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Consonance, Symmetry and Extended Outputs

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Abstract: In many real-world situations, performers always adopt different energy levels (strategies) to participate. Different from pre-existing results, this paper is devoted to proposing several generalized power outputs of the marginal index, the Banzhaf–Coleman index, and the Banzhaf–Owen index, respectively, by assigning different energy levels to all performers. Since these extended power outputs may not be efficacious, we further define the efficacious extensions of these power outputs, respectively. For each of these efficacious power outputs, we demonstrate that there exists a corresponding reduced game and related consonance property that can be used to characterize it. By focusing on the properties of symmetry and accordance, several axiomatic results are also introduced.

Keywords: power output; consonance; symmetry; accordance

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1. Introduction

Several indexes have been proposed as suitable mappings from weights and decision quota to power. The most widely used one is the Banzhaf–Coleman index. The Banzhaf–Coleman index, named after Banzhaf [1], is a power index proposed by the probability of changing a result of a vote where voting interests are not necessarily equally distributed among the voters. Based on the demands for accounting, economics, management sciences, and even political sciences, some more power indexes have been proposed, such as the Banzhaf–Owen index and the marginal index. Related results may be found in, e.g., Banzhaf [1], Owen [2], Dubey and Shapley [3], Moulin [4], Lehrer [5], van den Brink and van der Laan [6], and Wei et al. [7]. Briefly, the Banzhaf–Coleman index is a rule that gathers each performer's average marginal contribution from all coalitions in which it has participated. The Banzhaf–Owen index is a rule that gathers each performer's total marginal contribution from all coalitions in which it has participated. The marginal index gathers each performer's marginal contribution from the grand coalition. However, the three indexes do not necessarily allocate the usability over all the performers. Thus, the efficient Banzhaf–Coleman index, the efficient Banzhaf–Owen index, and the equal allocation of non-separable costs (EANSC) were introduced by Hwang and Liao [8] and Ransmeier [9], respectively. Based on the notion of the efficient Banzhaf–Coleman index (the efficient Banzhaf–Owen index, the EANSC), all performers receive its Banzhaf–Coleman index (Banzhaf–Owen, marginal index) and further allocate the remaining usability equally.

Under the axiomatic processes of cooperative games, consonance (consistency) is a crucial property of useful solutions. The notion behind this type of consonance is as follows: for a given game, performers might develop prospects of the game and may be willing to consent to the computation of its remunerations to be based on these prospects. The solution is consonant if it gives the same remunerations to performers in the original game as it does to performers of the imaginary reduced game. Thus, consonance is a requisite of the inner "robustness" of compromises. There exist three main types of reduction in the literature, the max-reduced game due to Davis–Maschler [10], the complement-reduced game due to Moulin [4], and the self-reduced game due to Hart–Mas-Colell [11].

On the other hand, symmetry has been applied widely and significantly in many fields of mathematics, such as control theory, convex analysis, data mining, dynamical system theory, functional analysis, game theory, mathematical economics, nonlinear ordinary and partial differential equations, signal processing, variational analysis, and so on. The symmetry property always can be adopted to analyze the importance or consideration relationships among performers of a game by applying a utility allocation or power distribution. Therefore, many solutions could be characterized by means of the symmetry property, such as the efficient Banzhaf–Coleman index, the efficient Banzhaf–Owen index, the EANSC, the Shapley value [12], and so on.

In a traditional game, each performer either entirely participates or does not participate at all with other performers, while under a multi-choice game, each performer could consent to take many finite different energy levels. It is known that the outputs of multi-choice games could be applied in many fields such as economics, management sciences, sports sciences, and so on. Hwang and Liao [13,14] and Liao [15] proposed several extensions of the max-reduced game and the complement-reduced game to characterize several core concepts in the context of multi-choice games. Liao [16,17] proposed several extensions of the complement-reduced game to characterize several extended EANSCs on multi-choice games. Later, Hwang and Liao [18] also proposed an extension of the self-reduced game to characterize a multi-choice generalization of the Shapley value.

The above-mentioned results raise one question:

- Could the power indexes be proposed under the multi-choice consideration?

The paper is devoted to investigating this question. The main results are as follows.

- In Sections 2 and 5, we extend the Banzhaf–Coleman index, the Banzhaf–Owen index, and the marginal index to multi-choice games, which we name the level-marginal output, the level-accumulated output, the level-average output, the *-level-marginal output, the *-level-accumulated output, and the *-level-average output, respectively. Since these outputs are not efficacious, we also consider the efficacious extensions of these outputs.
- Briefly, the max-reduced game is based on the "maximizing notion". For the "summing notion" instead of the "maximizing notion", we consider the sum-reduced game. For the "averaging notion" instead of the "maximizing notion", we consider the average-reduced game in Section 3.
- Inspired by Hart and Mas-Colell [11], we provide axiomatic results of the efficacious extensions of these outputs based on consonance related to the complement-reduction, the sum-reduction, and the average-reduction, respectively, in Sections 3 and 5.
- Inspired by Maschler and Owen [19], we further adopt the related properties of symmetry and accordance to characterize the efficacious extensions of these outputs in Sections 4 and 5. Further, some more interpretations are also provided throughout this paper.

2. Preliminaries

2.1. Definitions and Notation

Let UV be the universe of performers and $P \subseteq UV$ be a set of performers. Suppose that each performer $p \in P$ has $e_p \in \mathbb{N}$ levels at which it can actively participate. Let $e = (e_p)_{p \in P}$ be the vector that gives the amount of energy levels for each performer, at which it can participate. For $p \in UV$, we set $E_p = \{0, 1, \cdots, e_p\}$ to be the level repository of performer p, where the level 0 means not acting, and $E_p^+ = E_p \setminus \{0\}$. For $P \subseteq UV$, $P \neq \emptyset$, let $E^P = \prod_{p \in P} E_p$ be the product set of the level repositories for performers in P. Denote 0_P as the zero vector in $\mathbb{R}^P$.

A multi-choice game is denoted by (P, e, h), where $P \neq \emptyset$ is a finite collection of performers, e is the vector that gives the amount of energy levels for each performer, and $h : E^P \to \mathbb{R}$ is a map that apportions to each level vector $\omega = (\omega_p)_{p \in P} \in E^P$ the usability that the performers can accept when each performer p takes energy level $\omega_p \in E_p$ with $h(0_P) = 0$. A game (P, e, h) will be denoted by the map h if there is no confusion. Given a multi-choice game (P,e,h) and $\omega \in E^P$, we denote (P,ω,h) as the multi-choice subgame defined by restricting h to $\{\chi \in E^P \mid \chi_p \leq \omega_p \ \forall p \in P\}$.

Denote the family of all multi-choice games by Λ. Given $(P,e,h) \in \Lambda$, let $L^{P,e} = \{(p,q) \mid p \in P, q \in E_p^+\}$. An output on Λ is a map ρ apportioning to each $(P,e,h) \in \Lambda$ an element

$$\rho(P,e,h) = \Big(\rho_{p,q}(P,e,h)\Big)_{(p,q)\in L^{P,e}} \in \mathbb{R}^{L^{P,e}},$$

where $\rho_{p,q}(P,e,h)$ is the power output or payoff of the performer p when it takes level q to join game h. For convenience, one could define that $\rho_{p,0}(P,e,h) = 0$ for each $(P,e,h) \in \Lambda$ and for each $p \in P$.

