta \|\mathbf{v} - \mathbf{w}\|_V^2. \end{aligned}$$

Therefore, we attain the strong monotonicity of $\mathbf{\Phi}(t, \cdot)$ with constant β. □

(HΓ): For $l = 1, 2$, the operator $\mathbf{\Gamma}_l : (0, T) \to 2^{\mathcal{L}(V_l, V_l^*)}$ satisfies
(a) $\mathbf{\Gamma}_l$ is s-$\mathcal{H}$-continuous with compact values in $\mathcal{L}(V_l, V_l^*)$;
(b) each $\Gamma_l : (0, T) \to \mathcal{L}(V_l, V_l^*)$ with $\Gamma_l(t) \in \mathbf{\Gamma}_l(t)$ $\forall t \in (0, T)$, is such that $\Gamma_l \in L^2(0, T; \mathcal{L}(V_l, V_l^*))$.
(HJ): $J : (0, T) \times X_1 \times X_2 \to \mathbf{R}$ is locally Lipschitz, s.t.
(a) $J(\cdot, p_1, p_2)$ is measurable on $(0, T)$ for all $(p_1, p_2) \in X_1 \times X_2$;
(b) for $p_2 \in X_2$, $\exists \ell_1, \tau_1 \geq 0$ s.t.

$$\|\eta_1\|_{X_1^*} \leq \ell_1 + \tau_1 \|p_1\|_{X_1}, \ \forall p_1 \in X_1, \ \eta_1 \in \partial_1 J(t, p_1, p_2); \tag{24}$$

(c) for $p_1 \in X_1$, $\exists \ell_2, \tau_2 \geq 0$ s.t.

$$\|\eta_2\|_{X_2^*} \leq \ell_2 + \tau_2 \|p_2\|_{X_2}, \ \forall p_2 \in X_2, \ \eta_2 \in \partial_2 J(t, p_1, p_2); \tag{25}$$

(d) for $p_2 \in X_2$, $\partial_1 J(t, \cdot, p_2)$ is of relaxed monotonicity on X_1 that is, $\exists m_1 > 0$ s.t., $\forall p_1, q_1 \in X_1$, $\eta_1 \in \partial_1 J(t, p_1, p_2)$ and $\forall \xi_1 \in \partial_1 J(t, q_1, p_2)$,

$$\langle \eta_1 - \xi_1, p_1 - q_1 \rangle_{X_1^* \times X_1} \geq -m_1 \|p_1 - q_1\|_{X_1}^2; \tag{26}$$

(e) for $p_1 \in X_1$, $\partial_2 J(t, p_1, \cdot)$ is of relaxed monotonicity on X_2 that is, $\exists m_2 > 0$ s.t., $\forall p_2, q_2 \in X_2$, $\eta_2 \in \partial_2 J(t, p_1, p_2)$ and $\forall \xi_2 \in \partial_2 J(t, p_1, q_2)$,

$$\langle \eta_2 - \xi_2, p_2 - q_2 \rangle_{X_2^* \times X_2} \geq -m_2 \|p_2 - q_2\|_{X_2}^2. \tag{27}$$

Lemma 5 (see [24], Lemma 3.5)**.** *Let the locally Lipschitz $J(t,\cdot,\cdot)$ be regular on X. If conditions* **(HJ)** *(b) and* **(HJ)** *(c) hold, then $\partial J(t,\cdot,\cdot)$ is bounded on X and*

$$\|\eta\|_{X^*} \leq \ell + \tau\|\mathbf{p}\|_X, \quad \forall \mathbf{p} = (p_1, p_2) \in X,\ \eta \in \partial J(t, p_1, p_2), \tag{28}$$

with $\ell = \ell_1 + \ell_2$ and $\tau = \max\{\tau_1, \tau_2\}$. In addition, if conditions **(HJ)** *(d) and* **(HJ)** *(e) hold, then $\partial J(t,\cdot,\cdot)$ is of relaxed monotonicity on X with coefficient $m = \max\{m_1, m_2\}$.*

4. Main Results

Under quite mild assumptions, we will show that there exists only a solution to the GSTDHVI with VITs via an approach of an auxiliary vector inclusion problem (AVIP, for short) and by the Banach contraction mapping principle. It is worth mentioning that the techniques for the demonstrations of the following theorems had previously been applied in [6] for the subdifferential inclusion problems.

