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Qualitative Properties of Solutions of Second-Order Neutral Differential Equations

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Abstract: In this work, we consider a type of second-order functional differential equations and establish qualitative properties of their solutions. These new results complement and improve a number of results reported in the literature. Finally, we provide an example that illustrates our results.

Keywords: oscillation properties; neutral differential equations; second-order differential equations

MSC: 34C10; 34K11

1. Introduction

In this article, we consider the neutral differential equation

$$\left(a(y)\left(w'(y)\right)^{\gamma}\right)' + \sum_{j=1}^{m_2} q_j(y)x^{\beta_j}\left(\vartheta_j(y)\right) = 0, \quad y \geq y_0, \tag{1}$$

where $w(y) = x(y) + \sum_{i=1}^{m_1} p_i(y)x^{\alpha_i}\left(\varsigma_i(y)\right)$, α_i, γ and β_j, for all $i = 1, \cdots, m_1$ and $j = 1, \cdots, m_2$, are quotients of odd positive integers.

It is natural to ask why time-delayed systems are so important. Time delays are intrinsic in many real systems and, therefore, must be properly accounted for evolution models [1–4]. Recently there has been a considerable interest in dynamical systems both neutral and involving time-delays with applications ranging from Biology and Population Dynamics to Physics and Engineering, and from Economics to Medicine. For instance, some interesting studies have shown how delay differential equation can be used to solve cardiovascular models that have a discontinuous derivative [5]. Moreover, many researchers have studied the qualitative properties of delay mathematical models examining oscillation and nonoscillation properties of different delay logistic models and their modifications [6]. These studies are also concerned with the investigation of local and global stability. Mainly the oscillation properties are investigated for models with delayed feedback, hyperlogistic models, and models with varying capacity. For further details regarding the techniques and other applications to Biology, we refer the reader to [6] and the references therein.

For a recent review of the main results in the framework of asymptotic properties for second and higher-order functional differential equations (FDEs), we refer the reader to the interesting book [7].

For further details regarding oscillatory properties of differential equations, we suggest to read the papers [1–4,8–35].

Throughout this work, we assume that the following assumptions are fulfilled for Equation (1):

(A1) $\vartheta_j, \varsigma_i \in C([y_0,\infty),\mathbb{R}_+)$, $\varsigma_i \in C^2([y_0,\infty),\mathbb{R}_+)$, $\vartheta_j(y) < y$, $\varsigma_i(y) < y$, $\lim_{y\to\infty}\vartheta_j(y) = \infty$, $\lim_{y\to\infty}\varsigma_i(y) = \infty$ for all $i = 1,2,\cdots,m_1$ and $j = 1,2,\cdots,m_2$;
(A2) $a \in C^1([y_0,\infty),\mathbb{R}_+)$, $q_j \in C([y_0,\infty),\mathbb{R}_+)$; $0 \le q_j(y)$, for all $y \ge 0$ and $j = 1,2,\cdots,m_2$; $\sum_{j=1}^{m_2} q_j(y)$ is not identically zero in any interval $[b,\infty)$;
(A3) $\lim_{y\to\infty} A(y) = \infty$, where $A(y) = \int_{y_0}^{y} a^{-1/\gamma}(\eta)\,d\eta$;
(A4) $p_i : [y_0,\infty) \to \mathbb{R}^+$ are continuous functions for $i = 1,2,\cdots,m$;
(A5) there exists a differentiable function $\vartheta_0(y)$ satisfying the properties $0 < \vartheta_0(y) = \min_{j=1,\cdots,m_2}\{\vartheta_j(y) : y \ge y^* > y_0\}$ and $\vartheta_0'(y) \ge \vartheta_0$ for $y \ge y^* > y_0$, $\vartheta_0 > 0$.

Now we recall some basic definitions.

Definition 1. *A function $x(y) : [y_x,\infty) \to \mathbb{R}, y_x \ge y_0$ is said to be a solution of (1) if $x(y)$ and $a(y)\,(w'(y))^\gamma$ are continuously differentiable for all $y \in [y_x,\infty)$ and it satisfies Equation (1) for all $y \in [y_x,\infty)$.*

We assume that (1) admits a solution in the sense of Definition 1.

