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Fractional Hermite–Hadamard–Fejer Inequalities for a Convex Function with Respect to an Increasing Function Involving a Positive Weighted Symmetric Function

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Abstract: There have been many different definitions of fractional calculus presented in the literature, especially in recent years. These definitions can be classified into groups with similar properties. An important direction of research has involved proving inequalities for fractional integrals of particular types of functions, such as Hermite–Hadamard–Fejer (HHF) inequalities and related results. Here we consider some HHF fractional integral inequalities and related results for a class of fractional operators (namely, the weighted fractional operators), which apply to function of convex type with respect to an increasing function involving a positive weighted symmetric function. We can conclude that all derived inequalities in our study generalize numerous well-known inequalities involving both classical and Riemann–Liouville fractional integral inequalities.

Keywords: symmetric; weighted fractional operators; convex functions; Hermite–Hadamard–Fejer inequality

1. Introduction

First of all, we recall the basic notation in convex analysis. A set $\mathcal{V} \subset \mathbb{R}$ is said to be convex if

$$\varepsilon\,\vartheta_1 + (1-\varepsilon)\vartheta_2 \in \mathcal{V}$$

for each $\vartheta_1, \vartheta_2 \in \mathcal{V}$ and $\varepsilon \in [0,1]$. Based on a convex set $\mathcal{V}$, we say that a function $\hbar : \mathcal{V} \to \mathbb{R}$ is convex, if the inequality

$$\hbar(\varepsilon\,\vartheta_1 + (1-\varepsilon)\vartheta_2) \leq \varepsilon\,\hbar(\vartheta_1) + (1-\varepsilon)\hbar(\vartheta_2), \quad \forall\,\vartheta_1, \vartheta_2 \in \mathcal{V},\ \varepsilon \in [0,1] \tag{1}$$

holds. We say that $\hbar$ is concave if $-\hbar$ is convex.

Theory and application of convexity play an important role in the field of fractional integral inequalities due to the behavior of its properties and definition, especially in the past few years. There is a strong relationship between theories of convexity and symmetry. Whichever one we study, we can apply it to the other one; see, e.g., [1]. There are plenty of well-known integral inequalities that have been established for the convex functions (1) in the literature; for example, Ostrowski type integral inequalities [2], Simpson type integral inequalities [3], Hardy type integral inequalities [4], Olsen type integral inequalities [5], Gagliardo-Nirenberg type integral inequalities [6], Opial type type integral inequalities [7,8] and Rozanova type integral inequalities [9]. However, the most common integral inequalities are the Hermite-Hadamard type integral inequalities: the classical and fractional Hermite–Hadamard type integral inequalities [10,11] are, respectively, given by:

$$\hbar\left(\frac{\vartheta_1+\vartheta_2}{2}\right) \leq \frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\hbar(x)dx \leq \frac{\hbar(\vartheta_1)+\hbar(\vartheta_2)}{2}, \tag{2}$$

and

$$\hbar\left(\frac{\vartheta_1+\vartheta_2}{2}\right) \leq \frac{\Gamma(\ell+1)}{2(\vartheta_2-\vartheta_1)^{\ell}}\left[{}^{RL}\mathcal{J}^{\ell}_{\vartheta_1+}\hbar(\vartheta_2)+{}^{RL}\mathcal{J}^{\ell}_{\vartheta_2-}\hbar(\vartheta_1)\right] \leq \frac{\hbar(\vartheta_1)+\hbar(\vartheta_2)}{2}, \tag{3}$$

where $\hbar : \mathcal{V} \to \mathbb{R}$ is supposed to be a positive convex function, $\hbar \in L^1(\vartheta_1,\vartheta_2)$ with $\vartheta_1 < \vartheta_2$, and ${}^{RL}\mathcal{J}^{\ell}_{\vartheta_1+}$ and ${}^{RL}\mathcal{J}^{\ell}_{\vartheta_2-}$ stand for the left-sided and right-sided Riemann–Liouville fractional integrals of order $\ell > 0$, respectively, and these are defined by [12,13]:

$$\begin{aligned} {}^{RL}\mathcal{J}^{\ell}_{\vartheta_1+}\hbar(x) &= \frac{1}{\Gamma(\ell)}\int_{\vartheta_1}^{x}(x-\varepsilon)^{\ell-1}\hbar(\varepsilon)d\varepsilon, \quad x>\vartheta_1; \\ {}^{RL}\mathcal{J}^{\ell}_{\vartheta_2-}\hbar(x) &= \frac{1}{\Gamma(\ell)}\int_{x}^{\vartheta_2}(\varepsilon-x)^{\ell-1}\hbar(\varepsilon)d\varepsilon, \quad x<\vartheta_2. \end{aligned} \tag{4}$$

The HH type inequality (2) has been applied to numerous types of convex functions, including s-geometrically convex functions [14], GA-convex functions [15], MT-convex function [16] and (α,m)-convex functions [17], and many other types can be found in [18]. Besides, the HH type inequality (3) has been applied to a huge number of convex functions, such as F-convex functions [19], λ_{ψ}-convex functions [20], MT-convex functions [21] and (α,m)-convex functions [22], a new class of convex functions [23], and many other types can be found in the literature. Meanwhile, it has been applied to other models of fractional calculus, such as standard RL-fractional operators [24], conformable fractional operators [25,26], generalized fractional operators [27], ψ-RL-fractional operators [28,29], tempered fractional operators [30] and AB and Prabhakar fractional operators [31].

After growing the field of Hermite–Hadamard type inequalities (2) and (3), many classical and fractional integral inequalities have been established by many authors; for more details, see [24–31].

Definition 1 ([32])**.** *Let $g : [\vartheta_1,\vartheta_2] \to [0,\infty)$ be an integrable function; then we say g is symmetric with respect to $(\vartheta_1+\vartheta_2)/2$, if*

$$g(\vartheta_1+\vartheta_2-x) = g(x), \tag{5}$$

holds for each $x \in [\vartheta_1,\vartheta_2]$.

Based on this definition, the authors in [33,34] extended the HH-type inequalities (2) and (3) and they could deduce the so-called Hermite–Hadamard–Fejer (HHF) type inequalities, and their results were, respectively, as follows:

$$\hbar\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\int_{\vartheta_1}^{\vartheta_2}g(x)dx \leq \frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\hbar(x)g(x)dx \leq \frac{\hbar(\vartheta_1)+\hbar(\vartheta_2)}{2}\int_{\vartheta_1}^{\vartheta_2}g(x)dx, \tag{6}$$

and

$$\begin{aligned} \hbar\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\left[{}^{RL}\mathcal{J}^{\ell}_{\vartheta_1+}g(\vartheta_2)+{}^{RL}\mathcal{J}^{\ell}_{\vartheta_2-}g(\vartheta_1)\right] &\leq \left[{}^{RL}\mathcal{J}^{\ell}_{\vartheta_1+}(\hbar g)(\vartheta_2)+{}^{RL}\mathcal{J}^{\ell}_{\vartheta_2-}(\hbar g)(\vartheta_1)\right] \\ &\leq \frac{\hbar(\vartheta_1)+\hbar(\vartheta_2)}{2}\left[{}^{RL}\mathcal{J}^{\ell}_{\vartheta_1+}g(\vartheta_2)+{}^{RL}\mathcal{J}^{\ell}_{\vartheta_2-}g(\vartheta_1)\right], \end{aligned} \tag{7}$$

where $\hbar$ is as before and g is as defined in Definition 1.

Definition 2. *Let $(\vartheta_1, \vartheta_2) \subseteq \mathbb{R}$ and $\sigma(x)$ be an increasing positive and monotonic function on the interval $(\vartheta_1, \vartheta_2]$ with a continuous derivative $\sigma'(x)$ on the interval $(\vartheta_1, \vartheta_2)$ with $\sigma(0) = 0,\ 0 \in [\vartheta_1, \vartheta_2]$. Then, the left-side and right-side of the weighted fractional integrals of a function $\hbar$ with respect to another function $\sigma(x)$ on $[\vartheta_1, \vartheta_2]$ are defined by [35]:*

$$\begin{aligned} \left({}_{\vartheta_1+}\mathcal{J}_{w}^{\ell:\sigma}\hbar\right)(x) &= \frac{w^{-1}(x)}{\Gamma(\ell)}\int_{\vartheta_1}^{x}\sigma'(\varepsilon)(\sigma(x)-\sigma(\varepsilon))^{\ell-1}\hbar(\varepsilon)w(\varepsilon)d\varepsilon, \\ \left({}_{w}\mathcal{J}_{\vartheta_2-}^{\ell:\sigma}\hbar\right)(x) &= \frac{w^{-1}(x)}{\Gamma(\ell)}\int_{x}^{\vartheta_2}\sigma'(\varepsilon)(\sigma(\varepsilon)-\sigma(x))^{\ell-1}\hbar(\varepsilon)w(\varepsilon)d\varepsilon, \quad \ell > 0, \end{aligned} \tag{8}$$

where $w^{-1}(x) = \frac{1}{w(x)},\ w(x) \neq 0$.