We now define the AVIP, which is formulated below:

Find $\mathbf{p} \in \mathcal{V}$ s.t.

$$\varphi(t) \in \mathbf{\Phi}(t, \mathbf{p}(t)) + \Theta^* \circ \partial J(t, \Theta(\mathbf{p}(t))). \tag{29}$$

The important result for the AVIP is stated and proven below that will be applied for demonstrating that there exists only a solution to the GSTDHVI with VITs.

Theorem 2. *Assume that $\mathbf{\Phi}_l : (0,T) \times V_l \to 2^{V_l^*}$ and $\Psi_l : (0,T) \times V_k \to V_l^*$ are operators with images in V_l^* for $l,k = 1,2$ and $k \neq l$. Let $\Theta_l : V_l \to X_l$ be a linear bounded and compact operator, $J : (0,T) \times X_1 \times X_2 \to \mathbf{R}$ be regular and locally Lipschitz, and $\varphi_l \in \mathcal{V}_l$ for $l = 1,2$. If the conditions* **(HΦ)**, **(HΨ)**, *and* **(HJ)** *are valid, there exists only a solution $\mathbf{p}$ to the AVIP provided*

$$\alpha > d\|\Theta\|^2 \quad \text{and} \quad \beta > m\|\Theta\|^2, \tag{30}$$

with $\|\Theta\|$ being the norm of Θ in (2). In addition, the solution $\mathbf{p}$ satisfies

$$\|\mathbf{p}\|_{\mathcal{V}} \leq \gamma(1 + \|\varphi\|_{\mathcal{V}^*}) \quad \text{for some } \gamma > 0. \tag{31}$$

Proof. The proof of Theorem 2 is divided into two steps.

Step 1. We show the existence and uniqueness of solutions to the AVIP in $\mathcal{V}$. In fact, we construct a multi-valued mapping $M : V \to 2^{V^*}$ below:

$$M(\mathbf{v}) := \mathbf{\Phi}(t, \mathbf{v}) + \Theta^* \circ \partial J(t, \Theta\mathbf{v}), \quad \forall \mathbf{v} \in V.$$

Next, we derive the assertions on mapping M, successively.

(P1) M is of pseudomonotonicity on V.

First, according to Lemmas 3 and 5, we know that mapping $\mathbf{\Phi}(t,\cdot)$ is bounded and pseudomonotone, and mapping $\partial J(t,\cdot,\cdot)$ is of boundedness. Since Θ is a linear and bounded operator, M is bounded on V. In addition, it is clear that M has nonempty, closed, and convex values in V^* because $\partial J(t,\cdot,\cdot)$ has nonempty, closed and convex values in X^*. Thus, in view of Proposition 1, it is sufficient for us to show the generalized pseudo-monotonicity of M to attain the pseudo-monotonicity of M. In fact, assume that $\mathbf{v}^n \to \mathbf{v}$ weakly in V, $\xi^n \in M(\mathbf{v}^n)$ with $\xi^n \to \xi$ weakly in V^*, and

$$\limsup_{n\to\infty} \langle \xi^n, \mathbf{v}^n - \mathbf{v} \rangle_{V^* \times V} \leq 0. \tag{32}$$

Then, $\Theta\mathbf{v}^n \to \Theta\mathbf{v}$ in X and there exist $\Phi(t, \mathbf{v}^n) \in \mathbf{\Phi}(t, \mathbf{v}^n)$ and $\eta^n \in \partial J(t, \Theta\mathbf{v}^n) \subset X^*$, which are bounded in V^* and in X^*, respectively, by the boundedness of mappings $\mathbf{\Phi}(t,\cdot)$ and $\partial J(t,\cdot,\cdot)$, s.t.

