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On (ϕ, ψ)-Metric Spaces with Applications

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Abstract: The aim of this article is to introduce the notion of a (ϕ, ψ)-metric space, which extends the metric space concept. In these spaces, the symmetry property is preserved. We present a natural topology $\tau_{(\phi,\psi)}$ in such spaces and discuss their topological properties. We also establish the Banach contraction principle in the context of (ϕ, ψ)-metric spaces and we illustrate the significance of our main theorem by examples. Ultimately, as applications, the existence of a unique solution of Fredholm type integral equations in one and two dimensions is ensured and an example in support is given.

Keywords: (ϕ, ψ)-metric space; topological property; fixed point; Fredholm integral equation

MSC: 46T99; 47H10; 54H25

1. Introduction

Fixed-point technique offers a focal concept with many diverse applications in nonlinear analysis. It is an important theoretical tool in many fields and various disciplines such as topology, game theory, optimal control, artificial intelligence, logic programming, dynamical systems (and chaos), functional analysis, differential equations, and economics.

Recently, many important extensions (or generalizations) of the metric space notion have been investigated (as examples, see References [1–5]). In 1989, the class of of b-metric spaces has been introduced by Bakhtin [6], that is, the classical triangle inequality is relaxed in the right-hand term by a parameter $s \geq 1$. This class was formally defined by Czerwik [7] (see also References [8,9]) in 1993 with a view of generalizing the Banach contraction principle (BCP). The above class has been generalized by Mlaiki et al. [10] and Abdeljawad et al. [11], by introduction of control functions (see also Reference [12]). Fagin et al. [13] presented the notion of an s-relaxed metric. A 2-metric introduced by Gahler [14] is a function defined on $\Im \times \Im \times \Im$ (where $\Im$ is a nonempty set), and verifies some particular conditions. Gahler showed that a 2-metric generalizes the classical concept of a metric. While, different authors established that no relations exist between these two notions (see Reference [15]). Mustafa and Sims [16] initiated the class of G-metric spaces. Branciari [17] gave a new generalization of the metric concept by replacing the triangle inequality with a more general one involving four points. Partial metric spaces have been introduced by Matthews [18] (for related works, see References [19–21]) as a part of the discussion of denotational semantics in dataflow networks. Jleli and Samet [22] introduced the notion of a JS-metric, where the triangle inequality is replaced by a lim sup-condition. Very recently, Jleli and Samet [23] also introduced the concept of F -metric spaces. For this, denote by Ξ the set of functions $F:(0,\infty)\to(-\infty,\infty)$ satisfying the following conditions:

(F_1) F is non-decreasing;
(F_2) for each sequence $\{t_n\}\subset(0,\infty)$;

$$\lim_{n\to+\infty} F(t_n) = -\infty \, if \text{ and only } if \lim_{n\to+\infty} t_n = 0.$$

Definition 1 ([23]). *Let $\Im$ be a nonempty set and $D:\Im\times\Im\to[0,\infty)$ be a function. Assume that there exist a function $F\in\Xi$ and $\alpha\in[0,\infty)$ such that for $\sigma,\varsigma\in\Im$,*

(D_1) *$D(\sigma,\varsigma)=0$ if and only if $\sigma=\varsigma$;*
(D_2) *$D(\sigma,\varsigma)=D(\varsigma,\sigma)$;*
(D_3) *for each $n\in\mathbb{N}$ with $n\geq 2$, and for each $\{u_i\}_{i=1}^n\subset\Im$ with $(u_1,u_n)=(\sigma,\varsigma)$, we have,*

$$D(\sigma,\varsigma)>0\Rightarrow F(D(\sigma,\varsigma))\leq F\left(\sum_{i=1}^{n-1}D(u_i,u_{i+1})\right)+\alpha.$$

Then D is said to be a F-metric on $\Im$. The pair $(\Im,D)$ is said to be a F-metric space.

In this paper, we present a new generalization of the concept of metric spaces, namely, a $(\phi,\psi)-$metric space. We compare our concept with the existing generalizations in the literature. Next, we give a natural topology $\tau_{\phi,\psi}$ on these spaces, and study their topological properties. Moreover, we establish the BCP in the setting of (ϕ,ψ)-metric spaces. As applications, we ensure the existence of a unique solution of two Fredholm type integral equations.

2. On $(\phi,\psi)-$Metric Spaces

Definition 2. *Let D be the set of functions $\phi:(0,\infty)\to(0,\infty)$ such that:*

(ϕ_1) ϕ is non-decreasing;
(ϕ_2) for each positive sequence $\{t_n\}$,

$$\lim_{n\to\infty}\phi(t_n)=0 \text{ if and only if } \lim_{n\to\infty}t_n=0.$$

Let $\psi:(0,\infty)\to(0,\infty)$ be such that:

(i) *ψ is monotone increasing, that is, $\sigma<\varsigma\Rightarrow\psi(\sigma)\leq\psi(\varsigma)$;*
(ii) *$\psi(t)\leq t$ for every $t>0$.*

We denote by Ψ the set of functions satisfying (i)–(ii).

Now, we introduce the notion of (ϕ,ψ)-metric spaces.

Definition 3. *Let $\Im$ be a nonempty set and $d:\Im\times\Im\to[0,\infty)$ be a function. Assume that there exist two functions $\psi\in\Psi$ and $\phi\in D$ such that for all $\sigma,\varsigma\in\Im$, the following hold:*

(d_1) *$d(\sigma,\varsigma)=0$ if and only if $\sigma=\varsigma$;*
(d_2) *$d(\sigma,\varsigma)=d(\varsigma,\sigma)$;*
(d_3) *for each $n\in\mathbb{N}$, $n\geq 2$, and for each $\{\omega_i\}_{i=1}^n\subset\Im$ with $(\omega_1,\omega_n)=(\sigma,\varsigma)$, we have*

$$d(\sigma,\varsigma)>0\Rightarrow\phi(d(\sigma,\varsigma))\leq\psi\left(\phi\left(\sum_{i=1}^{n-1}d(\omega_i,\omega_{i+1})\right)\right).$$

Then d is named as a (ϕ, ψ)-metric on $\Im$. The pair $(\Im, d)$ is called a (ϕ, ψ)-metric space. It is known that property (d_2) states that this metric should measure the distances symmetrically.

Remark 1. *Any metric on $\Im$ is a (ϕ, ψ)-metric on $\Im$. Indeed, if d is a metric on $\Im$, then it satisfies (d_2) and (d_2). On the other hand, by the triangle inequality, for every $(\sigma, \varsigma) \in \Im \times \Im$, for each integer $n \geq 2$, and for each $\{\omega_i\}_{i=1}^{n} \subset \Im$ with $(\omega_1, \omega_n) = (\sigma, \varsigma)$,*

$$d(\sigma, \varsigma) \leq \sum_{i=1}^{n-1} d(\omega_i, \omega_{i+1}).$$

It yields that

$$d(\sigma, \varsigma) > 0 \Rightarrow e^{d(\sigma,\varsigma)} \leq e^{\left[\sum_{i=1}^{n-1} d(\omega_i,\omega_{i+1})\right]}.$$

That is,

$$d(\sigma, \varsigma)e^{d(\sigma,\varsigma)} \leq \sum_{i=1}^{n-1} d(\omega_i, \omega_{i+1}) \left(e^{\left[\sum_{i=1}^{n-1} d(\omega_i,\omega_{i+1})\right]}\right).$$

Thus,

$$\phi\left(d(\sigma, \varsigma)\right) \leq \psi\left(\phi\left(\sum_{i=1}^{n-1} d(\omega_i, \omega_{i+1})\right)\right).$$

Then (d_3) holds with $\phi(t) = te^t$ and $\psi(t) = t$.

