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Some Identities on the Poly-Genocchi Polynomials and Numbers

Dmitry V. Dolgy [1] and Lee-Chae Jang [2,*]

1 Kwangwoon Glocal Education Center, Kwangwoon University, Seoul 139-701, Korea; d_dol@mail.ru
2 Graduate School of Education, Konkuk University, Seoul 143-701, Korea
* Correspondence: Lcjang@konkuk.ac.kr

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Abstract: Recently, Kim-Kim (2019) introduced polyexponential and unipoly functions. By using these functions, they defined type 2 poly-Bernoulli and type 2 unipoly-Bernoulli polynomials and obtained some interesting properties of them. Motivated by the latter, in this paper, we construct the poly-Genocchi polynomials and derive various properties of them. Furthermore, we define unipoly Genocchi polynomials attached to an arithmetic function and investigate some identities of them.

Keywords: polylogarithm functions; poly-Genocchi polynomials; unipoly functions; unipoly Genocchi polynomials

MSC: 11B83; 11S80

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1. Introduction

The study of the generalized versions of Bernoulli and Euler polynomials and numbers was carried out in [1,2]. In recent years, various special polynomials and numbers regained the interest of mathematicians and quite a few results have been discovered. They include the Stirling numbers of the first and the second kind, central factorial numbers of the second kind, Bernoulli numbers of the second kind, Bernstein polynomials, Bell numbers and polynomials, central Bell numbers and polynomials, degenerate complete Bell polynomials and numbers, Cauchy numbers, and others (see [3–8] and the references therein). We mention that the study of a generalized version of the special polynomials and numbers can be done also for the transcendental functions like hypergeometric ones. For this, we let the reader refer to the papers [3,5,6,8,9]. The poly-Bernoulli numbers are defined by means of the polylogarithm functions and represent the usual Bernoulli numbers (more precisely, the values of Bernoulli polynomials at 1) when $k = 1$. At the same time, the degenerate poly-Bernoulli polynomials are defined by using the polyexponential functions (see [8]) and they are reduced to the degenerate Bernoulli polynomials if $k = 1$. The polyexponential functions were first studied by Hardy [10] and reconsidered by Kim [6,9,11,12] in view of an inverse to the polylogarithm functions which were studied by Zagier [13], Lewin [14], and Jaonquiѐre [15]. In 1997, Kaneko [16] introduced poly-Bernoulli numbers which are defined by the polylogaritm function.

Recently, Kim-Kim introduced polyexponential and unipoly functions [9]. By using these functions, they defined type 2 poly-Bernoulli and type 2 unipoly-Bernoulli polynomials and obtained several interesting properties of them.

In this paper, we consider poly-Genocchi polynomials which are derived from polyexponential functions. Similarly motivated, in the final section, we define unipoly Genocchi polynomials attached to an arithmetic function and investigate some identities for them. In addition, we give explicit expressions and identities involving those polynomials.

It is well known, the Bernoulli polynomials of order α are defined by their generating function as follows (see [1–3,17,18]):

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}, \quad . \tag{1}$$

We note that for $\alpha = 1$, $B_n(x) = B_n^{(1)}(x)$ are the ordinary Bernoulli polynomials. When $x = 0$, $B_n^{\alpha} = B_n^{\alpha}(0)$ are called the Bernoulli numbers of order α. The Genocchi polynomials $G_n(x)$ are defined by (see [19–24]).

$$\frac{2t}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \tag{2}$$

When $x = 0$, $G_n = G_n(0)$ are called the Genocchi numbers.

As is well-known, the Euler polynomials are defined by the generating function to be (see [1,4]).

