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The Problem of Invariance in Nonlinear Discrete-Time Dynamic Systems

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Abstract: The paper considers the problem of invariance with respect to the unknown input for discrete-time nonlinear dynamic systems. To solve the problem, the algebraic approaches, called algebra of functions and logic–dynamic approach, are used. Such approaches assume that description of the system may contain non-differentiable functions. Necessary and sufficient conditions of solvability the problem are obtained. Moreover, procedures which find the appropriate functions and matrices are developed. Some applications of such invariance in fault detection and isolation, disturbance decoupling problem, and fault-tolerant control are considered.

Keywords: nonlinear dynamic systems; invariance; unknown input; discrete-time systems; algebraic approaches

1. Introduction and Problem Statement

The problem of invariance with respect (IWR) to the unknown input in nonlinear dynamic systems is of wide theoretical and practical applications, in particular, in fault diagnosis [1–3], fault-tolerant control [4,5] and disturbance decoupling [6–9]. The problem of invariance in these branches is solved mainly for continuous-time systems based on the methods of linear algebra [10] and differential geometry developed in [11]. Such tools are of limited field of application in practice since actual systems may contain different non-differentiable function such as Coulomb friction, saturation, hysteresis and dead zone.

It is well-known from literature that the extensions of the differential geometric tools are not well developed for discrete-time systems in comparison with the continuous time case, see [12–14] (disturbance decoupling problem solution), [15] (transitivity and accessibility problems), [16] (non-interactive control). Such extensions are rather complicated and assume that systems under consideration are described by analytical functions. For this reason, such extensions are of limited field of application as well.

In this paper, we investigate the problem of IWR to the unknown input in discrete-time nonlinear dynamic systems. To overcome the difficulty with the above-mentioned practical applications, it is suggested to solve the problem of invariance based on algebra of functions and logic–dynamic approach (LDA). The advantage of these approaches, if compared to well-known linear algebraic [10] and differential geometric methods [10,17], is that they are applicable to non-differentiable systems.

The algebra of functions was developed in [18,19] on the basis of the pair algebra of partitions [20] elaborated for finite automata study. The algebra of functions is intended mainly for discrete-time nonlinear dynamic systems with non-differentiable nonlinearities. Since the algebra of functions is not well-known and demands complex analytical calculations, the webMathematica based software was developed [21].

The LDA was suggested in [22] for solving different problems in dynamic system theory. The advantage of the LDA is that under some limitations on the original system and a class of possible solutions it uses only methods of linear algebra. Moreover, the LDA can be applied to both continuous-time and discrete-time systems with non-differentiable nonlinearities. Shortcoming of the LDA is that functions transforming the system under consideration are required to be linear, see the relation $\varphi(x(t)) = \Phi x(t)$ in Section 4. Such a requirement imposes definite limitations on application of the LDA.

The contributions of this paper can be summarized as follows: (1) some relations describing the problem of invariance with respect to the unknown input in terms of the algebra of functions and the LDA are obtained; (2) links between the algebra of functions relations and the LDA relations are established since such links are interesting and useful for theory and practice, but not well known.

Consider a discrete-time nonlinear dynamic system described by the equations

$$x(t+1) = f(x(t), u(t), w(t)), \quad y(t) = h(x(t)). \tag{1}$$

Here $x \in X \subset R^n$, $u \in U \subset R^m$ and $y \in Y \subset R^l$ are vectors of state, input and output; $w(t) \in W \subset R^p$ is the unknown input; f and h are nonlinear functions. Note that f may be non-differentiable function.

The problem of invariance with respect to the unknown input is stated as follows: find a vector function $x_0 = \varphi(x)$, transforming system (1) into the system

$$x_0(t+1) = f_0(x_0(t), y(t), u(t)), \quad y_0(t) = h_0(x_0(t)) \tag{2}$$

which does not depend on the unknown input. Here $x_0 \in R^{n_0}$, $n_0 < n$, is the state vector, f_0 and h_0 are some functions to be determined. To solve this problem for system (1) and to develop a procedure of transformation, the algebra of functions will be used.

2. Algebra of Functions

The main definitions and concepts used in this paper are as follows [19,23,24]. Let S be a set of vector functions with the domain X. The elements of algebra of functions are vector functions on S with the following relations, operations and operators: (1) relation of partial preorder $\leq$, (2) two binary operations $\times$ and $\oplus$, (3) binary relation Δ, (4) two operators **m** and **M**.