$$\xi^n = \Phi(t, \mathbf{v}^n) + \Theta^*\eta^n. \tag{33}$$

We might assume, using the closedness of $\partial J(t,\cdot,\cdot)$ that $\eta^n \to \eta$ weakly with $\eta \in \partial J(t,\Theta\mathbf{v}) \subset X^*$. Meantime, noticing the compactness of Θ^*, we deduce that $\Theta^*\eta^n \to \Theta^*\eta \in \Theta^* \circ \partial J(t,\Theta\mathbf{v})$ and

$$\lim_{n\to\infty}\langle \Theta^*\eta^n, \mathbf{v}^n - \mathbf{v}\rangle_{V^*\times V} = 0. \tag{34}$$

Hence, combining (32)–(34) ensures that $\limsup_{n\to\infty}\langle \Phi(t,\mathbf{v}^n), \mathbf{v}^n - \mathbf{v}\rangle_{V^*\times V} \leq 0$. In the same inference way as the proof of Lemma 3, it is easy to check that $\Phi(t,\mathbf{v}^n) \to \Phi(t,\mathbf{v})$ weakly in V^* for some $\Phi(t,\mathbf{v}) \in \mathbf{\Phi}(t,\mathbf{v})$ and $\lim_{n\to\infty}\langle \Phi(t,\mathbf{v}^n), \mathbf{v}^n - \mathbf{v}\rangle_{V^*\times V} = 0$. Therefore, $\xi^n = \Phi(t,\mathbf{v}^n) + \Theta^*\eta^n \to \Phi(t,\mathbf{v}) + \Theta^*\eta = \xi \in M(\mathbf{v})$ weakly in V^* and

$$\lim_{n\to\infty}\langle \xi^n, \mathbf{v}^n - \mathbf{v}\rangle_{V^*\times V} = \lim_{n\to\infty}\langle \Phi(t,\mathbf{v}^n) + \Theta^*\eta^n, \mathbf{v}^n - \mathbf{v}\rangle_{V^*\times V} = 0.$$

Consequently,

$$\lim_{n\to\infty}\langle \xi^n, \mathbf{v}^n\rangle_{V^*\times V} = \lim_{n\to\infty}\langle \xi^n, \mathbf{v}^n - \mathbf{v} + \mathbf{v}\rangle_{V^*\times V} = \langle \xi, \mathbf{v}\rangle_{V^*\times V}.$$

Thus, M is generalized pseudomonotone on V.

(P2) M is coercive on V.

Suppose that $\mathbf{v} \in V$ and $\xi \in M(\mathbf{v})$. It is clear that $\exists\Phi(t,\mathbf{v}) \in \mathbf{\Phi}(t,\mathbf{v})$ and $\exists\eta \in \partial J(t,\Theta\mathbf{v})$ s.t. $\xi = \Phi(t,\mathbf{v}) + \Theta^*\eta$. Using Lemmas 4 and 5, we know that

$$\begin{aligned} \langle \xi, \mathbf{v}\rangle_{V^*\times V} &= \langle \Phi(t,\mathbf{v}), \mathbf{v}\rangle_{V^*\times V} + \langle \Theta^*\eta, \mathbf{v}\rangle_{V^*\times V} \\ &\geq \alpha\|\mathbf{v}\|_V^2 - \|\eta\|_{X^*}\|\Theta\|\|\mathbf{v}\|_V \\ &\geq (\alpha - d\|\Theta\|^2)\|\mathbf{v}\|_V^2 - c\|\Theta\|\|\mathbf{v}\|_V. \end{aligned} \tag{35}$$

This along with $d\|\Theta\|^2 < \alpha$ leads to the coercivity of M on V with $c(t) = (\alpha - d\|\Theta\|^2)t - c\|\Theta\|$.