Example 1. *Let $\Im = \mathbb{N}$ and let $d : \Im \times \Im \to [0, \infty)$ be defined by*

$$d(\sigma, \varsigma) = \begin{cases} |\sigma - \varsigma|, & \text{if } (\sigma, \varsigma) \notin [0, 2] \times [0, 2], \\ \frac{(\sigma-\varsigma)^2}{9} & \text{if } (\sigma, \varsigma) \in [0, 2] \times [0, 2], \end{cases}$$

for all $\sigma, \varsigma \in \Im$. It is easy to see that d satisfies (d_1) and (d_2). But, d does not verify the triangle inequality.

Indeed,

$$d(0, 2) = \frac{4}{9} > \frac{2}{9} = \frac{1}{9} + \frac{1}{9} = d(0, 1) + d(1, 2).$$

Hence, d is not a metric on $\Im$. Further, let $\sigma, \varsigma \in \Im$ such that $d(\sigma, \varsigma) > 0$. Let $\{\omega_i\}_{i=1}^{n} \subset \Im$ where $n \geq 2$ and $(\omega_1, \omega_n) = (\sigma, \varsigma)$. Consider,

$$I = \{1, 2, 3, ..., n-1 : (\omega_i, \omega_{i+1}) \in [0, 2] \times [0, 2]\},$$

and

$$J = \{1, 2, 3, ..., n-1\} \setminus I.$$

Hence, we have

$$\begin{aligned} \sum_{i=1}^{n-1} d(\omega_i, \omega_{i+1}) &= \sum_{i \in I} d(\omega_i, \omega_{i+1}) + \sum_{j \in J} d(\omega_j, \omega_{j+1}) \\ &= \sum_{i \in I} \frac{(\omega_{i+1} - \omega_i)^2}{9} + \sum_{j \in J} |\omega_{j+1} - \omega_j|. \end{aligned}$$

Now, we have two cases:

Case 1: If $(\sigma, \varsigma) \notin [0,2] \times [0,2]$, we have

$$\begin{aligned} d(\sigma,\varsigma) &= |\sigma - \varsigma| \le \sum_{i=1}^{n-1} |\omega_{i+1} - \omega_i|) \le \sum_{i=1}^{n-1} \frac{4}{3} |\omega_{i+1} - \omega_i|) \\ &= \sum_{i \in I} \frac{4|\omega_{i+1} - \omega_i|}{3} + \sum_{j \in J} \frac{4}{3} |\omega_{j+1} - \omega_j| \\ &\le \sum_{i \in I} \frac{4|\omega_{i+1} - \omega_i|}{3} + \sum_{j \in J} 4 |\omega_{j+1} - \omega_j| . \end{aligned}$$

Observe that

$$\frac{|\omega_{i+1} - \omega_i|}{3} \le \frac{(\omega_{i+1} - \omega_i)^2}{9}.$$

Thus, we get that

$$\begin{aligned} d(\sigma,\varsigma) &\le 4 \left[\sum_{i \in I} \frac{(\omega_{i+1} - \omega_i)^2}{9} + \sum_{j \in J} |\omega_{j+1} - \omega_j| \right] \\ &= 4 \sum_{i=1}^{n-1} d(\omega_i, \omega_{i+1}). \end{aligned}$$

Case 2: If $(\sigma, \varsigma) \in [0,2] \times [0,2]$, we have

$$\begin{aligned} d(\sigma,\varsigma) &= \frac{|\sigma - \varsigma|^2}{9} \le \frac{|\sigma - \varsigma|}{3} \\ &= \sum_{i \in I} \frac{|\omega_{i+1} - \omega_i|}{3} + \sum_{j \in J} \frac{|\omega_{j+1} - \omega_j|}{3} \\ &\le \sum_{i \in I} \frac{|\omega_{i+1} - \omega_i|}{3} + \sum_{j \in J} 3 |\omega_{j+1} - \omega_j| \\ &\le \sum_{i \in I} \frac{|\omega_{i+1} - \omega_i|^2}{3} + \sum_{j \in J} 3 |\omega_{j+1} - \omega_j| \\ &= \frac{1}{3} \left[\sum_{i \in I} \frac{|\omega_{i+1} - \omega_i|^2}{9} + \sum_{j \in J} |\omega_{j+1} - \omega_j| \right] \\ &= \frac{1}{3} \sum_{i=1}^{n-1} d(\omega_i, \omega_{i+1}). \end{aligned}$$

By combining the above, we conclude that for all $\sigma, \varsigma \in \Im$, for each integer $n \ge 2$, and for each $\{\omega_i\}_{i=1}^{n} \subset \Im$ with $(\omega_1, \omega_n) = (\sigma, \varsigma)$, we have

$$d(\sigma,\varsigma) > 0 \Rightarrow d(\sigma,\varsigma) \le \frac{1}{3} \sum_{i=1}^{n-1} d(\omega_i, \omega_{i+1}).$$

Therefore,

$$\begin{aligned} d(\sigma,\varsigma) e^{d(\sigma,\varsigma)} &\le \frac{1}{3} \sum_{i=1}^{n-1} d(\omega_i, \omega_{i+1}) e^{\left[\frac{1}{3} \sum_{i=1}^{n-1} d(\omega_i,\omega_{i+1})\right]} \\ &\le \frac{1}{3} \sum_{i=1}^{n-1} d(\omega_i, \omega_{i+1}) e^{\left[\sum_{i=1}^{n-1} d(\omega_i,\omega_{i+1})\right]}. \end{aligned}$$

It further implies that

$$d\left(\sigma,\varsigma\right)e^{d\left(\sigma,\varsigma\right)}\leq\frac{1}{3}\sum_{i=1}^{n-1}d(\omega_i,\omega_{i+1})e^{\left[\sum_{i=1}^{n-1}d(\omega_i,\omega_{i+1})\right]}.$$

Therefore, d is a (ϕ,ψ)-metric.

Remark 2. *It should be noted that the class of (ϕ,ψ)-metric spaces is effectively larger than the set of F-metric spaces. Indeed, a (ϕ,ψ)-metric is a $F-$metric by considering $\phi\left(t\right)=e^{f(t)}$ and $\psi\left(t\right)=e^{-\alpha}t$. We present an easy example to show that a (ϕ,ψ)-metric need not be a F -metric.*

Example 2. *Let $\Im=[0,1]$. Define $d:\Im\times\Im\rightarrow[0,\infty)$ as*

$$d\left(\sigma,\varsigma\right)=\left(\frac{\sigma-\varsigma}{6}\right)^2.$$

Clearly, d is a (ϕ,ψ) -metric on $\Im$ with $\phi(t)=t$ and $\psi(t)=\frac{t}{36}$. Assume that there are $F\in\Xi$ and $\alpha\in[0,\infty)$. Let $n\in N$ and $\omega_i=\frac{i}{n}$ for $i=0,2,...,n$. Using (D_3), we obtain

$$f(d(0,1))\leq f(d(0,\omega_1)+d(\omega_1,\omega_2)+...+d(\omega_{n-1},1))+\alpha,\ n\in\mathbb{N}.$$

Thus,

$$f(\frac{1}{36})\leq f(\frac{1}{36n})+\alpha,\ n\in\mathbb{N}.$$

Using (F_2), we get

$$\lim_{n\rightarrow\infty}f(\frac{1}{36n})+\alpha=-\infty,$$

which is a contradiction. Therefore, d is not a F-metric space on $\Im$.