Definition 2. *A solution $x(y)$ of (1) is said to be non-oscillatory if it is eventually positive or eventually negative; otherwise, it is said to be oscillatory.*

Definition 3. *The Equation (1) is said to be oscillatory if all of its solutions are oscillatory.*

In this paper, we restrict our attention in order to study oscillation and non-oscillation of (1). First of all, it is interesting to make a review in the context of functional differential equation.

In 1978, Brands [36] proved that for each bounded delay $\vartheta(y)$, the equation

$$x''(y) + q(y)x(y - \vartheta(y)) = 0$$

is oscillatory if and only if the equation

$$x''(y) + q(y)x(y) = 0$$

is oscillatory. In [37,38] Chatzarakis et al. considered a more general equation

$$\left(a(x')^\beta\right)'(y) + q(y)x^\beta(\vartheta(y)) = 0, \tag{2}$$

and established new oscillation criteria for (2) when $\lim_{y\to\infty} A(y) = \infty$ and $\lim_{y\to\infty} A(y) < \infty$.

Wong [39] has obtained the oscillation conditions of

$$(x(y) + px(y-\varsigma))'' + q(y)f\,(x(y-\vartheta)) = 0, \quad -1 < p < 0,$$

in which the neutral coefficient and delays are constants. In [40,41], the authors Baculíková and Džurina studied the equation

$$\left(a(y)\,(w'(y))^\gamma\right)' + q(y)x^\beta(\vartheta(y)) = 0, \quad w(y) = x(y) + p(y)x(\varsigma(y)), \quad y \ge y_0, \tag{3}$$

and established the oscillation of solutions of (3) using comparison techniques when $\gamma = \beta = 1$, $0 \le p(y) < \infty$ and $\lim_{y\to\infty} A(y) = \infty$. With the same technique, Baculíková and Džurina [42] considered (3) and obtained oscillation conditions of (3) considering the assumptions $0 \leq p(y) < \infty$ and $\lim_{y\to\infty} A(y) = \infty$. In [43], Tripathy et al. studied (3) and established several conditions of the solutions of (3) when considering the assumptions $\lim_{y\to\infty} A(y) = \infty$ and $\lim_{y\to\infty} A(y) < \infty$ for different values of the neutral coefficient p. In [44], Bohner et al. obtained sufficient conditions for the oscillation of the solutions of (3) when $\gamma = \beta$, $\lim_{y\to\infty} A(y) < \infty$ and $0 \leq p(y) < 1$. Grace et al. [15] studied the oscillation of (3) when $\gamma = \beta_j$, considering the assumptions $\lim_{y\to\infty} A(y) < \infty$, $\lim_{y\to\infty} A(y) = \infty$ and $0 \leq p(y) < 1$. In [45], Li et al. established sufficient conditions for the oscillation of the solutions of (3), under the assumptions $\lim_{y\to\infty} A(y) < \infty$ and $p(y) \geq 0$. Karpuz and Santra [46] considered the equation

$$\left(a(y)(x(y) + p(y)x(\varsigma(y)))'\right)' + q(y)f\left(x(\vartheta(y))\right) = 0,$$

When considering the assumptions $\lim_{y\to\infty} A(y) < \infty$ and $\lim_{y\to\infty} A(y) = \infty$, for different values of p.

2. Preliminary Results

To simplify our notation, for any positive, continuous and decreasing to zero function $\rho : [y_0, \infty) \to \mathbb{R}^+$, we set

$$\begin{aligned}
P(y) &= \left(1 - \sum_{i=1}^{m} \alpha_i p_i(y) - \frac{1}{\rho(y)} \sum_{i=1}^{m} (1-\alpha_i) p_i(y)\right);\\
Q_1(y) &= \sum_{j=1}^{m_2} q_j(y) P^{\beta_j}\left(\vartheta_j(y)\right);\\
Q_2(y) &= \sum_{j=1}^{m_2} q_j(y) P^{\beta_j}\left(\vartheta_j(y)\right) \rho^{\beta_j - 1}\left(\vartheta_j(y)\right);\\
Q_3(y) &= \sum_{j=1}^{m_2} q_j(y) P^{\beta_j}\left(\vartheta_j(y)\right) A^{\beta_j - 1}\left(\vartheta_j(y)\right);\\
Q_4(y) &= \sum_{j=1}^{m_2} q_j(y) P^{\beta_j}\left(\vartheta_j(y)\right) A^{\beta_j}(\vartheta_j(y));\\
U(y) &= \int_y^{\infty} \sum_{j=1}^{m_2} q_j(\zeta) x^{\beta_j}(\vartheta_j(\zeta))\, d\zeta\,.
\end{aligned}$$