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \tag{3}$$

For $n \geq 0$, the Stirling numbers of the first kind are defined by (see [5,7,25]),

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \tag{4}$$

where $(x)_0 = 1,\ (x)_n = x(x-1)\dots(x-n+1),\ (n \geq 1)$. From (4), it is easy to see that

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}. \tag{5}$$

In the inverse expression to (4), for $n \geq 0$, the Stirling numbers of the second kind are defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l. \tag{6}$$

From (6), it is easy to see that

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}. \tag{7}$$

2. The Poly-Genocchi Polynomials

For $k \in \mathbb{Z}$, by (2) and (14), we define the poly-Genocchi polynomials which are given by

$$\frac{2e_k(\log(1+t))}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{t^n}{n!}. \tag{8}$$

When $x = 0$, $G_n^{(k)} = G_n^{(k)}(0)$ are called the poly-Genocchi numbers. From (8), we see that

$$G_n^{(1)}(x) = G_n(x),\ (n \in \mathbb{N} \cup \{0\}) \tag{9}$$

are the ordinary Genocchi polynomials. From (2), (4) and (8) , we observe that

$$\begin{aligned}
&\sum_{n=0}^{\infty} G_n^{(k)} \frac{t^n}{n!} \\
&= \frac{2e_k(\log(1+t))}{e^t+1} \\
&= \frac{2}{e^t+1} \sum_{m=1}^{\infty} \frac{(\log(1+t))^m}{(m-1)!m^k} \\
&= \frac{2}{e^t+1} \sum_{m=0}^{\infty} \frac{(\log(1+t))^{m+1}}{m!(m+1)^k} \\
&= \frac{2}{e^t-1} \sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \sum_{l=m+1}^{\infty} S_1(l,m+1) \frac{t^l}{l!} \\
&= \frac{2t}{e^t+1} \sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \sum_{l=m}^{\infty} \frac{S_1(l+1,m+1)}{l+1} \frac{t^l}{l!} \\
&= \left(\sum_{j=0}^{\infty} G_j \frac{t^j}{j!} \right) \sum_{l=0}^{\infty} \left(\sum_{m=0}^{l} \frac{1}{(m+1)^{k-1}} \frac{S_1(l+1,m+1)}{l+1} \right) \frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \frac{1}{(m+1)^{k-1}} \frac{S_1(l+1,m+1)}{l+1} G_{n-l} \right) \frac{t^n}{n!}.
\end{aligned} \tag{10}$$

Therefore, by (10), we obtain the following theorem.

Theorem 1. *For $k \in \mathbb{Z}$ and $n \in \mathbb{N} \cup \{0\}$, we have*

$$G_n^{(k)} = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \frac{1}{(m+1)^{k-1}} \frac{S_1(l+1,m+1)}{l+1} G_{n-l}. \tag{11}$$

Corollary 1. *For $n \in \mathbb{N} \cup \{0\}$, we have*

$$G_n^{(1)} = G_n = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \frac{S_1(l+1,m+1)}{l+1} G_{n-l}. \tag{12}$$

Moreover,

$$\sum_{l=1}^{n} \sum_{m=0}^{l} \binom{n}{l} \frac{S_1(l+1,m+1)}{l+1} G_{n-l} = 0, \quad (n \in \mathbb{N}). \tag{13}$$

Kim-Kim ([9]) defined the polyexponential function by (see [6,9–12,26]).

$$e_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!n^k}, \tag{14}$$

In [18], it is well known that for $k \geq 2$,

$$\frac{d}{dx} e_k(x) = \frac{1}{x} e_{k-1}(x). \tag{15}$$

Thus, by (15), for $k \geq 2$, we get

$$e_k(x) = \int_0^x \frac{1}{t_1} \underbrace{\int_0^{t_1} \frac{1}{t_1} \cdots \int_0^{t_{k-2}}}_{(k-2)times} \frac{1}{t_{k-1}} (e^{t_{k-1}} - 1) d_{k-1} t dt_{k-1} \cdots dt_1. \tag{16}$$

From (16), we obtain the following equation.

$$\begin{aligned}
&\sum_{n=0}^{\infty} G_n^{(k)} \frac{x^n}{n!} \\
&= \frac{2}{e^x+1} e_k(\log(1+x)) \\
&= \frac{2}{e^x+1} \int_0^x \frac{1}{(1+t)\log(1+t)} e_{k-1}(\log(1+t))dt \\
&= \frac{2}{e^x+1} \int_0^x \frac{1}{(1+t_1)\log(1+t_1)} \\
&\quad \underbrace{\int_0^{t_1} \frac{1}{(1+t_2)\log(1+t_2)} \cdots \int_0^{t_{k-2}}}_{(k-2) times} \frac{t_{k-1}}{(1+t_{k-1})\log(1+t_{k-1})} dt_{k-1} dt_{k-2} \cdots dt_1, (k \geq 2).
\end{aligned} \tag{17}$$