For $\omega \in \mathbb{R}^P$, we define $S(\omega) = \{p \in P \mid \omega_p \neq 0\}$ and ω_T to be the restriction of ω at T for each $T \subseteq P$ (without loss of generality, one could suppose that $S(e) = P$ for all $(P,e,h) \in \Lambda$). Let $P \subseteq UV$, $p \in P$, and $\omega \in \mathbb{R}^P$. For convenience, we define the notation ω_{-p} to stand for $\omega_{P\setminus\{p\}}$ and let $\chi = (\omega_{-p}, q) \in \mathbb{R}^P$ be defined by $\chi_{-p} = \omega_{-p}$ and $\chi_p = q$. Let $t \in P$, ω_{-pt} stand for $\omega_{P\setminus\{p,t\}}$ and (ω_{-pt}, j, l) stand for $((\omega_{-p}, q)_{-t}, l)$.

In the context of multi-choice games, we present three generalizations of the marginal index, the Banzhaf–Coleman index, and the Banzhaf–Owen index as follows.

Definition 1.

- *The level-marginal output, Θ^M, is the output that associates with $(P,e,h) \in \Lambda$ and each $(p,q) \in L^{P,e}$ the value:*

$$\Theta^M_{p,q}(P,e,h) = h(e_{-p}, q) - h(e_{-p}, q-1),$$

where $\Theta^M_{p,q}(P,e,h)$ is the general marginal value of performer p from level $q-1$ to q in the grand coalition.

- *The level-accumulated output, Θ^{AC}, is the output that associates with $(P,e,h) \in \Lambda$ and each $(p,q) \in L^{P,e}$ the value:*

$$\Theta^{AC}_{p,q}(P,e,h) = \sum_{\substack{S\subseteq P\\ p\in S}} \Big[h\big((e_{-p},q)_S, 0_{P\setminus S}\big) - h\big((e_{-p},q-1)_S, 0_{P\setminus S}\big)\Big],$$

where $\Theta^{AC}_{p,q}(P,e,h)$ is total marginal value of performer p from level $q-1$ to q among all participating coalitions.

- *The level-average output, Θ^{AV}, is the output that associates with $(P,e,h) \in \Lambda$ and each $(p,q) \in L^{P,e}$ the value:*

$$\Theta^{AV}_{p,q}(P,e,h) = \frac{1}{2^{|P|-1}} \cdot \sum_{\substack{S\subseteq P\\ p\in S}} \Big[h\big((e_{-p},q)_S, 0_{P\setminus S}\big) - h\big((e_{-p},q-1)_S, 0_{P\setminus S}\big)\Big],$$

where $\Theta^{AV}_{p,q}(P,e,h)$ is the average marginal value of performer p from level $q-1$ to q among all participating coalitions.

Let $(P,e,h) \in \Lambda$. It is easy to confirm that the level-marginal output, the level-accumulated output, and the level-average output are not adequate in distributing the usability $h(e)$ of the grand coalition because they are not efficacious, i.e., they do not necessarily allocate the usability $h(e)$ over the performers in P. Formally,

- an output ρ conforms to efficacy (EIY) if $\sum\limits_{p\in P} \rho_{p,e_p}(P,e,h) = h(e)$ for every $(P,e,h) \in \Lambda$.

Therefore, we consider possible efficacious extensions of these outputs as follows.

Definition 2.

- *The efficacious marginal output* $\overline{\Theta^M}$ *is the output that associates with* $(P,e,h) \in \Lambda$ *and each* $(p,q) \in L^{P,e}$ *the value:*

$$\overline{\Theta^M_{p,q}}(P,e,h) = \Theta^M_{p,q}(P,e,h) + \frac{1}{|P|} \cdot \big[h(e) - \sum_{k \in P} \Theta^M_{k,e_k}(P,e,h)\big].$$

Under the output $\overline{\Theta^M_{p,q}}(P,e,h)$*, performers firstly receive its general marginal values and further distribute equally the remaining benefit.*

- *The efficacious accumulated output,* $\overline{\Theta^{AC}}$*, is the output that associates with* $(P,e,h) \in \Lambda$ *and each* $(p,q) \in L^{P,e}$ *the value:*

$$\overline{\Theta^{AC}_{p,q}}(P,e,h) = \Theta^{AC}_{p,q}(P,e,h) + \frac{1}{|P|} \cdot \big[h(e) - \sum_{k \in P} \Theta^{AC}_{k,e_k}(P,e,h)\big].$$

Under the output $\overline{\Theta^{AC}_{p,q}}(P,e,h)$*, performers firstly receive its total marginal values and further distribute equally the remaining benefit.*

- *The efficacious average output,* $\overline{\Theta^{AV}}$*, is the output that associates with* $(P,e,h) \in \Lambda$ *and each* $(p,q) \in L^{P,e}$ *the value:*

$$\overline{\Theta^{AV}_{p,q}}(P,e,h) = \Theta^{AV}_{p,q}(P,e,h) + \frac{1}{|P|} \cdot \big[h(e) - \sum_{k \in P} \Theta^{AV}_{k,e_k}(P,e,h)\big].$$

Under the output $\overline{\Theta^{AV}_{p,q}}(P,e,h)$*, performers firstly receive its average marginal values and further distribute equally the remaining benefit.*

Clearly, $\overline{\Theta^M}, \overline{\Theta^{AC}}$ *and* $\overline{\Theta^{AV}}$ *conform to EIY.*

2.2. Motivating and Practical Examples

The advantages of our methods are that these outputs of a multi-choice game always exist and result in a specific performer taking a specific level different from the usual attitude for multi-choice games, which results in a kind of global result for a specific performer by summarizing the result under all its energy levels.

Here, we provide a brief motivating example of multi-choice games in the setting of "management". Let $P = \{1, 2, \cdots, p\}$ be a set of all performers of a management system (P,e,h). The function h could be treated as a usability function that assigns to each level vector $\alpha = (\alpha_p)_{p \in P} \in E^P$ the worth that the performers can obtain when each performer p participates in the operation strategy $\alpha_p \in E_p$ in (P,e,h). Modeled in this way, the management system (P,e,h) could be considered as a multi-choice game, with h being each characteristic function and E_p being the set of all operation strategies of the performer p.

In the following, we also provide a practical application of the power distribution in a national parliament. Let $P = \{1, 2, \cdots, p\}$ be a set of all performers of a national parliament of a certain country. In the national parliament of a certain country, all the performers of the parliament are elected by voting or recommendation by parties. All performers have the power to propose, discuss, establish, and veto all bills. They dedicate different levels of attention and participation to different bills depending on their academic expertise and the public opinion they represent. The level of involvement is also closely associated with the alliance strategy formed for the interests of different political parties. For the aforementioned reasons, strategies adopted by each performer of the parliament show distinct levels of participation and certain amounts of ambiguity. The function h could be treated as a power function that assigns to each level vector $\alpha = (\alpha_p)_{p \in P} \in E^P$ the power that the performers can achieve when each performer p participates in operation strategy $\alpha_p \in E_p$. Modeled in this way, the national parliament operational system (P,e,h) could be considered as a multi-choice game, with h being each characteristic function and

E_p being the set of all operation strategies of the performer p. To evaluate the influence of each performer in the national parliament, using the power indicators we proposed, we first assess the influence each parliament performer has gained over previous bill meetings based on various strategies, which are the outputs mentioned in Definitions 1 and 2.