3. Topology of (ϕ,ψ)-Metric Spaces

Here, we study the natural topology defined on (ϕ,ψ) -metric spaces.

Definition 4. *Let $(\Im,d)$ be a (ϕ,ψ)-metric space and M be a subset of $\Im$. M is said to be (ϕ,ψ)-open if for each $\sigma\in M$, there is $r>0$ so that $B(\sigma,r)\subset M$, where*

$$B(\sigma,r)=\{\varsigma\in\Im:d(\sigma,\varsigma)<r\}.$$

A subset Z of $\Im$ is called (ϕ,ψ)-closed if $\Im\backslash Z$ is (ϕ,ψ)-open. We denote by $\tau_{(\phi,\psi)}$ the set of all (ϕ,ψ)-open subsets of $\Im$.

Proposition 1. *Let $(\Im,d)$ be a (ϕ,ψ) -metric space. Then $\tau_{(\phi,\psi)}$ is a topology on $\Im$.*

Proposition 2. *Let $(\Im,d)$ be a (ϕ,ψ)-metric space. Then, for each nonempty subset C of $\Im$, we have equivalence of the following assertions:*

(i) *C is (ϕ,ψ) -closed.*
(ii) *For any sequence $\{\sigma_n\}\subset\Im$, we have*

$$\lim_{n\longrightarrow\infty}d\left(\sigma_n,\sigma\right)=0,\ \sigma\in\Im\Rightarrow\sigma\in C.$$

Proof. Suppose that C is (ϕ,ψ)-closed. Let $\{\sigma_n\}$ be a sequence in C such that

$$\lim_{n\longrightarrow\infty}d\left(\sigma_n,\sigma\right)=0,\tag{1}$$

where $\sigma \in \Im$. Assume that $\sigma \in \Im \backslash C$. Since C is (ϕ, ψ)-closed, $\Im \backslash C$ is (ϕ, ψ)-open. Hence, there is $r > 0$ so that $B(\sigma, r) \subset \Im \backslash C$, that is, $B(\sigma, r) \cap C = \emptyset$. Also, by (1), there is $N \in \mathbb{N}$ so that

$$d(\sigma_n, \sigma) < r, n \geq N.$$

That is, $\sigma_n \in B(\sigma, r), n \geq N$. Hence, $\sigma_N \in B(\sigma, r) \cap C$. It is a contradiction, and so $\sigma \in C$. That is, $(i) \Rightarrow (ii)$ is proved. Conversely, assume that (ii) is verified. Let $\sigma \in \Im \backslash C$. We now show that there is some $r > 0$ so that $B(\sigma, r) \subset \Im \backslash C$. We argue by contradiction. assume that for each $r > 0$, there is $\sigma_r \in B(\sigma, r) \cap C$. Thus, for each $n \in \mathbb{N}$, there is $\sigma_n \in B(\sigma, \frac{1}{n}) \cap C$. Then $\{\sigma_n\} \subset C$ and

$$\lim_{n \longrightarrow \infty} d(\sigma_n, \sigma) = 0.$$

By (ii), we get $\sigma \in C$, which is a contradiction with $\sigma \in \Im \backslash C$. Thus, C is (ϕ, ψ)-closed and so $(ii) \Rightarrow (i)$. □

Proposition 3. *Let $(\Im, d)$ be a (ϕ, ψ)-metric space, $\alpha \in \Im$ and $r > 0$. Let $B(\alpha, r)$ be the subset of $\Im$ given as*

$$B(\alpha, r) = \{\sigma \in \Im : d(\alpha, \sigma) \leq r\}.$$

Assume that for each sequence $\{\sigma_n\} \subset \Im$, we have

$$\lim_{n \longrightarrow \infty} d(\sigma_n, \sigma) = 0, \ \sigma \in \Im \Rightarrow d(\sigma, \varsigma) \leq \limsup_{n \to \infty} d(\sigma, \varsigma), \ \varsigma \in \Im. \tag{2}$$

Then $B(\alpha, r)$ is (ϕ, ψ)-closed.

Proof. Let $\{\sigma_n\} \subset B(\alpha, r)$ be a sequence so that

$$\lim_{n \longrightarrow \infty} d(\sigma_n, \sigma) = 0, \ \sigma \in \Im.$$

From Proposition 2, we show that $\sigma \in B(\alpha, r)$. By using the definition of $B(\alpha, r)$, we obtain $d(\sigma_n, \sigma) \leq r$, $n \in \mathbb{N}$. Taking $\limsup_{n \to \infty}$, by (2), we get

$$d(\sigma, \varsigma) \leq \limsup_{n \to \infty} d(\sigma_n, \varsigma) \leq r,$$

which yields that $\sigma \in B(\alpha, r)$. Consequently, $B(\alpha, r)$ is (ϕ, ψ)-closed. □

Remark 3. *Proposition 3 gives only a sufficient condition ensuring that $B(\alpha, r)$ is (ϕ, ψ)-closed. An interesting problem is devoted to get a sufficient and necessary condition under which $B(\alpha, r)$ is (ϕ, ψ)-closed.*

Definition 5. *Let $(\Im, d)$ be a (ϕ, ψ)-metric space. Let C be a nonempty subset of $\Im$. Let $\overline{C}$ be the closure of C with respect to the topology $\tau_{(\phi,\psi)}$, that is, $\overline{C}$ is the intersection of all (ϕ, ψ)-closed subsets of $\Im$ containing C. Obviously, $\overline{C}$ is the smallest (ϕ, ψ)-closed subset containing C.*

Proposition 4. *Let $(\Im, d)$ be a (ϕ, ψ)-metric space. Let C be a nonempty subset of $\Im$. If $\sigma \in \overline{C}$, then $B(\sigma, r) \cap C \neq \emptyset$ for $r > 0$.*

Proof. Let $\psi \in \Psi$ and $\phi \in D$ be such that (d_3) holds. Define

$$C' = \{\sigma \in \Im : \text{ for every } r > 0, \text{ there is } c \in C : d(\sigma, \varsigma) < r\}.$$

By $(d1)$, it is easy to see that $C \subset C'$. Next, we will show that C' is (ϕ, ψ)-closed. Let $\{\sigma_n\}$ be a sequence in C' such that

$$\lim_{n \longrightarrow \infty} d(\sigma_n, \sigma) = 0, \ \sigma \in \Im. \tag{3}$$