Let us take $k = 2$. Then, by (2) and (16), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} G_n^{(2)} \frac{x^n}{n!} &= \frac{2}{e^x+1} \int_0^x \frac{t}{(1+t)\log(1+t)} dt \\
&= \frac{2}{e^x+1} \sum_{l=0}^{\infty} \frac{B_l^{(l)}}{l!} \int_0^x t^l dt \\
&= \frac{2}{e^x+1} \sum_{l=0}^{\infty} \frac{B_l^{(l)}}{l+1} \frac{x^{l+1}}{l!} \\
&= \frac{2x}{e^x+1} \sum_{l=0}^{\infty} \frac{B_l^{(l)}}{l+1} \frac{x^l}{l!} \\
&= \left(\sum_{m=0}^{\infty} G_m \frac{x^m}{m!} \right) \left(\sum_{l=0}^{\infty} \frac{B_l^{(l)}}{l+1} \frac{x^l}{l!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \frac{B_l^{(l)}}{l+1} G_{n-l} \right) \frac{x^n}{n!}.
\end{aligned} \tag{18}$$

Therefore, by (18), we obtain the following theorem.

Theorem 2. *Let $n \in \mathbb{N} \cup \{0\}$, we have*

$$G_n^{(2)} = \sum_{l=0}^{n} \binom{n}{l} \frac{B_l^{(l)}}{l+1} G_{n-l}. \tag{19}$$

From (3) and (16), we also get

$$\begin{aligned}
\sum_{n=0}^{\infty} G_n^{(2)} \frac{x^n}{n!} &= \frac{2}{e^x+1} \int_0^x \frac{t}{(1+t)\log(1+t)} dt \\
&= \frac{2}{e^x+1} \sum_{l=0}^{\infty} B_l^{(l)} \frac{x^{l+1}}{(l+1)!} \\
&= \frac{2}{e^x+1} \sum_{l=1}^{\infty} B_{l-1}^{(l-1)} \frac{x^l}{l!} \\
&= \left(\sum_{m=0}^{\infty} E_m \frac{x^m}{m!} \right) \left(\sum_{l=1}^{\infty} B_{l-1}^{(l-1)} \frac{x^l}{l!} \right) \\
&= \sum_{n=1}^{\infty} \left(\sum_{l=1}^{n} \binom{n}{l} B_{l-1}^{(l-1)} E_{n-l} \right) \frac{x^n}{n!}. \qquad (20)
\end{aligned}$$

Therefore, by (20), we obtain the following theorem.

Theorem 3. *Let $n \geq 1$, we have*

$$G_n^{(2)} = \sum_{l=1}^{n} \binom{n}{l} B_{l-1}^{(l-1)} E_{n-l}. \qquad (21)$$

From (8), we observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} G_n^{(k)}(x) \frac{t^n}{n!} &= \frac{2e_k(\log(1+t))}{e^t+1} e^{xt} \\
&= \left(\sum_{l=0}^{\infty} G_l^{(k)} \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} x^m \frac{t^m}{m!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} G_l^{(k)} x^{n-l} \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} G_{n-l}^{(k)} x^l \right) \frac{t^n}{n!}. \qquad (22)
\end{aligned}$$

From (22), we obtain the following theorem.