On the other hand, by the properties (P1) and (P2), we know that M is of surjectivity. Thus, $\exists\mathbf{p}_t \in V$ (due to its dependence on t), s.t. $\varphi(t) \in M(\mathbf{p}_t) = \mathbf{\Phi}(t,\mathbf{p}_t) + \Theta^* \circ \partial J(t,\Theta\mathbf{p}_t)$. This ensures the existence of solutions to the AVIP. To show the uniqueness of solutions to the AVIP, one supposes on the contrary that the AVIP enjoys two distinguished solutions $\mathbf{p}_t^1$ and $\mathbf{p}_t^2$ in V. Then, $\exists\Phi(t,\mathbf{p}_t^l) \in \mathbf{\Phi}(t,\mathbf{p}_t^l) \subset V^*$ and $\exists\eta^l \in \partial J(t,\Theta\mathbf{p}_t^l) \subset X^*$ for $l = 1,2$, s.t.

$$\Phi(t,\mathbf{p}_t^1) + \Theta^*\eta^1 = \varphi(t), \tag{36}$$

and

$$\Phi(t,\mathbf{p}_t^2) + \Theta^*\eta^2 = \varphi(t). \tag{37}$$

Subtracting (37) via (36) and multiplying the associated result by $\mathbf{p}_t^1 - \mathbf{p}_t^2$, we obtain that

$$\langle \Phi(t,\mathbf{p}_t^1) - \Phi(t,\mathbf{p}_t^2), \mathbf{p}_t^1 - \mathbf{p}_t^2\rangle_{V^*\times V} + \langle \Theta^*\eta^1 - \Theta^*\eta^2, \mathbf{p}_t^1 - \mathbf{p}_t^2\rangle_{V^*\times V} = 0. \tag{38}$$

Using Lemmas 4 and 5, one gets $(\beta - m\|\Theta\|^2)\|\mathbf{p}_t^1 - \mathbf{p}_t^2\|_V^2 \leq 0$. Thus, from the condition $\beta > m\|\Theta\|^2$, we have $\mathbf{p}_t^1 = \mathbf{p}_t^2$, which arrives at the contradiction. Thus, we get the uniqueness of solution $\mathbf{p}_t$ to the AVIP in V. In addition, by the coercivity of mapping M and in the same inference way as the proof of (35), it is easy to verify that $(\alpha - d\|\Theta\|^2)\|\mathbf{p}_t\|_V \leq c\|\Theta\| + \|\varphi(t)\|_{V^*}$. Consequently, $\exists\bar{\gamma} > 0$ s.t.

$$\|\mathbf{p}_t\|_V \leq \bar{\gamma}(1 + \|\varphi(t)\|_{V^*}). \tag{39}$$

Step 2. We show that there exists only a solution to the AVIP (29) and the estimation (31) holds. In the same inference way as the proof of ([6], Theorem 4.11), it is easy to check that the function $\mathbf{p} : t \mapsto \mathbf{p}_t \in V$, where $\mathbf{p}_t$ is only a solution in V to the AVIP for each t, is measurable and $\mathbf{p} \in \mathcal{V}$. Hence, there exists only a solution to the AVIP. In addition, from the estimation (39), it is clear that $\exists \gamma > 0$ s.t.

$$\|\mathbf{p}\|_{\mathcal{V}}^2 = \int_0^T \|\mathbf{p}(t)\|_V^2 dt \le \int_0^T (\tilde{\gamma}(1 + \|\varphi(t)\|_{V^*}))^2 dt \le (\gamma(1 + \|\varphi\|_{\mathcal{V}^*}))^2,$$

which leads to the estimation (31). □

By virtue of Theorem 2, we show below that there exists only a solution to the GSTDHVIP with VITs.

