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Some New Oscillation Results for Fourth-Order Neutral Differential Equations with Delay Argument

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Abstract: The aim of this paper is to study the oscillatory properties of 4th-order neutral differential equations. We obtain some oscillation criteria for the equation by the theory of comparison. The obtained results improve well-known oscillation results in the literate. Symmetry plays an important role in determining the right way to study these equation. An example to illustrate the results is given.

Keywords: oscillation; fourth-order; neutral differential equations

1. Introduction

Differential equations with neutral delay are used in many applications such as biological, physical, engineering and chemical applications [1]. Symmetry plays an important role in determining the right way to study these equations, see [2].

In the last few decades, there has been a constant interest to investigate the asymptotic property for oscillations of differential equations [3–19] and nonlinear neutral differential equations, see [20–32]. Oscillation of nonlinear differential equations with delay arguments has been further developed in recent years. For some this results, see [33–37].

In this work, we investigate the oscillation of fourth-order nonlinear differential equation with neutral delay

$$\left(r\left(y\right)\left(\varpi'''\left(y\right)\right)^{\gamma}\right)'+q\left(y\right)u^{\beta}\left(\varsigma\left(y\right)\right)=0,\ y\geq y_0, \tag{1}$$

where $\varpi\left(y\right):=u\left(y\right)+p\left(y\right)u\left(\zeta\left(y\right)\right)$. We assume that γ and β are quotient of odd positive integers, $r,p,q\in C[y_0,\infty)$, $r\left(y\right)>0$, $r'\left(y\right)\geq 0$, $q\left(y\right)>0$, $0\leq p\left(y\right)<p_0<\infty$, $\zeta,\varsigma\in C[y_0,\infty)$, $\zeta\left(y\right)\leq y$, $\lim_{y\to\infty}\zeta\left(y\right)=\lim_{y\to\infty}\varsigma\left(y\right)=\infty$ and

$$\int_{y_0}^{\infty}\frac{1}{r^{1/\gamma}\left(s\right)}\mathrm{d}s=\infty. \tag{2}$$

Definition 1. *If a solution u of (1) is neither eventually positive nor eventually negative, then it is said to be oscillatory. So, if all solutions are oscillate, then the equation is oscillatory.*

Several authors in [3–6,38] considered the equation

$$\left(r\left(y\right)\left(u^{\left(m-1\right)}\left(y\right)\right)^{\gamma}\right)' + q\left(y\right)u^{\beta}\left(\zeta\left(y\right)\right) = 0, \tag{3}$$

where $r'\left(u\right) > 0,\ m$ is an even and (2) holds. In [7,29], Zhang et al. studied the oscillation of (3) under the assumption that

$$\int_{y_0}^{\infty} r^{-1/\gamma}\left(s\right)\mathrm{d}s < \infty. \tag{4}$$

Moaaz et al. [23] established the oscillation of even-order neutral differential equation

$$\left(r\left(u\right)\left(\varpi^{\left(m-1\right)}\left(u\right)\right)^{\gamma}\right)' + q\left(u\right)u^{\gamma}\left(\varsigma\left(u\right)\right) = 0. \tag{5}$$

where m is an even and $\varpi\left(u\right) := u\left(u\right) + p\left(u\right)u\left(\zeta\left(u\right)\right)$.

Xing et al. [20] established the asymptotic properties of even-order neutral differential Equation (3) where $0 \leq p\left(u\right) < p_0 < \infty$.

Bazighifan et al. [27] studied the oscillation of neutral differential equation

$$\left(r\left(u\right)\left(\varpi'''\left(u\right)\right)^{\gamma}\right)' + \sum_{i=1}^{j} q_i\left(u\right)u^{\gamma}\left(\varsigma_i\left(u\right)\right) = 0,\ j \geq 1,$$

where $j \geq 1,\ \varsigma_i\left(u\right) \leq u$ and under the assumption (2).

Our aim in this work is to improve results in [20] and to complement results in [9].