By (3), there are some $\delta > 0$ and $N \in \mathbb{N}$ so that

$$d(\sigma_n, \sigma) < \frac{\delta}{2}, \text{ for } n \geq N.$$

Since $\sigma_N \in C$, there is $\alpha \in C$ so that

$$d(\sigma_N, \alpha) < \frac{\delta}{2}, \text{ for } n \geq N.$$

If $d(\sigma, \alpha) > 0$, by (d_3), we have

$$\begin{aligned} \phi(d(\sigma, \alpha)) &\leq \psi[\phi(d(\sigma_N, \sigma) + d(\sigma_N, \alpha))] \leq \psi[\phi(\delta)] \\ &< \phi(\delta). \end{aligned}$$

Hence,

$$\phi(d(\sigma, \alpha)) < \phi(\delta).$$

Using $(\phi 1)$, we get

$$d(\sigma, \alpha) < \delta.$$

Hence, in all cases, we obtain $d(\sigma, \alpha) < \delta$, which yields that $\sigma \in C'$. Then by Proposition 2, C' is (ϕ, ψ)-closed, which contains C. Then $\overline{C} \subset C'$. □

Definition 6. *Let $(\Im, d)$ be a (ϕ, ψ)-metric space. Let $\{\sigma_n\}$ be a sequence in $\Im$. We say that $\{\sigma_n\}$ is (ϕ, ψ)-convergent to $\sigma \in \Im$ if $\{\sigma_n\}$ is convergent to σ with respect to the topology $\tau_{(\phi,\psi)}$, that is, for each (ϕ, ψ)-open subset $\grave{O}_\sigma$ of $\Im$ containing σ, there is $N \in \mathbb{N}$ so that $\sigma_n \in \grave{O}_\sigma$ for any $n \geq N$. Here, σ is called the limit of $\{\sigma_n\}$.*

The next result comes directly by combining the above definition and the definition of $\tau_{(\phi,\psi)}$.

Proposition 5. *Let $(\Im, d)$ be a (ϕ, ψ)-metric space. Let $\{\sigma_n\}$ be a sequence in $\Im$ and $\sigma \in \Im$. We have equivalence of the following assertions:*

(i) $\{\sigma_n\}$ *is (ϕ, ψ)-convergent to σ.*
(ii) $\lim_{n \longrightarrow \infty} d(\sigma_n, \sigma) = 0.$

In the following, the limit of a (ϕ, ψ)-convergent sequence is unique.

Proposition 6. *Let $(\Im, d)$ be a (ϕ, ψ)-metric space. Let $\{\sigma_n\}$ be a sequence in $\Im$. Then*

$$\lim_{n \longrightarrow \infty} d(\sigma_n, \sigma) = \lim_{n \longrightarrow \infty} d(\sigma_n, \varsigma) = 0 \Rightarrow \sigma = \varsigma.$$

Proof. Let $\sigma, \varsigma \in \Im$ be so that

$$\lim_{n \longrightarrow \infty} d(\sigma_n, \sigma) = \lim_{n \longrightarrow \infty} d(\sigma_n, \varsigma) = 0.$$

Assume that $\sigma \neq \varsigma$. By (d_1), $d(\sigma, \varsigma) > 0$. Using (d_3), there are $\psi \in \Psi$ and $\phi \in D$ such that

$$\begin{aligned} \phi(d(\sigma, \varsigma)) &\leq \psi[\phi(d(\sigma_n, \sigma) + d(\sigma_n, \varsigma))] \\ &< \phi(d(\sigma_n, \sigma) + d(\sigma_n, \varsigma)), \end{aligned}$$

for every n. Next, in view of (d_2) and $(\phi 2)$,

$$\lim_{n\longrightarrow\infty}\phi\left(d\left(\sigma_n,\sigma\right)+d\left(\sigma_n,\varsigma\right)\right)=0,$$

and so $\phi\left(d\left(\sigma,\varsigma\right)\right)=0$, which is a contradiction, and so $\sigma=\varsigma$. □

Definition 7. *Let $(\Im,d)$ be a (ϕ,ψ)-metric space. Let $\{\sigma_n\}$ be a sequence in $\Im$. Then,*

(i) $\{\sigma_n\}$ *is (ϕ,ψ)-Cauchy if* $\lim_{n,m\longrightarrow\infty} d\left(\sigma_n,\sigma_m\right)=0$.

(ii) $(\Im,d)$ *is (ϕ,ψ)-complete, if any (ϕ,ψ)-Cauchy sequence in $\Im$ is (ϕ,ψ)-convergent to some element in $\Im$.*

Proposition 7. *Let $(\Im,d)$ be a (ϕ,ψ)-metric space. If $\{\sigma_n\}\subset\Im$ is (ϕ,ψ)-convergent, then it is (ϕ,ψ)-Cauchy.*

Proof. Let $\psi\in\Psi$ and $\phi\in D$ be such that (d_3) holds. Let $\sigma\in\Im$ be so that

$$\lim_{n\longrightarrow\infty} d\left(\sigma_n,\sigma\right)=0.$$

For any $\delta>0$, there is $N\in\mathbb{N}$ such that

$$d\left(\sigma_n,\sigma\right)+d\left(\sigma_m,\sigma\right)<\delta,\ n,m\geq N. \tag{4}$$

Let $m,n\geq N$. We consider the two following cases.

Case 1: If $\sigma_n=\sigma_m$. Here, by (d_1),

$$d\left(\sigma_n,\sigma_m\right)=0<\delta.$$

Case 2: If $\sigma_n\neq\sigma_m$. Here, from (4),

$$0<d\left(\sigma_n,\sigma\right)+d\left(\sigma_m,\sigma\right)<\delta.$$

One writes

$$\phi\left(d\left(\sigma_n,\sigma\right)+d\left(\sigma_m,\sigma\right)\right)<\phi\left(\delta\right).$$

It implies that

$$\psi\left(d\left(\sigma_n,\sigma\right)+d\left(\sigma_m,\sigma\right)\right)<\psi\left(\phi\left(\delta\right)\right).$$

Now, using (d_3), we obtain

$$\begin{aligned}\phi\left(d\left(\sigma_n,\sigma_m\right)\right) &\leq \psi\left(\phi d\left(\sigma_n,\sigma\right)+d\left(\sigma_m,\sigma\right)\right)<\psi\left(\phi\left(\delta\right)\right)\\ &< \phi\left(\delta\right),\end{aligned}$$

which implies from $(\phi 1)$ that

$$d\left(\sigma_n,\sigma_m\right)<\delta.$$

Hence,

$$d\left(\sigma_n,\sigma_m\right)<\delta,\ n,m\geq N.$$

Consequently,

$$\lim_{n,m\longrightarrow\infty} d\left(\sigma_n,\sigma_m\right)=0,$$

that is, $\{\sigma_n\}$ is (ϕ,ψ)-Cauchy. □

Now, we study the compactness on (ϕ,ψ)-metric spaces.