Here, we provide an application with real data as follows. Let $(P,e,h) \in \Delta$ with $P = \{i,j,k\}$ and $e = (2,1,1)$. Further, let $h(2,1,1) = 5$, $h(1,1,1) = 7$, $h(2,1,0) = 3$, $h(2,0,1) = 2$, $h(2,0,0) = 9$, $h(1,1,0) = 3$, $h(1,0,1) = -4$, $h(0,1,1) = 5$, $h(1,0,0) = -1$, $h(0,1,0) = 2$, $h(0,0,1) = -3$, and $h(0,0,0) = 0$. By Definitions 1 and 2,

$$\begin{array}{llllllll}
\Theta^{M}_{i,2}(P,e,h) = -2, & \Theta^{M}_{i,1}(P,e,h) = 2, & \Theta^{M}_{j,1}(P,e,h) = 3, & \Theta^{M}_{k,1}(P,e,h) = 2, \\
\Theta^{AC}_{i,2}(P,e,h) = 14, & \Theta^{AC}_{i,1}(P,e,h) = 1, & \Theta^{AC}_{j,1}(P,e,h) = 7, & \Theta^{AC}_{k,1}(P,e,h) = -5, \\
\Theta^{AV}_{i,2}(P,e,h) = \frac{7}{2}, & \Theta^{AV}_{i,1}(P,e,h) = \frac{1}{4}, & \Theta^{AV}_{j,1}(P,e,h) = \frac{7}{4}, & \Theta^{AV}_{k,1}(P,e,h) = \frac{-5}{4}, \\
\overline{\Theta^{M}_{i,2}}(P,e,h) = \frac{-4}{3}, & \overline{\Theta^{M}_{i,1}}(P,e,h) = \frac{8}{3}, & \overline{\Theta^{M}_{j,1}}(P,e,h) = \frac{11}{3}, & \overline{\Theta^{M}_{k,1}}(P,e,h) = \frac{8}{3}, \\
\overline{\Theta^{AC}_{i,2}}(P,e,h) = \frac{31}{3}, & \overline{\Theta^{AC}_{i,1}}(P,e,h) = \frac{-8}{3}, & \overline{\Theta^{AC}_{j,1}}(P,e,h) = \frac{10}{3}, & \overline{\Theta^{AC}_{k,1}}(P,e,h) = \frac{-26}{3}, \\
\overline{\Theta^{AV}_{i,2}}(P,e,h) = \frac{23}{6}, & \overline{\Theta^{AV}_{i,1}}(P,e,h) = \frac{7}{12}, & \overline{\Theta^{AV}_{j,1}}(P,e,h) = \frac{25}{12}, & \overline{\Theta^{AV}_{k,1}}(P,e,h) = \frac{-11}{12}.
\end{array}$$

Subsequently, we would like to show that the outputs mentioned in Definitions 1 and 2 could provide "optimal allocation mechanisms" among all performers, in the sense that this organization can get a payoff from each combination of the operation levels of all performers under multi-choice behavior.

3. Reductions, Consonance, and Characterizations

Here, we demonstrate that for these efficacious outputs, there exist corresponding reductions that can be used to characterize these efficacious outputs.

Definition 3. *Given an output ρ, $(P,e,h) \in \Lambda$ and $S \subseteq P$.*

- *The complement-reduced game $(S, e_S, h^{com}_{S,\rho})$ related to ρ and S is defined for every $\omega \in E^S$ by,*

$$h^{com}_{S,\rho}(\omega) = \begin{cases} 0 & \text{if } \omega = 0_S, \\ h(\omega, e_{P\setminus S}) - \sum\limits_{p \in P\setminus S} \rho_{p,e_p}(P,e,h) & \text{otherwise.} \end{cases}$$

- *The sum-reduced game $(S, e_S, h^{sum}_{S,\rho})$ related to ρ and S is defined for every $\omega \in E^S$ by,*

$$h^{sum}_{S,\rho}(\omega) = \begin{cases} 0 & \text{if } \omega = 0_S, \\ h(e) - \sum\limits_{p \in P\setminus S} \rho_{p,e_p}(P,e,h) & \text{if } \omega = e_S, \\ \sum\limits_{\substack{T \subseteq P\setminus S \\ T \neq \emptyset}} \left[h(\omega, e_T, 0_{P\setminus(S\cup T)}) - \sum\limits_{p \in T} \rho_{p,e_p}(P,e,h) \right] & \text{otherwise.} \end{cases}$$

- *The average-reduced game $(S, e_S, h^{ave}_{S,\rho})$ related to ρ and S is defined for every $\omega \in E^S$ by,*

$$h^{ave}_{S,\rho}(\omega) = \begin{cases} 0 & \text{if } \omega = 0_S, \\ h(e) - \sum\limits_{p \in P\setminus S} \rho_{p,e_p}(P,e,h) & \text{if } \omega = e_S, \\ \frac{1}{2^{|P\setminus S|}} \cdot \sum\limits_{\substack{T \subseteq P\setminus S \\ T \neq \emptyset}} \left[h(\omega, e_T, 0_{P\setminus(S\cup T)}) - \sum\limits_{p \in T} \rho_{p,e_p}(P,e,h) \right] & \text{otherwise.} \end{cases}$$

Let the subset $S \subseteq P$ be the collection of all performers who doubt the remuneration assigned by the output ρ. In the reduced game $(S, e_S, h^{sum}_{S,\rho})$ (or $(S, e_S, h^{ave}_{S,\rho})$), the performers of S are regathered to replay. Certainly, if each performer of S does not participate, the amount S can obtain is zero. At least one performer of $P \setminus S$ would be required to participate at full steam when some performers of S adopt nonzero levels to replay. If all the performers of S participate at full steam, all the performers of $P \setminus S$ also cooperate at full steam. Since the performers of $P \setminus S$ accept the remuneration assigned by the output ρ, they receive the deserved remuneration and leave.

The consonance requirement may be described informally as follows. Let ρ be a power output on Λ. For any group of performers in a game, one defines a "reduced game" among them by considering the amounts remaining after the rest of the performers are given the payoffs prescribed by ρ. Then, ρ is said to be consistent if, when it is applied to any reduced game, it always yields the same remunerations as in the original game. For each of these reduced games, there exists a corresponding consonance as follows. Let ρ be an output on Λ.

- ρ conforms to complement-consonance (ComCSA) if for every $(P, e, h) \in \Lambda$, for every $S \subseteq P$, and for every $(p, q) \in L^{S,e_S}$, $\rho_{p,q}(P, e, h) = \rho_{p,q}(S, e_S, h^{com}_{S,\rho})$.
- ρ conforms to sum-consonance (SumCSA) if for every $(P, e, h) \in \Lambda$, for every $S \subseteq P$, and for every $(p, q) \in L^{S,e_S}$, $\rho_{p,q}(P, e, h) = \rho_{p,q}(S, e_S, h^{sum}_{S,\rho})$.
- ρ conforms to average-consonance (AveCSA) if for every $(P, e, h) \in \Lambda$, for every $S \subseteq P$, and for every $(p, q) \in L^{S,e_S}$, $\rho_{p,q}(P, e, h) = \rho_{p,q}(S, e_S, h^{ave}_{S,\rho})$.
- ρ conforms to bilateral complement-consonance (BilComCSA) if for every $(P, e, h) \in \Lambda$, for every $S \subseteq P$ with $|S| = 2$, and for every $(p, q) \in L^{S,e_S}$, $\rho_{p,q}(P, e, h) = \rho_{p,q}(S, e_S, h^{com}_{S,\rho})$.
- ρ conforms to bilateral sum-consonance (BilSumCSA) if for every $(P, e, h) \in \Lambda$, for every $S \subseteq P$ with $|S| = 2$, and for every $(p, q) \in L^{S,e_S}$, $\rho_{p,q}(P, e, h) = \rho_{p,q}(S, e_S, h^{sum}_{S,\rho})$.
- ρ conforms to bilateral average-consonance (BilAveCSA) if for every $(P, e, h) \in \Lambda$, for every $S \subseteq P$ with $|S| = 2$, and for every $(p, q) \in L^{S,e_S}$, $\rho_{p,q}(P, e, h) = \rho_{p,q}(S, e_S, h^{ave}_{S,\rho})$.