Remark 1. *From the Definition 2, one can observe that*

- *If σ is specialized by $\sigma(x) = x$ and $w(x) = 1$, then the weighted fractional integral operators (8) reduce to the classical Riemann–Liouville fractional integral operators (4).*
- *If $w(x) = 1$, we get the fractional integral operators of a function $\hbar$ with respect to another function $\sigma(x)$, which is defined in [36,37] as follows:*

$$\begin{aligned} \left({}_{\vartheta_1+}\mathcal{J}^{\ell:\sigma}\hbar\right)(x) &:= \frac{1}{\Gamma(\ell)}\int_{\vartheta_1}^{x}\sigma'(\varepsilon)(\sigma(x)-\sigma(\varepsilon))^{\ell-1}\hbar(\varepsilon)d\varepsilon, \\ \left(\mathcal{J}_{\vartheta_2-}^{\ell:\sigma}\hbar\right)(x) &:= \frac{1}{\Gamma(\ell)}\int_{x}^{\vartheta_2}\sigma'(\varepsilon)(\sigma(\varepsilon)-\sigma(x))^{\ell-1}\hbar(\varepsilon)d\varepsilon, \quad \ell > 0. \end{aligned} \tag{9}$$

This study investigates several inequalities of HHF type via the weighted fractional operators (8) with positive weighted symmetric functions in the kernel.

The rest of the study is structured in the following way: In Section 2, we prove the necessary and auxiliary lemmas that are useful in the next section. Section 3 contains our main results which consists of proving several HHF fractional integral inequalities and some related results. In Section 4, we discuss our results and give the comparison between our results and the existing results, and we point out the future work. Section 5 is for the conclusions.

2. Auxiliary Results

Here, we shall prove analogues of the fractional HH inequalities (2) and (3) and HHF inequalities (6) and (7) for weighted fractional integrals with positive weighted symmetric function kernels. The main results here are Theorem 1 (a generalization of HH inequalities (2) and (3) and HHF inequality (6), and a reformulation of HHF inequality (7)) and Lemma 2 (a consequence of Theorem 1). First, we need the following fact.

Lemma 1. *(i) Let $w : [\vartheta_1, \vartheta_2] \to [0, \infty)$ be an integrable function and symmetric with respect to $(\vartheta_1 + \vartheta_2)/2,\ \vartheta_1 < \vartheta_2$; then we have*

$$w(\varepsilon\vartheta_1 + (1-\varepsilon)\vartheta_2) = w((1-\varepsilon)\vartheta_1 + \varepsilon\vartheta_2), \tag{10}$$

for each $\varepsilon \in [0, 1]$.

(ii) Let $w : [\vartheta_1, \vartheta_2] \to [0, \infty)$ be an integrable and symmetric function with respect to $(\vartheta_1 + \vartheta_2)/2,\ \vartheta_1 < \vartheta_2$; then we have for $\ell > 0$:

$$\begin{aligned}\left({}_{\sigma^{-1}(\vartheta_1)+}\mathcal{J}^{\ell:\sigma}(w\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_2)\right)&=\left(\mathcal{J}^{\ell:\sigma}_{\sigma^{-1}(\vartheta_2)-}(w\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_1)\right)\\&=\frac{1}{2}\left[\left({}_{\sigma^{-1}(\vartheta_1)+}\mathcal{J}^{\ell:\sigma}(w\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_2)\right)+\left(\mathcal{J}^{\ell:\sigma}_{\sigma^{-1}(\vartheta_2)-}(w\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_1)\right)\right].\quad(11)\end{aligned}$$

Proof. (i) Let $x=\varepsilon\vartheta_1+(1-\varepsilon)\vartheta_2$. It is clear that $x\in[\vartheta_1,\vartheta_2]$ for each $\varepsilon\in[0,1]$ and then $\vartheta_1+\vartheta_2-x=(1-\varepsilon)\vartheta_1+\varepsilon\vartheta_2$. Then, by using the assumptions and Definition 1, we get (10).

(ii) By using the symmetric property of w, we have

$$(w\circ\sigma(\varepsilon))=w(\sigma(\varepsilon))=w(\vartheta_1+\vartheta_2-\sigma(\varepsilon)),\quad\forall\,\varepsilon\in\left[\sigma^{-1}(\vartheta_1),\sigma^{-1}(\vartheta_2)\right].$$

From this and by setting $\sigma(x)=\vartheta_1+\vartheta_2-\sigma(\varepsilon)$, it follows that

$$\begin{aligned}\left({}_{\sigma^{-1}(\vartheta_1)+}\mathcal{J}^{\ell:\sigma}(w\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_2)\right)&=\frac{1}{\Gamma(\ell)}\int_{\sigma^{-1}(\vartheta_1)}^{\sigma^{-1}(\vartheta_2)}(\vartheta_2-\sigma(x))^{\ell-1}(w\circ\sigma)(x)\sigma'(x)dx\\&=\frac{1}{\Gamma(\ell)}\int_{\sigma^{-1}(\vartheta_1)}^{\sigma^{-1}(\vartheta_2)}(\sigma(\varepsilon)-\vartheta_1)^{\ell-1}w(\vartheta_1+\vartheta_2-\sigma(\varepsilon))\sigma'(\varepsilon)d\varepsilon\\&=\frac{1}{\Gamma(\ell)}\int_{\sigma^{-1}(\vartheta_1)}^{\sigma^{-1}(\vartheta_2)}(\sigma(\varepsilon)-\vartheta_1)^{\ell-1}(w\circ\sigma)(\varepsilon)\sigma'(\varepsilon)d\varepsilon\\&=\left(\mathcal{J}^{\ell:\sigma}_{\sigma^{-1}(\vartheta_2)-}(w\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_1)\right).\end{aligned}$$

This rearranges to the required (11). □

Remark 2. *Throughout this study* $w^{-1}(x)=\frac{1}{w(x)}$ *and* $\sigma^{-1}(x)$ *is the inverse of the function* $\sigma(x)$*.*

Example 1. *Consider the following integrable and positive weighted function*

$$w(x)=\begin{cases}2x+\frac{1}{2}, & 0\le x\le\frac{1}{2},\\-2x+\frac{5}{2}, & \frac{1}{2}\le x\le 1.\end{cases}$$

One can easily show that

$$w(1-x)=\begin{cases}2x+\frac{1}{2}, & 0\le x\le\frac{1}{2},\\-2x+\frac{5}{2}, & \frac{1}{2}\le x\le 1.\end{cases}$$

Thus, $w(x)=w(1-x)$ *and hence the given weighted function is symmetric on* $[0,1]$ *with respect to* $\frac{1}{2}$*.*

Theorem 1. *Let* $\hbar:[\vartheta_1,\vartheta_2]\subseteq[0,\infty)\to\mathbb{R}$ *be an* L^1 *convex function with* $0\le\vartheta_1<\vartheta_2$ *and* $w:[\vartheta_1,\vartheta_2]\to\mathbb{R}$ *be an integrable, positive and weighted symmetric function with respect to* $\frac{\vartheta_1+\vartheta_2}{2}$*. If* σ *is an increasing and positive function on* $[\vartheta_1,\vartheta_2)$ *and* $\sigma'(x)$ *is continuous on* $(\vartheta_1,\vartheta_2)$*, then, we have for* $\ell>0$*:*

$$\begin{aligned}&\hbar\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\left[\left({}_{\sigma^{-1}(\vartheta_1)+}\mathcal{J}^{\ell:\sigma}(w\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_2)\right)+\left(\mathcal{J}^{\ell:\sigma}_{\sigma^{-1}(\vartheta_2)-}(w\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_1)\right)\right]\\&\quad\le w(\vartheta_2)\left({}_{\sigma^{-1}(\vartheta_1)+}\mathcal{J}^{\ell:\sigma}_{w\circ\sigma}(\hbar\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_2)\right)+w(\vartheta_1)\left({}_{w\circ\sigma}\mathcal{J}^{\ell:\sigma}_{\sigma^{-1}(\vartheta_2)-}(\hbar\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_1)\right)\\&\quad\le\frac{\hbar(\vartheta_1)+\hbar(\vartheta_2)}{2}\left[\left({}_{\sigma^{-1}(\vartheta_1)+}\mathcal{J}^{\ell:\sigma}(w\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_2)\right)+\left(\mathcal{J}^{\ell:\sigma}_{\sigma^{-1}(\vartheta_2)-}(w\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_1)\right)\right].\quad(12)\end{aligned}$$

Proof. Since $\hbar$ is a convex function on $[\vartheta_1,\vartheta_2]$, we have

$$\hbar\left(\frac{x+y}{2}\right)\le\frac{\hbar(x)+\hbar(y)}{2},\quad\forall\,x,y\in[\vartheta_1,\vartheta_2].$$

Thus, for $x = \varepsilon\vartheta_1 + (1-\varepsilon)\vartheta_2$ and $y = (1-\varepsilon)\vartheta_1 + \varepsilon\vartheta_2,\ \varepsilon \in [0,1]$, it follows that

$$2\hbar\left(\frac{\vartheta_1+\vartheta_2}{2}\right) \le \hbar(\varepsilon\vartheta_1 + (1-\varepsilon)\vartheta_2) + \hbar((1-\varepsilon)\vartheta_1 + \varepsilon\vartheta_2). \tag{13}$$

By multiplying both sides of (13) by $\varepsilon^{\ell-1}w(\varepsilon\vartheta_1 + (1-\varepsilon)\vartheta_2)$, and then, by integrating the resulting inequality with respect to ε over $[0,1]$, we get

$$\begin{aligned}2\hbar\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\int_0^1 \varepsilon^{\ell-1}w(\varepsilon\vartheta_1+(1-\varepsilon)\vartheta_2)d\varepsilon \le \int_0^1 \varepsilon^{\ell-1}\hbar(\varepsilon\vartheta_1+(1-\varepsilon)\vartheta_2)w(\varepsilon\vartheta_1+(1-\varepsilon)\vartheta_2)d\varepsilon \\ + \int_0^1 \varepsilon^{\ell-1}\hbar((1-\varepsilon)\vartheta_1+\varepsilon\vartheta_2)w(\varepsilon\vartheta_1+(1-\varepsilon)\vartheta_2)d\varepsilon. \end{aligned} \tag{14}$$

