



Existence Results of Mild Solutions for the Fractional Stochastic Evolution Equations of Sobolev Type

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Article

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Abstract: In this paper, by utilizing the resolvent operator theory, the stochastic analysis method and Picard type iterative technique, we first investigate the existence as well as the uniqueness of mild solutions for a class of $\alpha \in (1,2)$ -order Riemann–Liouville fractional stochastic evolution equations of Sobolev type in abstract spaces. Then the symmetrical technique is used to deal with the $\alpha \in (1,2)$ -order Caputo fractional stochastic evolution equations of Sobolev type in abstract spaces. Two examples are given as applications to the obtained results.

Keywords: fractional resolvent family; existence and uniqueness; fractional stochastic evolution equations; Picard's iteration technique

1. Introduction

Since fractional differential equations can describe many problems in the fields of physical, biological and chemical and so on, some properties of solutions for the fractional differential equations have been considered by many authors, see [1–8]. In [2], when the nonlinearity satisfies non-Lipschitz conditions, Wang studied the existence of mild solutions of $\alpha \in (0, 1)$ -order fractional stochastic evolution equations with Caputo derivative in abstract spaces. Li et al. [3] obtained the existence as well as the uniqueness of weak solutions and strong solutions of an inhomogeneous Cauchy problem of order $\alpha \in (1, 2)$ involving Riemann–Liouville fractional derivatives via the technique of fractional resolvent.

Sobolev type (fractional) differential equation arises in various areas of physical problems, see [4,5], hence it has been investigated by researchers recently, see [4–7]. Fečkan et al. [5] proved the controllability results for $\alpha \in (0, 1)$ -order fractional functional evolution equations of Sobolev type in abstract spaces. By virtue of the characteristic solution operators, they obtained the exact controllability results via Schauder fixed point theorem. In [8], by using the characterizations of compact resolvent families, Ponce investigated the Cauchy problem for a class of fractional evolution equations. Furthermore, the stochastic perturbation is unavoidable in the natural systems. Therefore, it is important to consider stochastic effects in studying fractional differential systems. Recently, in [6], by means of the operator semigroup theory, fractional calculus and stochastic analysis technique, Benchaabane et al. established a group of sufficient conditions to guarantee the existence as well as the uniqueness of solutions for the $\alpha \in (0, 1)$ -order fractional stochastic evolution equations of Sobolev type. As far as we know, the existence as well as the uniqueness of mild solutions for the Sobolev type fractional stochastic evolution equations of order $\alpha \in (1, 2)$ have not been extensively discussed yet.

In the present work, we consider the existence as well as the uniqueness of mild solutions for two classes of the initial value problems (IVPs) of fractional stochastic equations of Sobolev type in a Hilbert space X

$$\begin{cases} {}^{L}D_{t}^{\alpha}(Su(t)) = Au(t) + f(t,u(t)) + \Sigma(t,u(t))\frac{dW(t)}{dt}, & t \in I := [0,b], \\ S(g_{2-\alpha} * u)(0) = \xi, & [S(g_{2-\alpha} * u)]'(0) = \eta \end{cases}$$
(1)

and

$$\begin{cases} {}^{C}D_{t}^{\alpha}(Su(t)) = Au(t) + f(t,u(t)) + \Sigma(t,u(t))\frac{dW(t)}{dt}, & t \in I, \\ Su(0) = \xi, & (Su)'(0) = \eta, \end{cases}$$
(2)

where $1 < \alpha < 2$ and b > 0 are constants, ${}^{L}D_{t}^{\alpha}$ and ${}^{C}D_{t}^{\alpha}$ denote, respectively, the α -order fractional derivative operators of Riemann–Liouville and Caputo, $A : \mathcal{D}(A) \subset X \to X$ is a densely defined and closed linear operator in $X, S : \mathcal{D}(S) \subset X \to X$ is also a closed linear operator in X, ζ and η are X-valued random variable, f, Σ, W and $g_{2-\alpha}$ will be specified later.

In the previous works, see [5,6], the authors often make the following assumptions on *A* and *S* when they investigate the Sobolev type differential equations.

- (i) $\mathcal{D}(S) \subset \mathcal{D}(A)$ and *S* is bijective;
- (ii) *S* has the compact and bounded inverse S^{-1} .

In this situation, $-AS^{-1}$ generates a semigroup $T(t) := e^{-AS^{-1}t}$ for $t \ge 0$ and $S : \mathcal{D}(S) \subset X \to X$ may be bounded.

In this paper, without assuming (i) and (ii) on *A* and *S* as well as any compactness conditions on *f* and Σ , we investigate the existence as well as the uniqueness of mild solutions of the IVPs (1) and (2). More precisely, we first present the concept of $(\alpha, \alpha - 1)$ -resolvent family and $(\alpha, 1)$ -resolvent family generated by the pair (A, S). With the help of $(\alpha, \alpha - 1)$ -resolvent family and $(\alpha, 1)$ -resolvent family and Laplace transform, the correct definitions of mild solutions of the IVPs (1) and (2) are presented. Under some essential conditions on *f* and Σ , we study the existence as well as the uniqueness of mild solutions of the IVPs (1) and (2) by virtue of the iteration technique of Picard type. We have to emphasize that we do not assume the compactness of the $(\alpha, \alpha - 1)$ -resolvent family and the $(\alpha, 1)$ -resolvent family in our main results.

2. Preliminaries

In this part, we first recall some definitions of fractional calculus. The definition of the fractional resolvent family is also given in this section. By using the fractional resolvent family and Laplace transform, the concepts of mild solution of the IVPs (1) and (2) are introduced, and an inequality is given in Lemma 1.