We shall employ the following lemmas:

Lemma 1. *[18] Assume that $u \in C^m\left(\left[y_0,\infty\right),\left(0,\infty\right)\right)$, then*

$$\frac{u\left(y\right)}{y^m/m!} \geq \frac{u'\left(y\right)}{y^{m-1}/\left(m-1\right)!}$$

where u satisfies $u^{\left(i\right)}\left(y\right) > 0,\ i = 0,1,...,m$, and $u^{\left(m+1\right)}\left(y\right) < 0$.

Lemma 2. *([22], Lemmas 1 and 2) Let $z_1, z_2 \geq 0$, then*

$$\left(z_1 + z_2\right)^{\beta} \leq 2^{\beta-1}\left(z_1^{\beta} + z_2^{\beta}\right),\ \text{for } \beta \geq 1$$

and

$$\left(z_1 + z_2\right)^{\beta} \leq z_1^{\beta} + z_2^{\beta},\ \text{for } \beta \leq 1.$$

where β is a positive real number.

Lemma 3. *([3], Lemma 2.2.3) Let $u \in C^m\left(\left[y_0,\infty\right),\left(0,\infty\right)\right)$. If $u^{\left(m\right)}\left(y\right)$ is of fixed sign and not identically zero on $\left[y_0,\infty\right)$ and that there exists a $y_1 \geq y_0$ such that $u^{\left(m-1\right)}\left(y\right)u^{\left(m\right)}\left(y\right) \leq 0$ for all $y \geq y_1$. If $\lim_{y\to\infty} u\left(y\right) \neq 0$, then for every $\mu \in \left(0,1\right)$ there exists $y_{\mu} \geq y_1$ such that*

$$u\left(y\right) \geq \frac{\mu}{\left(m-1\right)!}y^{m-1}\left|u^{\left(m-1\right)}\left(y\right)\right|,\ \text{for } y \geq y_{\mu}.$$

2. Main Results

Firstly, we will define the following notations:

$$\kappa := \begin{cases} 1 & \text{if } \beta \leq 1; \\ 2^{\beta-1} & \text{if } \beta > 1, \end{cases}$$

and

$$\widehat{q}\left(y\right) := \min\left\{q\left(\varsigma^{-1}\left(y\right)\right), q\left(\varsigma^{-1}\left(\zeta\left(y\right)\right)\right)\right\}.$$

Theorem 1. *Assume that*

$$\left(\varsigma^{-1}\left(y\right)\right)' \geq \varsigma_0 > 0 \ \text{ and } \ \zeta'\left(y\right) \geq \zeta_0 > 0. \tag{6}$$

If the differential inequality

$$\eta'\left(y\right) + \frac{1}{\kappa}\left(\frac{\mu y^3}{6r^{1/\gamma}\left(y\right)}\right)^{\beta}\left(\frac{\varsigma_0\zeta_0}{\zeta_0 + p_0^{\beta}}\right)^{\beta/\gamma}\widehat{q}\left(y\right)\eta^{\beta/\gamma}\left(\zeta^{-1}\left(\varsigma\left(y\right)\right)\right) \leq 0 \tag{7}$$

is oscillatory for some $\mu \in (0,1)$, then (1) is oscillatory.

Proof. Suppose that (1) has a nonoscillatory solution in $[y_0,\infty)$. Without loss of generality, we let u be an eventually positive solution of (1). Then, there exists a $y_1 \geq y_0$ such that $u\left(y\right) > 0$, $u\left(\zeta\left(y\right)\right) > 0$ and $u\left(\varsigma\left(y\right)\right) > 0$ for $y \geq y_1$. Since $r'\left(y\right) > 0$, we have

$$\varpi\left(y\right) > 0,\ \varpi'\left(y\right) > 0,\ \varpi'''\left(y\right) > 0,\ \varpi^{(4)}\left(y\right) < 0 \text{ and } \left(r\left(y\right)\left(\varpi'''\left(y\right)\right)^{\gamma}\right)' \leq 0, \tag{8}$$

for $y \geq y_1$. From (1), we get

$$\frac{1}{\left(\varsigma^{-1}\left(y\right)\right)'}\left(r\left(\varsigma^{-1}\left(y\right)\right)\left(\varpi'''\left(\varsigma^{-1}\left(y\right)\right)\right)^{\gamma}\right)' + q\left(\varsigma^{-1}\left(y\right)\right)u^{\beta}\left(y\right) = 0. \tag{9}$$