Theorem 4. *Let $n \in \mathbb{N}$, we have*

$$G_n^{(k)}(x) = \sum_{l=0}^{n} \binom{n}{l} G_{n-l}^{(k)} x^l. \qquad (23)$$

From (23), we observe that

$$\begin{aligned}
\frac{d}{dx}G_n^{(k)}(x) &= \sum_{l=1}^{n}\binom{n}{l}G_{n-l}^{(k)}lx^{l-1} \\
&= \sum_{l=0}^{n-1}\binom{n}{l+1}G_{n-l-1}^{(k)}(l+1)x^{l} \\
&= \sum_{l=0}^{n-1}\frac{n!}{(l+1)!(n-l-1)!}G_{n-1-l}^{(k)}(l+1)x^{l} \\
&= n\sum_{l=0}^{n-1}\frac{(n-1)!}{l!(n-1-l)!}G_{n-1-l}^{(k)}x^{l} \\
&= nG_{n-1}^{(k)}(x).
\end{aligned} \tag{24}$$

From (24), we obtain the following theorem.

Theorem 5. *Let $n \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{Z}$, we have*

$$\frac{d}{dx}G_n^{(k)}(x) = nG_{n-1}^{(k)}(x). \tag{25}$$

3. The Unipoly Genocchi Polynomials and Numbers

Let p be any arithmetic function which is real or complex valued function defined on the set of positive integers $\mathbb{N}$. Then, Kim-Kim ([9]) defined the unipoly function attached to polynomials by

$$u_k(x|p) = \sum_{n=1}^{\infty}\frac{p(n)x^n}{n^k}, \quad (k \in \mathbb{Z}). \tag{26}$$

It is well known that

$$u_k(x|1) = \sum_{n=1}^{\infty}\frac{x^n}{n^k} = Li_k(x) \tag{27}$$

is the ordinary polylogarithm function, and for $k \geq 2$,

$$\frac{d}{dx}u_k(x|p) = \frac{1}{x}u_{k-1}(x|p), \tag{28}$$

and

$$u_k(x|p) = \int_0^x \frac{1}{t}\underbrace{\int_0^t \frac{1}{t}\cdots\int_0^t \frac{1}{t}}_{(k-2)times}u_1(t|p)dtdt\cdots dt \tag{29}$$

By using (26), we define the unipoly Genocchi polynomials as follows:

$$\frac{2}{e^t+1}u_k(\log(1+t)|p)e^{xt} = \sum_{n=0}^{\infty}G_{n,p}^{(k)}(x)\frac{t^n}{n!}. \tag{30}$$

Let us take $p(n) = \frac{1}{(n-1)!}$. Then we have

$$\begin{aligned}\sum_{n=0}^{\infty} G_{n,p}^{(k)}(x)\frac{t^n}{n!} &= \frac{2}{e^t+1}u_k\left(\log(1+t)\,\middle|\,\frac{1}{(n-1)!}\right)e^{xt}\\ &= \frac{2}{e^t+1}\sum_{m=1}^{\infty}\frac{(\log(1+t))^m}{m^k(m-1)!}e^{xt}\\ &= \frac{2e_k(\log(1+t))}{e^t+1}e^{xt}\\ &= \sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{t^n}{n!}.\end{aligned} \tag{31}$$

Thus, by (31), we have the following theorem.

Theorem 6. *If we take $p(n) = \frac{1}{(n-1)!}$ for $n \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{Z}$, then we have*

$$G_{n,p}^{(k)}(x) = G_n^{(k)}(x). \tag{32}$$

From (4) and (30) with $x = 0$, we have

$$\begin{aligned}&\sum_{n=0}^{\infty} G_{n,p}^{(k)}\frac{t^n}{n!}\\ &= \frac{2}{e^t+1}\sum_{m=1}^{\infty}\frac{p(m)}{m^k}(\log(1+t))^m\\ &= \frac{2}{e^t+1}\sum_{m=0}^{\infty}\frac{p(m+1)(m+1)!}{(m+1)^k}\sum_{l=m+1}^{\infty}S_1(l,m+1)\frac{t^l}{l!}\\ &= \frac{2}{e^t+1}\sum_{m=0}^{\infty}\frac{p(m+1)(m+1)!}{(m+1)^k}\sum_{l=m}^{\infty}S_1(l+1,m+1)\frac{t^{l+1}}{(l+1)!}\\ &= \frac{2t}{e^t+1}\sum_{m=0}^{\infty}\frac{p(m+1)(m+1)!}{(m+1)^k}\sum_{l=m}^{\infty}S_1(l+1,m+1)\frac{t^l}{(l+1)!}\\ &= \left(\sum_{j=0}^{\infty}G_j\frac{t^j}{j!}\right)\sum_{l=0}^{\infty}\left(\sum_{m=0}^{l}\frac{p(m+1)(m+1)!}{(m+1)^k}\frac{S_1(l+1,m+1)}{l+1}\right)\frac{t^l}{l!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{p(m+1)(m+1)!}{(m+1)^k}\frac{S_1(l+1,m+1)}{l+1}G_{n-l}\right)\frac{t^n}{n!}.\end{aligned} \tag{33}$$