Given $\alpha, \beta \in S$, one says that $\alpha \leq \beta$ if a function γ exists such that $\beta(x) = \gamma(\alpha(x))$ for all $x \in X$. When $\alpha \leq \beta$ and $\beta \leq \alpha$, the functions α and β are called equivalent, denoted as $\alpha \cong \beta$.

The relation $\cong$ is reflexive, symmetric and transitive; therefore, this relation divides the set S into the classes of equivalence. Denote by $S\backslash \cong$ the set of all classes of equivalence; then the relation $\leq$ is partial order on this set. It can be shown that $S\backslash \cong$ is a lattice where every two elements α and β have a unique supremum (least upper bound) $\sup(\alpha, \beta)$ and a unique infimum (greatest lower bound) $\inf(\alpha, \beta)$. As is customary, we will operate not with $\sup(\alpha, \beta)$ and $\inf(\alpha, \beta)$ but with two binary operations $\oplus$ and $\times$ respectively.

In the simple cases, the definition $\alpha \oplus \beta = \sup(\alpha, \beta)$ is used to compute $\alpha \oplus \beta$. The rule for operation $\times$ is simple:

$$(\alpha \times \beta)(x) = (\alpha(x)\ \beta(x))^{\mathrm{T}}$$

Note that two special vector functions **0** and **1** exist; they correspond to the identity and constant functions, respectively, in the sense that for every vector function $\alpha \in S$, $\mathbf{0} \leq \alpha \leq \mathbf{1}$.

Consider illustrative example. Let $\alpha(x) = (x_1\ x_3 + x_4)^{\mathrm{T}}$ and $\beta(x) = (x_2\ x_3\ x_4)^{\mathrm{T}}$, then

$$(\alpha \times \beta)(x) = (x_1\ x_2\ x_3\ x_4)^{\mathrm{T}} = 0\,, (\alpha \oplus \beta)(x) = x_3 + x_4.$$

Clearly, $(\alpha \times \beta)(x) \leq \alpha(x) \leq (\alpha \oplus \beta)(x)$ and $(\alpha \times \beta)(x) \leq \beta(x) \leq (\alpha \oplus \beta)(x)$; the functions $\alpha(x)$ and $\beta(x)$ are incomparable.

Given $\alpha, \beta \in S$, then $(\alpha, \beta) \in \Delta$ if a function f_0 exists such that for all $x, u, w \in X \times U \times W$ $\beta(f(x, u, w)) = f_0(\alpha(x), u, w)$. When $(\alpha, \beta) \in \Delta$, one says that α and β form an ordered pair. Binary relation Δ is used to define the operators **m** and **M**.

Operator $\mathbf{m}(\alpha)$ is a function in S, satisfying the following conditions: (i) $(\alpha, \mathbf{m}(\alpha)) \in \Delta$, (ii) if $(\alpha, \beta) \in \Delta$, then $\mathbf{m}(\alpha) \leq \beta$.

Operator $\mathbf{M}(\beta)$ is a function in S, satisfying the following conditions: (i) $(\mathbf{M}(\beta), \beta) \in \Delta\ \Delta$, (ii) if $(\alpha, \beta) \in \Delta$, then $\alpha \leq \mathbf{M}(\beta)$.

It follows from the last definitions that given α, $\mathbf{m}(\alpha)$ is the minimal function, that forms a pair with α and given β, $\mathbf{M}(\beta)$ is the maximal function, that forms a pair with β.

Lemma 1 [19,23]. *Let α and β be some functions from S. Then*

(i). $\alpha \leq \beta \Leftrightarrow \alpha \cong \alpha \times \beta \Leftrightarrow \beta \cong \alpha \oplus \beta$,

(ii). $\alpha \leq \beta \Leftrightarrow \mathbf{M}(\alpha) \leq \mathbf{M}(\beta) \Leftrightarrow \mathbf{m}(\alpha) \leq \mathbf{m}(\beta)$,

(iii). $\mathbf{M}(\alpha \times \beta) \cong \mathbf{M}(\alpha) \times \mathbf{M}(\beta), \mathbf{m}(\alpha \oplus \beta) \cong \mathbf{m}(\alpha) \oplus \mathbf{m}(\beta)$.