Denote by $(\Omega, \mathcal{F}, {\mathcal{F}_t}_{t\geq 0}, \mathcal{P})$ the complete probability space involving a filtration ${\mathcal{F}_t}_{t\geq 0}$, which satisfies the usual conditions. On $(\Omega, \mathcal{F}, {\mathcal{F}_t}_{t\geq 0}, \mathcal{P}), {W(t)}_{t\geq 0}$ is a *Q*-wiener process with values in *X*, where *Q* is a bounded linear covariance operator and $trQ < +\infty$. Let $\ell_k \geq 0$ be a bounded sequence and ${e_k}_{k\geq 1}$ a complete orthonormal system of *X* satisfying $Qe_k = \ell_k e_k$ for $k = 1, 2, \cdots$. Let ${\beta_k}_{k\geq 1}$ be independent Brownian motions satisfying

$$\langle W(t), x \rangle = \sum_{k=1}^{\infty} \sqrt{\ell_k} \langle e_k, x \rangle \beta_k(t), \quad \forall x \in X, \ t \ge 0.$$

Further, let \mathcal{F}_t be the σ -algebra generated by $\{W(\theta) : 0 \le \theta \le t\}$. Put $L_2^0 := L_2(Q^{\frac{1}{2}}, Y)$. Then L_2^0 is a real separable Hilbert space endowed with $\|\varphi\|_{L_2^0}^2 = tr[\varphi Q \varphi^*]$. For $1 \le p \le +\infty$, denote by $L_p(\mathcal{F}, X)$ the set of strongly \mathcal{F} -measurable random variables with values in X. Then $L_p(\mathcal{F}, X)$ is a Banach space satisfying $E\|x\|^p \le +\infty$. Let $C(I, L_p(\mathcal{F}, X))$ be the Banach space of all continuous maps from I to $L_p(\mathcal{F}, X)$ satisfying the condition

$$\sup_{t\in I} E\|x(t)\|^p < +\infty.$$

Let $C_b^p \subset C(I, L_p(\mathcal{F}, X))$ be the closed subspace of $C(I, L_p(\mathcal{F}, X))$, which consist of \mathcal{F}_t -adapted and measurable processes u(t). Put

$$\|u\|_{\mathcal{C}^{p}_{b}} := (\sup_{t \in I} E \|u(t)\|^{p})^{\frac{1}{p}}, \quad \forall u \in \mathcal{C}^{p}_{b}, \ p \ge 2.$$
(3)

Then $(\mathcal{C}_b^p, \|\cdot\|_{\mathcal{C}_b^p})$ is a Banach space. ξ and η in the IVPs (1) and (2) are \mathcal{F}_0 -measurable and X-valued random variable independent of W.

Firstly, we recall a group of concepts of fractional calculus, see [9,10] for more details. For every $\nu \ge 0$, let

$$g_{\nu}(t) = \left\{ egin{array}{cc} rac{t^{
u-1}}{\Gamma(
u)}, & if \quad t > 0 \ 0, & else. \end{array}
ight.$$

We define the finite convolution of the functions *f* and *g* by

$$(f * g)(s) = \int_0^s f(s - \theta)g(\theta)d\theta.$$

Definition 1. For $\alpha > 0$, the α -order Riemann–Liouville fractional integral of the function $u \in L^1(I)$ is defined by

$$J_t^{\alpha} u(t) = (g_{\alpha} * u)(t), \quad t > 0.$$

Definition 2. For $\alpha > 0$, the α -order Riemann-Liouville fractional derivative is defined for all $u \in L^1(I)$ satisfying $g_{m-\alpha} * u \in W^{m,1}(I)$ by

$$^{L}D_{t}^{\alpha}u(t) = D_{t}^{m}(g_{m-\alpha} * u)(t), \quad t > 0,$$

where $D_t^m = \frac{d^m}{dt^m}$.

Definition 3. For $\alpha > 0$, the α -order Caputo fractional derivative of all $u \in L^1(I)$ is defined by

$$^{C}D_{t}^{\alpha}u(t)=J_{t}^{m-\alpha}\ ^{L}D_{t}^{m}u(t),\quad t>0,$$

If $u \in C^m(\mathbb{R}^+)$, for $\alpha \in (m-1, m)$, the α -order Caputo fractional derivative is defined by

$${}^{C}D_{t}^{\alpha}u = (g_{m-\alpha} * u^{(m)})(t), \quad t > 0.$$

Definition 4. Let the function u be defined on \mathbb{R}^+ . If the integral

$$\int_0^\infty e^{-\lambda\theta} u(\theta) d\theta$$

is convergence, then the Laplace transform of u is given by

$$\widehat{u}(\lambda) = \int_0^\infty e^{-\lambda\theta} u(\theta) d\theta.$$

Remark 1. If a function u is defined on \mathbb{R}^+ satisfying the conditions

(K1) u(t) is piecewise continuous on every bounded subset of $t \ge 0$;

(K2) There are $L^* \ge 0$ and $a \ge 0$ satisfying

$$|u(t)| \le L^* e^{at}, \quad \forall t \ge 0,$$

then the Laplace transform of u exists for $Re(\lambda) > a$.

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Hence from [10], since $\hat{g}_{\alpha}(\lambda) = \lambda^{-\alpha}$ for any $\alpha > 0$, by Remark 1 and the properties of the Laplace transform, we have

$$\widehat{LD}_{t}^{\alpha}u(\lambda) = \lambda^{\alpha}\widehat{u}(\lambda) - \sum_{n=0}^{m-1} (g_{m-\alpha} * u)^{(n)}(0)\lambda^{m-1-n}$$
(4)

and

$$\widehat{^{C}D_{t}^{\alpha}}u(\lambda) = \lambda^{\alpha}\widehat{u}(\lambda) - \sum_{n=0}^{m-1} u^{(n)}(0)\lambda^{\alpha-1-n}.$$
(5)

In the following, we establish the concept of the fractional resolvent family which is a basic concept in our main results, see [4,8] for more details. Let $\{\Pi(t)\}_{t\geq 0}$ be a strongly continuous family of $\mathcal{B}(X)$. If there exist $M \geq 1$ and $\omega > 0$ satisfying

$$\|\Pi(t)\| \le M e^{\omega t}, \quad \forall t \ge 0,$$

then it is said to be of type (M, ω) . Denote the set $\rho_S(A)$ by

 $\rho_{S}(A) := \{ \mu \in \mathbb{C} \mid (\mu S - A) : \mathcal{D}(A) \cap \mathcal{D}(S) \to X \text{ is invertible and } (\mu S - A)^{-1} \in \mathcal{B}(X, \mathcal{D}(A) \cap \mathcal{D}(S)) \}.$

Let $R(\lambda^{\alpha}S, A) := (\lambda^{\alpha}S - A)^{-1}$. According to the Definition 5 of [8], we present the following definition.