For the left hand side inequality, we make use of (11) to get

$$\begin{aligned} &\tfrac{\Gamma(\ell)}{2(\vartheta_2-\vartheta_1)^\ell}\left[\left({}_{\sigma^{-1}(\vartheta_1)+}\mathcal{J}^{\ell:\sigma}(w\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_2)\right) + \left(\mathcal{J}^{\ell:\sigma}_{\sigma^{-1}(\vartheta_2)-}(w\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_1)\right)\right] \\ &\qquad \tfrac{\Gamma(\ell)}{(\vartheta_2-\vartheta_1)^\ell}\left({}_{\sigma^{-1}(\vartheta_1)+}\mathcal{J}^{\ell:\sigma}(w\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_2)\right) \\ &\quad = \tfrac{1}{(\vartheta_2-\vartheta_1)^\ell}\int_{\sigma^{-1}(\vartheta_1)}^{\sigma^{-1}(\vartheta_2)}(\vartheta_2-\sigma(x))^{\ell-1}(w\circ\sigma)(x)\sigma'(x)dx \\ &\quad = \int_{\sigma^{-1}(\vartheta_1)}^{\sigma^{-1}(\vartheta_2)}\left(\tfrac{\vartheta_2-\sigma(x)}{\vartheta_2-\vartheta_1}\right)^{\ell-1}(w\circ\sigma)(x)\sigma'(x)\tfrac{dx}{\vartheta_2-\vartheta_1} \\ &= \int_0^1 \varepsilon^{\ell-1}w(\varepsilon\vartheta_1+(1-\varepsilon)\vartheta_2)d\varepsilon, \quad \left[\text{denoting } \varepsilon := \tfrac{\vartheta_2-\sigma(x)}{\vartheta_2-\vartheta_1}\right]. \end{aligned} \tag{15}$$

Now, we evaluate the weighted fractional operators as follows:

$$\begin{aligned} &w(\vartheta_2)\left({}_{\sigma^{-1}(\vartheta_1)+}\mathcal{J}^{\ell:\sigma}_{w\circ\sigma}(\hbar\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_2)\right) + w(\vartheta_1)\left({}_{w\circ\sigma}\mathcal{J}^{\ell:\sigma}_{\sigma^{-1}(\vartheta_2)-}(\hbar\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_1)\right) \\ &= w(\vartheta_2)\frac{(w\circ\sigma)^{-1}\left(\sigma^{-1}(\vartheta_2)\right)}{\Gamma(\ell)}\int_{\sigma^{-1}(\vartheta_1)}^{\sigma^{-1}(\vartheta_2)}(\vartheta_2-\sigma(x))^{\ell-1}(\hbar\circ\sigma)(x)(w\circ\sigma)(x)\sigma'(x)dx \\ &+ w(\vartheta_1)\frac{(w\circ\sigma)^{-1}\left(\sigma^{-1}(\vartheta_1)\right)}{\Gamma(\ell)}\int_{\sigma^{-1}(\vartheta_1)}^{\sigma^{-1}(\vartheta_2)}(\sigma(x)-\vartheta_1)^{\ell-1}(\hbar\circ\sigma)(x)(w\circ\sigma)(x)\sigma'(x)dx \\ &= \frac{(\vartheta_2-\vartheta_1)^\ell}{\Gamma(\ell)}\int_{\sigma^{-1}(\vartheta_1)}^{\sigma^{-1}(\vartheta_2)}\left(\frac{\vartheta_2-\sigma(x)}{\vartheta_2-\vartheta_1}\right)^{\ell-1}(\hbar\circ\sigma)(x)(w\circ\sigma)(x)\sigma'(x)\frac{dx}{\vartheta_2-\vartheta_1} \\ &+ \frac{(\vartheta_2-\vartheta_1)^\ell}{\Gamma(\ell)}\int_{\sigma^{-1}(\vartheta_1)}^{\sigma^{-1}(\vartheta_2)}\left(\frac{\sigma(x)-\vartheta_1}{\vartheta_2-\vartheta_1}\right)^{\ell-1}(\hbar\circ\sigma)(x)(w\circ\sigma)(x)\sigma'(x)\frac{dx}{\vartheta_2-\vartheta_1}, \end{aligned}$$

where

$$(w\circ\sigma)^{-1}\left(\sigma^{-1}(z)\right) = \frac{1}{(w\circ\sigma)\left(\sigma^{-1}(z)\right)} = \frac{1}{w(z)}, \quad z = \vartheta_1, \vartheta_2. \tag{16}$$

Setting $t_1 = \frac{\vartheta_2-\sigma(x)}{\vartheta_2-\vartheta_1}$ and $t_2 = \frac{\sigma(x)-\vartheta_1}{\vartheta_2-\vartheta_1}$, it follows that

$$w(\vartheta_2)\left({}_{\sigma^{-1}(\vartheta_1)+}\mathcal{J}^{\ell:\sigma}_{w\circ\sigma}(\hbar\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_2)\right)+w(\vartheta_1)\left({}_{w\circ\sigma}\mathcal{J}^{\ell:\sigma}_{\sigma^{-1}(\vartheta_2)-}(\hbar\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_1)\right)$$
$$=\frac{(\vartheta_2-\vartheta_1)^{\ell}}{\Gamma(\ell)}\left[\int_0^1 {t_1}^{\ell-1}\hbar(t_1\vartheta_1+(1-t_1)\vartheta_2)w(t_1\vartheta_1+(1-t_1)\vartheta_2)dt_1\right.$$
$$+\int_0^1 {t_2}^{\ell-1}\hbar((1-t_2)\vartheta_1+t_2\vartheta_2)w((1-t_2)\vartheta_1+t_2\vartheta_2)dt_2$$
$$=\frac{(\vartheta_2-\vartheta_1)^{\ell}}{\Gamma(\ell)}\left[\int_0^1 \varepsilon^{\ell-1}\hbar(\varepsilon\vartheta_1+(1-\varepsilon)\vartheta_2)w(\varepsilon\vartheta_1+(1-\varepsilon)\vartheta_2)d\varepsilon\right.$$
$$\left.+\int_0^1 \varepsilon^{\ell-1}\hbar((1-\varepsilon)\vartheta_1+\varepsilon\vartheta_2)\underbrace{w(\varepsilon\vartheta_1+(1-\varepsilon)\vartheta_2)}_{\text{by using (10)}}d\varepsilon\right]. \tag{17}$$

By making use of (15) and (17) in (14), we get

$$\hbar\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\left[{}^{RL}_{\vartheta_1+}\mathcal{J}^{\ell}w(\vartheta_2)+{}^{RL}\mathcal{J}^{\ell}_{\vartheta_2-}w(\vartheta_1)\right]\le w(\vartheta_2)\left({}_{\sigma^{-1}(\vartheta_1)+}\mathcal{J}^{\ell:\sigma}_{w\circ\sigma}(\hbar\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_2)\right)$$
$$+w(\vartheta_1)\left({}_{w\circ\sigma}\mathcal{J}^{\ell:\sigma}_{\sigma^{-1}(\vartheta_2)-}(\hbar\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_1)\right). \tag{18}$$

The first inequality of (12) is proved.

On the other hand, we will prove the second inequality of (12). By making use of the convexity of $\hbar$, we get

$$\hbar\left(\varepsilon\vartheta_1+(1-\varepsilon)\vartheta_2\right)+\hbar\left((1-\varepsilon)\vartheta_1+\varepsilon\vartheta_2\right)\le\hbar(\vartheta_1)+\hbar(\vartheta_2), \tag{19}$$

We multiply both sides of (19) by $\varepsilon^{\ell-1}w(\varepsilon\vartheta_1+(1-\varepsilon)\vartheta_2)$ and integrate with respect to ε over $[0,1]$ to get

$$\int_0^1 \varepsilon^{\ell-1}\hbar(\varepsilon\vartheta_1+(1-\varepsilon)\vartheta_2)w(\varepsilon\vartheta_1+(1-\varepsilon)\vartheta_2)d\varepsilon$$
$$+\int_0^1 \varepsilon^{\ell-1}\hbar((1-\varepsilon)\vartheta_1+\varepsilon\vartheta_2)w(\varepsilon\vartheta_1+(1-\varepsilon)\vartheta_2)d\varepsilon\le(\hbar(\vartheta_1)+\hbar(\vartheta_2))\int_0^1 \varepsilon^{\ell-1}w(\varepsilon\vartheta_1+(1-\varepsilon)\vartheta_2)d\varepsilon. \tag{20}$$

Then, by using (10) and (17) in (20), we get

$$w(\vartheta_2)\left({}_{\sigma^{-1}(\vartheta_1)+}\mathcal{J}^{\ell:\sigma}_{w\circ\sigma}(\hbar\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_2)\right)+w(\vartheta_1)\left({}_{w\circ\sigma}\mathcal{J}^{\ell:\sigma}_{\sigma^{-1}(\vartheta_2)-}(\hbar\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_1)\right)$$
$$\le\frac{\hbar(\vartheta_1)+\hbar(\vartheta_2)}{2}\left[{}^{RL}_{\vartheta_1+}\mathcal{J}^{\ell}w(\vartheta_2)+{}^{RL}\mathcal{J}^{\ell}_{\vartheta_2-}w(\vartheta_1)\right].$$