Computation of the operators $\mathbf{m}$ and $\mathbf{M}$. It is known from [19,23] that there exists the function γ satisfying the condition $(\alpha \times u(t)) \oplus f \cong \gamma(f)$; define $\mathbf{m}(\alpha) \cong \gamma$. When $\beta(f(x(t), u(t)))$ can be transformed into the form

$$\beta(f(x(t), u(t))) = \sum_{i=1}^{d} a_i(x(t)) b_i(u(t)),$$

where $a_1(x(t)), a_2(x(t)), \ldots, a_d(x(t))$ are arbitrary vector functions and $b_1(u(t)), b_2(u(t)), \ldots, b_d(u(t))$ are linearly independent functions, then $\mathbf{M}(\beta) = a_1 \times a_2 \times \ldots \times a_d$.

3. Problem Solution

To solve the problem of IWR to the unknown input for system (1), we at first find a vector function φ_0 with maximal number of functionally independent components such that the function $\varphi_0(f(x, u, w))$ is independent of the unknown function $w(t)$. The function φ_0 can be obtained by heuristic methods. Actually, this function provides some combination of the function $f(x, u, w)$ to achieve independence of the unknown input.

One says that the function φ is (h, f)-invariant if $\varphi(f(x, u, w)) = f_0(\varphi(x), h(x), u, w)$ for some vector function f_0. It is known [19,23] that φ is (h, f)-invariant if and only if $\varphi \times h \leq \mathbf{M}(\varphi)$ or $\mathbf{m}(\varphi \times h) \leq \varphi$.

Theorem 1. *System (2) is IWR to the unknown input $w(t)$ if and only if (h, f)-invariant function φ exist such that*

$$\varphi_0 \leq \varphi. \tag{3}$$

Proof. Let φ be (h, f)-invariant function satisfying (3). By definition of (h, f)-invariance, $\varphi(f(x, u, w)) = f_0(\varphi(x), h(x), u, w)$. Since $\varphi_0 \leq \varphi$, then $\gamma(\varphi_0) = \varphi$ for some function γ, therefore $\varphi(f(x, u, w)) = \gamma(\varphi_0(f(x, u, w)))$. By definition, $\varphi_0(f(x, u, w))$ is independent of the unknown function $w(t)$; as a result, the function $f_0(\varphi(x), h(x), u, w) = \gamma(\varphi_0(f(x, u, w)))$ is independent of $w(t)$ as well. On the other hand, let $\varphi(f(x, u, w))$ is independent of $w(t)$. Since the function φ_0 has the same property and has maximal number of functionally independent components, then $\gamma(\varphi_0) = \varphi$ for some function γ or $\varphi_0 \leq \varphi$. Since φ is (h, f)-invariant, then $\varphi(f(x, u, w)) = f_0(\varphi(x), h(x), u, w)$ for some function f_0. □

To construct system (2) of maximal dimension, the function φ should be minimal in terms of the partial preorder relation $\leq$. Such a function can be obtained as follows.

Theorem 2 [19,23]. *Given φ_0, compute recursively for $i \geq 0$, based on the formula*

$$\varphi_{i+1} = \varphi_i \oplus \mathbf{m}(\varphi_i \times h), \tag{4}$$

the sequence of vector functions $\varphi_0 \leq \varphi_1 \leq \ldots$ The sequence converges in a finite number of steps, since if $\varphi_i \neq \varphi_{i-1}$, the number of components of the function φ_i is less than that of the function φ_{i-1}, $i = 1, 2, \ldots$

This means that there exists a finite k such that $\varphi_{k+1} \cong \varphi_k$. *The function* $\varphi := \varphi_k$ *is minimal satisfying the condition* $\varphi_0 \leq \varphi$.

Hence, to solve the problem of IWR to the unknown input, one has to find the function φ_0 and then to use the recursive (4).