It follows from definition of ϖ and Lemma 2 that

$$\begin{aligned} \varpi^{\beta}\left(y\right) &= \left(u\left(y\right) + p\left(y\right)u\left(\zeta\left(y\right)\right)\right)^{\beta} \\ &\leq \kappa\left(u^{\beta}\left(y\right) + p_0^{\beta}u^{\beta}\left(\zeta\left(y\right)\right)\right). \end{aligned} \tag{10}$$

From (9) and (10), we obtain

$$\begin{aligned} 0 &= \frac{1}{\left(\varsigma^{-1}\left(y\right)\right)'}\left(r\left(\varsigma^{-1}\left(y\right)\right)\left(\varpi'''\left(\varsigma^{-1}\left(y\right)\right)\right)^{\gamma}\right)' + q\left(\varsigma^{-1}\left(y\right)\right)u^{\beta}\left(y\right) \\ &\quad + p_0^{\beta}\left(\frac{1}{\left(\varsigma^{-1}\left(\zeta\left(y\right)\right)\right)'}\left(r\left(\varsigma^{-1}\left(\zeta\left(y\right)\right)\right)\left(\varpi'''\left(\varsigma^{-1}\left(\zeta\left(y\right)\right)\right)\right)^{\gamma}\right)' + q\left(\varsigma^{-1}\left(\zeta\left(y\right)\right)\right)u^{\beta}\left(\zeta\left(y\right)\right)\right) \\ &= \frac{\left(r\left(\varsigma^{-1}\left(y\right)\right)\left(\varpi'''\left(\varsigma^{-1}\left(y\right)\right)\right)^{\gamma}\right)'}{\left(\varsigma^{-1}\left(y\right)\right)'} + p_0^{\beta}\frac{\left(r\left(\varsigma^{-1}\left(\zeta\left(y\right)\right)\right)\left(\varpi'''\left(\varsigma^{-1}\left(\zeta\left(y\right)\right)\right)\right)^{\gamma}\right)'}{\left(\varsigma^{-1}\left(\zeta\left(y\right)\right)\right)'} \\ &\quad + q\left(\varsigma^{-1}\left(y\right)\right)u^{\beta}\left(y\right) + p_0^{\beta}q\left(\varsigma^{-1}\left(\zeta\left(y\right)\right)\right)u^{\beta}\left(\zeta\left(y\right)\right) \\ &\geq \frac{\left(r\left(\varsigma^{-1}\left(y\right)\right)\left(\varpi'''\left(\varsigma^{-1}\left(y\right)\right)\right)^{\gamma}\right)'}{\left(\varsigma^{-1}\left(y\right)\right)'} + p_0^{\beta}\frac{\left(r\left(\varsigma^{-1}\left(\zeta\left(y\right)\right)\right)\left(\varpi'''\left(\varsigma^{-1}\left(\zeta\left(y\right)\right)\right)\right)^{\gamma}\right)'}{\left(\varsigma^{-1}\left(\zeta\left(y\right)\right)\right)'} + \frac{1}{\kappa}\widehat{q}\left(y\right)\varpi^{\beta}\left(y\right), \end{aligned}$$

which with (6) gives

$$\begin{aligned} &\frac{1}{\varsigma_0}\left(r\left(\varsigma^{-1}\left(y\right)\right)\left(\varpi'''\left(\varsigma^{-1}\left(y\right)\right)\right)^{\gamma}\right)' \\ &\quad + \frac{p_0^{\beta}}{\varsigma_0\zeta_0}\left(r\left(\varsigma^{-1}\left(\zeta\left(y\right)\right)\right)\left(\varpi'''\left(\varsigma^{-1}\left(\zeta\left(y\right)\right)\right)\right)^{\gamma}\right)' + \frac{1}{\kappa}\widehat{q}\left(y\right)\varpi^{\beta}\left(y\right) \leq 0. \end{aligned} \tag{11}$$