Definition 5. Let $A : \mathcal{D}(A) \subset X \to X$ and $S : \mathcal{D}(S) \subset X \to X$ be closed linear operator, $\mathcal{D}(A) \cap \mathcal{D}(S) \neq \{0\}$, and $\alpha > 0, \beta > 0$. If there are $\omega \ge 0$ and a strongly continuous function $C^S_{\alpha,\beta} : \mathbb{R}^+ \to \mathcal{B}(X)$ such that $C^S_{\alpha,\beta}(t)$ is of type (M, ω) , $\{\lambda^{\alpha} : \operatorname{Re}\lambda > \omega\} \subset \rho_S(A)$, and for $u \in X$,

$$\lambda^{\alpha-\beta}R(\lambda^{\alpha}S,A)u = \int_0^\infty e^{-\lambda\theta}C^S_{\alpha,\beta}(\theta)ud\theta, \quad Re\lambda > \omega,$$
(6)

Then the pair (A, S) generates an (α, β) -resolvent family $\{C_{\alpha,\beta}^{S}(t)\}_{t\geq 0}$ which is of type (M, ω) .

For $1 < \alpha < 2$, choosing $\beta = \alpha - 1 > 0$, by (6), we have

$$\lambda R(\lambda^{\alpha}S, A)u = \int_{0}^{\infty} e^{-\lambda\theta} C^{S}_{\alpha,\alpha-1}(\theta) u d\theta, \quad Re\lambda > \omega, \quad u \in X.$$
(7)

Then the pair (A, S) generates an $(\alpha, \alpha - 1)$ -resolvent family $\{C^{S}_{\alpha,\alpha-1}(t)\}_{t\geq 0}$ of type (M, ω) . Particularly, if we choose $\beta = 1$, then $\{C^{S}_{\alpha,\beta}(t)\}_{t\geq 0}$ changes to $\{C^{S}_{\alpha,1}(t)\}_{t\geq 0}$, which is the $(\alpha, 1)$ -resolvent family. It satisfies

$$\lambda^{\alpha-1}R(\lambda^{\alpha}S,A)u = \int_0^\infty e^{-\lambda\theta}C^S_{\alpha,1}(\theta)ud\theta, \quad Re\lambda > \omega, \ u \in X.$$

For α , β , $\gamma > 0$, Since $(\widehat{g_{\gamma} * C^{S}_{\alpha,\beta}})(\lambda) = \widehat{C^{S}_{\alpha,\beta+\gamma}}(\lambda)$, choosing $1 < \alpha < 2$, $\beta = \alpha - 1$, $\gamma = 1$, then by the properties of Laplace transform, we get

$$C^{S}_{\alpha,\alpha}(t) = (g_1 * C^{S}_{\alpha,\alpha-1})(t)$$
$$= \int_0^t C^{S}_{\alpha,\alpha-1}(s) ds.$$

Consequently, for all $u \in X$, we have

$$R(\lambda^{\alpha}S, A)u = \int_{0}^{\infty} e^{-\lambda\theta} C^{S}_{\alpha,\alpha}(\theta) u d\theta, \quad Re\lambda > \omega.$$
(8)

Since $\{C^{S}_{\alpha,\alpha-1}(t)\}_{t\geq 0}$ is of type (M,ω) , for any $t\in I$, we have

$$\|C^{S}_{\alpha,\alpha-1}(t)\| \leq Me^{\omega b}.$$

Without loss of generality, we put $M := \sup_{t \in I} \|C^S_{\alpha,\alpha-1}(t)\|$. Thus, $\|C^S_{\alpha,\alpha}(t)\| \le Mb$, $\forall t \in I$.

Let

$$\widehat{u}(\lambda) = \int_0^\infty e^{-\lambda\theta} u(\theta) d\theta,$$

$$\widehat{f}(\lambda) = \int_0^\infty e^{-\lambda\theta} f(\theta, u(\theta)) d\theta.$$

Applying the Laplace transform to the first equation of the IVP (1), by virtue of $\widehat{Su}(\lambda) = S\widehat{u}(\lambda)$ and (4), we have

Together this fact with (7) and (8), we obtain

$$\begin{aligned} \widehat{u}(\lambda) &= \lambda R(\lambda^{\alpha}S, A)\xi + R(\lambda^{\alpha}S, A)\eta, \\ &+ R(\lambda^{\alpha}S, A)\widehat{f}(\lambda) + R(\lambda^{\alpha}S, A)\int_{0}^{\infty} e^{-\lambda\theta}\Sigma(\theta, u(\theta))dW(\theta) \\ &= \int_{0}^{\infty} e^{-\lambda\theta}C_{\alpha,\alpha-1}^{S}(\theta)\xi d\theta + \int_{0}^{\infty} e^{-\lambda\theta}C_{\alpha,\alpha}^{S}(\theta)\eta d\theta, \\ &+ \int_{0}^{\infty} e^{-\lambda\theta}C_{\alpha,\alpha}^{S}(\theta)\widehat{f}(\lambda)d\theta + \int_{0}^{\infty} e^{-\lambda\theta}C_{\alpha,\alpha}^{S}(\theta)\bigg[\int_{0}^{\infty} e^{-\lambda t}\Sigma(t, u(t))dW(t)\bigg]d\theta. \end{aligned}$$