This completes the proof of our theorem. □

Remark 3. *Particularly, in Theorem 1, if we take*

(i) $\sigma(x)=x$, *then inequality* (12) *becomes*

$$\hbar\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\left[{}^{RL}_{\vartheta_1+}\mathcal{J}^{\ell}w(\vartheta_2)+{}^{RL}\mathcal{J}^{\ell}_{\vartheta_2-}w(\vartheta_1)\right]\le w(\vartheta_2)\left({}^{RL}_{\vartheta_1+}\mathcal{J}^{\ell}_{w}\hbar\right)(\vartheta_2)+w(\vartheta_1)\left({}^{RL}_{w}\mathcal{J}^{\ell}_{\vartheta_2-}\hbar\right)(\vartheta_1)$$
$$\le\frac{\hbar(\vartheta_1)+\hbar(\vartheta_2)}{2}\left[{}^{RL}_{\vartheta_1+}\mathcal{J}^{\ell}w(\vartheta_2)+{}^{RL}\mathcal{J}^{\ell}_{\vartheta_2-}w(\vartheta_1)\right], \tag{21}$$

where ${}^{RL}_{\vartheta_1+}\mathcal{J}^{\ell}_{w}$ *and* ${}^{RL}_{w}\mathcal{J}^{\ell}_{\vartheta_2-}$ *are the left and right weighted Riemann–Liouville fractional operators, defined by*

$$\left({}^{RL}_{\vartheta_1+}\mathcal{J}^{\ell}_{w}\hbar\right)(x) = \frac{w^{-1}(x)}{\Gamma(\ell)}\int_{\vartheta_1}^{x}(x-\varepsilon)^{\ell-1}\hbar(\varepsilon)w(\varepsilon)d\varepsilon,$$
$$\left({}^{RL}_{w}\mathcal{J}^{\ell}_{\vartheta_2-}\hbar\right)(x) = \frac{w^{-1}(x)}{\Gamma(\ell)}\int_{x}^{\vartheta_2}(\varepsilon-x)^{\ell-1}\hbar(\varepsilon)w(\varepsilon)d\varepsilon, \quad \ell>0,$$

respectively.

(ii) $\sigma(x)=x$ *and* $\ell=1$*; then inequality* (12) *reduces to inequality* (6).

(iii) $\sigma(x)=x$ *and* $w(x)=1$*; then inequality* (12) *reduces to inequality* (3).

(iv) $\sigma(x)=x$, $w(x)=1$ *and* $\ell=1$*; then inequality* (12) *reduces to inequality* (2).

Remark 4. *From Remark 3, we can observe that the HH inequality* (3) *and the HHF inequality* (6) *are essentially particular cases of our HHF inequality* (12)*. Additionally, the HHF inequality* (21) *can be seen as a reformulation of HHF inequality* (12)*, even though it is about weighted fractional and RL-fractional integrals rather than RL-fractional integrals explicitly.*

Lemma 2. *Let* $\hbar:[\vartheta_1,\vartheta_2]\subseteq[0,\infty)\to\mathbb{R}$ *be an* L^1 *function with* $\hbar'\in L^1$ *and* $0\le\vartheta_1<\vartheta_2$*, and* $w:[\vartheta_1,\vartheta_2]\to\mathbb{R}$ *be an integrable, positive and weighted symmetric function with respect to* $\frac{\vartheta_1+\vartheta_2}{2}$*. If* σ *is an increasing and positive function on* $[\vartheta_1,\vartheta_2)$ *and* $\sigma'(x)$ *is continuous on* $(\vartheta_1,\vartheta_2)$*, then, we have for* $\ell>0$*:*

$$\begin{aligned}
&\frac{\hbar(\vartheta_1)+\hbar(\vartheta_2)}{2}\left[\left({}_{\sigma^{-1}(\vartheta_1)+}\mathcal{J}^{\ell:\sigma}(w\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_2)\right)+\left(\mathcal{J}^{\ell:\sigma}_{\sigma^{-1}(\vartheta_2)-}(w\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_1)\right)\right]\\
&\quad-\left[w(\vartheta_2)\left({}_{\sigma^{-1}(\vartheta_1)+}\mathcal{J}^{\ell:\sigma}_{w\circ\sigma}(\hbar\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_2)\right)+w(\vartheta_1)\left({}_{w\circ\sigma}\mathcal{J}^{\ell:\sigma}_{\sigma^{-1}(\vartheta_2)-}(\hbar\circ\sigma)\right)\left(\sigma^{-1}(\vartheta_1)\right)\right]\\
&=\frac{1}{\Gamma(\ell)}\int_{\sigma^{-1}(\vartheta_1)}^{\sigma^{-1}(\vartheta_2)}\left[\int_{\sigma^{-1}(\vartheta_1)}^{\varepsilon}\sigma'(x)(\vartheta_2-\sigma(x))^{\ell-1}(w\circ\sigma)(x)dx\right.\\
&\qquad\left.-\int_{\varepsilon}^{\sigma^{-1}(\vartheta_2)}\sigma'(x)(\sigma(x)-\vartheta_1)^{\ell-1}(w\circ\sigma)(x)dx\right](\hbar'\circ\sigma)(\varepsilon)\sigma'(\varepsilon)d\varepsilon.
\end{aligned}\tag{22}$$

Proof. Setting

$$\begin{aligned}
&\frac{1}{\Gamma(\ell)}\int_{\sigma^{-1}(\vartheta_1)}^{\sigma^{-1}(\vartheta_2)}\left[\int_{\sigma^{-1}(\vartheta_1)}^{\varepsilon}\sigma'(x)(\vartheta_2-\sigma(x))^{\ell-1}(w\circ\sigma)(x)dx\right.\\
&\left.-\int_{\varepsilon}^{\sigma^{-1}(\vartheta_2)}\sigma'(x)(\sigma(x)-\vartheta_1)^{\ell-1}(w\circ\sigma)(x)dx\right](\hbar'\circ\sigma)(\varepsilon)\sigma'(\varepsilon)d\varepsilon\\
&=\frac{1}{\Gamma(\ell)}\int_{\sigma^{-1}(\vartheta_1)}^{\sigma^{-1}(\vartheta_2)}\left[\int_{\sigma^{-1}(\vartheta_1)}^{\varepsilon}\sigma'(x)(\vartheta_2-\sigma(x))^{\ell-1}(w\circ\sigma)(x)dx\right](\hbar'\circ\sigma)(\varepsilon)\sigma'(\varepsilon)d\varepsilon\\
&+\frac{-1}{\Gamma(\ell)}\int_{\sigma^{-1}(\vartheta_1)}^{\sigma^{-1}(\vartheta_2)}\left[\int_{\varepsilon}^{\sigma^{-1}(\vartheta_2)}\sigma'(x)(\sigma(x)-\vartheta_1)^{\ell-1}(w\circ\sigma)(x)dx\right](\hbar'\circ\sigma)(\varepsilon)\sigma'(\varepsilon)d\varepsilon\\
&:=\Xi_1+\Xi_2.
\end{aligned}$$

By integration by parts, making use of Lemma 1, and definitions (8) and (9), we obtain

$$\begin{aligned}
\Xi_1 &= \frac{1}{\Gamma(\ell)} \left(\int_{\sigma^{-1}(\vartheta_1)}^{\varepsilon} \sigma'(x)(\vartheta_2 - \sigma(x))^{\ell-1} (w \circ \sigma)(x)dx \right) (\hbar \circ \sigma)(\varepsilon)d\varepsilon \Bigg|_{t=\sigma^{-1}(\vartheta_1)}^{\sigma^{-1}(\vartheta_2)} \\
&- \frac{1}{\Gamma(\ell)} \int_{\sigma^{-1}(\vartheta_1)}^{\sigma^{-1}(\vartheta_2)} \sigma'(\varepsilon)(\vartheta_2 - \sigma(\varepsilon))^{\ell-1} (w \circ \sigma)(\varepsilon)(\hbar \circ \sigma)(\varepsilon)d\varepsilon \\
&= \left(\frac{1}{\Gamma(\ell)} \int_{\sigma^{-1}(\vartheta_1)}^{\sigma^{-1}(\vartheta_2)} \sigma'(x)(\vartheta_2 - \sigma(x))^{\ell-1} (w \circ \sigma)(x)dx \right) \hbar(\vartheta_2) \\
&- \underbrace{w(\vartheta_2) \frac{(w \circ \sigma)^{-1} \left(\sigma^{-1}(\vartheta_2)\right)}{\Gamma(\ell)}}_{\text{by using (16)}} \int_{\sigma^{-1}(\vartheta_1)}^{\sigma^{-1}(\vartheta_2)} \sigma'(\varepsilon)(\vartheta_2 - \sigma(\varepsilon))^{\ell-1} (w \circ \sigma)(\varepsilon)(\hbar \circ \sigma)(\varepsilon)d\varepsilon \\
&= \hbar(\vartheta_2) \left({}_{\sigma^{-1}(\vartheta_1)+}\mathcal{J}^{\ell:\sigma}(w \circ \sigma) \right) \left(\sigma^{-1}(\vartheta_2)\right) - w(\vartheta_2) \left({}_{\sigma^{-1}(\vartheta_1)+}\mathcal{J}^{\ell:\sigma}_{w\circ\sigma}(\hbar \circ \sigma) \right) \left(\sigma^{-1}(\vartheta_2)\right) \\
&= \frac{\hbar(\vartheta_2)}{2} \left[\left({}_{\sigma^{-1}(\vartheta_1)+}\mathcal{J}^{\ell:\sigma}(w \circ \sigma) \right) \left(\sigma^{-1}(\vartheta_2)\right) + \left(\mathcal{J}^{\ell:\sigma}_{\sigma^{-1}(\vartheta_2)-}(w \circ \sigma) \right) \left(\sigma^{-1}(\vartheta_1)\right) \right] \\
&- w(\vartheta_2) \left({}_{\sigma^{-1}(\vartheta_1)+}\mathcal{J}^{\ell:\sigma}_{w\circ\sigma}(\hbar \circ \sigma) \right) \left(\sigma^{-1}(\vartheta_2)\right).
\end{aligned}$$