Since $\varpi'(y) > 0$, we get that $\lim_{y\to\infty} \varpi(y) \neq 0$. Thus, from Lemma 3, we get

$$\varpi(y) \geq \frac{\mu}{6} y^3 \varpi'''(y). \tag{12}$$

Combining (11) and (12), we see that

$$\begin{aligned} &\frac{1}{\varsigma_0}\left(r\left(\varsigma^{-1}(y)\right)\left(\varpi'''\left(\varsigma^{-1}(y)\right)\right)^{\gamma}\right)' + \frac{p_0^{\beta}}{\varsigma_0\zeta_0}\left(r\left(\varsigma^{-1}(\zeta(y))\right)\left(\varpi'''\left(\varsigma^{-1}(\zeta(y))\right)\right)^{\gamma}\right)' \\ &\quad + \frac{1}{\kappa}\widehat{q}(y)\left(\frac{\mu}{6}y^3\right)^{\beta}\left(\varpi'''(y)\right)^{\beta} \leq 0. \end{aligned} \tag{13}$$

If we set

$$\eta(y) := \frac{1}{\varsigma_0} r\left(\varsigma^{-1}(y)\right)\left(\varpi'''\left(\varsigma^{-1}(y)\right)\right)^{\gamma} + \frac{p_0^{\beta}}{\varsigma_0\zeta_0} r\left(\varsigma^{-1}(\zeta(y))\right)\left(\varpi'''\left(\varsigma^{-1}(\zeta(y))\right)\right)^{\gamma},$$

then it is easy to see that

$$\eta\left(\zeta^{-1}(\varsigma(y))\right) \leq \left(\frac{\zeta_0 + p_0^{\beta}}{\varsigma_0\zeta_0}\right) r(y)\left(\varpi'''(y)\right)^{\gamma}.$$

Thus, from (13), we get that η is a positive solution of

$$\eta'(y) + \frac{1}{\kappa}\left(\frac{\mu y^3}{6 r^{1/\gamma}(y)}\right)^{\beta}\left(\frac{\varsigma_0\zeta_0}{\zeta_0 + p_0^{\beta}}\right)^{\beta/\gamma}\widehat{q}(y)\,\eta^{\beta/\gamma}\left(\zeta^{-1}(\varsigma(y))\right) \leq 0.$$

which is a contradiction. The proof is complete. □

Theorem 2. *Assume that (6) holds. If the differential inequality*

$$\vartheta'(y) + \frac{1}{\kappa}\left(\frac{\mu y^3}{6 r^{1/\gamma}(y)}\right)^{\beta}\left(\frac{\varsigma_0\zeta_0}{\zeta_0 + p_0^{\beta}}\right)\widehat{q}(y)\,\vartheta^{\beta/\gamma}(\varsigma(y)) \leq 0 \tag{14}$$

is oscillatory for some $\mu \in (0,1)$, then (1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 1, we get (13). If we set $\vartheta(y) := r\left(\varsigma^{-1}(y)\right)\left(\varpi'''\left(\varsigma^{-1}(y)\right)\right)^{\gamma}$, then ϑ is a positive solution of (14), which is a contradiction. The proof is complete. □

Corollary 1. *Let $\gamma = \beta$ and (6) holds. If $\xi(y) \leq y$ and*

$$\liminf_{y\to\infty} \int_{\xi(y)}^{y} \frac{s^{3\gamma}}{r(s)}\widehat{q}(s)\,\mathrm{d}s > \left(\frac{\zeta_0 + p_0^{\gamma}}{\varsigma_0\zeta_0}\right)\frac{\kappa 6^{\gamma}}{\mathrm{e}}, \tag{15}$$

where $\xi(y) = \zeta^{-1}(\varsigma(y))$ or $\varsigma(y)$, then (1) is oscillatory.