Let us assume that $P(y)$ and $U(y)$ are non-negative in $[y_0, \infty)$. Moreover, it is worth pointing out that the inequality $P(y) \geq 0$ implies $p_i(y) \to 0$ since $\rho(y) \to 0$.

We need the following technical Lemmas in order to obtain the main results.

Lemma 1 ([47]). *If a_1 and b_1 are nonnegative numbers, then*

$${a_1}^{\alpha_1} {b_1}^{1-\alpha_1} \leq \alpha_1 a_1 + (1-\alpha_1) b_1 \quad \text{for} \quad 0 < \alpha_1 \leq 1,$$

where equality holds if and only if $a_1 = b_1$.

Lemma 2. *Let* (A1)–(A4) *hold for $y \geq y_0$. If a solution x of* (1) *is eventually positive, then w satisfies*

$$w(y) > 0, \quad w'(y) > 0, \quad \text{and} \quad \left(a(w')^{\gamma}\right)'(y) \leq 0 \quad \text{for} \quad y \geq y_1. \tag{4}$$

Proof. Let the solution x be eventually positive. Hence, $w(y) > 0$ and there exists a $y_0 \geq 0$ such that $x(y) > 0$, $x(\varsigma_i(y)) > 0$ and $x\left(\vartheta_j(y)\right) > 0$ for all $y \geq y_0$ and for all $i = 1, 2, \cdots, m_1$ and $j = 1, 2, \cdots, m_2$. From (1), it follows that

$$\left(a(y)\left(w'(y)\right)^{\gamma}\right)' = -\sum_{j=1}^{m_2} q_j(y) x^{\beta_j}\left(\vartheta_j(y)\right) \leq 0 \quad \text{for } y \geq y_0.$$

Therefore, $a(y)\left(w'(y)\right)^{\gamma}$ is non-increasing for $y \geq y_0$. Assume that $a(y)\left(w'(y)\right)^{\gamma} \leq 0$ for $y \geq y_1 > y_0$. Hence,

$$a(y)\left(w'(y)\right)^{\gamma} \leq a(y_1)\left(w'(y_1)\right)^{\gamma} < 0 \quad \text{for all } y \geq y_1,$$

that is,

$$w'(y) \leq \left(\frac{a(y_1)}{a(y)}\right)^{1/\gamma} w'(y_1) \quad \text{for } y \geq y_1.$$

Integrating from y_1 to y, we have

$$w(y) \leq w(y_1) + \left(a(y_1)\right)^{1/\gamma} w'(y_1) A(y) \to -\infty$$

as $y \to \infty$ due to (A3), which is a contradiction to $w(y) > 0$.

Therefore, $a(y)\left(w'(y)\right)^{\gamma} > 0$ for all $y \geq y_1$. From $a(y)\left(w'(y)\right)^{\gamma} > 0$ and $a(y) > 0$, it follows that $w'(y) > 0$. This completes the proof. □

Lemma 3. *Let* (A1)–(A4) *hold for* $y \geq y_0$. *If a solution* x *of* (1) *is eventually positive, then* w *satisfies*

$$w(y) \geq \left(a(y)\right)^{1/\gamma} w'(y) A(y) \quad \text{for } y \geq y_1$$

and

$$\frac{w(y)}{A(y)} \text{ is decreasing for } y \geq y_1.$$

Proof. Proceeding as in the proof of the Lemma 2, we obtain (4) for $y \geq y_1$. Because $a(y)\left(w'(y)\right)^{\gamma}$ is decreasing, we have

$$\begin{aligned} w(y) &\geq \int_{y_1}^{y} \left(a(\eta)\right)^{1/\gamma} w'(\eta) \frac{1}{\left(a(\eta)\right)^{1/\gamma}}\, d\eta \\ &\geq \left(a(y)\right)^{1/\gamma} w'(y) \int_{y_1}^{y} \frac{1}{\left(a(\eta)\right)^{1/\gamma}}\, d\eta \\ &\geq \left(a(y)\right)^{1/\gamma} w'(y) A(y). \end{aligned}$$