Next, we demonstrate that the output $\overline{\Theta^M}$ ($\overline{\Theta^{AC}}$, $\overline{\Theta^{AV}}$) conforms to BilComCSA (BilSumCSA, BilAveCSA).

Lemma 1.

1. *The output* $\overline{\Theta^M}$ *conforms to BilComCSA.*
2. *The output* $\overline{\Theta^{AC}}$ *conforms to BilSumCSA.*
3. *The output* $\overline{\Theta^{AV}}$ *conforms to BilAveCSA.*

Proof of Lemma 1. Given $(P, e, h) \in \Lambda$ with $|P| \geq 2$ and $S = \{p, k\}$ for some $p, k \in P$, $p \neq k$. To confirm 1, by the definition of $\overline{\Theta^M}$, for every $(p, q) \in L^{S,e_S}$,

$$\overline{\Theta^M_{p,q}}(S, e_S, h^{com}_{S,\overline{\Theta^M}}) = \Theta^M_{p,q}(S, e_S, h^{com}_{S,\overline{\Theta^M}}) + \frac{1}{|S|} \cdot \Big[h^{com}_{S,\overline{\Theta^M}}(e_S) - \sum_{t \in S} \Theta^M_{t,e_t}(S, e_S, h^{com}_{S,\overline{\Theta^M}})\Big]. \quad (1)$$

By the definitions of Θ and $h^{com}_{S,\overline{\Theta^M}}$, for every $q \in E^+_p$,

$$\begin{aligned} \Theta^M_{p,q}(S, e_S, h^{com}_{S,\overline{\Theta^M}}) &= h^{com}_{S,\overline{\Theta^M}}(e_k, q) - h^{com}_{S,\overline{\Theta^M}}(e_k, q-1) \\ &= h(e_{-p}, q) - \sum_{t \in P \setminus S} \overline{\Theta^M_{t,e_t}}(P, e, h) - h(e_{-p}, q-1) + \sum_{t \in P \setminus S} \overline{\Theta^M_{t,e_t}}(P, e, h) \\ &= h(e_{-p}, q) - h(e_{-p}, q-1) \\ &= \Theta^M_{p,q}(P, e, h). \end{aligned} \quad (2)$$

Hence, by Equations (1) and (2) and the definitions of $h^{com}_{S,\overline{\Theta^M}}$ and $\overline{\Theta^M}$,

$$\begin{aligned}
\overline{\Theta^M_{p,q}}(S,e_S,h^{com}_{S,\overline{\Theta^M}}) &= \Theta^M_{p,q}(P,e,h) + \tfrac{1}{|S|}\cdot\Big[h^{com}_{S,\overline{\Theta^M}}(e_S) - \sum_{t\in S}\Theta^M_{t,e_t}(P,e,h)\Big] \\
&= \Theta^M_{p,q}(P,e,h) + \tfrac{1}{|S|}\cdot\Big[h(e) - \sum_{t\in P\setminus S}\overline{\Theta^M_{t,e_t}}(P,e,h) - \sum_{t\in S}\Theta^M_{t,e_t}(P,e,h)\Big] \\
&= \Theta^M_{p,q}(P,e,h) + \tfrac{1}{|S|}\cdot\Big[\sum_{t\in S}\overline{\Theta^M_{t,e_t}}(P,e,h) - \sum_{t\in S}\Theta^M_{t,e_t}(P,e,h)\Big] \\
&\textbf{(by EIY of } \overline{\Theta^M}\textbf{)} \\
&= \Theta^M_{p,q}(P,e,h) + \tfrac{1}{|S|}\cdot\Big[\tfrac{|S|}{|P|}\cdot\Big[h(e) - \sum_{t\in P}\Theta^M_{t,e_t}(P,e,h)\Big]\Big] \\
&\textbf{(by Definition 1)} \\
&= \Theta^M_{p,q}(P,e,h) + \tfrac{1}{|P|}\cdot\Big[h(e) - \sum_{t\in P}\Theta^M_{t,e_t}(P,e,h)\Big] \\
&= \overline{\Theta^M_{p,q}}(P,e,h).
\end{aligned}$$

Hence, the output $\overline{\Theta^M}$ conforms to BilComCSA. The proofs of 2 and 3 are similar. □

Inspired by Hart–Mas-Colell [11], we adopt the criterion approach to axiomatize the efficacious extensions of these outputs by means of the properties of the corresponding bilateral consonance and the corresponding criterion for games. Let ρ be an output on Λ.

- Marginal-criterion for games (MCG): for every $(P,e,h)\in\Lambda$ with $|P|\leq 2$, $\rho(P,e,h) = \overline{\Theta^M}(P,e,h)$.
- Accumulated-criterion for games (ACCG): for every $(P,e,h)\in\Lambda$ with $|P|\leq 2$, $\rho(P,e,h) = \overline{\Theta^{AC}}(P,e,h)$.
- Average-criterion for games (AVCG): for every $(P,e,h)\in\Lambda$ with $|P|\leq 2$, $\rho(P,e,h) = \overline{\Theta^{AV}}(P,e,h)$.

Remark 1. *Clearly, $\overline{\Theta^M}$, $\overline{\Theta^{AC}}$, and $\overline{\Theta^{AV}}$ conform to MCG, ACCG, and AVCG, respectively. It is also easy to verify that if an output ρ conforms to MCG (ACCG, AVCG) and BilComCSA (BilSumCSA, BilAveCSA), then $\rho_{p,e_p}(\{p\},e_p,h) = h(e_p)$ for every $(\{p\},e_p,h)\in\Lambda$ (the technique is similar to Hart and Mas-Colell [11]: page 599).*

Lemma 2. *Let ρ be an output on Λ.*

1. *If ρ conforms to MCG and BilComCSA, then it also conforms to EIY.*
2. *If ρ conforms to ACCG and BilSumCSA, then it also conforms to EIY.*
3. *If ρ conforms to ACCG and BilAveCSA, then it also conforms to EIY.*

Proof of Lemma 2. To confirm 1, suppose that ρ conforms to MCG and BilComCSA. Let $(P,e,h)\in\Lambda$. If $|P| = 1$, then ρ conforms to EIY by Remark 2. It is trivial that ρ conforms to EIY by MCG if $|P| = 2$. Suppose $|P| > 2$. Since BilComCSA and ρ conforms to EIY for one-person games, it is easy to derive that ρ conforms to EIY. The proofs of 2 and 3 are similar. □

Subsequently, we characterize the output $\overline{\Theta^M}$ ($\overline{\Theta^{AC}}$, $\overline{\Theta^{AV}}$) by means of BilComCSA (BilSumCSA, BilAveCSA).

Theorem 1. *Let ρ be an output on Λ.*

1. *ρ conforms to MCG and BilComCSA if and only if $\rho = \overline{\Theta^M}$.*
2. *ρ conforms to ACCG and BilSumCSA if and only if $\rho = \overline{\Theta^{AC}}$.*
3. *ρ conforms to ACCG and BilAveCSA if and only if $\rho = \overline{\Theta^{AV}}$.*

Proof of Theorem 1. By Lemma 1, $\overline{\Theta^M}$ conforms to BilComCSA, $\overline{\Theta^{AC}}$ conforms to BilSumCSA, and $\overline{\Theta^{AV}}$ conforms to BilAveCSA. Clearly, $\overline{\Theta^M}$, $\overline{\Theta^{AC}}$, and $\overline{\Theta^{AV}}$ conform to MCG, ACCG, and AVCG, respectively.

