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Ulam Stability of a Functional Equation in Various Normed Spaces

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Abstract: In this note, we study the Ulam stability of a general functional equation in four variables. Since its particular case is a known equation characterizing the so-called bi-quadratic mappings (i.e., mappings which are quadratic in each of their both arguments), we get in consequence its stability, too. We deal with the stability of the considered functional equations not only in classical Banach spaces, but also in 2-Banach and complete non-Archimedean normed spaces. To obtain our outcomes, the direct method is applied.

Keywords: Ulam stability; functional equation; bi-quadratic mapping; 2-norm; non-Archimedean norm

MSC: 39B82

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1. Introduction

Let us recall that one of the most known and prominent functional equations is the quadratic equation

$$q(x+y)+q(x-y)=2q(x)+2q(y). \tag{1}$$

This functional equation, which is also called the Jordan–von Neumann equation, is useful, among others, in some characterizations of inner product spaces. For more information about it and its applications we refer the reader to, for example, [1,2]. Let us finally mention that by a quadratic mapping we mean each solution of Equation (1).

Let X and Y be linear spaces over fields $\mathbb{F}$ and $\mathbb{K}$, respectively. Assume, moreover, that $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in \mathbb{F}$ and $C_{11}, C_{12}, C_{21}, C_{22} \in \mathbb{K}$ are given scalars.

We consider the following functional equation in four variables

$$\begin{aligned} &f(a_1(x_1+x_2), b_1(y_1+y_2)) + f(a_2(x_1+x_2), b_2(y_1-y_2)) + \\ &f(a_3(x_1-x_2), b_3(y_1+y_2)) + f(a_4(x_1-x_2), b_4(y_1-y_2)) = \\ &C_{11}f(x_1,y_1) + C_{12}f(x_1,y_2) + C_{21}f(x_2,y_1) + C_{22}f(x_2,y_2), \end{aligned} \tag{2}$$

where $f : X^2 \to Y$ and $x_1, x_2, y_1, y_2 \in X$.

Example 1. *Equation* (2) *with* $a_1 = b_1 = \ldots = a_4 = b_4 = 1$ *and* $C_{11} = C_{12} = C_{21} = C_{22} = 4$ *leads to the functional equation*

$$\begin{aligned} &f(x_1+x_2, y_1+y_2) + f(x_1+x_2, y_1-y_2) + \\ &f(x_1-x_2, y_1+y_2) + f(x_1-x_2, y_1-y_2) = \\ &4f(x_1,y_1) + 4f(x_1,y_2) + 4f(x_2,y_1) + 4f(x_2,y_2), \end{aligned} \tag{3}$$

which was investigated in [3]. This equation characterizes the so-called bi-quadratic mappings, i.e., functions $f : X^2 \to Y$ which are quadratic in each of their arguments.

The question about an error we commit replacing an object possessing some properties by an object fulfilling them only approximately is natural and interesting in many scientific investigations. To deal with it one can use the notion of the Ulam stability.

As it is well-known, the problem of the stability of homomorphisms was posed by S.M. Ulam in 1940. A year later, its solution in the case of Banach spaces (let us mention here that Ulam asked about metric groups) was presented by D.H. Hyers. Another very important example is a question concerning the stability of isometries. This problem was investigated for instance in [4–7] (see also [8] for more information and references on this subject).

Let us recall that an equation is called Ulam stable provided, roughly speaking, each mapping fulfilling our equation "approximately" is "near" to its actual solution.

In recent years, the Ulam type stability of various objects has been studied by many researchers (for more information on this notion as well as its applications we refer the reader to [1,8–13]). In particular, the stability of Equation (3) was investigated in [3].

In this note, the Ulam stability of Equation (2) is shown. Moreover, we apply our main results (Theorems 1 and 2) to get some stability outcomes on functional Equation (3).