Proof. It is well-known (see, e.g., ([17], Theorem 2.1.1)) that condition (15) implies the oscillation of (7) and (14). □

Theorem 3. *Assume that $p_0 < 1$ and $\varsigma(y) \leq y$. If the equation*

$$\psi'(y) + (1 - p_0)^{\beta}\left(\frac{\mu\varsigma^3(y)}{6 r^{1/\gamma}(\varsigma(y))}\right)^{\beta} q(y)\,\psi^{\beta/\gamma}(\varsigma(y)) = 0 \tag{16}$$

is oscillatory for some $\mu \in (0,1)$, then (1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 1, we get (8). From definition of ϖ, we get

$$\begin{aligned} u(y) &\geq \varpi(y) - p_0 u(\zeta(y)) \geq \varpi(y) - p_0 \varpi(\zeta(y)) \\ &\geq (1-p_0)\varpi(y), \end{aligned}$$

which with (1) gives

$$\left(r(y)\left(\varpi'''(y)\right)^{\gamma}\right)' + q(y)(1-p_0)^{\beta}\varpi^{\beta}(\varsigma(y)) \leq 0. \tag{17}$$

From Lemma 3, we obtain

$$\varpi(y) \geq \frac{\mu}{6} y^3 \varpi'''(y). \tag{18}$$

Combining (17) and (18), we get

$$\left(r(y)\left(\varpi'''(y)\right)^{\gamma}\right)' + q(y)(1-p_0)^{\beta}\left(\frac{\mu}{6}\varsigma^3(y)\right)^{\beta}\left(\varpi'''(\varsigma(y))\right)^{\beta} \leq 0.$$

Hence, if we set $\psi := r(\varpi''')^{\gamma}$, then we get that ψ is a positive solution of the inequality

$$\psi'(y) + (1-p_0)^{\beta}\left(\frac{\mu\varsigma^3(y)}{6r^{1/\gamma}(\varsigma(y))}\right)^{\beta} q(y)\,\psi^{\beta/\gamma}(\varsigma(y)) \leq 0.$$

In view of ([19], Corollary 1), the associated delay differential Equation (16) also has a positive solution, which is a contradiction. The proof is complete. □

Corollary 2. *Let $\gamma = \beta$, $p_0 < 1$ and $\varsigma(y) \leq y$. If*

$$\liminf_{y\to\infty} \int_{\varsigma(y)}^{y} \frac{\varsigma^{3\gamma}(s)}{r(\varsigma(s))} q(s)\,\mathrm{d}s > \frac{6^{\gamma}}{(1-p_0)^{\gamma}\,\mathrm{e}}, \tag{19}$$

then (1) is oscillatory.

Proof. It is well-known (see, e.g., ([17], Theorem 2.1.1)) that condition (19) implies the oscillation of (16). □

Theorem 4. *Assume that $p_0 < 1$ and $\varsigma(y) \leq y$. If there exists a positive functions ρ, $\delta \in C^1([y_0,\infty))$ such that*

$$\int_{y_0}^{\infty}\left(\Psi(s) - \frac{2^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{r(s)\left(\rho'(s)\right)^{\gamma+1}}{\mu_1^{\gamma}s^{2\gamma}\rho^{\gamma}(s)}\right)\mathrm{d}s = \infty \tag{20}$$

and

$$\int_{y_0}^{\infty}\left(\tau(s) - \frac{\left(\delta'(s)\right)^2}{4\delta(s)}\right)\mathrm{d}s = \infty, \tag{21}$$

for some $\mu_1, \mu_2 \in (0,1)$ and every $M_1, M_2 > 0$, where

$$\Psi(y) := M_1^{\beta-\gamma}\rho(y)\,q(y)(1-p_0)^{\beta}\left(\frac{\varsigma(y)}{y}\right)^{3\beta}$$

and

$$\tau(y) := (1-p_0)^{\beta/\gamma}\delta(y)\,M_2^{(\beta-\gamma)/\gamma}\int_y^{\infty}\left(\frac{1}{r(z_1)}\int_{z_1}^{\infty} q(s)\frac{\varsigma^{\beta}(s)}{s^{\beta}}\mathrm{d}s\right)^{1/\gamma}\mathrm{d}z_1,$$

then (1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 3, we find (8) and (17). From (8), we have ϖ'' is of one sign.