Analogously, one can get

$$\begin{aligned}
\Xi_2 &= \frac{-1}{\Gamma(\ell)} \left(\int_{\varepsilon}^{\sigma^{-1}(\vartheta_2)} \sigma'(x)(\sigma(x) - \vartheta_1)^{\ell-1} (w \circ \sigma)(x)dx \right) (\hbar \circ \sigma)(\varepsilon)d\varepsilon \Bigg|_{t=\sigma^{-1}(\vartheta_1)}^{\sigma^{-1}(\vartheta_2)} \\
&- \frac{1}{\Gamma(\ell)} \int_{\sigma^{-1}(\vartheta_1)}^{\sigma^{-1}(\vartheta_2)} \sigma'(\varepsilon)(\sigma(\varepsilon) - \vartheta_1)^{\ell-1} (w \circ \sigma)(\varepsilon)(\hbar \circ \sigma)(\varepsilon)d\varepsilon \\
&= \left(\frac{1}{\Gamma(\ell)} \int_{\sigma^{-1}(\vartheta_1)}^{\sigma^{-1}(\vartheta_2)} \sigma'(x)(\sigma(x) - \vartheta_1)^{\ell-1} (w \circ \sigma)(x)dx \right) \hbar(\vartheta_1) \\
&- \underbrace{w(\vartheta_1) \frac{(w \circ \sigma)^{-1} \left(\sigma^{-1}(\vartheta_1)\right)}{\Gamma(\ell)}}_{\text{by using (16)}} \int_{\sigma^{-1}(\vartheta_1)}^{\sigma^{-1}(\vartheta_2)} \sigma'(\varepsilon)(\sigma(\varepsilon) - \vartheta_1)^{\ell-1} (w \circ \sigma)(\varepsilon)(\hbar \circ \sigma)(\varepsilon)d\varepsilon \\
&= \frac{\hbar(\vartheta_1)}{2} \left[\left({}_{\sigma^{-1}(\vartheta_1)+}\mathcal{J}^{\ell:\sigma}(w \circ \sigma) \right) \left(\sigma^{-1}(\vartheta_2)\right) + \left(\mathcal{J}^{\ell:\sigma}_{\sigma^{-1}(\vartheta_2)-}(w \circ \sigma) \right) \left(\sigma^{-1}(\vartheta_1)\right) \right] \\
&- w(\vartheta_1) \left({}_{w\circ\sigma}\mathcal{J}^{\ell:\sigma}_{\sigma^{-1}(\vartheta_2)-}(\hbar \circ \sigma) \right) \left(\sigma^{-1}(\vartheta_1)\right).
\end{aligned}$$

Thus, we can deduce

$$\begin{aligned}
\Xi_1 + \Xi_2 &= \frac{\hbar(\vartheta_1) + \hbar(\vartheta_2)}{2} \left[\left({}_{\sigma^{-1}(\vartheta_1)+}\mathcal{J}^{\ell:\sigma}(w \circ \sigma) \right) \left(\sigma^{-1}(\vartheta_2)\right) + \left(\mathcal{J}^{\ell:\sigma}_{\sigma^{-1}(\vartheta_2)-}(w \circ \sigma) \right) \left(\sigma^{-1}(\vartheta_1)\right) \right] \\
&- \left[w(\vartheta_2) \left({}_{\sigma^{-1}(\vartheta_1)+}\mathcal{J}^{\ell:\sigma}_{w\circ\sigma}(\hbar \circ \sigma) \right) \left(\sigma^{-1}(\vartheta_2)\right) + w(\vartheta_1) \left({}_{w\circ\sigma}\mathcal{J}^{\ell:\sigma}_{\sigma^{-1}(\vartheta_2)-}(\hbar \circ \sigma) \right) \left(\sigma^{-1}(\vartheta_1)\right) \right],
\end{aligned}$$

which completes the proof. □

Remark 5. *Particularly, in Lemma 2, if we take:*

(i) $\sigma(x) = x$, *then equality (12) becomes*

$$\frac{\hbar(\vartheta_1)+\hbar(\vartheta_2)}{2}\left[{}^{RL}_{\vartheta_1+}\mathcal{J}^{\ell} w(\vartheta_2) + {}^{RL}\mathcal{J}^{\ell}_{\vartheta_2-} w(\vartheta_1)\right] - \left[w(\vartheta_2)\left({}^{RL}_{\vartheta_1+}\mathcal{J}^{\ell}_{w}\hbar\right)(\vartheta_2) + w(\vartheta_1)\left({}^{RL}_{w}\mathcal{J}^{\ell}_{\vartheta_2-}\hbar\right)(\vartheta_1)\right] = \frac{1}{\Gamma(\ell)}\int_{\vartheta_1}^{\vartheta_2}\left[\int_{\vartheta_1}^{\varepsilon}(\vartheta_2-x)^{\ell-1}w(x)dx - \int_{\varepsilon}^{\vartheta_2}(x-\vartheta_1)^{\ell-1}w(x)dx\right]\hbar'(\varepsilon)d\varepsilon, \quad (23)$$

where ${}^{RL}_{\vartheta_1+}\mathcal{J}^{\ell}_{w}$ *and* ${}^{RL}_{w}\mathcal{J}^{\ell}_{\vartheta_2-}$ *are as defined in Remark 3.*

(ii) $\sigma(x) = x$ *and* $w(x) = 1$, *then equality (12) becomes*

$$\frac{\hbar(\vartheta_1)+\hbar(\vartheta_2)}{2} - \frac{\Gamma(\ell+1)}{2(\vartheta_2-\vartheta_1)^{\ell}}\left[{}^{RL}_{\vartheta_1+}\mathcal{J}^{\ell} w(\vartheta_2) + {}^{RL}\mathcal{J}^{\ell}_{\vartheta_2-} w(\vartheta_1)\right] = \frac{\vartheta_2-\vartheta_1}{2}\int_0^1\left[\varepsilon^{\ell} - (1-\varepsilon)^{\ell}\right]\hbar'(\varepsilon\vartheta_1 + (1-\varepsilon)\vartheta_2)d\varepsilon,$$

which is already established in ([11], lemma 2).

(iii) $\sigma(x) = x$, $w(x) = 1$ *and* $\ell = 1$, *we obtain*

$$\frac{\hbar(\vartheta_1)+\hbar(\vartheta_2)}{2} - \frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\hbar(x)dx = \frac{\vartheta_2-\vartheta_1}{2}\int_0^1[1-2\varepsilon]\,\hbar'(\varepsilon\vartheta_1+(1-\varepsilon)\vartheta_2)d\varepsilon, \quad (24)$$

which is already established in ([38] lemma 2.1).

Remark 6. *From Remark 5 (i), we can observe that our result Lemma 2 is essentially a reformulation of the result of ([34], lemma 2.4), even though it is about weighted fractional and RL-fractional integrals rather than RL-fractional integrals explicitly. Additionally, from Remark 5 (ii) and (iii), we can observe that the results of ([11], lemma 2) and ([38], lemma 2.1) are basically particular cases of our result Lemma 2.*

3. Main Results

In view of Lemma 2, we can obtain the following HHF inequalities.

Theorem 2. *Let* $\hbar : [\vartheta_1, \vartheta_2] \subseteq [0,\infty) \to \mathbb{R}$ *be an* L^1 *function with* $\hbar' \in L^1$ *and* $0 \le \vartheta_1 < \vartheta_2$, *and* $w : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ *be an integrable, positive and weighted symmetric function with respect to* $\frac{\vartheta_1+\vartheta_2}{2}$. *If* $|\hbar'|$ *is convex on* $[\vartheta_1, \vartheta_2]$, σ *is an increasing and positive function on* $[\vartheta_1, \vartheta_2)$, *and* $\sigma'(x)$ *is continuous on* $(\vartheta_1, \vartheta_2)$. *Then, we have for* $\ell > 0$:

$$|\Xi_1 + \Xi_2| \le \frac{\|w\circ\sigma\|_\infty}{(\vartheta_2-\vartheta_1)\Gamma(\ell+1)}\left[A_\sigma(\ell;\vartheta_1,\vartheta_2)|\hbar'(\vartheta_1)| + B_\sigma(\ell;\vartheta_1,\vartheta_2)|\hbar'(\vartheta_2)|\right], \quad (25)$$

where Ξ_1 *and* Ξ_2 *are defined as in the proof of Lemma 2, and*

$$A_\sigma(\ell;\vartheta_1,\vartheta_2) := \vartheta_2\left[\frac{(\vartheta_2-\vartheta_1)^{\ell+1} - \left(\vartheta_2 - \sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)\right)^{\ell+1}}{\ell+1} - \frac{\left(\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right) - \vartheta_1\right)^{\ell+1}}{\ell+1}\right] - \vartheta_2\left[\frac{(\vartheta_2-\vartheta_1)^{\ell+1} - \left(\vartheta_2 - \sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)\right)^{\ell+1}}{\ell+1}\right] + \left[\frac{(\vartheta_2-\vartheta_1)^{\ell+2} - \left(\vartheta_2 - \sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)\right)^{\ell+2}}{\ell+2}\right] + \frac{\vartheta_1}{\ell+1}\left(\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right) - \vartheta_1\right)^{\ell+1} + \frac{1}{\ell+2}\left(\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right) - \vartheta_1\right)^{\ell+2}$$