Theorem 3. *Suppose that the hypothesis* **(HΓ)** *holds and all conditions in Theorem 2 are valid. Then, there exists at least one solution* $\mathbf{p} \in \mathcal{V}$ *to the GSTDHVI with VITs. In addition, if the above equality (6) or (7) holds, then there is only a solution to the GSTDHVI with VITs.*

Proof. Using Lemma 1, we can show the existence of solutions to the DVIP with VIT, which hence leads to the existence of solutions to the GSTDHVI with VITs. Meantime, using Lemma 2, we know the uniqueness of solutions to the DVIP with VIT, which hence leads to the uniqueness of solutions to the GSTDHVI with VITs. In the same inference way as the proof of ([6], Theorem 4.13), we could obtain the desired conclusion by using the Banach contraction mapping principle. Therefore, we omit the details of proof here. □

Next, let the GTDHVI indicate the generalized time-dependent hemivariational inequality. We first provide a particular example for Theorem 3. Put $V_1 = V_2 = W$, $X_1 = X_2 = Y$, and set $\mathbf{\Phi}_1 = \mathbf{\Phi}_2 = \bar{\mathbf{\Phi}}$, $\Theta_1 = \Theta_2 = \bar{\Theta}$, $\mathbf{\Gamma}_1 = \mathbf{\Gamma}_2 = \bar{\mathbf{\Gamma}}$, and $\varphi_1 = \varphi_2 = \bar{\varphi}$. Assume further that $\Psi_1 = \Psi_2$ is a zero operator, where its images are always zero vector $\theta \in W^*$ and J is chosen as in (6) with $\phi = \psi = j$ on Y by $J(t, y_1, y_2) := j(t, y_1) + j(t, y_2)\ \forall y_1, y_2 \in Y$. Then, the GSTDHVI with VITs reverts to the GTDHVI with VIT below:

Find $w \in L^2(0, T; W)$ s.t. for some $\bar{\Phi}(t, w(t)) \in \bar{\mathbf{\Phi}}(t, w(t))$ and $\bar{\Gamma}(t) \in \mathbf{\Gamma}(t)$,

$$\begin{aligned} &\langle \bar{\Phi}(t, w(t)), v\rangle_{W^* \times W} + j^\circ(t, \bar{\Theta}(w(t)); \bar{\Theta}v) + \langle \textstyle\int_0^t \bar{\Gamma}(t-s)w(s)ds, v\rangle_{W^* \times W} \\ &\quad \ge \langle \bar{\varphi}(t), v\rangle_{W^* \times W}, \quad \forall v \in W. \end{aligned} \tag{40}$$

Utilizing Theorem 3, one can readily obtain the result below that there exists only a solution to the GTDHVI with VIT.

Theorem 4. *Suppose that W and Y are reflexive Banach spaces such that W is separable.*

Assume that $\bar{\mathbf{\Phi}} : (0, T) \times W \to 2^{W^*}$ is an operator with images in W^*, $\bar{\Theta} : W \to Y$ is a linear bounded mapping with compactness, $\bar{\mathbf{\Gamma}} : (0, T) \to 2^{\mathcal{L}(W, W^*)}$ is an operator with images of linear continuous mappings of W into W^*, and $j : (0, T) \times Y \to \mathbf{R}$ is of regularity and of local Lipschitz continuity. Let the following conditions on the mappings in the GTDHVI with VIT be valid:

(a) $\bar{\mathbf{\Phi}}(t, \cdot) : W \to 2^{W^*}$ is w-$\mathcal{H}$-continuous;

(b) $\bar{\mathbf{\Phi}}(t, \cdot) : W \to 2^{W^*}$ is of boundedness, of pseudo-monotonicity, of coercivity with constant $\alpha > 0$ and of strong monotonicity with coefficient $\beta > 0$;

(c) $j(\cdot, x) : (0, T) \to \mathbf{R}$ is measurable for each $x \in Y$;

(d) $\exists \ell, \tau \ge 0$ s.t. $\|\eta\|_{Y^*} \le \ell + \tau\|y\|_Y\ \ \forall y \in Y,\ \eta \in \partial j(t, y)$;