Let us finally mention that as the concept of the nearness of two mappings may be obviously understood in various ways, we deal with the stability of the mentioned functional equations in three types of spaces. Let us also point out that in two of them non-standard measures of the distance occur (the ones given by a 2-norm and a non-Archimedean norm, respectively).

In what follows, $\mathbb{N}$ stands, as usual, for the set of all positive integers and we put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

2. Results

In this section, we present our main results. Roughly speaking, they show that Equation (2) is Ulam stable in three classes of spaces.

2.1. Stability in Banach Spaces

We start with Banach spaces.

Theorem 1. *Assume that Y is a Banach space, $\varepsilon > 0$ and*

$$|C_{11} + C_{12} + C_{21} + C_{22}| > 1. \tag{4}$$

If $f : X^2 \to Y$ is a function satisfying

$$f(x, 0) = 0 = f(0, y), \qquad x, y \in X \tag{5}$$

and

$$\begin{aligned} &\|f(a_1(x_1 + x_2), b_1(y_1 + y_2)) + f(a_2(x_1 + x_2), b_2(y_1 - y_2))+ \\ &f(a_3(x_1 - x_2), b_3(y_1 + y_2)) + f(a_4(x_1 - x_2), b_4(y_1 - y_2))- \\ &C_{11} f(x_1, y_1) - C_{12} f(x_1, y_2) - C_{21} f(x_2, y_1) - C_{22} f(x_2, y_2)\| \le \varepsilon \end{aligned} \tag{6}$$

for $x_1, x_2, y_1, y_2 \in X$, then there is a mapping $F : X^2 \to Y$ fulfilling Equation (2) and

$$\|f(x, y) - F(x, y)\| \le \frac{\varepsilon}{|C_{11} + C_{12} + C_{21} + C_{22}| - 1}, \qquad x, y \in X. \tag{7}$$

Proof. Put

$$C := C_{11} + C_{12} + C_{21} + C_{22}.$$

Let us first note that (5) and (6) with $x_2 = x_1$ and $y_2 = y_1$ give

$$\|f(2a_1x_1, 2b_1y_1) - Cf(x_1, y_1)\| \le \varepsilon, \qquad (x_1, y_1) \in X^2,$$

and consequently

$$\left\|\frac{f((2a_1)^{k+1}x_1, (2b_1)^{k+1}y_1)}{C^{k+1}} - \frac{f((2a_1)^k x_1, (2b_1)^k y_1)}{C^k}\right\| \le \frac{\varepsilon}{|C|^{k+1}}, \qquad (x_1, y_1) \in X^2,\ k \in \mathbb{N}_0. \quad (8)$$

Fix $l, p \in \mathbb{N}_0$ such that $l < p$. Then

$$\left\|\frac{f((2a_1)^p x_1, (2b_1)^p y_1)}{C^p} - \frac{f((2a_1)^l x_1, (2b_1)^l y_1)}{C^l}\right\| \le \sum_{j=l}^{p-1} \frac{\varepsilon}{|C|^{j+1}}, \qquad (x_1, y_1) \in X^2, \quad (9)$$

and thus for each $(x_1, y_1) \in X^2$, $\left(\frac{f((2a_1)^k x_1, (2b_1)^k y_1)}{C^k}\right)_{k \in \mathbb{N}_0}$ is a Cauchy sequence. Using the fact that Y is a Banach space we conclude that this sequence is convergent, which allows us to define

$$F(x_1, y_1) := \lim_{k \to \infty} \frac{f((2a_1)^k x_1, (2b_1)^k y_1)}{C^k}, \qquad (x_1, y_1) \in X^2. \quad (10)$$

Putting now $l = 0$ and letting $p \to \infty$ in (9) we see that

$$\|f(x_1, y_1) - F(x_1, y_1)\| \le \frac{\varepsilon}{|C| - 1}, \qquad (x_1, y_1) \in X^2,$$

i.e., condition (7) is satisfied.