When the problem of fault detection and isolation is considered, one needs to generate co-called residual $r(t)$ as a mismatch between system (1) behavior and the model (2) behavior, based on the outputs $y(t) = h(x(t))$ and $y_0(t) = h_0(x_0(t))$, respectively. Such a mismatch is presented in the form $r(t) = \rho(y(t)) - y_0(t)$ for some function ρ. When faults are absent, $r(t) = 0$ or $\rho(y(t)) = y_0(t)$. It can be shown that the last equality is equivalent to the functional relation

$$\rho(h) = h_0(\varphi) \neq \mathbf{1}. \tag{5}$$

Theorem 3. *If* $h \oplus \varphi \neq \mathbf{1}$, *the relation (5) is true for some nontrivial functions* ρ *and* h_0.

Proof. Let $h \oplus \varphi \neq \mathbf{1}$, then $h \oplus \varphi \geq h$ and $h \oplus \varphi \geq \varphi$ by the definition of operation $\oplus$. By definition of the partial order relation $\leq$, the nontrivial functions ρ and h_0 exist such that $\rho(h) = h \oplus \varphi = h_0(\varphi) \neq \mathbf{1}$. □

Given ρ, to construct system (2) of minimal dimension, satisfying the conditions (3) and (5), (h, f)-invariant function φ should be maximal in terms of the partial preorder relation $\leq$ Such a function can be obtained based on the following Algorithm.

Algorithm

Step 1. Set $\beta_0 := \rho(h)$ and $i := 0$.
Step 2. Compute the function $\gamma_i = \mathbf{M}(\beta_i)$.
Step 3. If the components of the vector function γ_i can be expressed in terms of the function $h \times \beta_0 \times \ldots \times \beta_i$, then go to Step 5. Otherwise, go to Step 4.
Step 4. Find the vector function β_{i+1} with minimal number of components such that $h \times \beta_0 \times \ldots \times \beta_{i+1} \leq \gamma_i$, set $i := i+1$ and go to Step 2.
Step 5. Define $\varphi := \beta_0 \times \ldots \times \beta_i$.

Solution of the problem of IWR to the unknown input may be simplified significantly when the function φ is sought in a class of linear functions. Actually, this restricts a set of possible solutions, but allows to solve the problem for systems with non-differential nonlinearities by methods of linear algebra. Note that if the problem of invariance for system (1) has a solution with linear function φ, such a solution can be found by the logic–dynamic approach described below [22].

4. Logic–Dynamic Approach

To use the LDA, one has to present system (1) in the form

$$x(t+1) = Fx(t) + Gu(t) + \Psi(x(t), u(t)) + Lw(t), \quad y(t) = Hx(t), \tag{6}$$

where

$$\Psi(x(t), u(t)) = C\begin{pmatrix} \varphi_1(A_1 x(t), u(t)) \\ \cdots \\ \varphi_q(A_q x(t), u(t)) \end{pmatrix}; \tag{7}$$

matrices F and G are used to describe the linear dynamic part of the system (6); H, H_*, C and L are constant matrices, the functions $\varphi_1, \ldots, \varphi_q$ may be non-differentiable, $A_1, \ldots, A_q$ are constant matrices. The model (6) can be obtained from the original system (1) by some simple transformations [22].

Specifically, the linear part with the matrices F and G, is separated from the nonlinear part (7) containing non-differentiable functions $\varphi_1, \ldots, \varphi_q$ and matrices $C, A_1, \ldots, A_q$.

By analogy with (2), a system IWR to the unknown input is described by

$$x_0(t+1) = F_0 x_0(t) + G_0 u(t) + Jy(t) + C_0 \begin{pmatrix} \varphi_1(A_{01} z_0(t), u(t)) \\ \cdots \\ \varphi_q(A_{0q} z_0(t), u(t)) \end{pmatrix}, \quad y(t) = H_0 x_0(t), \tag{8}$$

where $x_0 \in R^{n_0}$, $z_0 = (x_0^{\mathrm{T}} y^{\mathrm{T}})^{\mathrm{T}}$, $n_0 \leq n$, $F_0, G_0, J, C_0, A_{01}, \ldots, A_{0q}$ are matrices to be determined.

Assuming initially $q = 1$, we construct system (8). The LDA, which is used for solving this problem, contains three main steps [22].

Step 1. Remove the term $\Psi(x(t), u(t))$ from the original system (6).

Step 2. Solve the problem of IWS to the unknown input for the linear part under some linear restriction. This restriction is necessary to find out whether or not the nonlinear term can be designed based on the linear solution.