Again, using the previous inequality, we have

$$\left(\frac{w(y)}{A(y)}\right)' = \frac{\left(a(y)\right)^{1/\gamma} w'(y) A(y) - w(y)}{\left(a(y)\right)^{1/\gamma} A^2(y)} \leq 0.$$

We conclude that $\frac{w(y)}{A(y)}$ is decreasing for $y \geq y_1$. This completes the proof. □

Lemma 4. *Let* (A1)–(A4) *hold for* $y \geq y_0$. *If a solution* x *of* (1) *is eventually positive, then* w *satisfies*

$$x(y) \geq P(y) w(y) \quad \textit{for} \quad y \geq y_1. \tag{5}$$

Proof. Let the solution x be eventually positive. Hence, $w(y) > 0$, and there exists a $y_0 \geq 0$, such that

$$\begin{aligned} x(y) &= w(y) - \sum_{i=1}^{m} p_i(y) x^{\alpha_i}(\varsigma_i(y)) \\ &\geq w(y) - \sum_{i=1}^{m} p_i(y) w^{\alpha_i}(\varsigma_i(y)) \\ &\geq w(y) - \sum_{i=1}^{m} p_i(y) w^{\alpha_i}(y) \\ &\geq w(y) - \sum_{i=1}^{m} p_i(y)\left(\alpha_i w(y) - (1-\alpha_i)\right) \\ &= \left(1 - \sum_{i=1}^{m} \alpha_i p_i(y)\right) w(y) - \sum_{i=1}^{m} (1-\alpha_i) p_i(y) \end{aligned} \tag{6}$$

using the Lemma 1. Since $w(y)$ is positive and increasing and $\rho(y)$ is positive and decreasing to zero, there is a $y_0 \geq y_1$ such that

$$w(y) \geq \rho(y) \quad \text{for} \quad y \geq y_1. \tag{7}$$

Using (7) in (6), we obtain

$$x(y) \geq P(y) w(y).$$

This completes the proof. □

Lemma 5. *Let* (A1)–(A4) *hold for* $y \geq y_0$. *If a solution* x *of* (1) *is eventually positive, then there exist* $y_1 > y_0$ *and* $\delta > 0$, *such that*

$$0 < w(y) \leq \delta A(y) \text{ and} \tag{8}$$

$$A(y) U^{1/\gamma}(y) \leq w(y) \tag{9}$$

hold for all $y \geq y_1$.

Proof. Let the solution x be eventually positive. Then there exists a $y_0 > 0$ such that $x(y) > 0$, $x(\varsigma_i(y)) > 0$ and $x(\vartheta_j(y)) > 0$ for all $y \geq y_0$ and for all $i = 1, 2, \cdots, m_1$ and $i = 1, 2, \cdots, m_2$. So, there exists $y_1 > y_0$, such that Lemma 2 holds true and w satisfy (4) for $y \geq y_1$. From $a(y)\left(w'(y)\right)^{\gamma} > 0$ and being non-increasing, we have

$$w'(y) \leq \left(\frac{a(y_1)}{a(y)}\right)^{1/\gamma} w'(y_1) \quad \text{for } y \geq y_1\,.$$

Integrating this inequality from y_1 to y,

$$w(y) \leq w(y_1) + \left(a(y_1)\right)^{1/\gamma} w'(y_1) A(y)\,.$$

Because $\lim_{y\to\infty} A(y) = \infty$, there exists a positive constant δ, such that (8) holds. On the other hand, $\lim_{y\to\infty} a(y)\left(w'(y)\right)^{\gamma}$ exists and integrating (1) from y to ξ, we obtain

$$a(\xi)\left(w'(\xi)\right)^{\gamma} - a(y)\left(w'(y)\right)^{\gamma} = -\int_y^{\xi} \sum_{j=1}^{m_2} q_j(\eta) x^{\beta_j}(\vartheta_j(\eta))\, d\eta\,.$$