Definition 8. *Let $(\Im,d)$ be a (ϕ,ψ)-metric space. Let C be a nonempty subset of $\Im$. then C is called (ϕ,ψ)-compact if C is compact with respect to the topology $\tau_{(\phi,\psi)}$ on $\Im$.*

Proposition 8. *Let $(\Im, d)$ be a (ϕ, ψ)-metric space. Let C be a nonempty subset of $\Im$. Then, we have equivalent of the following assertions:*

(i) *C is (ϕ, ψ)-compact.*
(ii) *For each sequence $\{\sigma_n\} \subset C$, there is a subsequence $\{\sigma_{n(k)}\}$ of $\{\sigma_n\}$ so that*

$$\lim_{k \longrightarrow \infty} d\left(\sigma_{n(k)}, \sigma\right) = 0.$$

Proof. Assume that C is (ϕ, ψ)-compact. Note that the set of decreasing sequences of nonempty (ϕ, ψ)-closed subsets of C has a nonempty intersection. Let $\{\sigma_n\}$ be a sequence in C. For any $n \in \mathbb{N}$, let $Z_n = \{\sigma_m : m \geq n\}$. Clearly, $Z_{n+1} \subset Z_n$ for each $n \in \mathbb{N}$. This implies that $\{\overline{Z}_n\}_{n \in \mathbb{N}}$ is a decreasing sequence of nonempty (ϕ, ψ)-closed subsets of Z. Thus, there is $\sigma \in \cap_{n \in \mathbb{N}} \overline{Z}_n$. Given an arbitrary element $\varepsilon > 0$. Since $\sigma \in \overline{Z_0}$, by Proposition 4, there are $n_0 \geq 0$ and $\sigma_{n_0} \in C$ so that $d\left(\sigma_{n_0}, \sigma\right) < \varepsilon$. Continuing in this direction, for any $k \in \mathbb{N}$, there are $n(k) \geq k$ and $\sigma_{n(k)} \in C$ so that

$$d\left(\sigma_{n(k)}, \sigma\right) < \varepsilon.$$

Consequently,

$$\lim_{k \longrightarrow \infty} d\left(\sigma_{n(k)}, \sigma\right) = 0.$$

Since C is (ϕ, ψ)-compact, one says that C is (ϕ, ψ)-closed, and $\sigma \in C$.Hence, we established that $(i) \Rightarrow (ii)$. Conversely, suppose that (ii) is satisfied. Let $\psi \in \Psi$ and $\phi \in D$ such that (d_3) is satisfied.

First, we claim that

$$\forall r > 0, \exists (\sigma_0), i = 1, ..., n \subset C : C \subset \underset{i=1,...,n}{\cup} B(\sigma_i, r). \tag{5}$$

We argue by contradiction. Suppose there is $r > 0$ so that for any finite number of elements $(\sigma_0), i = 1, ..., n \subset C$,

$$C \subsetneq \underset{i=1,...,n}{\cup} B(\sigma_i, r).$$

Let $\sigma_1 \in C$ be a fixed element. Then

$$C \subsetneq B(\sigma_1, r).$$

That is, there is $\sigma_2 \in C$ so that $d\left(\sigma_1, \sigma_2\right) \geq r$. Also,

$$C \subsetneq B(\sigma_1, r) \cup B(\sigma_2, r).$$

So there is $\sigma_3 \in C$ so that $d\left(\sigma_i, \sigma_3\right) \geq r$ for $i = 1, ..., n$. Continuing in this direction and by induction, we build a sequence $\{\sigma_n\} \subset C$ so that $d\left(\sigma_n, \sigma_m\right) \geq r$, $n, m \in \mathbb{N}$. Note that we could bot extract from $\{\sigma_n\}$ any (ϕ, ψ)-Cauchy subsequence, and so (from Proposition 7), any (ϕ, ψ)-convergent subsequence. We get so a contradiction with (ii), which proves (5). Next, let $\{\grave{O}_i\}_{i \in I}$ be an arbitrary family of (ϕ, ψ)-open subsets of $\Im$ so that

$$C \subset \cup_{i \in I} \grave{O}_i. \tag{6}$$

We claim that

$$\forall r_0 > 0 : \forall \sigma \in C, \exists i \in I : B(\sigma, r_0) \subset \grave{O}_i. \tag{7}$$

We argue by contradiction. Assume that for every $r > 0$, there is $\sigma_r \in C$ so that $B(\sigma_r, r) \subsetneq \grave{O}_i$, for all $i \in I$. Particularly, for all $n \in \mathbb{N}$, there is $\sigma_n \in C$ so that $B(\sigma_n, \frac{1}{n}) \subsetneq \grave{O}_i$ for all $i \in I$. By (ii), we build a subsequence $\left\{\sigma_{n(k)}\right\}$ from $\{\sigma_n\}$ so that

$$\lim_{k \longrightarrow \infty} d\left(\sigma_{n(k)}, \sigma\right) = 0, \tag{8}$$

for some $\sigma \in C$. Moreover, using (6), there is $j \in I$ so that $\sigma \in \Im$. In view of the fact that $\grave{O}_j$ is a (ϕ, ψ)-open subset of $\Im$, there is $r_0 > 0$ so that $B(\sigma, r_0) \subset \grave{O}_j$. Now, for each $n(k) \in \mathbb{N}$ and for every $q \in B(\sigma_{n(k)}, \frac{1}{n(k)})$, one writes

$$\begin{aligned} d(\sigma, q) &> 0 \Rightarrow \phi(d(\sigma, q)) \leq \psi\left(\phi\left(d\left(\sigma, \sigma_{n(k)}\right) + d\left(\sigma_{n(k)}, q\right)\right)\right) \\ &< \psi\left(\phi\left(d\left(\sigma, \sigma_{n(k)}\right) + \frac{1}{n(k)}\right)\right) \\ & \quad \phi\left(d\left(\sigma, \sigma_{n(k)}\right) + \frac{1}{n(k)}\right). \end{aligned}$$

Using (8) and $(\phi 2)$, there is $K \in \mathbb{N}$ so that

$$\phi\left(d\left(\sigma, \sigma_{n(k)}\right) + \frac{1}{n(k)}\right) < \phi(r_0)$$

for each $k \geq K$. It yields that

$$d(\sigma, q) > 0 \Rightarrow \phi(d(\sigma, q)) < \phi(r_0).$$

Consequently, by $(\phi 1)$, we find that $d(\sigma, q) < r_0$. Hence, we get

$$B(\sigma_{n(k)}, \frac{1}{n(k)}) \subset B(\sigma, r_0),$$

for $n(k) \in \mathbb{N}$. Thus,

$$B(\sigma_{n(k)}, \frac{1}{n(k)}) \subset \grave{O}_j, n(k) \in \mathbb{N}.$$

We get a contradiction with respect to

$$B(\sigma_{n(k)}, \frac{1}{n(k)}) \subsetneq \grave{O}_i, n(k) \in \mathbb{N}.$$

for all $i \in I$. Then (7) holds. Further, by (5), there is $\left\{\sigma_p\right\}_{p=1,\ldots,n} \subset C$ so that

$$C \subset \bigcup_{p=1,\ldots,n} B(\sigma_p, r_0).$$

But by (7), for any $p = 1, \ldots, n$, there exists $i(p) \in I$ such that $B(\sigma_p, r_0) \subset \grave{O}_{i(p)}$, which yields

$$C \subset \bigcup_{p=1,\ldots,n} \grave{O}_{i(p)}.$$

Thus, C is (ϕ, ψ)-compact, and so (ii)$\Rightarrow$(i). $\square$

Definition 9. *Let $(\Im, d)$ be a (ϕ, ψ)-metric space. Let C be a nonempty subset of $\Im$. The subset C is said to be sequentially (ϕ, ψ)-compact, if for each sequence , there are a subsequence $\left\{\sigma_{n(k)}\right\}$ of $\{\sigma_n\}$ and $\sigma \in C$ so that*

$$\lim_{k \longrightarrow \infty} d\left(\sigma_{n(k)}, \sigma\right) = 0.$$

Definition 10. *Let $(\Im, d)$ be a (ϕ, ψ)-metric space. Let C be a nonempty subset of $\Im$. The subset C is called (ϕ, ψ)-totally bounded if*

$$\forall r > 0, \exists(\sigma_0), i = 1, ..., n \subset C : C \subset \bigcup_{i=1,...,n} B(\sigma_i, r).$$

Due to the proof of Proposition 8, we may state the following proposition.