Since

$$\begin{split} \int_{0}^{\infty} e^{-\lambda\theta} C_{\alpha,\alpha}^{S}(\theta) \widehat{f}(\lambda) d\theta &= \int_{0}^{\infty} e^{-\lambda\theta} C_{\alpha,\alpha}^{S}(\theta) \bigg[\int_{0}^{\infty} e^{-\lambda t} f(t,u(t)) dt \bigg] d\theta \\ &= \int_{0}^{\infty} \int_{0}^{\infty} e^{-\lambda(\theta+t)} C_{\alpha,\alpha}^{S}(\theta) f(t,u(t)) d\theta dt \\ &= \int_{0}^{\infty} \int_{t}^{\infty} e^{-\lambda\theta} C_{\alpha,\alpha}^{S}(\theta-t) f(t,u(t)) d\theta dt \\ &= \int_{0}^{\infty} e^{-\lambda\theta} \int_{0}^{\theta} C_{\alpha,\alpha}^{S}(\theta-t) f(t,u(t)) dt d\theta, \end{split}$$

and (similarly)

$$\int_0^\infty e^{-\lambda\theta} C^S_{\alpha,\alpha}(\theta) \left[\int_0^\infty e^{-\lambda t} \Sigma(t,u(t)) dW(t) \right] d\theta = \int_0^\infty e^{-\lambda\theta} \left[\int_0^\theta C^S_{\alpha,\alpha}(\theta-t) \Sigma(t,u(t)) dW(t) \right] d\theta,$$

by the properties of Laplace transform, we get

$$u(t) = C_{\alpha,\alpha-1}^{S}(t)\xi + C_{\alpha,\alpha}^{S}(t)\eta + \int_{0}^{t} C_{\alpha,\alpha}^{S}(t-\theta)f(\theta, u(\theta))d\theta + \int_{0}^{t} C_{\alpha,\alpha}^{S}(t-\theta)\Sigma(\theta, u(\theta))dW(\theta).$$
(9)

Thus, based on the above discussion, the mild solution of the IVP (1) is defined below.

Definition 6. A stochastic process $u \in C(I, L_p(\mathcal{F}, X))$ is called a mild solution of the IVP (1) if it satisfies the integral Equation (9).

Similarly, we can define the mild solution of the IVP (2) by applying (5).

Definition 7. A stochastic process $u \in C(I, L_p(\mathcal{F}, X))$ is called a mild solution of the IVP (2) if it satisfies the integral equation

$$u(t) = C_{\alpha,1}^{S}(t)\xi + K_{\alpha,1}^{S}(t)\eta + \int_{0}^{t} H_{\alpha,1}^{S}(t-\theta)f(\theta,u(\theta))d\theta + \int_{0}^{t} H_{\alpha,1}^{S}(t-\theta)\Sigma(\theta,u(\theta))dW(\theta),$$
(10)

where $\{C_{\alpha,1}^{S}(t)\}_{t\geq 0}$ is the $(\alpha, 1)$ -resolvent family generated by (A, S), and

$$\begin{split} K^{S}_{\alpha,1}(t) &= \int_{0}^{t} C^{S}_{\alpha,1}(\theta) d\theta, \quad \forall t \geq 0, \\ H^{S}_{\alpha,1}(\zeta) &= \frac{1}{\Gamma(\alpha-1)} \int_{0}^{\zeta} (\zeta-\theta)^{\alpha-2} C^{S}_{\alpha,1}(\theta) d\theta, \quad \forall \zeta \geq 0 \end{split}$$

At last, we recall an inequality, which cites from the Proposition 1.9 of [11–14].

Lemma 1. If $\Sigma : I \times \Omega \to L_2^0$ is a strongly measurable mapping satisfying $\int_0^T E \|\Sigma(\theta)\|_{L_2^0}^p d\theta < +\infty$ for some $p \ge 2$, then

$$E\left\|\int_0^t \Sigma(\theta) dW(\theta)\right\|^p \le L_{\Sigma} \int_0^t E\|\Sigma(\theta)\|_{L_2^0}^p d\theta, \quad \forall t \in I,$$

where $L_{\Sigma} > 0$ is a constant involving p and b.

3. Main Results

In this part, by utilizing the iteration technique of Picard type, we will prove the existence as well as the uniqueness of mild solutions of the IVPs (1) and (2). To this end, the following assumptions are needed.

Hypothesis 1 (H1). $f : I \times X \to X$ and $\Sigma : I \times X \to L_2^0$ are continuous functions and there is a function $\Phi : I \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\max\left\{E\|f(t,x)\|^{p}, \ E\|\Sigma(t,x)\|_{L^{0}_{2}}^{p}\right\} \leq \Phi(t,E\|x\|^{p}), \quad x \in L_{p}(\mathcal{F},X), \ t \in I.$$

Hypothesis 2 (H2). *The function* $\Phi : I \times \mathbb{R}^+ \to \mathbb{R}^+$ *satisfies the assumptions:*

- (*i*) For every $u \in [0, \infty)$, $\Phi(\cdot, u)$ is locally integrable.
- *(ii)* For every $t \in I$, $\Phi(t, \cdot)$ is nondecreasing and continuous.
- (iii) For all $C_1 > 0, C_2 \ge 0$, the equation

$$u(t) = C_1 + C_2 \int_0^t \Phi(s, u(s)) ds$$

has a global solution on I.