Taking the limit as $\xi \to \infty$, we get

$$a(y)\left(w'(y)\right)^{\gamma} \geq \int_{y}^{\infty} \sum_{j=1}^{m_2} q_j(\eta) x^{\beta_j}(\vartheta_j(\eta))\, d\eta, \tag{10}$$

that is,

$$w'(y) \geq \left[\frac{1}{a(y)} \int_{y}^{\infty} \sum_{j=1}^{m_2} q_j(\eta) x^{\beta_j}(\vartheta_j(\eta))\, d\eta\right]^{1/\gamma}.$$

Therefore,

$$\begin{aligned} w(y) &\geq \int_{y_1}^{y} \left[\frac{1}{a(\eta)} \int_{\eta}^{\infty} \sum_{j=1}^{m_2} q_j(s) x^{\beta_j}(\vartheta_j(s))\, ds\right]^{1/\gamma} d\eta \\ &\geq \int_{y_1}^{y} \left[\frac{1}{a(\eta)} \int_{y}^{\infty} \sum_{j=1}^{m_2} q_j(s) x^{\beta_j}(\vartheta_j(s))\, ds\right]^{1/\gamma} d\eta \\ &= A(y) \left[\int_{y}^{\infty} \sum_{j=1}^{m_2} q_j(s) x^{\beta_j}(\vartheta_j(s))\, ds\right]^{1/\gamma}. \end{aligned}$$

This completes the proof. □

3. Qualitative Properties of Solutions of (1)

Theorem 1. *Assume that there exists a constant δ_1, quotient of odd positive integers, such that $0 < \beta_j < \delta_1 < \gamma$, and* (A1)–(A4) *hold for $y \geq y_0$. If*

(A6) $\int_0^\infty Q_4(\eta)\, d\eta = \infty$

holds, then every solution of (1) *is oscillatory.*

Proof. Let the solution x of (1) be eventually positive. Accordingly, there exists a $y_0 > 0$ such that $x(y) > 0$, $x(\varsigma_i(y)) > 0$ and $x\left(\vartheta_j(y)\right) > 0$ for all $y \geq y_0$, $i = 1, 2, \cdots, m_1$ and $j = 1, 2, \cdots, m_2$. Applying Lemmas 2 and 5 for $y \geq y_1 > y_0$ we conclude that w satisfy (4), (5), (8) and (9) for all $y \geq y_1$. We can find a $y_1 > 0$, such that

$$w(y) \geq A(y) U^{1/\gamma}(y) \geq 0 \quad \text{for } y \geq y_1. \tag{11}$$

Using (5) and (8), $\beta_j - \delta_1 < 0$ and (11), we have

$$\begin{aligned} x^{\beta_j}(y) &\geq P^{\beta_j}(y) w^{\beta_j - \delta_1}(y) w^{\delta_1}(y) \geq P^{\beta_j}(y) (\delta A(y))^{\beta_j - \delta_1} w^{\delta_1}(y) \\ &\geq P^{\beta_j}(y) \left(\delta A(y)\right)^{\beta_j - \delta_1} \left(A(y) U^{1/\gamma}(y)\right)^{\delta_1} = P^{\beta_j}(y) \delta^{\beta_j - \delta_1} A^{\beta_j}(y) U^{\delta_1/\gamma}(y) \end{aligned}$$

for $y \geq y_2$. Since, $U'(y) = -\sum_{j=1}^{m_2} q_j(y) x^{\beta_j}(\vartheta_j(y)) \leq 0$, $y \geq y_2$, that is, U is non-increasing, then the last inequality becomes

$$\begin{aligned} x^{\beta_j}\left(\vartheta_j(\eta)\right) &\geq P^{\beta_j}\left(\vartheta_j(\eta)\right) \delta^{\beta_j - \delta_1} A^{\beta_j}\left(\vartheta_j(\eta)\right) U^{\delta_1/\gamma}(\vartheta_j(\eta)) \\ &\geq P^{\beta_j}\left(\vartheta_j(\eta)\right) \delta^{\beta_j - \delta_1} A^{\beta_j}\left(\vartheta_j(\eta)\right) U^{\delta_1/\gamma}(\eta). \end{aligned} \tag{12}$$