Proposition 9. *Let $(\Im, d)$ be a (ϕ, ψ)-metric space. Let C be a nonempty subset of $\Im$.*

(i) C is (ϕ, ψ)-compact if and only if C is sequentially (ϕ, ψ)-compact.
(ii) If C is (ϕ, ψ)-compact, then C is (ϕ, ψ)-totally bounded.

4. Banach Contraction Principle on (ϕ, ψ)-Metric Spaces

In this section, we prove a new version of the BCP in the context of (ϕ, ψ)-metric spaces.

Theorem 1. *Let $(\Im, d)$ be a complete (ϕ, ψ)-metric space and $T : \Im \to \Im$ be a self-mapping. Suppose that there exists $\lambda \in (0, 1)$ such that for all $\sigma, \varsigma \in \Im$,*

$$d\left(T(\sigma), T(\varsigma)\right) \leq \lambda d(\sigma, \varsigma). \tag{9}$$

Then T has a unique fixed point in $\Im$.

Proof. Let $\sigma_0 \in \Im$. Define the sequence $\{\sigma_n\}$ in $\Im$ by

$$\sigma_{n+1} = T(\sigma_n), \text{ where } n \in \mathbb{N}.$$

If for some n, $d(\sigma_n, \sigma_{n+1}) = 0$, then σ_n is a fixed point of T. Without restriction of the generality, we may suppose that $d(\sigma_n, \sigma_{n+1}) > 0$ for all n. Using (9), we get

$$\begin{aligned} d(\sigma_n, \sigma_{n+1}) &\leq \lambda d(\sigma_{n-1}, \sigma_n) \leq \lambda^2 d(\sigma_{n-2}, \sigma_{n-1}) \\ &\leq ... \leq \lambda^n d(\sigma_0, \sigma_1), \end{aligned}$$

for all $n \in \mathbb{N}$. Thus,

$$\sum_{i=n}^{m-1} d(\sigma_i, \sigma_{i+1}) \leq \frac{\lambda^n}{1-\lambda} d(\sigma_0, \sigma_1), \quad m > n.$$

Hence, by $(\phi 1)$, we have

$$\phi\left(\sum_{i=n}^{m-1} d(\sigma_i, \sigma_{i+1})\right) \leq \phi\left(\frac{\lambda^n}{1-\lambda} d(\sigma_0, \sigma_1)\right), \quad m > n.$$

Since ψ is monotone increasing, we obtain for $m > n$,

$$\begin{aligned} \psi\left(\phi\left(\sum_{i=n}^{m-1} d(\sigma_i, \sigma_{i+1})\right)\right) &\leq \psi\left(\phi\left(\frac{\lambda^n}{1-\lambda} d(\sigma_0, \sigma_1)\right)\right) \\ &< \phi\left(\frac{\lambda^n}{1-\lambda} d(\sigma_0, \sigma_1)\right). \end{aligned}$$

Since
$$\lim_{n\longrightarrow\infty}\frac{\lambda^n}{1-\lambda}d\left(\sigma_0,\sigma_1\right)=0,$$
by $(\phi 2)$, we have
$$\lim_{n\longrightarrow\infty}\phi\left(\frac{\lambda^n}{1-\lambda}d\left(\sigma_0,\sigma_1\right)\right)=0. \tag{10}$$
Using (d_3), we obtain
$$\begin{aligned} d\left(\sigma_n,\sigma_m\right) &> 0,\ m>n\Rightarrow\phi\left(d\left(\sigma_n,\sigma_m\right)\right)\leq\psi\left(\phi\left(\sum_{i=n}^{m-1}d\left(\sigma_i,\sigma_{i+1}\right)\right)\right)\\ &< \phi\left(\frac{\lambda^n}{1-\lambda}d\left(\sigma_0,\sigma_1\right)\right). \end{aligned}$$
It implies that
$$\phi\left(d\left(\sigma_n,\sigma_m\right)\right)<\phi\left(\frac{\lambda^n}{1-\lambda}d\left(\sigma_0,\sigma_1\right)\right).$$
By using (10), we obtain
$$\lim_{n,m\longrightarrow\infty}\phi\left(d\left(\sigma_n,\sigma_m\right)\right)=0.$$
Then from $(\phi 2)$, we have
$$\lim_{n,m\longrightarrow\infty}d\left(\sigma_n,\sigma_m\right)=0.$$
Therefore, $\{\sigma_n\}$ is a (ϕ,ψ)-Cauchy sequence in $\Im$. Since $\Im$ is (ϕ,ψ)-complete, we can find $\sigma^*\in\Im$ such that
$$\lim_{n\to\infty}d\left(\sigma_n,\sigma^*\right)=0. \tag{11}$$
Next, we prove that $T\left(\sigma^*\right)=\sigma^*$. We argue by contradiction. Assume that $d\left(T\left(\sigma^*\right),\sigma^*\right)>0$. By using (d_3), we obtain
$$\begin{aligned} \phi\left(d\left(T\left(\sigma^*\right),\sigma^*\right)\right) &\leq \psi\left(\phi\left(d\left(T\left(\sigma^*\right),T\left(\sigma_n\right)\right)+d\left(T\left(\sigma_n\right),\sigma^*\right)\right)\right)\\ &< \phi\left(d\left(T\left(\sigma^*\right),T\left(\sigma_n\right)\right)+d\left(T\left(\sigma_n\right),\sigma^*\right)\right), \end{aligned}$$
for $n\in\mathbb{N}$. By (9) and $(\phi 1)$,
$$d\left(T\left(\sigma^*\right),\sigma^*\right)<\lambda d\left(\sigma^*,\sigma_n\right)+d\left(\sigma_{n+1},\sigma^*\right).$$
By using $(\phi 2)$ and (11), we get
$$\lim_{n\to\infty}\phi\left(\lambda d\left(\sigma^*,\sigma_n\right)+d\left(\sigma_{n+1},\sigma^*\right)\right)=0,$$
which is a contradiction. Therefore, $d\left(T\left(\sigma^*\right),\sigma^*\right)=0$ and $T\left(\sigma^*\right)=\sigma^*$. Thus, T has a fixed point $\sigma^*\in\Im$. Next, we prove that T has at most one fixed point. Assume that σ^* and ζ^* are two fixed points of T such that $\sigma^*\neq\zeta^*$. Then from (9), we have
$$0<d\left(\sigma^*,\varsigma^*\right)=d\left(T\left(\sigma^*\right),T\left(\varsigma^*\right)\right)\leq\lambda d\left(\sigma^*,\varsigma^*\right)<d\left(\sigma^*,\varsigma^*\right).$$
It is a contradiction. Hence, T has a unique fixed point in $\Im$. $\square$