Hypothesis 3 (H3). *There exists a function* $\Psi : I \times \mathbb{R}^+ \to \mathbb{R}^+$ *satisfying*

$$\max\left\{E\|\Sigma(t,x)-\Sigma(t,y)\|_{L_{2}^{0}}^{p}, E\|f(t,x)-f(t,y)\|^{p}\right\} \leq \Psi(t,E\|x-y\|^{p}), \quad \forall \ x,y \in L_{p}(\mathcal{F},X), \ t \in I.$$

Hypothesis 4 (H4). *The function* $\Psi : I \times \mathbb{R}^+ \to \mathbb{R}^+$ *satisfies the assumptions:*

- (*i*) For each $u \in [0, \infty)$, $\Psi(\cdot, u)$ is locally integrable.
- (ii) For $t \in I$, the function $\Psi(t, \cdot)$ is nondecreasing and continuous.
- (iii) $\Psi(t,0) = 0$ and if a monotone nondecreasing and nonnegative function $\vartheta(t), t \in I$ satisfies

$$\begin{cases} \vartheta(t) \leq \tau \int_0^t \Psi(\theta, \vartheta(\theta)) d\theta, & t \in I, \\ \vartheta(0) = 0, \end{cases}$$

where $\tau > 0$ is a constant, then $\vartheta(t) \equiv 0$ for all $t \in I$.

Remark 2. If $\Psi(t, u) = Lu, u \ge 0$, where L > 0, the condition **Hypothesis 3** (H3) implies the global Lipschitz condition. Hence, the condition **Hypothesis 3** (H3) includes some existing cases.

Remark 3. If u(t) is a global solution on I of the IVP of the first-order ordinary differential equation

$$\begin{cases} u'(t) = C_2 \Phi(t, u(t)), & t \in I, \\ u(0) = C_1, \end{cases}$$

where $C_1 > 0, C_2 \ge 0$ are constants, then the assumption **Hypothesis 2** (H2)(*iii*) holds.

We will use Picard type approximate technique to prove our main results. For this purpose, we define the sequence of stochastic process $\{u_n\}_{n\geq 0}$ as follows:

$$\begin{cases} u_0(t) = C^S_{\alpha,\alpha-1}(t)\xi + C^S_{\alpha,\alpha}(t)\eta, & t \in I, \\ u_{n+1}(t) = C^S_{\alpha,\alpha-1}(t)\xi + C^S_{\alpha,\alpha}(t)\eta + Q_1(u_n)(t) + Q_2(u_n)(t), & t \in I, n \ge 0, \end{cases}$$
(11)

where

$$Q_1(u_n)(t) := \int_0^t C^S_{\alpha,\alpha}(t-\theta) f(\theta, u_n(\theta)) d\theta,$$
$$Q_2(u_n)(t) := \int_0^t C^S_{\alpha,\alpha}(t-\theta) \Sigma(\theta, u_n(\theta)) dW(\theta).$$

Lemma 2. Let (A, S) be the pair which generates an $(\alpha, \alpha - 1)$ -resolvent family $\{C_{\alpha,\alpha-1}^{S}(t)\}_{t\geq 0}$ of type (M, ω) . If the **Hypothesis 1** (H1) and **Hypothesis 2** (H2) hold, the sequence $\{u_n\}_{n\geq 0}$ is well-defined. Moreover, there is a constant C > 0 satisfying

$$\sup_{n\geq 0} \|u_n\|_{\mathcal{C}_b^p}^p \leq C.$$
(12)

Proof of Lemma 2. By (11), we have

$$E \|u_{n+1}(t)\|^{p} \leq 4^{p-1} E \|C_{\alpha,\alpha-1}^{S}(t)\xi\|^{p} + 4^{p-1} E \|C_{\alpha,\alpha}^{S}(t)\eta\|^{p} + 4^{p-1} E \|Q_{1}(u_{n})(t)\|^{p} + 4^{p-1} E \|Q_{2}(u_{n})(t)\|^{p}, \quad t \in I, \ n \geq 0.$$

By applying Hölder inequality and Lemma 1, we get

$$\begin{split} E\|Q_1(u_n)(t)\|^p &\leq M^p b^p E\|\int_0^t f(\theta, u_n(\theta))d\theta\|^p \\ &\leq M^p b^{2p-1}\int_0^t E\|f(\theta, u_n(\theta))\|^p d\theta \\ &\leq M^p b^{2p-1}\int_0^t \Phi(\theta, E\|u_n(\theta))\|^p)d\theta \end{split}$$

and

$$E \|Q_2(u_n)(t)\|^p \leq M^p b^p L_{\Sigma} \int_0^t E \|\Sigma(\theta, u_n(\theta))\|_{L_2^0}^p d\theta$$

$$\leq M^p b^p L_{\Sigma} \int_0^t \Phi(\theta, E \|u_n(\theta))\|^p) d\theta.$$

Together these facts with the monotonicity of Φ , by (3), we conclude that

$$\|u_{n+1}\|_{\mathcal{C}_{t}^{p}}^{p} \leq C_{1} + C_{2} \int_{0}^{t} \Phi(\theta, \|u_{n}\|_{\mathcal{C}_{\theta}^{p}}^{p}) d\theta,$$
(13)

where $C_1 = 4^{p-1}M^p(E\|\xi\|^p + b^p E\|\eta\|^p)$, $C_2 = 4^{p-1}M^p b^p(b^{p-1} + L_{\Sigma})$.