Since

$$\left(U^{1-\delta_1/\gamma}(y)\right)' = \left(1 - \frac{\delta_1}{\gamma}\right) U^{-\delta_1/\gamma}(y) U'(y), \tag{13}$$

integrating (13) from y_2 to y and using the fact that $U \geq 0$, we find

$$\begin{aligned}
\infty > U^{1-\delta_1/\gamma}(y_2) &\geq \left(1 - \frac{\delta_1}{\gamma}\right)\left[-\int_{y_2}^{y} U^{-\delta_1/\gamma}(\eta)U'(\eta)\,d\eta\right] \\
&= \left(1 - \frac{\delta_1}{\gamma}\right)\left[\int_{y_2}^{y} U^{-\delta_1/\gamma}(\eta)\left(\sum_{j=1}^{m_2} q_j(\eta)x^{\beta_j}(\vartheta_j(\eta))\right)d\eta\right] \\
&\geq \left(1 - \frac{\delta_1}{\gamma}\right)\left[\int_{y_2}^{y}\sum_{j=1}^{m_2}\frac{1}{\delta^{\delta_1-\beta_j}} q_j(\eta)P^{\beta_j}\left(\vartheta_j(\eta)\right)A^{\beta_j}(\vartheta_j(\eta))\,d\eta\right]
\end{aligned}$$

which contradicts (A6) as $y \to \infty$. This completes the proof. □

Theorem 2. *Assume that there exists a constant δ_2, quotient of odd positive integers, such that $\gamma < \delta_2 < \beta_j$. Furthermore, assume that* (A1)–(A5) *hold for $y \geq y_0$ and $a(y)$ is non-decreasing. If*

(A7) $\int_0^\infty \left[\frac{1}{a(\eta)}\int_\eta^\infty Q_1(\zeta)\,d\zeta\right]^{1/\gamma} d\eta = \infty$

holds, then every solution of (1) *is oscillatory.*

Proof. Let x be an eventually positive solution of (1). Subsequently, there exists a $y_0 > 0$ such that $x(y) > 0$, $x(\varsigma_i(y)) > 0$ and $x\left(\vartheta_j(y)\right) > 0$ for all $y \geq y_0$, $i = 1, 2, \cdots, m_1$ and $j = 1, 2, \cdots, m_2$. Applying Lemmas 2 and 4 for $y \geq y_1 > y_0$ we conclude that w satisfy (4), w is increasing and $x(y) \geq P(y)w(y)$ for all $y \geq y_1$. Accordingly,

$$x^{\beta_j}(y) \geq P^{\beta_j}(y)w^{\beta_j}(y) \geq P^{\beta_j}(y)w^{\beta_j-\delta_2}(y)w^{\delta_2}(y) \geq P^{\beta_j}(y)w^{\beta_j-\delta_2}(y_1)w^{\delta_2}(y)$$

implies that

$$x^{\beta_j}\left(\vartheta_j(y)\right) \geq P^{\beta_j}\left(\vartheta_j(y)\right)w^{\beta_j-\delta_2}(y_1)w^{\delta_2}\left(\vartheta_j(y)\right) \qquad \text{for } y \geq y_2 > y_1\,. \tag{14}$$

Using (10) and (14), we have

$$\begin{aligned}
a(y)\left(w'(y)\right)^\gamma &\geq w^{\beta_j-\delta_2}(y_1)\left[\int_y^\infty \sum_{j=1}^{m_2} q_j(\eta)P^{\beta_j}\left(\vartheta_j(\eta)\right)d\eta\right] w^{\delta_2}(\vartheta_j(y)) \\
&\geq w^{\beta_j-\delta_2}(y_1)\left[\int_y^\infty \sum_{j=1}^{m_2} q_j(\eta)P^{\beta_j}\left(\vartheta_j(\eta)\right)d\eta\right] w^{\delta_2}(\vartheta_0(y))
\end{aligned} \tag{15}$$

for $y \geq y_2$. Being $a(y)\left(w'(y)\right)^\gamma$ non-increasing and $\vartheta_0(y) \leq y$, we have

$$a(\vartheta_0(y))\left(w'(\vartheta_0(y))\right)^\gamma \geq a(y)\left(w'(y)\right)^\gamma.$$