To demonstrate the uniqueness of 1, suppose that ρ conforms to MCG and BilComCSA. By Lemma 2, ρ conforms to EIY. Let $(P,e,h) \in \Lambda$. It is trivial that $\rho(P,e,h) = \overline{\Theta^M}(P,e,h)$ by MCG if $|P| \leq 2$. The case $|P| > 2$: Let $p \in P$ and $S = \{p,k\}$ for some $k \in P \setminus \{p\}$, then for every $q \in E_p^+$ and for every $l \in E_k^+$,

$$\begin{aligned}
\rho_{p,q}(P,e,h) - \rho_{k,l}(P,e,h) &= \rho_{p,q}(S,e_S,h_{S,\rho}^{com}) - \rho_{k,l}(S,e_S,h_{S,\rho}^{com}) \\
&\textbf{(by BilComCSA of } \rho\textbf{)} \\
&= \overline{\Theta^M_{p,q}}(S,e_S,h_{S,\rho}^{com}) - \overline{\Theta^M_{k,l}}(S,e_S,h_{S,\rho}^{com}) \\
&\textbf{(by MCG of } \rho\textbf{)} \\
&= \Theta^M_{p,q}(S,e_S,h_{S,\rho}^{com}) - \Theta^M_{k,l}(S,e_S,h_{S,\rho}^{com}) \\
&\textbf{(by Definition 2)} \\
&= \big[h_{S,\rho}^{com}(e_k,q) - h_{S,\rho}^{com}(e_k,q-1)\big] - \big[h_{S,\rho}^{com}(e_p,l) - h_{S,\rho}^{com}(e_p,l-1)\big] \\
&\textbf{(by Definition 1)} \\
&= \Big[h(e_{-p},q) - \sum_{t\in P\setminus S}\rho_{t,e_t}(P,e,h) - h(e_{-p},q-1) + \sum_{t\in P\setminus S}\rho_{t,e_t}(P,e,h)\Big] \\
&\quad - \Big[h(e_{-k},l) - \sum_{t\in P\setminus S}\rho_{t,e_t}(P,e,h) - h(e_{-k},l-1) + \sum_{t\in P\setminus S}\rho_{t,e_t}(P,e,h)\Big] \\
&\textbf{(by Definition of } h_{S,\rho}^{com}\textbf{)} \\
&= \big[h(e_{-p},q) - h(e_{-p},q-1)\big] - \big[h(e_{-k},l) - h(e_{-k},l-1)\big].
\end{aligned} \tag{3}$$

Similarly, $\overline{\Theta^M_{p,q}}(P,e,h) - \overline{\Theta^M_{k,l}}(P,e,h) = \big[h(e_{-p},q) - h(e_{-p},q-1)\big] - \big[h(e_{-k},l) - h(e_{-k},l-1)\big]$. Hence, by Equation (3), for every $(p,q),(k,l) \in L^{P,e}$ with $p \neq k$,

$$\rho_{p,q}(P,e,h) - \rho_{k,l}(P,e,h) = \overline{\Theta^M_{p,q}}(P,e,h) - \overline{\Theta^M_{k,l}}(P,e,h),$$

i.e., there exists $d \in \mathbb{R}$ such that for every $(p,q) \in L^{P,e}$, $\rho_{p,q}(P,e,h) - \overline{\Theta^M_{p,q}}(P,e,h) = d$. Therefore,

$$\begin{aligned}
|P| \cdot d &= \sum_{t\in P}\big[\rho_{t,e_t}(P,e,h) - \overline{\Theta^M_{t,e_t}}(P,e,h)\big] \\
&= h(e) - h(e) \ \ \textbf{(by EIY of } \rho \textbf{ and } \overline{\Theta^M}\textbf{)} \\
&= 0.
\end{aligned}$$

That is, $d = 0$. Therefore, $\rho_{p,q}(P,e,h) = \overline{\Theta^M_{p,q}}(P,e,h)$ for every $(p,q) \in L^{P,e}$.

To demonstrate the uniqueness of 2, suppose ρ conforms to ACCG and BilSumCSA. By Lemma 2, ρ conforms to EIY. Let $(P,e,h) \in \Lambda$. If $|P| \leq 2$, it is trivial that $\rho(P,e,h) = \overline{\Theta^{AC}}(P,e,h)$ by ACCG. The case $|P| > 2$: Let $p \in P$ and $S = \{p,k\}$ for some $k \in P \setminus \{p\}$, then for every $q \in E_p^+$ and for every $l \in E_k^+$,

$$\begin{aligned}
\rho_{p,q}(P,e,h) - \rho_{k,l}(P,e,h) &= \rho_{p,q}(S,e_S,h_{S,\rho}^{sum}) - \rho_{k,l}(S,e_S,h_{S,\rho}^{sum}) \\
&\textbf{(by BilSumCSA of } \rho\textbf{)} \\
&= \overline{\Theta^{AC}_{p,q}}(S,e_S,h_{S,\rho}^{sum}) - \overline{\Theta^{AC}_{k,l}}(S,e_S,h_{S,\rho}^{sum}) \\
&\textbf{(by ACCG of } \rho\textbf{)} \\
&= \Theta^{AC}_{p,q}(S,e_S,h_{S,\rho}^{sum}) - \Theta^{AC}_{k,l}(S,e_S,h_{S,\rho}^{sum}) \\
&\textbf{(by Definition 2)} \\
&= \big[h_{S,\rho}^{sum}(e_k,q) - h_{S,\rho}^{sum}(e_k,q-1) + h_{S,\rho}^{sum}(0,q) - h_{S,\rho}^{sum}(0,q-1)\big] \\
&\quad - \big[h_{S,\rho}^{sum}(e_p,l) - h_{S,\rho}^{sum}(e_p,l-1) + h_{S,\rho}^{sum}(0,l) - h_{S,\rho}^{sum}(0,l-1)\big] \\
&\textbf{(by Definition 1)} \\
&= \big[h(e_{-p},q) - h(e_{-p},0) + h(e_{-ik},j,0) - h(e_{-ik},q-1,0)\big] \\
&\quad - \big[h(e_{-k},l) - h(e_{-k},0) + h(e_{-ik},0,l) - h(e_{-ik},0,l-1)\big]. \\
&\textbf{(by Definition of } h_{S,\rho}^{sum}\textbf{)}
\end{aligned} \tag{4}$$

Similarly,

$$\begin{aligned}\overline{\Theta^{AC}_{p,q}}(P,e,h) - \overline{\Theta^{AC}_{k,l}}(P,e,h) \;&= \big[h(e_{-p},q) - h(e_{-p},0) + h(e_{-ik},j,0) - h(e_{-ik},q-1,0)\big] \\ &\quad - \big[h(e_{-k},l) - h(e_{-k},0) + h(e_{-ik},0,l) - h(e_{-ik},0,l-1)\big].\end{aligned}$$

Hence, by Equation (4), for every $(p,q),(k,l) \in L^{P,e}$ with $p \neq k$,

$$\rho_{p,q}(P,e,h) - \rho_{k,l}(P,e,h) = \overline{\Theta^{AC}_{p,q}}(P,e,h) - \overline{\Theta^{AC}_{k,l}}(P,e,h).$$

Therefore, there exists $d \in \mathbb{R}$ such that for every $(p,q) \in L^{P,e}$,

$$\rho_{p,q}(P,e,h) - \overline{\Theta^{AC}_{p,q}}(P,e,h) = d.$$

By EIY of ρ and $\overline{\Theta^{AC}}$,

$$\begin{aligned}|P| \cdot d \;&= \sum_{t\in P}\big[\rho_{t,e_t}(P,e,h) - \overline{\Theta^{AC}_{t,e_t}}(P,e,h)\big] \\ &= h(e) - h(e) \\ &= 0.\end{aligned}$$

Hence, $d = 0$. That is, $\rho_{p,q}(P,e,h) = \overline{\Theta^{AC}_{p,q}}(P,e,h)$ for every $(p,q) \in L^{P,e}$. The proof of 3 is similar to the proof of 2, so we omit it. □

The following examples are to demonstrate that each of the properties appearing in Theorem 1 is logically independent of the rest of the properties.