$$+\vartheta_2\left[\frac{(\vartheta_2-\vartheta_1)^{\ell+1}-\left(\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)-\vartheta_1\right)^{\ell+1}}{\ell+1}-\frac{\left(\vartheta_2-\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)\right)^{\ell+1}}{\ell+1}\right]$$
$$-\vartheta_1\left[\frac{(\vartheta_2-\vartheta_1)^{\ell+1}-\left(\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)-\vartheta_1\right)^{\ell+1}}{\ell+1}\right]-\left[\frac{(\vartheta_2-\vartheta_1)^{\ell+2}-\left(\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)-\vartheta_1\right)^{\ell+2}}{\ell+2}\right]$$
$$+\frac{\vartheta_2}{\ell+1}\left(\vartheta_2-\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)\right)^{\ell+1}-\frac{1}{\ell+2}\left(\vartheta_2-\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)\right)^{\ell+2},$$

and

$$B_\sigma(\ell;\vartheta_1,\vartheta_2):=\vartheta_1\left[\frac{\left(\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)-\vartheta_1\right)^{\ell+1}}{\ell+1}-\frac{(\vartheta_2-\vartheta_1)^{\ell+2}-\left(\vartheta_2-\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)\right)^{\ell+2}}{\ell+2}\right.$$
$$\left.-\frac{(\vartheta_2-\vartheta_1)^{\ell+1}-\left(\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)-\vartheta_1\right)^{\ell+1}}{\ell+1}+\frac{\left(\vartheta_2-\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)\right)^{\ell+1}}{\ell+1}\right]$$
$$+\frac{\vartheta_2}{\ell+1}\left[(\vartheta_2-\vartheta_1)^{\ell+1}-\left(\vartheta_2-\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)\right)^{\ell+1}\right]$$
$$-\frac{1}{\ell+2}\left[(\vartheta_2-\vartheta_1)^{\ell+2}-\left(\vartheta_2-\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)\right)^{\ell+2}\right]$$

$$-\frac{\vartheta_1}{\ell+1}\left(\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)-\vartheta_1\right)^{\ell+1}-\frac{1}{\ell+2}\left(\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)-\vartheta_1\right)^{\ell+2}$$
$$+\frac{\vartheta_1}{\ell+1}\left[(\vartheta_2-\vartheta_1)^{\ell+1}-\left(\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)-\vartheta_1\right)^{\ell+1}\right]$$
$$+\frac{1}{\ell+2}\left[(\vartheta_2-\vartheta_1)^{\ell+2}-\left(\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)-\vartheta_1\right)^{\ell+2}\right]$$
$$-\frac{\vartheta_2}{\ell+1}\left(\vartheta_2-\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)\right)^{\ell+1}+\frac{1}{\ell+2}\left(\vartheta_2-\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)\right)^{\ell+2}.$$

Proof. By using Lemma 2 and properties of modulus, we have

$$|\Xi_1+\Xi_2|\leq\frac{1}{\Gamma(\ell)}\int_{\sigma^{-1}(\vartheta_1)}^{\sigma^{-1}(\vartheta_2)}\left|\int_{\sigma^{-1}(\vartheta_1)}^{\varepsilon}\sigma'(x)(\vartheta_2-\sigma(x))^{\ell-1}(w\circ\sigma)(x)dx\right.$$
$$\left.-\int_{\varepsilon}^{\sigma^{-1}(\vartheta_2)}\sigma'(x)(\sigma(x)-\vartheta_1)^{\ell-1}(w\circ\sigma)(x)dx\right|\left|(\hbar'\circ\sigma)(\varepsilon)\right||\sigma'(\varepsilon)|d\varepsilon. \quad (26)$$

Since $|\hbar'|$ is convex on $[\vartheta_1,\vartheta_2]$, we get for $\varepsilon\in\left[\sigma^{-1}(\vartheta_1),\sigma^{-1}(\vartheta_2)\right]$:

$$\left|(\hbar'\circ\sigma)(\varepsilon)\right|=\left|\hbar'\left(\frac{\vartheta_2-\sigma(\varepsilon)}{\vartheta_2-\vartheta_1}\vartheta_1+\frac{\sigma(\varepsilon)-\vartheta_1}{\vartheta_2-\vartheta_1}\vartheta_2\right)\right|\leq\frac{\vartheta_2-\sigma(\varepsilon)}{\vartheta_2-\vartheta_1}|\hbar'(\vartheta_1)|+\frac{\sigma(\varepsilon)-\vartheta_1}{\vartheta_2-\vartheta_1}|\hbar'(\vartheta_2)|. \quad (27)$$

Additionally, since $w:[\vartheta_1,\vartheta_2]\to\mathbb{R}$ is symmetric weighted function with respect to $\frac{\vartheta_1+\vartheta_2}{2}$, so we can write

$$
\begin{aligned}
&\int_{\varepsilon}^{\sigma^{-1}(\vartheta_2)} \sigma'(x)(\sigma(x)-\vartheta_1)^{\ell-1}(w\circ\sigma)(x)dx \\
&= \int_{\sigma^{-1}(\vartheta_1)}^{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)-\varepsilon} \sigma'(x)(\vartheta_2-\sigma(x))^{\ell-1}(w\circ\sigma)(\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)-x)dx \\
&= \int_{\sigma^{-1}(\vartheta_1)}^{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)-\varepsilon} \sigma'(x)(\vartheta_2-\sigma(x))^{\ell-1}(w\circ\sigma)(x)dx.
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
&\left|\int_{\sigma^{-1}(\vartheta_1)}^{\varepsilon} \sigma'(x)(\vartheta_2-\sigma(x))^{\ell-1}(w\circ\sigma)(x)dx - \int_{\varepsilon}^{\sigma^{-1}(\vartheta_2)} \sigma'(x)(\sigma(x)-\vartheta_1)^{\ell-1}(w\circ\sigma)(x)dx\right| \\
&= \left|\int_{\varepsilon}^{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)-\varepsilon} \sigma'(x)(\vartheta_2-\sigma(x))^{\ell-1}(w\circ\sigma)(x)dx\right| \\
&\leq \begin{cases} \int_{\varepsilon}^{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)-\varepsilon} |\sigma'(x)| \left|(\vartheta_2-\sigma(x))^{\ell-1}(w\circ\sigma)(x)\right| dx, & \varepsilon \in \left[\sigma^{-1}(\vartheta_1), \frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right]; \\ \int_{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)-\varepsilon}^{\varepsilon} |\sigma'(x)| \left|(\vartheta_2-\sigma(x))^{\ell-1}(w\circ\sigma)(x)\right| dx, & \varepsilon \in \left[\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}, \sigma^{-1}(\vartheta_2)\right]. \end{cases}
\end{aligned} \tag{28}
$$

By applying the inequalities (26)–(28), we have

$$
\begin{aligned}
|\Xi_1+\Xi_2| \leq{}& \frac{1}{\Gamma(\ell)} \int_{\sigma^{-1}(\vartheta_1)}^{\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}} \left(\int_{\varepsilon}^{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)-\varepsilon} |\sigma'(x)| \left|(\vartheta_2-\sigma(x))^{\ell-1}(w\circ\sigma)(x)\right| dx \right) \\
&\times \left(\frac{\vartheta_2-\sigma(\varepsilon)}{\vartheta_2-\vartheta_1}|\hbar'(\vartheta_1)| + \frac{\sigma(\varepsilon)-\vartheta_1}{\vartheta_2-\vartheta_1}|\hbar'(\vartheta_2)| \right) \sigma'(\varepsilon)d\varepsilon \\
&+ \frac{1}{\Gamma(\ell)} \int_{\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}}^{\sigma^{-1}(\vartheta_2)} \left(\int_{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)-\varepsilon}^{\varepsilon} |\sigma'(x)| \left|(\vartheta_2-\sigma(x))^{\ell-1}(w\circ\sigma)(x)\right| dx \right) \\
&\times \left(\frac{\vartheta_2-\sigma(\varepsilon)}{\vartheta_2-\vartheta_1}|\hbar'(\vartheta_1)| + \frac{\sigma(\varepsilon)-\vartheta_1}{\vartheta_2-\vartheta_1}|\hbar'(\vartheta_2)| \right) \sigma'(\varepsilon)d\varepsilon.
\end{aligned} \tag{29}
$$

After simple calculations of integrals arising from inequality (29), we can obtain the desired result (25). □

Remark 7. *Particularly, in Theorem 2, if we take*

(i) $\sigma(x)=x$, *we have*

$$
\begin{aligned}
&\Bigg| \frac{\hbar(\vartheta_1)+\hbar(\vartheta_2)}{2}\left[{}^{RL}_{\vartheta_1+}\mathcal{J}^{\ell} w(\vartheta_2) + {}^{RL}\mathcal{J}^{\ell}_{\vartheta_2-} w(\vartheta_1)\right] \\
&\qquad - \left[w(\vartheta_2)\left({}^{RL}_{\vartheta_1+}\mathcal{J}^{\ell}_{w}\hbar\right)(\vartheta_2) + w(\vartheta_1)\left({}^{RL}_{w}\mathcal{J}^{\ell}_{\vartheta_2-}\hbar\right)(\vartheta_1) \right] \Bigg| \\
&\qquad\qquad \leq \frac{\|w\|_{\infty}(\vartheta_2-\vartheta_1)^{\ell+1}}{\Gamma(\ell+2)}\left(1-\frac{1}{2^{\ell}}\right)\left[|\hbar'(\vartheta_1)|+|\hbar'(\vartheta_2)|\right].
\end{aligned} \tag{30}
$$

(ii) $\sigma(x)=x$ *and* $w(x)=1$, *we get*

$$
\begin{aligned}
&\left| \frac{\hbar(\vartheta_1)+\hbar(\vartheta_2)}{2} - \frac{\Gamma(\ell+1)}{2(\vartheta_2-\vartheta_1)^{\ell}}\left[{}^{RL}_{\vartheta_1+}\mathcal{J}^{\ell}\hbar(\vartheta_2) + {}^{RL}\mathcal{J}^{\ell}_{\vartheta_2-}\hbar(\vartheta_1)\right] \right| \\
&\qquad \leq \frac{\vartheta_2-\vartheta_1}{2(\ell+1)}\left(1-\frac{1}{2^{\ell}}\right)\left[|\hbar'(\vartheta_1)|+|\hbar'(\vartheta_2)|\right],
\end{aligned} \tag{31}
$$

which is already established in ([11] Theorem 3).