Step 3. Supplement the solution, obtained at Step 2, by the transformed nonlinear term.

It should be noted that the LDA can be applied to the continuous-time systems as well as to the discrete-time ones. This is possible due to a linear nature of the solution at Step 1 and a linear nature of the restriction at Step 2. Moreover, note that a transformation at Step 3 does not transform the nonlinear functions $\varphi_1, \ldots, \varphi_q$ themselves, but transforms their arguments based on the relation (9) and the matrix C into C_0.

We will assume that $x_0(t) = \varphi(x(t)) = \Phi x(t)$ for some matrix Φ of maximal rank satisfying the following conditions [22]:

$$\Phi F = F_0 \Phi + JH, \quad G_0 = \Phi G, \quad \Phi L = 0.$$

One can show that the relations $C_0 = \Phi C$ and

$$A = A_0 \begin{pmatrix} \Phi \\ H \end{pmatrix}, \tag{9}$$

corresponding to the term $\Psi(x(t), u(t))$, are true [22]. Clearly, the last relation is equivalent to

$$\operatorname{rank}\begin{pmatrix} \Phi \\ H \end{pmatrix} = \operatorname{rank}\begin{pmatrix} \Phi \\ H \\ A \end{pmatrix}. \tag{10}$$

When $q > 1$, the matrix A in (9) and (10) is replaces by A_i, $i = 1, \ldots, q$. Note that (10) is precisely restriction which is checked at Step 2.

5. Solvability Conditions of Invariance

Before constructing system (8), it is worth to check whether or not such a system exists. To make the appropriate checking, note that analog of the function φ_0 is the matrix L_0 of maximal rank satisfying the condition $L_0 L = 0$. Analog of the condition $\varphi_0 \leq \varphi$ is the relation $\Phi L = 0$ which is equivalent to $\Phi = N L_0$ for some matrix N.

Replace the matrix Φ in the equation $\Phi F = F_0 \Phi + J_0 H$ by $\Phi = N L_0$ and transform the result: $N L_0 F - F_0 N L_0 - J_0 H = 0$. The obtained equation has a nontrivial solution with some matrices F_0 and J_0 when the rows of the matrices$L_0 F$ and $(L_0^{\mathrm{T}}\, H^{\mathrm{T}})^{\mathrm{T}}$ are linearly dependent that is equivalent to the rank condition

$$\operatorname{rank}\begin{pmatrix} L_0 F \\ L_0 \\ H \end{pmatrix} < \operatorname{rank}(L_0 F) + \operatorname{rank}\begin{pmatrix} L_0 \\ H \end{pmatrix}. \tag{11}$$

Considering analogously the Equation (8), we get the necessary condition

$$\operatorname{rank}\begin{pmatrix} L_0 \\ H \end{pmatrix} = \operatorname{rank}\begin{pmatrix} L_0 \\ H \\ A \end{pmatrix}. \tag{12}$$

Note that the relations (11) and (12) are necessary solvability conditions, i.e., if one of the conditions (11) and (12) is not true, the system IWR to the unknown input does not exist.

6. Problem Solution

To design system (8), we assume that the matrices F_0 and H_0 are sought in the form

$$F_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \; H_0 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

As a result, the equation $\Phi F = F_0 \Phi + JH$ is replaced by k equations:

$$\Phi_i F = \Phi_{i+1} + J_{0i} H, \; i = 1, \dots, n_0 - 1, \; \Phi_{n_0} F = J_{n_0} H \tag{13}$$

where Φ_i and J_i denote the i-th rows of the matrices Φ and J, respectively, $i = 1, \dots, n_0$, n_0 is the number of the matrix Φ rows.

It was shown in [22] that (13) and the condition $\Phi L = 0$ can be transformed into the single equation

$$(\Phi_1 - J_1 - J_2 \dots - J_k)(W^{(k)} \; L^{(k)}) = 0, \; k = 1, 2, \dots \tag{14}$$

where

$$W^{(k)} = \begin{pmatrix} F^k \\ HF^{k-1} \\ \dots \\ H \end{pmatrix}, \; L^{(k)} = \begin{pmatrix} L & FL & \cdots & F^{k-1}L \\ 0 & HL & \cdots & HF^{k-1}L \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

To construct the system of maximal dimension, set $k := n - p$ and check the condition

$$\operatorname{rank}(W^{(k)} \; L^{(k)}) < lk + n. \tag{15}$$

When (15) is true, then there exists the row $(\Phi_1 - J_1 \dots - J_k)$ such that (14) has a solution. Then one calculates the matrix Φ based on (13) and checks the condition (10). If it is satisfied, find the matrix A_0 from (9), set $n_0 := k$, $G_0 := \Phi G$, and $C_0 := \Phi C$. Thus, system (8) IWR to the unknown input $w(k)$ has been constructed.