In the case where $\varpi''(y) > 0$, we define

$$\eta(y) := \rho(y) \frac{r(y)(\varpi'''(y))^{\gamma}}{\varpi^{\gamma}(y)} > 0.$$

By differentiating and using (17), we get

$$\eta'(y) \leq \frac{\rho'(y)}{\rho(y)}\eta(y) - \rho(y) q(y)(1-p_0)^{\beta} \frac{\varpi^{\beta}(\varsigma(y))}{\varpi^{\gamma}(y)} - \gamma\rho(y) \frac{r(y)(\varpi'''(y))^{\gamma}}{\varpi^{\gamma+1}(y)}\varpi'(y). \tag{22}$$

By Lemma 1, we find $\varpi(y) \geq \frac{y}{3}\varpi'(y)$, and hence,

$$\frac{\varpi(\varsigma(y))}{\varpi(y)} \geq \frac{\varsigma^3(y)}{y^3}. \tag{23}$$

Using Lemma 3, we get

$$\varpi'(y) \geq \frac{\mu_1}{2} y^2 \varpi'''(y), \tag{24}$$

for all $\mu_1 \in (0,1)$. Thus, by (22)–(24), we obtain

$$\begin{aligned}\eta'(y) \quad \leq \quad & \frac{\rho'(y)}{\rho(y)}\eta(y) - \rho(y) q(y)(1-p_0)^{\beta} \varpi^{\beta-\gamma}(y) \left(\frac{\varsigma(y)}{y}\right)^{3\beta} \\ & -\gamma\mu_1 \frac{y^2}{2r^{1/\gamma}(y)\rho^{1/\gamma}(y)} \eta^{\frac{\gamma+1}{\gamma}}(y).\end{aligned}$$

Since $\varpi'(y) > 0$, there exist a $y_2 \geq y_1$ such that

$$\varpi(y) > M, \tag{25}$$

for all $y \geq y_2$ and a constant $M > 0$. Using the inequality

$$Ex - Fx^{(\gamma+1)/\gamma} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} E^{\gamma+1} F^{-\gamma}, \quad F > 0,$$

with $E = \rho'(y)/\rho(y)$, $F = \gamma\mu y^2/2r^{1/\gamma}(y)\rho_1^{1/\gamma}(y)$ and $u = \eta$, we find

$$\eta'(y) \leq -\Psi(y) + \frac{2^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{r(y)(\rho'(y))^{\gamma+1}}{\mu_1^{\gamma} y^{2\gamma} \rho^{\gamma}(y)}.$$

This implies that

$$\int_{y_1}^{y} \left(\Psi(s) - \frac{2^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{r(s)(\rho'(s))^{\gamma+1}}{\mu_1^{\gamma} s^{2\gamma}\rho^{\gamma}(s)}\right) \mathrm{d}s \leq \eta(y_1),$$

which contradicts (20).

For $\varpi''(y) < 0$, integrating (17) from y to z, we obtain

$$r(z)(\varpi'''(z))^{\gamma} - r(y)(\varpi'''(y))^{\gamma} \leq -\int_{y}^{z_1} q(s)(1-p_0)^{\beta} \varpi^{\beta}(\varsigma(s))\,\mathrm{d}s. \tag{26}$$

From Lemma 1, we see that $\varpi(y) \geq y\varpi'(y)$, and hence,

$$\varpi(\varsigma(y)) \geq \frac{\varsigma(y)}{y}\varpi(y). \tag{27}$$

For (26), letting $z \to \infty$ and using (27), we get

$$r(y)\left(\varpi'''(y)\right)^{\gamma} \geq (1-p_0)^{\beta}\varpi^{\beta}(y)\int_y^{\infty} q(s)\frac{\varsigma^{\beta}(s)}{s^{\beta}}\mathrm{d}s. \tag{28}$$