(iii) $\sigma(x)=x$, $w(x)=1$ *and* $\ell=1$, *we obtain*

$$\left|\frac{\hbar(\vartheta_1)+\hbar(\vartheta_2)}{2}-\frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\hbar(x)dx\right|\leq\frac{\vartheta_2-\vartheta_1}{8}\left[|\hbar'(\vartheta_1)|+|\hbar'(\vartheta_2)|\right], \tag{32}$$

which is already established in ([38] Theorem 2.2).

Remark 8. *Again, from Remark 7 (i), we can observe that our result Lemma 2 is essentially a reformulation of the result of ([34], Theorem 2.8), even though it is about weighted fractional and RL-fractional integrals rather than RL-fractional integrals explicitly. In addition, from Remark 7 (ii) and (iii), we can observe that the results of ([11], Theorem 3) and ([38], Theorem 2.2) are basically particular cases of our result Lemma 2.*

Theorem 3. *Let* $\hbar:[\vartheta_1,\vartheta_2]\subseteq[0,\infty)\to\mathbb{R}$ *be an* L^1 *function with* $\hbar'\in L^1$ *and* $0\leq\vartheta_1<\vartheta_2$, *and* $w:[\vartheta_1,\vartheta_2]\to\mathbb{R}$ *be an integrable, positive and weighted symmetric function with respect to* $\frac{\vartheta_1+\vartheta_2}{2}$. *If* $|\hbar'|^q$, $q\geq 1$ *is convex on* $[\vartheta_1,\vartheta_2]$, σ *is an increasing and positive function on* $[\vartheta_1,\vartheta_2)$ *and* $\sigma'(x)$ *is continuous on* $(\vartheta_1,\vartheta_2)$. *Then, we have for* $\ell>0$:

$$|\Xi_1+\Xi_2|\leq\frac{\|w\circ\sigma\|_\infty}{(\vartheta_2-\vartheta_1)^{\frac{1}{q}}\Gamma(\ell+1)}\left(C_\sigma(\ell;\vartheta_1,\vartheta_2)\right)^{1-\frac{1}{q}}\times\left[D_\sigma(\ell;\vartheta_1,\vartheta_2)|\hbar'(\vartheta_1)|^q+E_\sigma(\ell;\vartheta_1,\vartheta_2)|\hbar'(\vartheta_2)|^q\right]^{\frac{1}{q}}, \tag{33}$$

where

$$C_\sigma(\ell;\vartheta_1,\vartheta_2):=\frac{1}{\ell}\left[\frac{(\vartheta_2-\vartheta_1)^{\ell+1}-\left(\vartheta_2-\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)\right)^{\ell+1}}{\ell+1}-C_\sigma^{(1)}(\ell;\vartheta_1,\vartheta_2)+C_\sigma^{(2)}(\ell;\vartheta_1,\vartheta_2)-\frac{\left(\vartheta_2-\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)\right)^{\ell+1}}{\ell+1}\right];$$

$$C_\sigma^{(1)}(\ell;\vartheta_1,\vartheta_2):=\int_{\vartheta_1}^{\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)}\left[\vartheta_2-\sigma\left(\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)-\sigma^{-1}(x)\right)\right]^\ell dx;$$

$$C_\sigma^{(2)}(\ell;\vartheta_1,\vartheta_2):=\int_{\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)}^{\vartheta_2}\left[\vartheta_2-\sigma\left(\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)-\sigma^{-1}(x)\right)\right]^\ell dx;$$

$$D_\sigma(\ell;\vartheta_1,\vartheta_2):=\frac{1}{\ell}\left[\frac{(\vartheta_2-\vartheta_1)^{\ell+2}-\left(\vartheta_2-\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)\right)^{\ell+2}}{\ell+2}-\vartheta_2\,C_\sigma^{(1)}(\ell;\vartheta_1,\vartheta_2)-D_\sigma^{(1)}(\ell;\vartheta_1,\vartheta_2)+\vartheta_2\,C_\sigma^{(2)}(\ell;\vartheta_1,\vartheta_2)-\frac{\left(\vartheta_2-\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)\right)^{\ell+2}}{\ell+2}-D_\sigma^{(2)}(\ell;\vartheta_1,\vartheta_2)\right];$$

$$D_\sigma^{(1)}(\ell;\vartheta_1,\vartheta_2):=\int_{\vartheta_1}^{\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)}x\left[\vartheta_2-\sigma\left(\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)-\sigma^{-1}(x)\right)\right]^\ell dx;$$

$$D_\sigma^{(2)}(\ell;\vartheta_1,\vartheta_2):=\int_{\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)}^{\vartheta_2}x\left[\vartheta_2-\sigma\left(\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)-\sigma^{-1}(x)\right)\right]^\ell dx,$$

and

$$E_\sigma(\ell;\vartheta_1,\vartheta_2) := \frac{1}{\ell}\Bigg\{\frac{\vartheta_2}{\ell+1}\left[(\vartheta_2-\vartheta_1)^{\ell+1}-\left(\vartheta_2-\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)\right)^{\ell+1}\right]$$
$$-\frac{1}{\ell+2}\left[(\vartheta_2-\vartheta_1)^{\ell+2}-\left(\vartheta_2-\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)\right)^{\ell+2}\right]$$
$$-\frac{\vartheta_1}{\ell+1}\left[(\vartheta_2-\vartheta_1)^{\ell+1}-\left(\vartheta_2-\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)\right)^{\ell+1}\right]-D_\sigma^{(1)}(\ell;\vartheta_1,\vartheta_2)+\vartheta_1 C_\sigma^{(1)}(\ell;\vartheta_1,\vartheta_2)$$
$$+D_\sigma^{(2)}(\ell;\vartheta_1,\vartheta_2)+\vartheta_1\left[\frac{1}{\ell+1}\left(\vartheta_2-\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)\right)^{\ell+1}-C_\sigma^{(2)}(\ell;\vartheta_1,\vartheta_2)\right]$$
$$-\frac{\vartheta_2}{\ell+1}\left(\vartheta_2-\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)\right)^{\ell+1}+\frac{1}{\ell+2}\left(\vartheta_2-\sigma\left(\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}\right)\right)^{\ell+2}\Bigg\}.$$

Proof. By using Lemma 2, the well–known power mean inequality, inequality (28), convexity of $|\hbar'|^q$ and properties of modulus, we can deduce

$$|\Xi_1+\Xi_2| \leq \frac{1}{\Gamma(\ell)}\left[\int_{\sigma^{-1}(\vartheta_1)}^{\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}}\left(\int_{\varepsilon}^{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)-\varepsilon}\left|\sigma'(x)(\vartheta_2-\sigma(x))^{\ell-1}(w\circ\sigma)(x)\right|dx\right)\sigma'(\varepsilon)d\varepsilon\right.$$
$$\left.+\int_{\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}}^{\sigma^{-1}(\vartheta_2)}\left(\int_{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)-\varepsilon}^{\varepsilon}\left|\sigma'(x)(\vartheta_2-\sigma(x))^{\ell-1}(w\circ\sigma)(x)\right|dx\right)\sigma'(\varepsilon)d\varepsilon\right]^{1-\frac{1}{q}}$$

$$\times\left[\int_{\sigma^{-1}(\vartheta_1)}^{\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}}\left(\int_{\varepsilon}^{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)-\varepsilon}\left|\sigma'(x)(\vartheta_2-\sigma(x))^{\ell-1}(w\circ\sigma)(x)\right|dx\right)|(\hbar'\circ\sigma)(\varepsilon)|^q\sigma'(\varepsilon)d\varepsilon\right.$$
$$\left.+\int_{\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}}^{\sigma^{-1}(\vartheta_2)}\left(\int_{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)-\varepsilon}^{\varepsilon}\left|\sigma'(x)(\vartheta_2-\sigma(x))^{\ell-1}(w\circ\sigma)(x)\right|dx\right)|(\hbar'\circ\sigma)(\varepsilon)|^q\sigma'(\varepsilon)d\varepsilon\right]^{\frac{1}{q}}$$

$$\leq \frac{\|w\circ\sigma\|_\infty}{\Gamma(\ell)}\left[\int_{\sigma^{-1}(\vartheta_1)}^{\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}}\left(\int_{\varepsilon}^{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)-\varepsilon}\left|\sigma'(x)\right|(\vartheta_2-\sigma(x))^{\ell-1}dx\right)\sigma'(\varepsilon)d\varepsilon\right.$$
$$\left.+\int_{\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}}^{\sigma^{-1}(\vartheta_2)}\left(\int_{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)-\varepsilon}^{\varepsilon}\left|\sigma'(x)\right|(\vartheta_2-\sigma(x))^{\ell-1}dx\right)\sigma'(\varepsilon)d\varepsilon\right]^{1-\frac{1}{q}}$$