Let us next observe that from (6) we get

$$\begin{aligned}
&\Bigg\|\tfrac{f((2a_1)^k a_1(x_1+x_2), (2b_1)^k b_1(y_1+y_2))}{C^k} + \tfrac{f((2a_1)^k a_2(x_1+x_2), (2b_1)^k b_2(y_1-y_2))}{C^k} + \\
&\tfrac{f((2a_1)^k a_3(x_1-x_2), (2b_1)^k b_3(y_1+y_2))}{C^k} + \tfrac{f((2a_1)^k a_4(x_1-x_2), (2b_1)^k b_4(y_1-y_2))}{C^k} - \\
&C_{11}\tfrac{f((2a_1)^k x_1, (2b_1)^k y_1)}{C^k} - C_{12}\tfrac{f((2a_1)^k x_1, (2b_1)^k y_2)}{C^k} - \\
&C_{21}\tfrac{f((2a_1)^k x_2, (2b_1)^k y_1)}{C^k} - C_{22}\tfrac{f((2a_1)^k x_2, (2b_1)^k y_2)}{C^k}\Bigg\| \le \tfrac{\varepsilon}{|C|^k}.
\end{aligned}$$

for $x_1, x_2, y_1, y_2 \in X$ and $k \in \mathbb{N}_0$. Letting now $k \to \infty$ and applying definition (10) we deduce that

$$\begin{aligned}
&\|F(a_1(x_1 + x_2), b_1(y_1 + y_2)) + F(a_2(x_1 + x_2), b_2(y_1 - y_2)) + \\
&F(a_3(x_1 - x_2), b_3(y_1 + y_2)) + F(a_4(x_1 - x_2), b_4(y_1 - y_2)) - \\
&C_{11}F(x_1, y_1) - C_{12}F(x_1, y_2) - C_{21}F(x_2, y_1) - C_{22}F(x_2, y_2)\| \le 0
\end{aligned}$$

for $x_1, x_2, y_1, y_2 \in X$, and thus we see that the mapping $F : X^2 \to Y$ is a solution of functional Equation (2). □

2.2. Stability in 2-Banach Spaces

Next, we deal with 2-Banach spaces.

Let us recall (see for example [14,15]) that the a 2-normed space was defined by S. Gähler in 1964. Assume that X is an at least two-dimensional real linear space. We say that a mapping $\|\cdot,\cdot\| : X \times X \to [0,\infty)$ is a 2-norm provided it satisfies the following four conditions:

$$\|x,y\| = 0 \iff x \text{ and } y \text{ are linearly dependent},$$
$$\|x,y\| = \|y,x\|,$$
$$\|x,y+z\| \le \|x,y\| + \|x,z\|,$$
$$\|\alpha x,y\| = |\alpha|\|x,y\|$$

for any $\alpha \in \mathbb{R}$, $x,y,z \in X$. By a linear 2-normed space we mean a pair $(X,\|\cdot,\cdot\|)$.

Now, we quote some useful definitions and a few known properties of 2-norms.

Let $(x_k)_{k\in\mathbb{N}}$ be a sequence of elements of a linear 2-normed space $(X,\|\cdot,\cdot\|)$. It is said to be a Cauchy sequence if there exist linearly independent $y,z \in X$ with

$$\lim_{n,m\to\infty} \|x_n - x_m, y\| = 0 = \lim_{n,m\to\infty} \|x_n - x_m, z\|. \tag{11}$$

On the other hand, $(x_n)_{n\in\mathbb{N}}$ is called convergent provided there is an $x \in X$ for which

$$\lim_{k\to\infty} \|x_k - x, y\| = 0, \qquad y \in X.$$

In the latter case we say that the element x is the limit of the sequence $(x_k)_{k\in\mathbb{N}}$ and denote it by $\lim_{k\to\infty} x_k$. Obviously each convergent sequence possesses a unique limit. Moreover, the limit has the standard properties.

By a 2-Banach space we mean a linear 2-normed space such that each its Cauchy sequence is convergent.