Integrating (28) from y to ∞, we get

$$\varpi''(y) \leq -(1-p_0)^{\beta/\gamma}\varpi^{\beta/\gamma}(y)\int_y^{\infty}\left(\frac{1}{r(z_1)}\int_{z_1}^{\infty} q(s)\frac{\varsigma^{\beta}(s)}{s^{\beta}}\mathrm{d}s\right)^{1/\gamma}\mathrm{d}z_1, \tag{29}$$

for all $\mu_2 \in (0,1)$. Now, we define

$$\vartheta(y) = \delta(y)\frac{\varpi'(y)}{\varpi(y)}.$$

Then $\vartheta(y) > 0$ for $y \geq y_1$. By using (25) and (29), we obtain

$$\begin{aligned}
\vartheta'(y) &= \frac{\delta'(y)}{\delta(y)}\vartheta(y) + \delta(y)\frac{\varpi''(y)}{\varpi(y)} - \delta(y)\left(\frac{\varpi'(y)}{\varpi(y)}\right)^2 \\
&\leq \frac{\delta'(y)}{\delta(y)}\vartheta(y) - \frac{1}{\delta(y)}\vartheta^2(y) \\
&\quad -(1-p_0)^{\beta/\gamma}\delta(y)\varpi^{\beta/\gamma-1}(y)\int_y^{\infty}\left(\frac{1}{r(z_1)}\int_{z_1}^{\infty} q(s)\frac{\varsigma^{\beta}(s)}{s^{\beta}}\mathrm{d}s\right)^{1/\gamma}\mathrm{d}z_1.
\end{aligned}$$

Thus, we find

$$\vartheta'(y) \leq -\tau(y) + \frac{\delta'(y)}{\delta(y)}\vartheta(y) - \frac{1}{\delta(y)}\vartheta^2(y),$$

and so

$$\vartheta'(y) \leq -\tau(y) + \frac{(\delta'(y))^2}{4\delta(y)}.$$

Then, we obtain

$$\int_{y_1}^{y}\left(\tau(s) - \frac{(\delta'(y))^2}{4\delta(y)}\right)\mathrm{d}s \leq \vartheta(y_1),$$

which contradicts (21). This completes the proof. □

Example 1. *Consider the differential equation*

$$\left(\left((u+p_0u(\delta y))'''\right)^{\gamma}\right)' + \frac{q_0}{y^{3\gamma+1}}u(\lambda y) = 0,\ y \geq 1, \tag{30}$$

where $\delta, \lambda \in (0,1]$ *and* $p_0, q_0 > 0$. *Let* $\gamma = \beta$, $r(y) = 1$, $p(y) = p_0$, $\zeta(y) = \delta y$, $\varsigma(y) = \lambda y$ *and* $q(y) = q_0/y^{3\gamma+1}$. *Hence, it is easy to see that*

$$\widehat{q}(y) = q_0\lambda^{3\gamma+1}\frac{1}{y^{3\gamma+1}}.$$

Using Corollary 1, the Equation (30) is oscillatory if

$$q_0 \ln \frac{1}{\lambda} > \kappa \left(\frac{\delta + p_0^\gamma}{\delta} \right) \frac{6^\gamma}{\lambda^{3\gamma} \mathrm{e}}. \tag{31}$$

From Corollary 2, if

$$q_0 \ln \frac{1}{\lambda} > \frac{1}{(1-p_0)^\gamma} \frac{6^\gamma}{\lambda^{3\gamma} \mathrm{e}}, \tag{32}$$

then (30) is oscillatory.

Finally, if we set $\rho(s) := y^{3\gamma}$ and $\delta(y) := y^2$, then we have

$$\Psi(y) = q_0 (1-p_0)^\gamma \lambda^{3\gamma} \frac{1}{s}$$

and

$$\tau(y) := \frac{1}{2} \left(\frac{q_0}{3\gamma} \right)^{1/\gamma} (1-p_0) \lambda.$$

Thus, from Theorem 4, Equation (30) is oscillatory if

$$q_0 (1-p_0)^\gamma \lambda^{3\gamma} > 2^\gamma 3^{\gamma+1} \left(\frac{\gamma}{\gamma+1} \right)^{\gamma+1} \tag{33}$$

and

$$q_0 > \left(\frac{2}{(1-p_0)\lambda} \right)^\gamma 3\gamma. \tag{34}$$

3. Conclusions

In this article, we studied the oscillatory properties of 4th-order differential equations. New oscillation criteria are established. We used Riccati technique and the theory of comparison to prove that every solution of (1) is oscillatory.

Further, we shall study Equation (1) under the condition $\int_{y_0}^{\infty} \frac{1}{r^{1/\gamma}(s)} \mathrm{d}s < \infty$, in the future work.

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