$$\times\left[\int_{\sigma^{-1}(\vartheta_1)}^{\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}}\left(\int_{\varepsilon}^{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)-\varepsilon}\left|\sigma'(x)\right|(\vartheta_2-\sigma(x))^{\ell-1}dx\right)|(\hbar'\circ\sigma)(\varepsilon)|^q\sigma'(\varepsilon)d\varepsilon\right.$$
$$\left.+\int_{\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}}^{\sigma^{-1}(\vartheta_2)}\left(\int_{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)-\varepsilon}^{\varepsilon}\left|\sigma'(x)\right|(\vartheta_2-\sigma(x))^{\ell-1}dx\right)|(\hbar'\circ\sigma)(\varepsilon)|^q\sigma'(\varepsilon)d\varepsilon\right]^{\frac{1}{q}}$$

$$
\leq \frac{\|w\circ\sigma\|_\infty}{(\vartheta_2-\vartheta_1)^{\frac{1}{q}}\Gamma(\ell)}\left(C_\sigma(\ell;\vartheta_1,\vartheta_2)\right)^{1-\frac{1}{q}}\left[\int_{\sigma^{-1}(\vartheta_1)}^{\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}}\left(\int_{\varepsilon}^{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)-\varepsilon}\left|\sigma'(x)\right|(\vartheta_2-\sigma(x))^{\ell-1}dx\right)\right.
$$
$$
\times\left(\frac{\vartheta_2-\sigma(\varepsilon)}{\vartheta_2-\vartheta_1}|\hbar'(\vartheta_1)|^q+\frac{\sigma(\varepsilon)-\vartheta_1}{\vartheta_2-\vartheta_1}|\hbar'(\vartheta_2)|^q\right)\sigma'(\varepsilon)d\varepsilon
$$
$$
+\int_{\frac{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)}{2}}^{\sigma^{-1}(\vartheta_2)}\left(\int_{\sigma^{-1}(\vartheta_1)+\sigma^{-1}(\vartheta_2)-\varepsilon}^{\varepsilon}\left|\sigma'(x)\right|(\vartheta_2-\sigma(x))^{\ell-1}dx\right)
$$
$$
\left.\times\left(\frac{\vartheta_2-\sigma(\varepsilon)}{\vartheta_2-\vartheta_1}|\hbar'(\vartheta_1)|^q+\frac{\sigma(\varepsilon)-\vartheta_1}{\vartheta_2-\vartheta_1}|\hbar'(\vartheta_2)|^q\right)\sigma'(\varepsilon)d\varepsilon\right]^{\frac{1}{q}}. \quad (34)
$$

After simple calculations of integrals arising from inequality (34), one can obtain the desired result (33). □

Remark 9. *Particularly, in Theorem 3, if we take:*

(i) $\sigma(x)=x$, *we get*

$$
\left|\frac{\hbar(\vartheta_1)+\hbar(\vartheta_2)}{2}\left[{}^{RL}_{\vartheta_1+}\mathcal{J}^{\ell}w(\vartheta_2)+{}^{RL}\mathcal{J}^{\ell}_{\vartheta_2-}w(\vartheta_1)\right]\right.
$$
$$
\left.-\left[w(\vartheta_2)\left({}^{RL}_{\vartheta_1+}\mathcal{J}^{\ell}_{w}\hbar\right)(\vartheta_2)+w(\vartheta_1)\left({}^{RL}_{w}\mathcal{J}^{\ell}_{\vartheta_2-}\hbar\right)(\vartheta_1)\right]\right|
$$
$$
\leq\frac{\|w\|_\infty}{(\vartheta_2-\vartheta_1)^{\frac{1}{q}}\Gamma(\ell+1)}\left(C(\ell;\vartheta_1,\vartheta_2)\right)^{1-\frac{1}{q}}\left[D(\ell;\vartheta_1,\vartheta_2)|\hbar'(\vartheta_1)|^q+E(\ell;\vartheta_1,\vartheta_2)|\hbar'(\vartheta_2)|^q\right]^{\frac{1}{q}}, \quad (35)
$$

where

$$
C(\ell;\vartheta_1,\vartheta_2):=\frac{2(\vartheta_2-\vartheta_1)^{\ell+1}}{\ell(\ell+1)}\left(1-\frac{1}{2^\ell}\right),
$$
$$
D(\ell;\vartheta_1,\vartheta_2):=\frac{1}{\ell}\left[\frac{1}{\ell+1}\left((\vartheta_2-\vartheta_1)^{\ell+2}-2\vartheta_2\left(\frac{\vartheta_2-\vartheta_1}{2}\right)^{\ell+1}\right)-\frac{2}{\ell+2}\left(\frac{\vartheta_2-\vartheta_1}{2}\right)^{\ell+2}\right],
$$

and

$$
E(\ell;\vartheta_1,\vartheta_2):=\frac{(\vartheta_2-\vartheta_1)^{\ell+2}}{\ell(\ell+1)}\left(1-\frac{1}{2^\ell}\right).
$$

(ii) $\sigma(x)=x$ *and* $w(x)=1$, *we get*

$$
\left|\frac{\hbar(\vartheta_1)+\hbar(\vartheta_2)}{2}-\frac{\Gamma(\ell+1)}{2(\vartheta_2-\vartheta_1)^\ell}\left[{}^{RL}_{\vartheta_1+}\mathcal{J}^{\ell}\hbar(\vartheta_2)+{}^{RL}\mathcal{J}^{\ell}_{\vartheta_2-}\hbar(\vartheta_1)\right]\right|
$$
$$
\leq\frac{1}{(\vartheta_2-\vartheta_1)^{\frac{1}{q}}\Gamma(\ell)}\left(C(\ell;\vartheta_1,\vartheta_2)\right)^{1-\frac{1}{q}}\left[D(\ell;\vartheta_1,\vartheta_2)|\hbar'(\vartheta_1)|^q+E(\ell;\vartheta_1,\vartheta_2)|\hbar'(\vartheta_2)|^q\right]^{\frac{1}{q}}, \quad (36)
$$

where $C(\ell;\vartheta_1,\vartheta_2)$, $D(\ell;\vartheta_1,\vartheta_2)$ *and* $E(\ell;\vartheta_1,\vartheta_2)$ *are defined as above.*

(iii) $\sigma(x) = x$, $w(x) = 1$ *and* $\ell = 1$, *we obtain*

$$\left|\frac{\hbar(\vartheta_1)+\hbar(\vartheta_2)}{2} - \frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\hbar(x)dx\right| \leq \frac{1}{(\vartheta_2-\vartheta_1)^{\frac{1}{q}}}\left(\frac{(\vartheta_2-\vartheta_1)^2}{2}\right)^{1-\frac{1}{q}} \times \left[\frac{(\vartheta_2-\vartheta_1)^2}{12}(2\vartheta_2-5\vartheta_1)|\hbar'(\vartheta_1)|^q + \frac{3(\vartheta_2-\vartheta_1)^3}{4}|\hbar'(\vartheta_2)|^q\right]^{\frac{1}{q}}. \quad (37)$$

Remark 10. *The specific results are different from those obtained in [11,34,38] according to Remark 9.*

4. Discussion

We have considered the weighted fractional operators. In our present investigation, we have established new fractional HHF integral inequalities involving the weighted fractional operators associated with positive symmetric functions. The HHF fractional integral inequality (7) has been applied to other class of convex functions, such as p-convex functions [39], generalized convex functions [40], (η_1, η_2)-convex functions [41] and many others that can be found in the literature. Thus, the results obtained here can be also be applied to the above class of convex functions.

It is worthwhile to mention that there are three well-known versions of fractional Hermite–Hadamard integral inequalities. The first version was established by Sarikaya et al. in [11] and their result is given in (3). The other versions consist of

$$\hbar\left(\frac{\vartheta_1+\vartheta_2}{2}\right) \leq \frac{2^{\ell-1}\Gamma(\ell+1)}{(\vartheta_2-\vartheta_1)^\ell}\left[{}^{RL}\mathcal{J}^{\ell}_{\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+}\hbar(\vartheta_2) + {}^{RL}\mathcal{J}^{\ell}_{\left(\frac{\vartheta_1+\vartheta_2}{2}\right)-}\hbar(\vartheta_1)\right] \leq \frac{\hbar(\vartheta_1)+\hbar(\vartheta_2)}{2}, \quad (38)$$

and

$$\hbar\left(\frac{\vartheta_1+\vartheta_2}{2}\right) \leq \frac{2^{\ell-1}\Gamma(\ell+1)}{(\vartheta_2-\vartheta_1)^\ell}\left[{}^{RL}\mathcal{J}^{\ell}_{\vartheta_1+}\hbar\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + {}^{RL}\mathcal{J}^{\ell}_{\vartheta_2-}\hbar\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right] \leq \frac{\hbar(\vartheta_1)+\hbar(\vartheta_2)}{2}; \quad (39)$$

these were already established by Sarikaya and Yaldiz [42], and Mohammed and Brevik [1], respectively. We believe that the results in this study are very generic and can be extended to give further potentially interesting and useful integral inequalities involving other versions of fractional integral inequalities (38) and (39).

5. Conclusions

Integral inequality forms a significant branch of mathematical analysis, which has been combined with all models of fractional calculus but never before with weighted fractional calculus models. For this reason, in this study we have considered the Hermite–Hadamard–Fejer integral inequalities in the context of fractional calculus with positive weighted symmetric function kernels.

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