We will also use the following known facts.

Remark 1. *Assume that $(X,\|\cdot,\cdot\|)$ is a 2-normed space and $(x_k)_{k\in\mathbb{N}}$ is a sequence of elements of X. Then:*

(i) *if $x \in X$ and*

$$\|x,y\| = 0, \qquad y \in X,$$

then $x = 0$;

(ii) *if the sequence $(x_k)_{k\in\mathbb{N}}$ is convergent, then*

$$\lim_{k\to\infty} \|x_k, y\| = \Big\|\lim_{k\to\infty} x_k, y\Big\|, \qquad y \in X.$$

Next, we show the Ulam stability of Equation (2). For some other recent stability outcomes on various functional equations in 2-Banach spaces we refer the reader to [14–18].

Theorem 2. *Assume that Y is a 2-Banach space, $\varepsilon > 0$ and condition (4) holds true. If $f : X^2 \to Y$ is a function satisfying (5) and*

$$\begin{aligned} &\|f(a_1(x_1+x_2), b_1(y_1+y_2)) + f(a_2(x_1+x_2), b_2(y_1-y_2)) + \\ &f(a_3(x_1-x_2), b_3(y_1+y_2)) + f(a_4(x_1-x_2), b_4(y_1-y_2)) - \\ &C_{11} f(x_1,y_1) - C_{12} f(x_1,y_2) - C_{21} f(x_2,y_1) - C_{22} f(x_2,y_2), z\| \le \varepsilon \end{aligned} \tag{12}$$

for $x_1, x_2, y_1, y_2 \in X$ *and* $z \in Y$*, then there is a mapping* $F : X^2 \to Y$ *fulfilling Equation* (2) *and*

$$\|f(x, y) - F(x, y), z\| \le \frac{\varepsilon}{|C_{11} + C_{12} + C_{21} + C_{22}| - 1} \tag{13}$$

for $x, y \in X$ *and* $z \in Y$*.*

Proof. Let C be as in the proof of Theorem 1 and fix $l, p \in \mathbb{N}_0$ with $l < p$. One can show that

$$\left\|\frac{f((2a_1)^p x_1, (2b_1)^p y_1)}{C^p} - \frac{f((2a_1)^l x_1, (2b_1)^l y_1)}{C^l}, z\right\| \le \sum_{j=l}^{p-1} \frac{\varepsilon}{|C|^{j+1}}, \qquad (x_1, y_1) \in X^2,\ z \in Y, \tag{14}$$

and therefore for each $(x_1, y_1) \in X^2$, $\left(\frac{f((2a_1)^k x_1,(2b_1)^k y_1)}{C^k}\right)_{k\in\mathbb{N}_0}$ is a Cauchy sequence. By the fact that Y is a 2-Banach space we infer that this sequence is convergent, and thus we can define the function $F : X^2 \to Y$ by (10).

Next, putting $l = 0$ and letting $p \to \infty$ in (14), and using Remark 1 we see that

$$\|f(x_1, y_1) - F(x_1, y_1), z\| \le \frac{\varepsilon}{|C| - 1}, \qquad (x_1, y_1) \in X^2,\ z \in Y,$$

which means that condition (13) is satisfied.

Now, observe that by (12) we have

$$\begin{aligned}
&\Bigg\|\tfrac{f((2a_1)^k a_1(x_1+x_2),(2b_1)^k b_1(y_1+y_2))}{C^k} + \tfrac{f((2a_1)^k a_2(x_1+x_2),(2b_1)^k b_2(y_1-y_2))}{C^k} + \\
&\tfrac{f((2a_1)^k a_3(x_1-x_2),(2b_1)^k b_3(y_1+y_2))}{C^k} + \tfrac{f((2a_1)^k a_4(x_1-x_2),(2b_1)^k b_4(y_1-y_2))}{C^k} - \\
&C_{11}\tfrac{f((2a_1)^k x_1,(2b_1)^k y_1)}{C^k} - C_{12}\tfrac{f((2a_1)^k x_1,(2b_1)^k y_2)}{C^k} - \\
&C_{21}\tfrac{f((2a_1)^k x_2,(2b_1)^k y_1)}{C^k} - C_{22}\tfrac{f((2a_1)^k x_2,(2b_1)^k y_2)}{C^k}, z\Bigg\| \le \tfrac{\varepsilon}{|C|^k}.
\end{aligned}$$