Example 1. *Define an output ρ on Λ by for every $(P,e,h) \in \Lambda$ and for every $(p,q) \in L^{P,e}$,*

$$\rho_{p,q}(P,e,h) = 0.$$

It is easy to confirm that ρ conforms to BilComCSA, BilSumCSA, and BilAveCSA, but it does not satisfy MCG, ACCG, and AVCG.

Example 2. *Define an output ρ on Λ by for every $(P,e,h) \in \Lambda$ and for every $(p,q) \in L^{P,e}$,*

$$\rho_{p,q}(P,e,h) = \begin{cases} \overline{\Theta^{E}_{p,q}}(P,e,h) & \text{, if } |P| \le 2 \\ 0 & \text{, otherwise.}\end{cases}$$

It is easy to confirm that ρ conforms to MCG, but it does not satisfy BilComCSA.

Example 3. *Define an output ρ on Λ by for every $(P,e,h) \in \Lambda$ and for every $(p,q) \in L^{P,e}$,*

$$\rho_{p,q}(P,e,h) = \begin{cases} \overline{\Theta^{AC}_{p,q}}(P,e,h) & \text{, if } |P| \le 2 \\ 0 & \text{, otherwise.}\end{cases}$$

It is easy to confirm that ρ conforms to ACCG, but it does not satisfy BilSumCSA.

Example 4. *Define an output ρ on Λ by for every $(P,e,h) \in \Lambda$ and for every $(p,q) \in L^{P,e}$,*

$$\rho_{p,q}(P,e,h) = \begin{cases} \overline{\Theta^{AV}_{p,q}}(P,e,h) & \text{, if } |P| \le 2 \\ 0 & \text{, otherwise.}\end{cases}$$

It is easy to confirm that ρ conforms to AVCG, but it does not satisfy BilAveCSA.

4. Symmetry, Accordance, and Characterizations

In this section, we characterize these efficacious outputs by means of the related properties of symmetry and accordance. Thus, some more properties are needed. Let ρ be an output.

- ρ conforms to marginal symmetry (MASYM) if for every $(P,e,h) \in \Lambda$ with $\Theta^{M}_{p,k_p}(P,e,h) = \Theta^{M}_{q,k_q}(P,e,h)$ for some $(p,k_p),(q,k_q) \in L^{P,e}$ and for every $\omega \in E^{P\setminus\{p,q\}}$, $\rho_{p,k_p}(P,e,h) = \rho_{q,k_q}(P,e,h)$.
- ρ conforms to accumulated symmetry (ACSYM) if for every $(P,e,h) \in \Lambda$ with $\Theta^{AC}_{p,k_p}(P,e,h) = \Theta^{AC}_{q,k_q}(P,e,h)$ for some $(p,k_p),(q,k_q) \in L^{P,e}$ and for every $\omega \in E^{P\setminus\{p,q\}}$, $\rho_{p,k_p}(P,e,h) = \rho_{q,k_q}(P,e,h)$.
- ρ conforms to average symmetry (AVSYM) if for every $(P,e,h) \in \Lambda$ with $\Theta^{AV}_{p,k_p}(P,e,h) = \Theta^{AV}_{q,k_q}(P,e,h)$ for some $(p,k_p),(q,k_q) \in L^{P,e}$ and for every $\omega \in E^{P\setminus\{p,q\}}$, $\rho_{p,k_p}(P,e,h) = \rho_{q,k_q}(P,e,h)$.
- ρ conforms to accordance (ADE) if for every $(P,e,h),(P,e,\zeta) \in \Lambda$ with $h(\omega) = \zeta(\omega) + \sum_{p\in S(\omega)} b_{p,\omega_p}$ for some $b \in \mathbb{R}^{L^{P,e}}$ and for every $\omega \in E^P$, $\rho(P,e,h) = \rho(P,e,\zeta) + b$.

MASYM (ACSYM, AVSYM) means that the remunerations of two performers should be equal if the general (total, average) marginal values of these two performers are coincidental. ADE can be treated as an extreme weakness of additivity. By Definition 2, it is easy to confirm that the efficacious marginal (accumulated, average) output conforms to MASYM (ACSYM, AVSYM) and ADE.

Next, we demonstrate that the property MCG (ACCG, AXCG) could be replaced by the efficacy, marginal (accumulated, average) symmetry, accordance, and bilateral complement (sum, average)-consonance simultaneously.

Lemma 3.

1. *If an output ρ conforms to EIY, MASYM, and ADE, then ρ conforms to MCG.*
2. *If an output ρ conforms to EIY, ACSYM, and ADE, then ρ conforms to ACCG.*
3. *If an output ρ conforms to EIY, AVSYM, and ADE, then ρ conforms to AVCG.*

Proof of Lemma 3. Assume that an output ρ conforms to EIY, MASYM, and ADE. Given $(P,e,h) \in \Lambda$ with $P = \{p,q\}$ for some $i \neq j$, we define a game (P,e,ζ) to be that for every $\omega \in E^P$,

$$\zeta(\omega) = h(\omega) - \sum_{p\in S(\omega)} \Theta^{M}_{i,\omega_p}(P,e,h).$$

By the definition of ζ,

$$\begin{aligned}
\Theta^{M}_{i,1}(P,e,\zeta) &= \zeta(1,e_q) - \zeta(0,e_q) \\
&= h(1,e_q) - \Theta^{M}_{i,1}(P,e,h) - \Theta^{M}_{q,e_q}(P,e,h) - h(0,e_q) + \Theta^{M}_{q,e_q}(P,e,h) \\
&= h(1,e_q) - h(0,e_q) - \Theta^{M}_{i,1}(P,e,h) \\
&= \Theta^{M}_{i,1}(P,e,h) - \Theta^{M}_{i,1}(P,e,h) \\
&= 0.
\end{aligned}$$

Further,

$$\begin{aligned}
\Theta^{M}_{i,2}(P,e,\zeta) &= \zeta(2,e_q) - \zeta(1,e_q) \\
&= h(2,e_q) - \Theta^{M}_{i,2}(P,e,h) - \Theta^{M}_{q,e_q}(P,e,h) - h(1,e_q) + \Theta^{M}_{i,1}(P,e,h) \\
&\quad + \Theta^{M}_{q,e_q}(P,e,h) \\
&= h(2,e_q) - h(1,e_q) - \Theta^{M}_{i,2}(P,e,h) - \Theta^{M}_{q,e_q}(P,e,h) + 0 - \Theta^{M}_{q,e_q}(P,e,h) \\
&= \Theta^{M}_{i,2}(P,e,h) - \Theta^{M}_{i,2}(P,e,h) \\
&= 0.
\end{aligned}$$

Similarly, $\Theta^M_{p,k_p}(P,e,\zeta) = 0$ for every $k_p \in E^+_p$. Since $\Theta^M_{i,0}(P,e,\zeta) = 0$, we have that $\Theta^M_{p,k_p}(P,e,\zeta) = 0$ for every $k_p \in E_p$. Similarly, $\Theta^M_{q,k_q}(P,e,\zeta) = 0$ for every $k_q \in E_q$. Since $\Theta^M_{p,e_p}(P,e,\zeta) = 0 = \Theta^M_{q,e_q}(P,e,\zeta)$, by the MASYM of ρ, $\rho_{p,e_p}(P,e,\zeta) = \rho_{q,e_q}(P,e,\zeta)$. By the EIY of ρ,

$$\zeta(e) = \rho_{p,e_p}(P,e,\zeta) + \rho_{q,e_q}(P,e,\zeta) = 2 \cdot \rho_{p,e_p}(P,e,\zeta). \tag{5}$$