In view of **Hypothesis 2** (*H*2)(iii), the solution $u(\cdot)$ of the integral equation

$$u(t) = C_1 + C_2 \int_0^t \Phi(\theta, u(\theta)) d\theta$$
(14)

global exists on *I*. In the following, we prove $||u_n||_{C_t^p} \le u(t)$ for all $t \in I, n \ge 0$ by utilizing the induction method. Indeed,

$$\|u_0\|_{\mathcal{C}^p_t}^p \le 2^{p-1}M^p E \|\xi\|^p + 2^{p-1}M^p b^p E \|\eta\|^p \le C_1 \le u(t), \quad \forall t \in I.$$

Let $||u_n||_{\mathcal{C}^p_t} \le u(t)$ for all $t \in I, n \ge 0$. By means of (13) and (14), we obtain

$$u(t) - \|u_{n+1}\|_{\mathcal{C}_t^p}^p \ge C_2 \int_0^t \left(\Phi(\theta, u(\theta)) - \Phi(\theta, \|u_n\|_{\mathcal{C}_\theta^p}^p)\right) d\theta \ge 0, \quad \forall t \in I.$$

This implies that (12) holds with C := u(b) and the sequence $\{u_n\}_{n \ge 0}$ is well-defined. \Box

Theorem 1. Let (A, S) be the pair which generates an $(\alpha, \alpha - 1)$ -resolvent family $\{C^{S}_{\alpha,\alpha-1}(t)\}_{t\geq 0}$ of type (M, ω) . Suppose that the **Hypothesis 1** (H1)-**Hypothesis 4** (H4) hold, then there is a unique mild solution of the IVP (1) on I.

Proof of Theorem 1. By the **Hypothesis 3** (*H*3), Lemma 1 and (11), for any $m \ge n \ge 0$, we have

$$\begin{split} E \|u_m(t) - u_n(t)\|^p &\leq 2^{p-1} E \|Q_1(u_{m-1})(t) - Q_1(u_{n-1})(t)\|^p \\ &+ 2^{p-1} E \|Q_2(u_{m-1})(t) - Q_2(u_{n-1})(t)\|^p \\ &\leq 2^{p-1} M^p b^{2p-1} \int_0^t E \|f(\theta, u_{m-1}(\theta)) - f(\theta, u_{n-1}(\theta))\|^p d\theta \\ &+ 2^{p-1} M^p b^p L_{\Sigma} \int_0^t E \|\Sigma(\theta, u_{m-1}(\theta)) - \Sigma(\theta, u_{n-1}(\theta))\|_{L^2_2}^p d\theta \\ &\leq 2^{p-1} M^p b^p (b^{p-1} + L_{\Sigma}) \int_0^t \Psi(\theta, E \|u_{m-1}(\theta) - u_{n-1}(\theta)\|^p) d\theta, \quad \forall t \in I. \end{split}$$

By the monotonicity of Ψ and (3), we can obtain

$$\|u_{m} - u_{n}\|_{\mathcal{C}_{t}^{p}}^{p} \leq \tau \int_{0}^{t} \Psi(\theta, \|u_{m-1} - u_{n-1}\|_{\mathcal{C}_{\theta}^{p}}^{p}) d\theta,$$
(15)

where $\tau := 2^{p-1} M^p b^p (b^{p-1} + L_{\Sigma})$. Let

$$\phi_n(t) = \sup_{m \ge n} \|u_m - u_n\|_{\mathcal{C}_t^p}^p, \quad \forall t \in I$$

By (15), we have

$$\phi_n(t) \le \tau \int_0^t \Psi(\theta, \phi_{n-1}(\theta)) d\theta.$$
(16)

Since $\{\phi_n(t)\}_{n\geq 0}$ is monotone and uniformly bounded due to Lemma 2, we know that there exists a function $\phi(t)$ satisfying

$$\lim_{n\to\infty}\phi_n(t)=\phi(t), \quad \forall t\in I.$$

Taking $n \to \infty$ in the inequality (16), by the continuity of Ψ and dominated convergence theorem, we deduce that

$$\phi(t) \leq \tau \int_0^t \Psi(\theta, \phi(\theta)) d\theta.$$

Hence, $\phi(t) \equiv 0$ for all $t \in I$ in view of **Hypothesis 4** (*H*4). Particularly, $\phi(b) = 0$. Consequently, we obtain

$$0 \leq \|u_m - u_n\|_{\mathcal{C}_b^p}^p \leq \phi_n(b) \to \phi(b) = 0.$$

Then $\{u_n\}_{n\geq 0}$ is a Cauchy sequence in \mathcal{C}_b^p . Since \mathcal{C}_b^p is complete, we put

$$u^*(t) := \lim_{n \to \infty} u_n(t), \quad \forall t \in I.$$

Then taking $n \to \infty$ in the second equality of (11), by the continuity of f, Σ and dominated convergence theorem, we can obtain

$$u^{*}(t) = C^{S}_{\alpha,\alpha-1}(t)\xi + C^{S}_{\alpha,\alpha}(t)\eta + \int_{0}^{t} C^{S}_{\alpha,\alpha}(t-\theta)f(\theta, u^{*}(\theta))d\theta + \int_{0}^{t} C^{S}_{\alpha,\alpha}(t-\theta)\Sigma(\theta, u^{*}(\theta))dW(\theta), \ \forall t \in I.$$

Therefore, the IVP (1) has a mild solution u^* belongs to C_b^p due to Definition 6.

Next, we prove the uniqueness. Let the IVP (1) have mild solutions u^* and v^* . By a similar method as above, we obtain

$$\left\|u^*-v^*\right\|_{\mathcal{C}^p_t}^p \leq \tau \int_0^t \Psi(\theta, \left\|u^*-v^*\right\|_{\mathcal{C}^p_\theta}^p) d\theta.$$

Hence $||u^* - v^*||_{\mathcal{C}^p_t}^p \equiv 0$ for all $t \in I$. Thus, $u^* \equiv v^*$ and the proof is completed. \Box

For the IVP (2), by (10), we define the sequence of stochastic process $\{v_n\}_{n\geq 1}$ by

$$\begin{cases} v_0(t) = C_{\alpha,1}^S(t)\xi + K_{\alpha,1}^S(t)\eta, & t \in I, \\ v_{n+1}(t) = C_{\alpha,1}^S(t)\xi + K_{\alpha,1}^S(t)\eta + Q_1(v_n)(t) + Q_2(v_n)(t), & t \in I, n \ge 0, \end{cases}$$

where

$$Q_1(v_n)(t) = \int_0^t H^S_{\alpha,1}(t-\theta)f(\theta, v_n(\theta))d\theta,$$
$$Q_2(v_n)(t) = \int_0^t H^S_{\alpha,1}(t-\theta)\Sigma(\theta, v_n(\theta))dW(\theta).$$

By utilizing similar techniques as in the proof of Lemma 2 and Theorem 1, the following conclusions are obtained.