for $x_1, x_2, y_1, y_2 \in X$, $z \in Y$ and $k \in \mathbb{N}_0$. Consequently, letting $k \to \infty$ and using (10) and Remark 1 we finally conclude that the function F fulfils Equation (2). □

2.3. Stability in Complete Non-Archimedean Normed Spaces

Finally, we will consider the case of complete non-Archimedean normed spaces. In order to do this, let us first recall (see for instance [10,14,19,20]) some basic definitions and facts concerning such spaces.

A field $\mathbb{F}$ equipped with a mapping $|\cdot| : \mathbb{F} \to [0, \infty)$, which is called a valuation, satisfying

$$|r| = 0 \iff r = 0,$$

$$|rs| = |r||s|, \qquad r, s \in \mathbb{F}$$

and

$$|r + s| \le \max\{|r|, |s|\}, \qquad r, s \in \mathbb{F}$$

is said to be a non-Archimedean field.

Let $\mathbb{F}$ be a field. The mapping $|\cdot| : \mathbb{F} \to [0, \infty)$ given by

$$|x| := \begin{cases} 0, & x = 0, \\ 1, & x \neq 0 \end{cases}$$

is a valuation, which is called trivial. However, the most important examples of non-Archimedean fields are p-adic numbers. The reason is that they appear in physicists' research connected with quantum physics, p-adic strings and superstrings.

Let us also mention that in any non-Archimedean field we have

$$|1| = |-1| = 1$$

and

$$|n| \leq 1, \qquad n \in \mathbb{N}_0.$$

Assume that X is a linear space over a non-Archimedean field $\mathbb{F}$ equipped with a non-trivial valuation $|\cdot|$. A mapping $\|\cdot\| : X \to [0, \infty)$ fulfilling the following conditions:

$$\|x\| = 0 \iff x = 0,$$

$$\|rx\| = |r|\|x\|, \qquad r \in \mathbb{F}, x \in X$$

and

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}, \qquad x, y \in X$$

is said to be a non-Archimedean norm. By a non-Archimedean normed space we mean a pair $(X, \|\cdot\|)$.

It is well-known that in any non-Archimedean normed space the mapping $d : X \times X \to [0, \infty)$ given by

$$d(x, y) := \|x - y\|, \qquad x, y \in X$$

is a metric on X. Moreover, the addition, the scalar multiplication as well as the non-Archimedean norm are continuous functions.

Let $(x_k)_{k \in \mathbb{N}}$ be a sequence of elements of a non-Archimedean normed space. It is known that it is Cauchy if and only if

$$\lim_{k \to \infty} (x_{k+1} - x_k) = 0.$$

We can now formulate our last outcome concerning the Ulam stability of Equation (2). Let us mention that some other results on the Ulam type stability of several functional equations in non-Archimedean spaces can be found for instance in [10,14,19,20].

Theorem 3. *Assume that Y is a complete non-Archimedean normed space, $\varepsilon > 0$ and condition (4) holds true. If $f : X^2 \to Y$ is a function fulfilling (5) and (6) for any $x_1, x_2, y_1, y_2 \in X$, then there exists a solution $F : X^2 \to Y$ of Equation (2) such that*

$$\|f(x, y) - F(x, y)\| \leq \frac{\varepsilon}{|C_{11} + C_{12} + C_{21} + C_{22}|}, \qquad x, y \in X. \tag{15}$$

Proof. Assume that C is as in the proof of Theorem 1.