Therefore, by Equation (5) and the definition of ζ,

$$\rho_{p,e_p}(P,e,\zeta) = \frac{\zeta(e)}{2} = \frac{1}{2} \cdot \left[h(e) - \Theta^M_{p,e_p}(P,e,h) - \Theta^M_{q,e_q}(P,e,h)\right].$$

By the ADE of ρ,

$$\begin{aligned} \rho_{p,k_p}(P,e,h) &= \rho_{p,k_p}(P,e,\zeta) + \Theta^M_{p,k_p}(P,e,h) \\ &= \tfrac{1}{2} \cdot \left[h(e) - \Theta^M_{p,e_p}(P,e,h) - \Theta^M_{q,e_q}(P,e,h)\right] + \Theta^M_{p,k_p}(P,e,h) \\ &= \overline{\Theta^M}_{p,k_p}(P,e,h). \end{aligned}$$

Similarly, $\rho_{q,k_q}(P,e,h) = \overline{\Theta^M}_{q,k_q}(P,e,h)$ for every $k_q \in E_q$. Hence, ρ conforms to MCG. The proofs of the results of 2 and 3 are similar. □

Subsequently, we characterize the efficacious marginal (accumulated, average) output by means of the efficacy, marginal (accumulated, average) symmetry, accordance, and bilateral complement (sum, average)-consonance.

Theorem 2.

1. *An output ρ conforms to EIY, MASYM, ADE, and BilComCSA if and only if $\rho = \overline{\Theta^M}$.*
2. *An output ρ conforms to EIY, ACSYM, ADE, and BilSumCSA if and only if $\rho = \overline{\Theta^{AC}}$.*
3. *An output ρ conforms to EIY, AVSYM, ADE, and BilAveCSA if and only if $\rho = \overline{\Theta^{AV}}$.*

Proof of Theorem 2. It is easy to confirm that the efficacious marginal (accumulated, average) output conforms to MASYM (ACSYM, AVSYM) and ADE by Definition 2. The rest of the proofs follow from Theorem 1 and Lemmas 1 and 3. □

5. Different Generalizations and Related Results

In this section, we consider different generalizations by applying different marginal values.

Definition 4.

- *The *-efficacious marginal output, $\overline{\Theta^{*M}}$, is the output on Λ that associates with $P,e,h \in \Lambda$ and all $(p,q) \in L^{P,e}$ the value:*

$$\overline{\Theta^{*M}_{p,q}}(P,e,h) = \Theta^{*M}_{p,q}(P,e,h) + \frac{1}{|P|} \cdot \left[h(e) - \sum_{k\in P} \Theta^M_{k,e_k}(P,e,h)\right],$$

  where $\Theta^{*M}_{p,q}(P,e,h) = h(e_{-p},e_p) - h(e_{-p},q-1)$ is the general marginal value of performer p from level $q-1$ to e_p in the grand coalition.
- *The *-efficacious accumulated output, $\overline{\Theta^{*AC}}$, is the output on Λ that associates with $P,e,h \in \Lambda$ and all $(p,q) \in L^{P,e}$ the value:*

$$\overline{\Theta^{*AC}_{p,q}}(P,e,h) = \Theta^{*AC}_{p,q}(P,e,h) + \frac{1}{|P|} \cdot \left[h(e) - \sum_{k\in P} \Theta^{AC}_{k,e_k}(P,e,h)\right],$$

where $\Theta^{*AC}_{p,q}(P,e,h) = \sum_{\substack{S\subseteq P\\ p\in S}} \Big[h\big((e_{-p},e_p)_S,0_{P\setminus S}\big) - h\big((e_{-p},q-1)_S,0_{P\setminus S}\big)\Big]$ *is the total marginal value of performer* p *from level* $q-1$ *to* e_p *among all participating coalitions.*

- *The *-efficacious average output,* $\overline{\Theta^{*AV}}$*, is the output on* Λ *that associates with* $P,e,h \in \Lambda$ *and all* $(p,q) \in L^{P,e}$ *the value:*

$$\overline{\Theta^{*AV}_{p,q}}(P,e,h) = \Theta^{*AV}_{p,q}(P,e,h) + \frac{1}{|P|}\cdot\Big[h(e) - \sum_{k\in P}\Theta^{AV}_{k,e_k}(P,e,h)\Big],$$

where
$\Theta^{*AV}_{p,q}(P,e,h) = \frac{1}{2^{|P|-1}}\cdot\sum_{\substack{S\subseteq P\\ p\in S}} \Big[h\big((e_{-p},e_p)_S,0_{P\setminus S}\big) - h\big((e_{-p},q-1)_S,0_{P\setminus S}\big)\Big]$ *is the average marginal value of performer* p *from level* $q-1$ *to* e_p *among all participating coalitions.*

Clearly, $\overline{\Theta^{*M}}$, $\overline{\Theta^{*AC}}$ *and* $\overline{\Theta^{*AV}}$ *conform to EIY. The main dissimilarity among* $\overline{\Theta^{M}}$ *(*$\overline{\Theta^{AC}}$*,* $\overline{\Theta^{AV}}$*) and* $\overline{\Theta^{*M}}$ *(*$\overline{\Theta^{*AC}}$*,* $\overline{\Theta^{*AV}}$*) is the different definition of "marginal values".*

To present the main results of this section, some more properties are needed. Let ρ be an output on Λ.

- *-marginal-criterion for games (*MCG): for every $(P,e,h)\in\Lambda$ with $|P|\le 2$, $\rho(P,e,h) = \overline{\Theta^{*M}}(P,e,h)$.
- *-accumulated-criterion for games (*ACCG): for every $(P,e,h)\in\Lambda$ with $|P|\le 2$, $\rho(P,e,h) = \overline{\Theta^{*AC}}(P,e,h)$.
- *-average-criterion for games (*AVCG): for every $(P,e,h)\in\Lambda$ with $|P|\le 2$, $\rho(P,e,h) = \overline{\Theta^{*AV}}(P,e,h)$.

Remark 2. *Clearly,* $\overline{\Theta^{*M}}$, $\overline{\Theta^{*AC}}$ *and* $\overline{\Theta^{*AV}}$ *conform to *MCG, *ACCG, and *AVCG respectively. It is easy to derive that if an output* ρ *conforms to *MCG (*ACCG, *AVCG) and BilComCSA (BilSumCSA, BilAveCSA), then* $\rho_{p,e_p}(\{p\},e_p,v) = h(e_p)$ *for every* $(\{p\},e_p,h)\in\Lambda$.

Similar to the related results introduced throughout Section 3, several axiomatic results of $\overline{\Theta^{*M}}$, $\overline{\Theta^{*AC}}$, and $\overline{\Theta^{*AV}}$ also can be provided as follows.

Lemma 4.

1. *The output* $\overline{\Theta^{*M}}$ *conforms to BilComCSA.*
2. *The output* $\overline{\Theta^{*AC}}$ *conforms to BilSumCSA.*
3. *The output* $\overline{\Theta^{*AV}}$ *conforms to BilAveCSA.*

Proof of Lemma 4. The proof is similar to Lemma 1. □

Lemma 5. *Let* ρ *be an output on* Λ.

1. *If* ρ *conforms to *MCG and BilComCSA, then it also conforms to EIY.*
2. *If* ρ *conforms to *ACCG and BilSumCSA, then it also conforms to EIY.*
3. *If* ρ *conforms to *AVCG and BilAveCSA, then it also conforms to EIY.*

Proof of Lemma 5. The proof is similar to Lemma 2. □

Theorem 3. *Let* ρ *be an output on* Λ.