Therefore, by comparing the coefficients on both sides of (33), we obtain the following theorem.

Remark 1. *Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Then, we have*

$$G_{n,p}^{(k)} = \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{p(m+1)(m+1)!}{(m+1)^k}\frac{S_1(l+1,m+1)}{l+1}G_{n-l}. \tag{34}$$

In particular,

$$G_{n,\frac{1}{(n-1)!}}^{(k)} = \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{G_{n-l}}{(m+1)^{k-1}}\frac{S_1(l+1,m+1)}{l+1} \tag{35}$$

arrives at (11).

From (30), we easily obtain the following theorem.

Theorem 7. *Let $n \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{Z}$. Then, we have*

$$G_{n,p}^{(k)}(x) = \sum_{l=0}^{n} \binom{n}{l} G_{n-l,p}^{(k)} x^l. \tag{36}$$

From (36), we easily obtain the following theorem.

Theorem 8. *Let $n \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{Z}$. Then, we have*

$$\frac{d}{dx} G_{n,p}^{(k)}(x) = n G_{n-1,p}^{(k)}(x). \tag{37}$$

Finally, by (4) and (30), we observe that

$$\begin{aligned}
&\sum_{n=0}^{\infty} G_{n,p}^{(k)} \frac{t^n}{n!} \\
&= \frac{2}{e^t+1} \sum_{m=1}^{\infty} \frac{p(m)}{m^k} \frac{m!}{m!} (\log(1+t))^m \\
&= \frac{2}{e^t+1} \sum_{m=1}^{\infty} \frac{p(m+1)}{(m+1)^k} \frac{(m+1)!}{(m+1)!} (\log(1+t))^{m+1} \\
&= \sum_{j=0}^{\infty} Ej \frac{t^j}{j!} \sum_{m=0}^{\infty} \frac{p(m+1)(m+1)!}{(m+1)^k} \sum_{l=m+1}^{\infty} S_1(l, m+1) \frac{t^l}{l!} \\
&= \sum_{j=0}^{\infty} Ej \frac{t^j}{j!} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{p(m+1)(m+1)!}{(m+1)^k} \sum_{l=m}^{\infty} S_1(l+1, m+1) \frac{t^{l+1}}{(l+1)!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \frac{p(m+1)(m+1)!}{(m+1)^k} \frac{S_1(l+1, m+1)}{l+1} E_{n-l} \right) \frac{t^n}{n!}.
\end{aligned} \tag{38}$$

From (37) , we obtain the following theorem.

Theorem 9. *Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have*

$$G_{n,p}^{(k)} = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \frac{p(m+1)(m+1)!}{(m+1)^k} \frac{S_1(l+1, m+1)}{l+1} E_{n-l}. \tag{39}$$

4. Conclusions

In 2019, Kim-Kim considered the polyexponential functions and poly-Bernoulli polynomials. In the same view as these functions and polynomials, we defined the poly-Genocchi polynomials (Equation (8)) and obtained some identities (Theorem 1 and Corollary 1). In particular, we observed explicit poly-Genocchi numbers for $k = 2$ (Theorems 2, 3 and 4). Furthermore, by using the unipoly functions, we defined the unipoly Genocchi polynomials (Equation (30)) and obtained some their properties (Theorems 6 and 7). Finally, we obtained the derivative of the unipoly Genocchi polynomials (Theorem 8) and gave the identity indicating the relationship of unipoly Genocchi polynomials and Euler polynomials (Theorem 9). It is recommended that our readers look at references [27–31] if they want to know the applications related to this paper.

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