Let us first note that inequality (8) is satisfied, and therefore (see the remarks before Theorem 3) for each $(x_1, y_1) \in X^2$, $\left(\frac{f((2a_1)^k x_1, (2b_1)^k y_1)}{C^k}\right)_{k \in \mathbb{N}_0}$ is a Cauchy sequence. The fact that the space Y is complete now shows that this sequence is convergent, and thus we can define the mapping $F : X^2 \to Y$ by (10).

Next, using induction, we get

$$\left\| \frac{f((2a_1)^k x_1, (2b_1)^k y_1)}{C^k} - f(x_1, y_1) \right\| \le \frac{\varepsilon}{|C|}, \qquad x_1, y_1 \in X, k \in \mathbb{N}.$$

Letting now $k \to \infty$ and applying definition (10) we conclude that condition (15) holds true.

Finally, proceeding as in the proof of Theorem 1, we show that F is a solution of functional Equation (2). □

3. Discussion

In the previous section, the Ulam stability of Equation (2) in three classes of spaces has been proved. Now, we present two consequences of the obtained results, which deal with the stability of functional Equation (3). Recall also that this equation and its stability were investigated in [3].

Let us first note that Theorem 1 with $a_1 = b_1 = \ldots = a_4 = b_4 = 1$ and $C_{11} = C_{12} = C_{21} = C_{22} = 4$ gives the following.

Corollary 1. *Assume that Y is a Banach space and $\varepsilon > 0$. If $f : X^2 \to Y$ is a function satisfying condition (5) and*

$$\|f(x_1 + x_2, y_1 + y_2) + f(x_1 + x_2, y_1 - y_2) +$$

$$f(x_1 - x_2, y_1 + y_2) + f(x_1 - x_2, y_1 - y_2) -$$

$$4f(x_1, y_1) - 4f(x_1, y_2) - 4f(x_2, y_1) - 4f(x_2, y_2)\| \le \varepsilon$$

for $x_1, x_2, y_1, y_2 \in X$, then there is a solution $F : X^2 \to Y$ of Equation (3) such that

$$\|f(x, y) - F(x, y)\| \le \frac{\varepsilon}{15}, \qquad x, y \in X.$$

Similarly, an obvious consequence of Theorem 2 is the following outcome on the stability of Equation (3) in 2-Banach spaces.

Corollary 2. *Assume that Y is a 2-Banach space and $\varepsilon > 0$. If $f : X^2 \to Y$ is a function satisfying condition (5) and*

$$\|f(x_1 + x_2, y_1 + y_2) + f(x_1 + x_2, y_1 - y_2) +$$

$$f(x_1 - x_2, y_1 + y_2) + f(x_1 - x_2, y_1 - y_2) -$$

$$4f(x_1, y_1) - 4f(x_1, y_2) - 4f(x_2, y_1) - 4f(x_2, y_2), z\| \le \varepsilon$$

for $x_1, x_2, y_1, y_2 \in X$ and $z \in Y$, then there is a solution $F : X^2 \to Y$ of Equation (3) such that

$$\|f(x, y) - F(x, y), z\| \le \frac{\varepsilon}{15}, \qquad x, y \in X,\ z \in Y.$$

4. Conclusions

It seems that a natural and interesting problem is a question about a general solution of functional Equation (2). Let us note that for $a_1 = b_1 = \ldots = a_4 = b_4 = 1$ and $C_{11} = C_{12} = C_{21} = C_{22} = 4$ this solution is already known, as then (2) means (3) and the latter equation was solved in [3]. What about other cases?

Let us finally point out that to obtain the presented stability outcomes we have applied the direct method, which is also called the Hyers method. However, it is easily seen and standard that similar results can be proved via the fixed point method, too. These methods were applied for study of the stability of various equations, not only the functional ones. More information and references both on them as well as on a few other efficient approaches to the Ulam stability of functional equations can be found, for instance, in [21].

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