Corollary 1. *Let $(\Im, d)$ be a (ϕ, ψ)-metric space. Suppose there exist a continuous comparison function $\psi \in \Psi$ and $\phi \in D$ so that (d_3) holds. Let $S : B(\sigma_0, r) \to \Im$ be a given mapping, where $\sigma_0 \in \Im$ and $r > 0$. Assume that:*

(i) *Suppose that for each sequence $\{\sigma_n\} \subset \Im$, we have*

$$\lim_{n\to\infty} d(\sigma_n, \sigma) = 0 \Rightarrow d(\sigma, \varsigma) \le \lim_{n\to\infty} \sup d(\sigma_n, \varsigma), \ \varsigma \in \Im;$$

(ii) *$(\Im, d)$ is (ϕ, ψ)-complete;*
(iii) *There exists $\lambda \in (0, 1)$ such that*

$$d(S(\sigma), S(\varsigma)) \le \lambda d(\sigma, \varsigma), \ (\sigma, \varsigma) \in B(\sigma_0, r) \times B(\sigma_0, r);$$

(iv) *There exists $0 < \varepsilon < r$ such that*

$$\phi(\lambda\varepsilon + d(S\sigma_0, \sigma_0)) \le \phi(\varepsilon).$$

Then S has a fixed point.

Proof. Consider $0 < \varepsilon < r$ such that (iv) is satisfied. First, we will show that

$$S(B(\sigma_0, \varepsilon)) \subset B(\sigma_0, \varepsilon).$$

Let $\sigma \in B(\sigma_0, \varepsilon)$, that is, $d(\sigma_0, \sigma) \le \varepsilon$. Assume that $d(S\sigma, \sigma_0) > 0$. By (d_3),

$$\phi(d(S\sigma, \sigma_0)) \le \psi(\phi(d(S\sigma, S\sigma_0) + d(S\sigma_0, \sigma_0))).$$

Using (iii), we obtain

$$\begin{aligned} \phi(d(S\sigma, \sigma_0)) &\le \psi(\phi(d(S\sigma, S\sigma_0) + d(S\sigma_0, \sigma_0))) \\ &\le \psi(\phi(\lambda d(\sigma, \sigma_0) + d(S\sigma_0, \sigma_0))) \\ &\le \psi(\phi(\lambda\varepsilon + d(S\sigma_0, \sigma_0))) \\ &< \phi(\lambda\varepsilon + d(S\sigma_0, \sigma_0)) \\ &\le \phi(\varepsilon). \end{aligned}$$

Hence, by $(\phi 1)$, we have $d(S\sigma, \sigma_0) \le \varepsilon$, which yields $S(\sigma) \in B(\sigma_0, \varepsilon)$. Therefore,

$$S(B(\sigma_0, \varepsilon)) \subset B(\sigma_0, \varepsilon).$$

Further, the mapping $S : B(\sigma_0, \varepsilon) \to B(\sigma_0, \varepsilon)$ is well-defined, and the Banach contraction condition holds. Next, since the condition of Proposition 3 is satisfied, it is known that $B(\sigma_0, \varepsilon)$ is (ϕ, ψ)-closed, so from (i), it is (ϕ, ψ)-complete. Finally, the result is deduced by using Theorem 1. □

5. Solving a Nonlinear Fredholm Integral Equation

This section is devoted to discusses the existence and uniqueness of a solution of a Fredholm type integral equation of the 2nd kind [24–29]. Consider the equation below:

$$\sigma(\mu) = \beta(\mu) + \int_u^v \Omega(\mu, \ell)\Re(\mu, \ell, \sigma(\ell))\, d\ell, \ \mu \in [u, v]. \tag{12}$$

Let $\Theta = C[u, v]$ be the set of all continuous functions defined on $[u, v]$. For $\sigma, \zeta \in \Theta$ and $q > 1$, define $d : \Theta \times \Theta \to [0, \infty)$ by

$$d(\sigma, \zeta) = \left(\frac{1}{6} \sup_{\mu \in [u,v]} |\sigma(\mu) - \zeta(\mu)| \right)^q.$$

Then (Θ, d) is a complete $(\phi, \psi)-$metric space with $\phi(\rho) = \rho$ and $\psi(\rho) = \frac{\rho}{6^q}$.

To study the existence of a solution for the problem (12), we state and prove the theorem below.

Theorem 2. *Consider the problem (12) via the assumptions below:*

($\dagger_1$) *$\Re : [u, v] \times [u, v] \times \mathbb{R} \to \mathbb{R}$, $\Omega : [u, v] \times [u, v] \to \mathbb{R}$, and $\beta : [u, v] \to \mathbb{R}$ are continuous functions;*

($\dagger_2$) *For $\mu \in [u, v]$, we have*

$$\sup_{\mu \in [u,v]} \int_u^v \Omega(\mu, \ell) d\ell \leq 1;$$

($\dagger_3$) *For $q > 1$, consider*

$$|\Re(\mu, \ell, \sigma(\ell)) - \Re(\mu, \ell, \zeta(\ell))| \leq \frac{1}{\sqrt[q]{3}} |\sigma(\ell) - \zeta(\ell)|.$$

Then the nonlinear integral equation (12) has a unique solution in Θ.

Proof. Define the operator $T : C[u, v] \to C[u, v]$ by

$$T\sigma(\mu) = \beta(\mu) + \int_u^v \Omega(\mu, \ell) \Re(\mu, \ell, \sigma(\ell)) \, d\ell, \ \mu \in [u, v]. \tag{13}$$

The solution of problem (12) is a fixed point for the operator (13). By hypotheses $(\dagger_1) - (\dagger_3)$, we have

$$\begin{aligned}
& d\left(T\sigma(\mu), T\zeta(\mu)\right) \\
= & \left(\frac{1}{6} \sup_{\mu \in [u,v]} |T\sigma(\mu) - T\zeta(\mu)|\right)^q \\
= & \frac{1}{6^q} \left(\sup_{\mu \in [u,v]} \left| \int_u^v \Omega(\mu, \ell) \Re(\mu, \ell, \sigma(\ell)) \, d\ell - \int_u^v \Omega(\mu, \ell) \Re(\mu, \ell, \zeta(\ell)) \, d\ell \right| \right)^q \\
\leq & \frac{1}{6^q} \left(\sup_{\mu \in [u,v]} \int_u^v \Omega(\mu, \ell) \left|\Re(\mu, \ell, \sigma(\ell)) - \Re(\mu, \ell, \zeta(\ell))\right| dv \right)^q \\
\leq & \frac{1}{6^q} \left(\sup_{\mu \in [u,v]} \int_u^v \Omega(\mu, \ell) \right)^q \times \sup_{\ell \in [u,v]} \left(\frac{1}{\sqrt[q]{3}} |\sigma(\ell) - \zeta(\ell)|\right)^q \\
\leq & \frac{1}{3} \sup_{\ell \in [u,v]} \left(\frac{1}{6} |\sigma(\ell) - \zeta(\ell)|\right)^q \\
= & \lambda d\left(\sigma(\mu), \zeta(\mu)\right).
\end{aligned}$$

Thus, the condition (9) of Theorem 1 holds with $\lambda = \frac{1}{3}$. Therefore, all hypotheses of Theorem 1 are fulfilled. So the problem (12) has a unique solution in Θ. □

The example below supports Theorem 2.