Lemma 3. Let the pair (A, S) generate an $(\alpha, 1)$ -resolvent family $\{C_{\alpha,1}^{S}(t)\}_{t\geq 0}$ of type (M, ω) . If the **Hypothesis 1** (H1) and **Hypothesis 2** (H2) hold, the sequence $\{v_n\}_{n\geq 0}$ is well-defined. Moreover, there is a constant $\overline{C} > 0$ satisfying

$$\sup_{n\geq 0} \|v_n\|_{\mathcal{C}^p_b}^p \leq \overline{C}.$$

Theorem 2. Let the pair (A, S) generate an $(\alpha, 1)$ -resolvent family $\{C_{\alpha,1}^{S}(t)\}_{t\geq 0}$ of type (M, ω) . If the **Hypothesis 1** (H1)-**Hypothesis 4** (H4) hold, there is a unique mild solution of the IVP (2) on I.

Remark 4. In Theorems 1 and 2, we do not assume the compactness of fractional resolvent families $\{C_{\alpha,\alpha-1}^{S}(t)\}_{t\geq 0}$ and $\{C_{\alpha,1}^{S}(t)\}_{t\geq 0}$ as well as any compact conditions on f and Σ . Hence our results extend some results of [4,8].

Remark 5. In Theorems 1 and 2, we do not assume the existence, boundedness and compactness of S^{-1} , which are essential assumptions of [5,6]. So, the operator S in the IVP (1) and (2) may be unbounded. Therefore, our results improve the ones of [5,6].

Remark 6. *By employing the symmetrical technique of Theorem 1 and 2, we can study the fractional evolution systems in the following form*

$$\begin{bmatrix}
^{L}D_{t}^{\alpha}(Su(t)) = Au(t) + S[f(t, u(t)) + \Sigma(t, u(t))]\frac{dW(t)}{dt}], \quad t \in I, \\
^{S}(g_{2-\alpha} * u)(0) = S\xi, \quad [S(g_{2-\alpha} * u)]'(0) = S\eta$$
(17)

and

$${}^{C}D_{t}^{\alpha}(Su(t)) = Au(t) + S[f(t, u(t)) + \Sigma(t, u(t))\frac{dW(t)}{dt}], \quad t \in I,$$

$$Su(0) = S\xi, \quad (Su)'(0) = S\eta,$$
(18)

where $1 < \alpha < 2$, ${}^{L}D_{t}^{\alpha}$, $C D_{t}^{\alpha}$, A and S are defined as in the IVP (1) and (2), $\xi, \eta \in \mathcal{D}(S)$, f, Σ, W and $g_{2-\alpha}$ are appropriate functions.

In this case, similar to (6) and Definition 5, for any $\alpha > 0$, $\beta > 0$, let (A, S) generate an (α, β) -resolvent family $\{C^{S}_{\alpha,\beta}(t)\}_{t\geq 0}$ of type (M, ω) satisfying

$$\lambda^{\alpha-\beta}R(\lambda^{\alpha}S,A)Su = \int_0^{\infty} e^{-\lambda\theta}C^S_{\alpha,\beta}(\theta)ud\theta, \quad Re\lambda > \omega, \ u \in \mathcal{D}(S).$$

If the functions $f : I \times X \to \mathcal{D}(S) \subset X$ and $\Sigma : I \times X \to \mathcal{D}(S) \subset L_2^0$ satisfy the **Hypothesis 1** (H1)–**Hypothesis 4** (H4), the fractional evolution systems (17) and (18) have a unique mild solution on I, respectively.

4. Applications

Let $X := L^2[0, \pi]$. We consider the Caputo fractional partial differential equation with initial-boundary conditions of order $\alpha \in (1, 2)$

$$CD_{t}^{\alpha}[(I - \frac{\partial^{2}}{\partial s^{2}})u(t,s)] = \frac{\partial^{4}}{\partial s^{4}}u(t,s) + h(t,s,u(t,s)) + \sigma(t,s,u(t,s))\frac{dW(t)}{dt}, (t,s) \in [0,1] \times [0,\pi],$$

$$u(t,0) = u(t,\pi) = 0, \quad t \in [0,1],$$

$$[(I - \frac{\partial^{2}}{\partial s^{2}})u](0,s) = \xi(s), \quad s \in [0,\pi],$$

$$[(I - \frac{\partial^{2}}{\partial s^{2}})u]_{t}(0,s) = \eta(s), \quad s \in [0,\pi].$$
(19)

We define $A : \mathcal{D}(A) \subset X \to X$ and $S : \mathcal{D}(S) \subset X \to X$ by

$$Au := -\frac{\partial^4}{\partial s^4}u, \quad \forall u \in \mathcal{D}(A),$$
$$Su := (I - \frac{\partial^2}{\partial s^2})u, \quad \forall u \in \mathcal{D}(S),$$

where $\mathcal{D}(A) = \mathcal{D}(S) = \{u \in X : u \in W^{4,2}[0, \pi], \text{ and } u(0) = u(\pi) = 0\}$. From [4,5,7], we have

$$\begin{split} Au &= -\sum_{\ell=1}^{\infty} \ell^4 \langle u, e_{\ell} \rangle e_{\ell}, \quad u \in \mathcal{D}(A), \\ Su &= -\sum_{\ell=1}^{\infty} (1+\ell^2) \langle u, e_{\ell} \rangle e_{\ell}, \quad u \in \mathcal{D}(S), \end{split}$$