If (15) is not true, set $k := k - 1$ and continue checking (15) and (10). If (15) and (10) are not true for all k, then the system IWR to the unknown input does not exist and the problem has no solution. Since the dimension n_0 is maximal, the matrix Φ is an analog of the function φ from Theorem 2.

By analogy with general case, in some applications it is necessary to take into account the output function $y_0(t) = H_0 x_0(t)$ and the requirement $y_0(t) = Ry(t)$ for all $t \geq 0$ and some matrix R. It can be shown that this is equivalent to the relation

$$RH = H_0 \Phi, \tag{16}$$

which is analog of (5). Replace the matrix Φ in (16) by $\Phi = NL_0$ and obtain the equation $RH - H_0NL_0 = 0$ which has a nontrivial solution when

$$\text{rank}\begin{pmatrix} H \\ L_0 \end{pmatrix} < \text{rank}(H) + \text{rank}(L_0). \tag{17}$$

Note that (17) is necessary solvability condition additional to (11) and (12). When the matrix Φ has been obtained, the matrices R and H_0 can be found from (16).

To construct system (8) of minimal dimension, one takes $k := 1$ and checks the condition (15). When (15) is satisfied, one calculates the matrix Φ and checks the condition (10). If it is satisfied, find the matrix A_0 from (9), set $n_0 := k$, $G_0 := \Phi G$ and $C_0 := \Phi C$. If (15) is not true, set $k := k + 1$ and continue checking (15) and (10).

7. Applications

The property "invariance with respect to the unknown inputs" has many different practical applications. Consider some of them related to the fault diagnosis, disturbance decoupling and fault-tolerant control. Here, the unknown inputs are interpreted as the disturbances and faults.

In the fault diagnosis process, the residual is generated as a result of mismatch between the original system behavior and the reference model behavior. Then a decision is made by evaluation of this residual. System (2) or (8) in the fault diagnosis process is used as a reference model. The residual $r(t)$ is generated in the form

$$r(t) = \rho(y(t)) - y_0(t) \text{ or } r(t) = Ry(t) - y_0(t). \tag{18}$$

Different tools for fault diagnosis have been developed: diagnostic observers, parity relations and identification [1–3].

The main goal of fault detection process is to construct system (2) or (8) of minimal dimension IWR to the disturbances such that the relation (5) or (16) holds. Consider the simple practical example of an electric servo–actuator described by the equations

$$\begin{aligned}
&x_1(t+1) = x_1(t) + k_1x_2(t),\\
&x_2(t+1) = x_2(t) + k_2x_3(t) + k_3 sign(x_2(t)) + w(t),\\
&x_3(t+1) = k_4x_2(t) + k_5x_3(t) + k_6u(t),\\
&y_1(t) = x_1(t), \;\; y_2(t) = x_3(t).
\end{aligned}$$

Using the LDA model (6), we obtain

$$F = \begin{pmatrix} 1 & k_1 & 0 \\ 0 & 1 & k_2 \\ 0 & k_4 & k_5 \end{pmatrix}, G = \begin{pmatrix} 0 \\ 0 \\ k_6 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, L = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, C = \begin{pmatrix} 0 \\ k_3 \\ 0 \end{pmatrix}, A = (0\,1\,0),\; \varphi = sign(Ax).$$

It can be shown that $\Phi = (k_4\;0\; -k_1)$, $R = (k_4 \; -k_1)$, $J = (k_4 - k_1k_5)$, $G_0 = -k_1k_6$, $C_0 = 0$. The diagnostic observer description is given by

$$x_*(t+1) = k_4y_1(t) - k_1k_5y_2(t) - k_1k_6u(t), \quad y_*(t) = x_*(t),$$

the residual is generated as follows: $r(t) = k_4y_1(t) - k_1y_2(t) - y_*(t)$. The observer is invariant with respect to the disturbances $w(t)$ and allows to detect the faults in sensors and deviation of the coefficients $k_1,\ k_4,\ k_5,\ k_6$ from their nominal values.