Using the last inequality in (15), dividing by $a(\vartheta_0(y))w^{\delta_2}(\vartheta_0(y)) > 0$, and then operating the power $1/\gamma$ on both sides, we obtain

$$\frac{w'(\vartheta_0(y))}{w^{\delta_2/\gamma}(\vartheta_0(y))} \geq \left[\frac{w^{\beta_j-\delta_2}(y_1)}{a(\vartheta_0(y))}\int_y^\infty \sum_{j=1}^{m_2} q_j(\eta)P^{\beta_j}\left(\vartheta_j(\eta)\right)d\eta\right]^{1/\gamma}$$

for $y \geq y_2$. Multiplying the left-hand side by $\vartheta_0'(y)/\vartheta_0 \geq 1$ and integrating from y_2 to y, we find

$$\begin{aligned} &\frac{1}{\vartheta_0}\int_{y_2}^{y}\frac{w'(\vartheta_0(\eta))\vartheta_0'(\eta)}{w^{\delta_2/\gamma}(\vartheta_0(\eta))}\,d\eta \\ &\geq w^{(\beta_j-\delta_2)/\gamma}(y_1)\int_{y_2}^{y}\left[\frac{1}{a(\vartheta_0(\eta))}\int_{\eta}^{\infty}\sum_{j=1}^{m_2}q_j(\zeta)P^{\beta_j}\left(\vartheta_j(\zeta)\right)\,d\zeta\right]^{1/\gamma}d\eta \quad y \geq y_2\,. \end{aligned} \tag{16}$$

Because $\gamma < \delta_2$, $a(\vartheta_0(\eta)) \leq a(\eta)$ and

$$\frac{1}{\vartheta_0(1-\delta_2/\gamma)}\left[w^{1-\delta_2/\gamma}(\vartheta_0(\eta))\right]_{\eta=y_2}^{y} \leq \frac{1}{\vartheta_0(\delta_2/\gamma-1)}w^{1-\delta_2/\gamma}(\vartheta_0(y_2))\,,$$

it follows that (16) becomes

$$\int_{y_2}^{\infty}\left[\frac{1}{a(\eta)}\int_{\eta}^{\infty}\sum_{j=1}^{m_2}q_j(\zeta)P^{\beta_j}\left(\vartheta_j(\zeta)\right)\,d\zeta\right]^{1/\gamma}d\eta < \infty$$

which contradicts (A7). This contradiction implies that the solution x cannot be eventually positive. The case where x is eventually negative is very similar and we omit it here. □

Remark 1. *Theorems 1 and 2 hold for any index i and j (i.e., for $i \neq j$ and $i = j$).*

We conclude the paper presenting an example that shows the effectiveness and the feasibility of the main results.

Example 1. *Consider the differential equation*

$$\left[(y+1)\left(x(y)+\frac{1}{y^2}x^{\frac{1}{3}}\left(\frac{y}{2}\right)+\frac{1}{y^4}x^{\frac{3}{5}}\left(\frac{y}{3}\right)\right)'\right]'+y^{12}x^3\left(\frac{y}{2}\right)+y^{13}x^3\left(\frac{y}{3}\right)=0 \quad \text{for} \quad y \geq 2, \tag{17}$$

where $a(y) :\equiv y+1$, $q_j(y) :\equiv y^{j+11}$, $\vartheta_j(y) :\equiv \frac{y}{i+1}$, $\vartheta_0'(y) > \frac{1}{3} = \vartheta_0$, $\beta_j = 3 > \gamma = 1$, $p_i(y) :\equiv \frac{1}{y^{2i}}$, $\alpha_i :\equiv \frac{2i-1}{2i+1}$ and $\varsigma_i(y) :\equiv \frac{y}{i+1}$ for $i = 1,2$; $j = 1,2$ and $y \geq 2$. All of the assumptions of Theorem 2 can be verified with the index $i = 1,2$ and $\rho(y) = \frac{1}{y^2}$. Hence, due to Theorem 1, every solution of (17) is oscillatory.

4. Conclusions

In this work, we have undertaken the problem by taking a second order nonlinear neutral differential equation with sublinear neutral terms and established sufficient conditions for the oscillation of (1).

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