1. ρ *conforms to *MCG and BilComCSA if and only if* $\rho = \overline{\Theta^{*M}}$.
2. ρ *conforms to *ACCG and BilSumCSA if and only if* $\rho = \overline{\Theta^{*AC}}$.
3. ρ *conforms to *AVCG and BilAveCSA if and only if* $\rho = \overline{\Theta^{*AV}}$.

Proof of Theorem 3. The proof is similar to Theorem 1. □

The following examples are to demonstrate that each of the properties applied in Theorem 3 is logically independent of the rest of the properties.

Example 5. *Define an output ρ on Λ by for every $(P,e,h) \in \Lambda$ and for every $(p,q) \in L^{P,e}$,*

$$\rho_{p,q}(P,e,h) = 0.$$

*It is easy to confirm that ρ conforms to BilComCSA, BilSumCSA, and BilAveCSA, but it does not satisfy *MCG, *ACCG, and *ACCG.*

Example 6. *Define an output ρ on Λ by for every $(P,e,h) \in \Lambda$ and for every $(p,q) \in L^{P,e}$,*

$$\rho_{p,q}(P,e,h) = \begin{cases} \overline{\Theta^{*M}_{p,q}}(P,e,h) & \text{, if } |P| \leq 2 \\ 0 & \text{, otherwise.} \end{cases}$$

*It is easy to confirm that ρ conforms to *MCG, but it does not satisfy BilComCSA.*

Example 7. *Define an output ρ on Λ by for every $(P,e,h) \in \Lambda$ and for every $(p,q) \in L^{P,e}$,*

$$\rho_{p,q}(P,e,h) = \begin{cases} \overline{\Theta^{*AC}_{p,q}}(P,e,h) & \text{, if } |P| \leq 2 \\ 0 & \text{, otherwise.} \end{cases}$$

*It is easy to confirm that ρ conforms to *ACCG, but it does not satisfy BilSumCSA.*

Example 8. *Define an output ρ on Λ by for every $(P,e,h) \in \Lambda$ and for every $(p,q) \in L^{P,e}$,*

$$\rho_{p,q}(P,e,h) = \begin{cases} \overline{\Theta^{*AV}_{p,q}}(P,e,h) & \text{, if } |P| \leq 2 \\ 0 & \text{, otherwise.} \end{cases}$$

*It is easy to confirm that ρ conforms to *AVCG, but it does not satisfy BilAveCSA.*

Similar to Section 3, some more properties are defined as follows. Let ρ be an output.

- ρ conforms to *-marginal symmetry (*MASYM) if for every $(P,e,h) \in \Lambda$ with $\Theta^{*M}_{p,k_p}(P,e,h) = \Theta^{*M}_{q,k_q}(P,e,h)$ for some $(p,k_p),(q,k_q) \in L^{P,e}$ and for every $\omega \in E^{P\setminus\{p,q\}}$, $\rho_{p,k_p}(P,e,h) = \rho_{q,k_q}(P,e,h)$.
- ρ conforms to *-accumulated symmetry (*ACSYM) if for every $(P,e,h) \in \Lambda$ with $\Theta^{*AC}_{p,k_p}(P,e,h) = \Theta^{*AC}_{q,k_q}(P,e,h)$ for some $(p,k_p),(q,k_q) \in L^{P,e}$ and for every $\omega \in E^{P\setminus\{p,q\}}$, $\rho_{p,k_p}(P,e,h) = \rho_{q,k_q}(P,e,h)$.
- ρ conforms to *-average symmetry (*AVSYM) if for every $(P,e,h) \in \Lambda$ with $\Theta^{*AV}_{p,k_p}(P,e,h) = \Theta^{*AV}_{q,k_q}(P,e,h)$ for some $(p,k_p),(q,k_q) \in L^{P,e}$ and for every $\omega \in E^{P\setminus\{p,q\}}$, $\rho_{p,k_p}(P,e,h) = \rho_{q,k_q}(P,e,h)$.

By Definition 4, it is easy to confirm that the *-efficacious marginal (accumulated, average) output conforms to *MASYM (*ACSYM, *AVSYM) and ADE. Next, we characterize the *-efficacious marginal (accumulated, average) output by means of related properties of the efficacy, *-marginal (accumulated, average) symmetry, accordance, and bilateral complement (sum, average)-consonance.

Lemma 6.

1. *If an output ρ conforms to EIY, *MASYM, and ADE, then ρ conforms to *MCG.*
2. *If an output ρ conforms to EIY, *ACSYM, and ADE, then ρ conforms to *ACCG.*
3. *If an output ρ conforms to EIY, *AVSYM, and ADE, then ρ conforms to *ACCG.*

Proof of Lemma 6. The proof is similar to Lemma 3. □

Theorem 4.

1. *An output ρ conforms to EIY, *MASYM, ADE, and BilComCSA if and only if $\rho = \overline{\Theta^{*M}}$.*
2. *An output ρ conforms to EIY, *ACSYM, ADE, and BilSumCSA if and only if $\rho = \overline{\Theta^{*AC}}$.*
3. *An output ρ conforms to EIY, *AVSYM, ADE, and BilAveCSA if and only if $\rho = \overline{\Theta^{*AV}}$.*

Proof of Theorem 4. By Definition 4, it is easy to confirm that the *-efficacious marginal (accumulated, average) output conforms to *MASYM (*ACSYM, *AVSYM) and ADE. The rest of the proofs follow from Theorem 2 and Lemmas 4 and 6. □

6. Conclusions

Different from pre-existing results, several main results of this paper are as follows.

- Based on the multi-choice consideration, we propose several extensions of the Banzhaf–Coleman index, the Banzhaf–Owen index, and the marginal index on multi-choice games. Since these extensions are not efficacious, we also consider the efficacious extensions of these outputs.
- In order to present the rationality of these efficacious outputs, we adopt the complement-reduction, the sum-reduction, the average-reduction and related consonance property to characterize these efficacious outputs.
- By applying the related properties of symmetry and accordance, alternative axiomatic results of these efficacious outputs are also proposed.

Liao [16,17] proposed two efficacious extensions of the marginal index and related results by considering the maximal marginal values and replication on multi-choice games, respectively. By analyzing the performers and the energy levels simultaneously, we propose several extensions of the EANSC, the Banzhaf–Coleman index, and the Banzhaf–Owen index on multi-choice games. One should also compare our results with these existing results. There are several major dissimilarities between previous results and ours:

- Liao [16] defined the maximal EANSC to compute a kind of global value for a specific performer by adopting the maximal marginal values of performers among total levels. Differing from the extended EANSC of Liao [16], we also focus on the Banzhaf–Coleman index and the Banzhaf–Owen index and consider the outputs, the reduction, and several properties by simultaneously analyzing the performers and the energy levels. The other major dissimilarity is the fact that we offer the related properties of symmetry and accordance to characterize the outputs introduced in this paper. The related properties of symmetry and accordance were not present in Liao [16].
- Liao [17] proposed the duplicate EANSC to compute a kind of global value for a specific performer by considering the replicated behavior of performers among total levels. Differing from the extended EANSC of Liao [17], we also focus on the Banzhaf–Coleman index and the Banzhaf–Owen index and consider the outputs, the reduction, and several properties by simultaneously analyzing the performers and the energy levels. The other major dissimilarity is the fact that we offer the related properties of symmetry and accordance to characterize the outputs introduced in this paper. The related properties of symmetry and accordance were not present in Liao [17].

Here, we extend the Banzhaf–Coleman index, the Banzhaf–Owen index, and the marginal index to multi-choice games. Some might wonder whether some more traditional output concepts can also be extended to multi-choice games. We leave this to the readers to explore in future studies.

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