where $e_{\ell}(s) = \sqrt{\frac{2}{\pi}} \sin \ell s$, $\ell \in \mathbb{N}$ is the eigenvector of *A* associated with the eigenvalue $-\ell^4$. From [4], we obtain that (A, S) generates an $(\alpha, 1)$ -resolvent family $C^S_{\alpha, 1}(t)(t \ge 0)$ expressed by

$$C^{S}_{\alpha,1}(t)u = \sum_{n=1}^{\infty} \rho^{n}_{\alpha}(t) \langle u, e_{n} \rangle e_{n}, \quad \forall u \in X,$$

where

$$\rho_{\alpha}^{n}(t) = \sum_{n=1}^{\infty} \frac{(-1)^{k} n^{4k} t^{\alpha k}}{(1+n^{2})^{k} \Gamma(\alpha k+1)}$$

By [4], we know that $C_{\alpha,1}^{S}(t)(t \ge 0)$ is of type (2, 1).

Let u(t)(s) = u(t,s) and

$$f(t, u(t))(s) = h(t, s, u(t, s)),$$

$$\Sigma(t, u(t))(s) = \sigma(t, s, u(t, s)).$$

Then the problem (19) can be abstracted as the IVP (2). If we assume that the **Hypothesis 1** (H1)–**Hypothesis 4** (H4) hold, by Theorem 2, there is a unique mild solution of the IVP (19) on [0,1].

Analogously, we can investigate the Riemann-Liouville fractional partial differential equation with initial-boundary conditions of order $\alpha \in (1, 2)$

Similarly, (A, S) can generate an $(\alpha, \alpha - 1)$ -resolvent family $C^{S}_{\alpha,\alpha-1}(t)$ ($t \ge 0$), which is of type (2, 1). Under the same assumptions, we can obtain that there is a unique mild solution of the IVP(20) on [0, 1] due to Theorem 1.

5. Conclusions

In the present work, we investigate the existence as well as the uniqueness of mild solutions for the IVPs of Sobolev type fractional evolution equations involving $\alpha \in (1, 2)$ -order Riemann–Liouville or Caputo fractional derivatives. By using the stochastic analysis method, the Laplace transform and the fractional resolvent family, we first present the concept of mild solutions to the concerned problems.

Then the existence as well as the uniqueness theorems are proved by using an iteration technique of the Picard type. At the end of this paper, two examples are provided as applications of the abstract results.

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Nomenclature

The next list describes several symbols in the manuscript.

- *X*, *Y* real separable Hilbert spaces
- $\mathcal{B}(X, Y)$ the space of all bounded linear operators from X to Y
- $\mathcal{B}(X)$ the space of all bounded linear operators from *X* to *X*
- *m* the smallest integer which is bigger than or equal to any $\alpha > 0$
- $\Gamma(\cdot)$ the Gamma function
- \mathbb{R}^+ $[0, +\infty)$

 \mathbb{C} the complex number

References

- 1. Wang, J.R.; Zhou, Y.; Medved, M. On the solvability and optimal controls of fractional integrodifferential evolution systems with infinite delay. *J. Optim. Theory Appl.* **2012**, *152*, 31–50. [CrossRef]
- 2. Wang, J.R. Approximate mild solutions of fractional stochastic evolution equations in Hilbert spaces. *Appl. Math. Comput.* **2015**, 256, 315–323. [CrossRef]
- 3. Li, K.X.; Peng, J.G.; Jia, J.X. Cauchy problems for fractional differential equations with Riemann-Liouville fractional derivatives. *J. Funct. Anal.* **2012**, *263*, 476–510. [CrossRef]
- 4. Chang, Y.K.; Pei, Y.T.; Ponce, R. Existence and optimal controls for fractional stochastic evolution equations of Sobolev type via fractional resolvent operators. *J. Optim. Theory Appl.* **2019**, *182*, 558–572. [CrossRef]
- 5. Fečkan, M.; Wang, J.R.; Zhou, Y. Controllability of fractional functional evolution equations of Sobolev type via characteristic solution operators. *J. Optim. Theory Appl.* **2013**, 156, 79–95. [CrossRef]
- 6. Benchaabane, A.; Sakthivel, R. Sobolev-type fractional stochastic differential equations with non-Lipschitz coefficients. *J. Comput. Appl. Math.* **2017**, *312*, 65–73. [CrossRef]
- Lightbourne, J.; Rankin, S. A partial functional differential equations of Sobolev type. *J. Math. Anal. Appl.* 1983, 93, 328–337. [CrossRef]
- 8. Ponce, R. Existence of mild solutions to nonlocal fractional Cauchy problems via compactness. *Abstr. Appl. Anal.* **2016**, 2016. [CrossRef]
- 9. Zhou, Y. Fractional Evolution Equations and Inclusions: Analysis and Control; Elsevier: New York, NY, USA, 2016.
- 10. Bazhlekova, E. *Fractional Evolution Equations in Banach Spaces;* University Press Facilities, Eindhoven University of Technology: Eindhoven, The Netherlands, 2001.
- 11. Ichikawa, A. Stability of semilinear stochastic evolution equations. *J. Math. Anal. Appl.* **1982**, *90*, 12–44. [CrossRef]
- 12. Fan, Z.B. Characterization of compactness for resolvents and its applications. *Appl. Math. Comput.* **2014**, 232, 60–67. [CrossRef]
- 13. IMahmudov, N. Controllability of linear stochastic systems in Hilbert space. *J. Math. Anal. Appl.* **2003**, *288*, 197–211. [CrossRef]
- 14. Yang, H.; Agarwal, R.; Liang, Y. Controllability for a class of integro-differential evolution equations involving non-local initial conditions. *Int. J. Control* **2017**, *90*, 2567–2574. [CrossRef]



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