When the problem of fault isolation is solved, a bank of such systems is constructed where some faults are considered as the unknown inputs for each system which is constructed to be IWR to the disturbances and faults, considered as the unknown inputs and sensitive to other faults. As a result, we obtain selective sensitivity to different faults that allows to develop fault isolation process based on so-called matrix of syndromes [2].

Sliding mode observers are often used to solve the fault identification problem [3,25,26]. In [3,25], such observers are constructed on the basis of the original system and then IWR to the disturbances is achieved. In contrast to this approach, we suggest at first to construct system (8) of minimal dimension IWR to the disturbances and then design sliding mode observer based on this system [26]. This allows to reduce sliding mode observer complexity and relaxed the limitation imposed on the original system.

The disturbance decoupling problem can be stated as follows. The purpose is to find a dynamic measurement feedback in such a way that the output-to-be-controlled $y_*(t) = h_*(x(t))$, for $t \geq 0$, of the closed-loop system does not depend on the disturbances (unknown inputs) $w(t)$. This problem for nonlinear control systems has been studied in [6–9,23]. Except [23] the papers [6–9] study the continuous-time case, the solvability conditions are provided in the papers [6–8] on the basis of differential geometric tools.

To solve the disturbance decoupling problem for the initial system (1) or (6), system (2) or (8) of maximal dimension, IWR to the unknown inputs, are constructed at first under some restriction imposed by the function $h_*(x)$. Then this system is transformed into special compensator which generates the control for the initial system. The details for system (1) can be found in [23], for system (6) in [27,28].

Fault-tolerant control allows to meet the design purposes when the faults occur or if impossible, to redefine the attainable design purposes [4,5]. Active approaches in fault-tolerant control are fault accommodation and plant reconfiguration. The purpose of fault accommodation is to find a new control law which can attain the predefined control purposes. In system reconfiguration, either the controller or the faulty plant is reconfigured when the faults occur. Both approaches are based on system (2) or (8) of maximal dimension without the relations (5) and (16). Such a system is considered as a dynamic part of the compensator and then it is supplemented by static part, generating a new control. In [24], the problem of faulty plant reconfiguration has been solved based on the disturbance decoupling problem solution.

8. Conclusions

The paper deals with the problem of IWR to the unknown input in discrete-time nonlinear dynamic systems. So-called algebra of functions and logic–dynamic approach are used to solve the problem. The algebra of functions produces a solution in general form but demands analytical calculations. The advantage of the LDA is that only methods of linear algebra are used to solve the problems and the considered system may contain non-differential nonlinearities such as Coulomb friction, backlash and hysteresis. Moreover, the LDA methods can be applied both to the discrete-time and the continuous-time systems.

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References

1. Ding, S. *Data-driven Design of Fault Diagnosis and Fault-tolerant Control Systems*; Springer: London, UK, 2014.
2. Blanke, M.; Kinnaert, M.; Lunze, J.; Staroswiecki, M. *Diagnosis and Fault-Tolerant Control*; Springer: Berlin, Germany, 2006.
3. Alwi, H.; Edwards, C.; Tan, C. *Fault Detection and Fault-Tolerant Control Using Sliding Modes*; Springer: London, UK, 2011.
4. Tabatabaeipour, S.; Stroustrup, M.; Bak, T. Fault-tolerant control of discrete-time LPV systems using virtual actuators. *Int. J. Robust Nonlinear Control* **2015**, *25*, 707–734.
5. Puig, V. Fault diagnosis and fault tolerant control using set-membership approaches: Application to real case studies. *Int. J. Appl. Math. Comput. Sci.* **2010**, *20*, 619–635. [CrossRef]