Example 3. *The following problem:*

$$\sigma(\mu) = \frac{1}{36} \int_0^1 \ell^2 \sigma(\ell) d\ell, \ \mu \in [0, 1], \tag{14}$$

has a solution in $C[0, 1]$.

Proof. Define the operator $T : C[0,1] \to C[0,1]$ by $T\sigma(\mu) = \frac{1}{36}\int_0^1 \ell^2\sigma(\ell)d\ell$. Customize $\Omega(\mu,\ell) = \frac{\ell}{6}$, $\beta(\mu) = 0$ and $\Re(\mu,\ell,\sigma(\ell)) = \frac{\ell\sigma(\ell)}{6}$ in Theorem 2. Note that

- $\Re$ and Ω are continuous functions;
- For $\mu \in [0,1]$, we have

$$\sup_{\mu\in[u,v]}\int_u^v \Omega(\mu,\ell)d\ell = \sup_{\mu\in[0,1]}\int_0^1 \frac{\ell}{6}d\ell = \frac{1}{12} < 1;$$

- Take $q = 2$. For $\ell \in [0,1]$, we get

$$\begin{aligned} |\Re(\mu,\ell,\sigma(\ell)) - \Re(\mu,\ell,\zeta(\ell))| &= \left|\frac{\ell\sigma(\ell)}{6} - \frac{\ell\zeta(\ell)}{6}\right| \\ &= \frac{\ell}{6}|\sigma(\ell) - \zeta(\ell)| \\ &\le \frac{1}{\sqrt{3}}|\sigma(\ell) - \zeta(\ell)|. \end{aligned}$$

Therefore, the stipulations of Theorem 2 are justified, hence the mapping T has a unique fixed point in $C[0,1]$, which is the unique solution of the equation (14). □

6. Solving a Two-Dimensional Nonlinear Fredholm Integral Equation

In many problems in engineering and mechanics under a suitable transformation, two-dimensional Fredholm integral equations of the second kind appear. For example, in the calculation of plasma physics, it is usually required to solve some Fredholm integral equations, see References [30–32].

Now, consider the two-dimensional Fredholm integral equation of the shape:

$$\zeta(r,j) = e(r,j) + \int_0^1\int_0^1 \Omega(r,j,f,g)\daleth(r,j,\zeta(f,g))dfdg; \quad (r,j) \in [0,1]^2, \tag{15}$$

where e, Ω and $\daleth$ are given continuous functions defined on $L^2(C([0,1]\times[0,1]))$ and ζ is a function in $L^2(C([0,1]\times[0,1]))$.

Let $\nabla = C([0,1])$ be the set of all real valued continuous functions on $[0,1]$. Consider the same distance of the above section, then for $\sigma,\zeta \in \nabla$, the pair (∇,d) is a complete $(\phi,\psi)-$metric space with $\phi(\rho) = \rho$ and $\psi(\rho) = \frac{\rho}{6^q}$.

Now, we consider the problem (15) under the hypotheses below:

($\ddagger_1$) $\Omega : [0,1]^4 \to \mathbb{R}$, and $\daleth : [0,1]^2 \times \mathbb{R} \to \mathbb{R}$ and $e : [0,1]^2 \to \mathbb{R}$ are continuous functions;
($\ddagger_2$) for all $\sigma,\zeta \in \nabla$, there is a constant $\kappa < 1$ such that

$$|\daleth(r,j,\sigma(f,g)) - \daleth(r,j,\zeta(f,g))| \le \frac{1}{\sqrt[q]{2\kappa}}|\sigma(h,g) - \zeta(h,g)|,\ q > 1;$$

($\ddagger_3$) we have $\int_0^1\int_0^1 \Omega(r,j,f,g)dfdg \le \kappa$.

Our related theorem in this part is listed as follows.

Theorem 3. *The problem (15) has a unique solution in $L^2(C([0,1]\times[0,1]))$ if the hypotheses $(\ddagger_1) - (\ddagger_3)$ hold.*

Proof. Define the operator $T : \nabla \to \nabla$ by

$$T\left(\zeta(\tau,\mu)\right) = e(r,j) + \int_0^1\int_0^1 \Omega(r,j,f,g)\daleth(r,j,\zeta(f,g))dfdg,\ (a,b) \in [0,1]\times[0,1], \tag{16}$$

then for $q > 1$, we get

$$\begin{aligned}
&\frac{1}{6^q}\left|T\left(\sigma(r,j)\right) - T\left(\zeta(r,j)\right)\right|^q \\
=\ & \frac{1}{6^q}\left|\int_0^1\int_0^1 \Omega(r,j,f,g)\daleth(r,j,\sigma(f,g))dfdg - \int_0^1\int_0^1 \Omega(r,j,f,g)\daleth(r,j,\zeta(f,g))dfdg\right|^q \\
\leq\ & \frac{1}{6^q}\left(\int_0^1\int_0^1 \Omega(r,j,f,g)\left|\daleth(r,j,\sigma(f,g)) - \daleth(r,j,\zeta(f,g))\right|dfdg\right)^q \\
\leq\ & \frac{1}{6^q}\left(\int_0^1\int_0^1 \Omega(r,j,f,g)dfdg\right)^q \left(\left|\daleth(r,j,\sigma(f,g)) - \daleth(r,j,\zeta(f,g))\right|\right)^q \\
\leq\ & \frac{1}{6^q}\kappa^q\left(\frac{1}{\sqrt[q]{2}\kappa}\left|\sigma(h,g)-\zeta(h,g)\right|\right)^q \\
=\ & \frac{1}{2}\left(\frac{1}{6}\left|\sigma(h,g)-\zeta(h,g)\right|\right)^q.
\end{aligned}$$

Taking the supremum, we get

$$\begin{aligned}
d(T\sigma,T\zeta) &= \left(\frac{1}{6}\sup_{\mu\in[u,v]}\left|T\left(\sigma(r,j)\right) - T\left(\zeta(r,j)\right)\right|\right)^q \\
&\leq \frac{1}{2}\left(\frac{1}{6}\sup_{\mu\in[u,v]}\left|\sigma(h,g)-\zeta(h,g)\right|\right)^q \\
&= \lambda d(\sigma,\zeta).
\end{aligned}$$

Thus, from Theorem 1, the operator (16) has a unique fixed point in $L^2(C([0,1]\times[0,1]))$, which is considered as the unique solution of the problem (15). □

7. Conclusions

In this manuscript, we initiated the concept a (ϕ,ψ)-metric space. It is a generalization of the metric space setting. We also presented its topological structure natural topology. The Banach contraction principle in this class has been established. Moreover, we gave some examples and applications in support of the introduced new concepts and presented results. As perspectives, it is an open problem to treat the cases of Kannan, Chatterjea, Hardy-Rogers, Ćirić and Suzuki type contractions. Also, it would be interesting to investigate the case of common fixed points.

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