(e) $\partial j(t, \cdot)$ is of relaxed monotonicity, that is, $\exists m > 0$ s.t. $\forall y_1, y_2 \in Y$,

$$\langle \eta_1 - \eta_2, y_1 - y_2\rangle_{Y^* \times Y} \ge -m\|y_1 - y_2\|_Y^2\ \ \forall \eta_1 \in \partial j(t, y_1), \eta_2 \in \partial j(t, y_2);$$

(f) $\bar{\mathbf{\Gamma}}$ is s-$\mathcal{H}$-continuous with compact-values in $\mathcal{L}(W, W^*)$;

(g) the mapping $\bar{\Gamma} \in L^2(0, T; \mathcal{L}(W, W^*))$ and $\bar{\varphi} \in L^2(0, T; W)$.

Then, there exists only a solution to the GTDHVI with VIT provided $\alpha > d\|\bar{\Theta}\|^2$ and $\beta > m\|\bar{\Theta}\|^2$.

Remark 1. *The GSTDHVI with VITs and the corresponding DVIP with VIT are more general and more advantageous than the STDHVI with VITs and the corresponding DVIP with VIT in Xiao et al. [24], respectively. Theorem 2 on the AVIP exhibits a powerful role in the proof of the conclusion that only a solution to the GSTDHVI with VITs exists. In the proof of Theorem 2, to demonstrate that only a solution to the AVIP without VIT exists, we introduce the new notion of w-$\mathcal{H}$-continuity (resp., s-$\mathcal{H}$-continuity) for multi-valued operators and impose new mild restrictions on multi-valued operators $\mathbf{\Phi}_l$ and $\mathbf{\Gamma}_l$ $(l = 1, 2)$, e.g., the hypotheses (a)–(d) in* **(HΦ)** *and hypotheses (a)–(b) in* **(HΓ)**. *In the proof of Lemma 3, we make use of Proposition 1 to derive the pseudo-monotonicity of multi-valued operator $\mathbf{\Phi}(t, \cdot)$. However, there is an assumption of boundedness for multi-valued operators in Proposition 1. Note that the pseudo-monotonicity of multi-valued operator $\mathbf{\Phi}_l(t, \cdot)$ can not guarantee its boundedness. Meantime, we make use of Nadler's result [27] and the w-$\mathcal{H}$-continuity and pseudo-monotonicity of $\mathbf{\Phi}_l(t, \cdot)$, to obtain the generalized pseudo-monotonicity of $\mathbf{\Phi}(t, \cdot)$. Without doubt, the approach of the proof of Theorem 2 is very different from that of the proof in ([24], Theorem 2). All in all, Theorems 2–4 extend, improve, and develop Theorems 2–4 in [24] to a great extent, respectively. In addition, whenever $\bar{\Theta} = \Theta \circ i$, with Θ being linear bounded and i being the compact embedding mapping in the time-dependent subdifferential inclusion problem of [6], Theorem 4.3 in [24] reverts to Theorem 4.13 in [6].*

5. Conclusions

In this paper, under quite mild assumptions, it is shown that there exists only a solution to the GSTDHVI with VITs via demonstrating that there exists only a solution to the corresponding DVIP with VIT. Our results generalize, improve, and develop the corresponding ones in very recent literature. It is worth pointing out that, inspired by the problem put forward in [20], Xiao et al. [24] introduced and studied the STDHVI with VITs, and showed that only a solution to the STDHVI with VITs under quite appropriate conditions exists.

Put forth as the above, a particular case of our main theorem is an extension of ([24], Theorem 4) for the solvability of the TDHVI with VIT. However, a special case of the one in [24] is also an extension of ([6], Theorem 4.13) for the solvability of a time-dependent subdifferential inclusion with VIT. An HVI is referred to as parabolic or evolutionary HVI if it involves the time derivative of unknown function. To the extent of our knowledge, it would be very meaningful and quite valuable to explore under what conditions the theorems in this article are still valid for a generalized parabolic or evolutionary SHVIs with VITs.

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