6. Andiarti, R.; Moog, C.H. Output feedback disturbance decoupling in nonlinear systems. *IEEE Trans. Autom. Control* **1996**, *41*, 1683–1689. [CrossRef]
7. Battilotti, S. A sufficient condition for nonlinear disturbance decoupling with stability via measurement feedback. In Proceedings of the 36th Conference on Decision & Control, San Diago, CA, USA, 10–12 December 1997; pp. 3509–3514.
8. Isidori, A.; Krener, A.J.; Gori-Giorgi, C.; Monaco, S. Nonlinear decoupling via feedback: A differential gemetric approach. *IEEE Trans. Autom. Control* **1981**, *26*, 331–345. [CrossRef]
9. Xia, X.; Moog, C.H. Disturbance decoupling by measurement feedback for SISO nonlinear systems. *IEEE Trans. Autom. Control* **1999**, *44*, 1425–1429. [CrossRef]
10. Wonham, W. *Linear Multivariable Control: A Geometric Approach*; Springer: Berlin, Germany, 1979.
11. Isidori, A. *Nonlinear Control Systems*; Springer: London, UK, 1995.
12. Grizzle, J. Controlled invariance for discrete-time nonlinear systems with an application to the disturbance decoupling problem. *IEEE Trans. Autom. Control* **1985**, *30*, 868–873. [CrossRef]
13. Aranda-Bricaire, E.; Kotta, U. A geometric solution to the dynamic disturbance decoupling for discrete-time nonlinear systems. *Kybernetika* **2004**, *40*, 197–206.
14. Kaldmae, A.; Kotta, U. Disturbance decoupling for discrete-time nonlinear systems by static measurement feedback. In Proceedings of the 18th International Conference on Process Control, Tatranska Lomnica, Slovakia, 14–17 June 2011; pp. 135–140.
15. Albertini, F.; Sontag, E. Discrete-time transitivity and accessibility: Analytical systems. *SIAM J. Control Optim.* **1993**, *31*, 1599–1622. [CrossRef]
16. Califano, C.; Monaco, S.; Normand-Cyrot, D. Nonlinear non-interactive control with stability in discrete-time framework. *Int. J. Control* **2002**, *75*, 11–22. [CrossRef]
17. Conte, G.; Moog, C.H.; Perdon, A.M. *Algebraic Methods for Nonlinear Control Systems. Theory and Applications*; Springer: Berlin, Germany, 2007.
18. Zhirabok, A.; Shumsky, A. *The Algebraic Methods for Analysis of Nonlinear Dynamic Systems*; Dalnauka: Vladivostok, Russia, 2008. (In Russian)
19. Shumsky, A.; Zhirabok, A. Unified approach to the problem of full decoupling via output feedback. *Eur. J. Control* **2010**, *16*, 313–325. [CrossRef]
20. Hartmanis, J.; Stearns, R. *The Algebraic Structure Theory of Sequential Machines*; Prentice-Hall: New York, NY, USA, 1966.
21. Available online: http://webmathematica.cc.ioc.ee/webmathematica/NLControl/main/index.html (accessed on 1 July 2014).
22. Zhirabok, A.; Shumsky, A.; Solyanik, S.; Suvorov, A. Fault detection in nonlinear systems via linear methods. *Int. J. Appl. Math. Comput. Sci.* **2017**, *27*, 261–272. [CrossRef]
23. Kaldmae, A.; Kotta, U.; Shumsky, A.; Zhirabok, A. Measurement feedback disturbance decoupling in discrete-time nonlinear systems. *Automatica* **2013**, *49*, 2887–2891. [CrossRef]
24. Kaldmae, A.; Kotta, U.; Jiang, B.; Shumsky, A.; Zhirabok, A. Faulty plant reconfiguration based on disturbance decoupling methods. *Asian J. Control* **2016**, *8*, 858–867. [CrossRef]
25. Yan, Z.; Edwards, C. Nonlinear robust fault reconstruction and estimation using a sliding modes observer. *Automatica* **2007**, *43*, 1605–1614. [CrossRef]
26. Zhirabok, A.; Zuev, A.; Shumsky, A. Diagnosis of linear dynamic systems: An approach based on sliding mode observers. *Autom. Remote Control* **2020**, *81*, 211–225. [CrossRef]
27. Zhirabok, A. Disturbance decoupling problem: Logic-dynamic approach-based solution. *Symmetry* **2019**, *11*, 555. [CrossRef]
28. Bobko, E.; Zhirabok, A.; Shumsky, A. Method of fault accommodation in technical systems. *J. Comput. Syst. Sci. Int.* **2016**, *55*, 735–749. [CrossRef]